Hopf algebras, quantum groups and topological field theory

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Christoph Schweigert

Hamburg University

Department of Mathematics

Section Algebra and Number Theory

and

Center for Mathematical Physics

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The current version of these notes can be found under
`http://www.math.uni-hamburg.de/home/schweigert/skripten/hskript.pdf`
as a pdf file.

Please send comments and corrections to `christoph.schweigert@uni-hamburg.de`!

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1 Introduction

1.1 Braided vector spaces

Let us study the following ad hoc problem:

Definition 1.1.1
Let $\mathbb{K}$ be a field. A braided vector space is a $\mathbb{K}$-vector space V, together with an invertible $\mathbb{K}$-linear map

$$c : V \otimes V \to V \otimes V$$

which obeys the equation

$$(c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c)$$

in $\mathrm{End}(V \otimes V \otimes V)$.

Remark 1.1.2.
Let $(v_i)_{i \in I}$ be a $\mathbb{K}$-basis of V. This allows us to describe $c \in \mathrm{End}(V \otimes V)$ by a family $(c_{ij}^{kl})_{i,j,k,l \in I}$ of scalars:

$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l \, .$$

If c is invertible, then c describes a braided vector space, if and only if the following equation holds:

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn} \quad \text{for all} \quad l, m, n, i, j, k \in I \, .$$

This is a complicated set of non-linear equations, called the Yang-Baxter equation. In this lecture, we will see how to find solutions to this equation (and why this is an interesting problem at all).

Examples 1.1.3.

(i) For any $\mathbb{K}$-vector space V denote by

$$\begin{aligned} \tau_{V,V} : V \otimes V &\to V \otimes V \\ v_1 \otimes v_2 &\mapsto v_2 \otimes v_1 \end{aligned}$$

the map that switches the two copies of V. The pair (V, τ) is a braided vector space, since the following relation holds in the symmetric group S_3 for transpositions $\tau_{i,i+1}$ exchanging i and $i+1$:

$$\tau_{12}\tau_{23}\tau_{12} = \tau_{23}\tau_{12}\tau_{23} \, .$$

(ii) Let V be finite-dimensional with ordered basis $(e_1, \ldots, e_n)$. We choose some $q \in \mathbb{K}^\times$ and define $c \in \mathrm{End}(V \otimes V)$, by

$$c(e_i \otimes e_j) = \begin{cases} q \; e_i \otimes e_i & \text{if } \; i = j \\ e_j \otimes e_i & \text{if } \; i < j \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{if } \; i > j \, . \end{cases}$$

For $n = \dim_{\mathbb{K}} V = 2$, the vector space $V \otimes V$ has the basis $(e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2, e_2 \otimes e_1)$ which leads to the following matrix representation for c:

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix} .$$

The reader should check by direct calculation that the pair (V, c) is a braided vector space. Moreover, we have

$$(c - q\,\mathrm{id}_{V\otimes V})(c + q^{-1}\mathrm{id}_{V\otimes V}) = 0\,.$$

For $q = 1$, one recovers example (i). For this reason, example (ii) is called a one-parameter deformation of example (i).

1.2 Braid groups

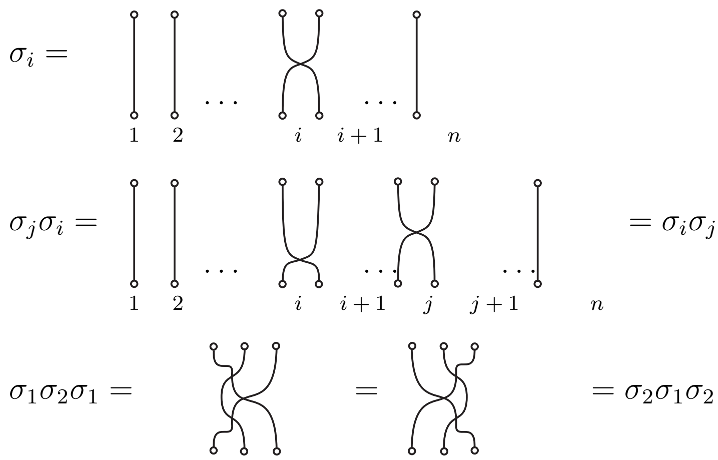
Definition 1.2.1
Fix an integer $n \geq 3$. The <u>braid group</u> B_n on n strands is the group with $n-1$ generators $\sigma_1 \ldots \sigma_{n-1}$ and relations

$$\begin{array}{rclll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for} & |i-j| > 1. \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for} & 1 \leq i \leq n-2 \end{array}$$

We define for $n = 2$ the braid group B_2 as the free group with one generator and we let $B_0 = B_1 = \{1\}$ be the trivial group.

Remarks 1.2.2.

(i) The following pictures explain the name braid group: the generators are depicted as



(ii) These pictures are made more precise by the following definition:
Definition 1.2.3

(i) A <u>braid</u> with n strands is a continuous embedding of n closed intervals $[0, 1]$ into $\mathbb{C} \times [0, 1]$ whose image L_f has the following properties:

(i) The boundary of L_f is the set $\{1, 2, \ldots n\} \times \{0, 1\}$

(ii) For any $s \in [0, 1]$, the intersection $L_f \cap (\mathbb{C} \times \{s\})$ contains precisely n different points.

(ii) Braids can be concatenated.

(iii) There is an equivalence relation on the set of braids, called <u>isotopy</u> such that the set of equivalence classes with a composition derived from the concatenation of braids is isomorphic to the braid group.

(iii) There is a canonical surjection from the braid group to the symmetric group:

$$\begin{array}{rcl} \pi : B_n & \to & S_n \\ \sigma_i & \mapsto & \tau_{i,i+1} \,. \end{array}$$

There is an important difference between the symmetric group S_n and the braid group B_n: in the symmetric group S_n the additional relation $\tau_{i,i+1}^2 = \mathrm{id}$ holds. (For a description of the symmetric group in terms of generators and relations, we refer e.g. to [JS, Example I.A.10 (4)].) In contrast to the symmetric group, the braid group is an infinite group without any non-trivial torsion elements, i.e. without elements of finite order. The kernel of the surjection π is called the pure braid group.

Let (V, c) be a braided vector space. For $1 \leq i \leq n-1$, define a linear automorphism of the n-th tensor power $V^{\otimes n}$ by

$$c_i := \begin{cases} c \otimes \mathrm{id}_{V^{\otimes(n-2)}} & \text{for} \quad i = 1 \\ \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} & \text{for} \quad 1 < i < n-1 \\ \mathrm{id}_{V^{\otimes(n-2)}} \otimes c & \text{for} \quad i = n-1 \,. \end{cases}$$

We deduce from the axioms of a braided vector space that this defines for any $n \in \mathbb{N}$ a linear representation of the braid group B_n on the vector space $V^{\otimes n}$:

Proposition 1.2.4.
Let (V, c) with $c \in$ Aut $(V \otimes V)$ be a braided vector space. We have then for any $n > 0$ a unique homomorphism of groups

$$\begin{array}{rcl} \rho_n^c : B_n & \to & \mathrm{Aut}\ (V^{\otimes n}) \\ \sigma_i & \mapsto & c_i \quad \text{for} \quad i = 1, 2, \ldots n-1 \,. \end{array}$$

Proof.
The relation $c_i c_j = c_j c_i$ for $|i - j| \geq 2$ holds, since the linear maps c_i and c_j act on different tensorands of the tensor product. The relation $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$ is part of the axioms of a braided vector space in definition 1.1.1. □

Let us explain one reason why the braid group is interesting: consider the subset $Y_n \subset \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ consisting of all n-tuples $(z_1, \ldots, z_n) \in \mathbb{C}^n$ of pairwise distinct points, i.e. such that

$$z_i \neq z_j \quad \text{for all pairs} \quad i \neq j \,.$$

The symmetric group S_n acts on Y_n by permutation of entries. The orbit space $X_n = Y_n/S_n$ is called the configuration space of n different points in the complex plane $\mathbb{C}$. Fix the point $p = (1, 2, \ldots n) \in Y_n$ and the quotient topology on X_n.

Theorem 1.2.5 (Artin[1]).
The fundamental group $\pi_1(X_n, p)$ of the configuration space X_n is isomorphic to the braid group B_n.

Proof.
We only give a group homomorphism

$$B_n \to \pi_1(X_n, p) \,.$$

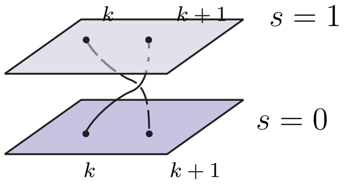
[1]Vienna 1989 - Hamburg 1962, Professor in Hamburg 1923-37 and 1958-62

We assign to the generator $\sigma_k \in B_n$ the continuous path in the configuration space X_n described by the map

$$f = (f_1, \ldots, f_n) : [0,1] \to \mathbb{C}^n$$

given by

$$\begin{aligned} f_j(s) &= j \qquad \text{for} \qquad j \neq k \;\text{ and }\; j \neq i+1 \\ f_k(s) &= \frac{1}{2}(2k+1-e^{\mathrm{i}\pi s}) \\ f_{k+1}(s) &= \frac{1}{2}(2k+1+e^{\mathrm{i}\pi s}) \end{aligned}$$



Since we identified points, this describes a closed path in the configuration space X_n. Denote the class of f in the fundamental group $\pi_1(X_n, p)$ by $\hat{\sigma}_k$. One verifies that the classes $\hat{\sigma}_k$ obey the relations of the braid group. Hence there is a unique homomorphism

$$B_n \to \pi_1(X, p)\,.$$

We omit in these lectures the proof that the homomorphism is even an isomorphism and refer e.g. to [GM, Section 3]. □

In physics, the braid group appears in the description of (quasi-)particles in low-dimensional quantum field theories. In this case, more general statistics than Bose or Fermi statistics is possible.

One of our goals is to present a general mathematical framework in which representations of the braid group can be produced. This framework will incidentally allow to describe a variety of physical phenomena:

- Universality classes of low-dimensional gapped systems.
- Candidates for implementations of quantum computing.
- Quantum groups also describe symmetries in a variety of integrable systems, including in particular sectors of Yang-Mills theories.

It also produces representation theoretic structures that arise in many fields of mathematics, ranging from algebraic topology to number theory. In particular, it is clear that when one closes a braid, one obtains a knot, hence there is a relation to knot theory.

2 Hopf algebras and their representation categories

2.1 Algebras and modules

Definition 2.1.1

1. *Let $\mathbb{K}$ be a field. A unital $\mathbb{K}$-algebra is a pair (A, μ) consisting of a $\mathbb{K}$-vector space A and a $\mathbb{K}$-linear map*
$$\mu : A \otimes A \to A$$
such that there is a $\mathbb{K}$-linear map
$$\eta : \mathbb{K} \to A ,$$
called the unit, such that
   (a) $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$ *(associativity)*
   (b) $\mu \circ (\eta \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$ *(unitality)*

   In the first identity, the identification $(A \otimes A) \otimes A \cong A \otimes (A \otimes A)$ of tensor products of vector spaces is tacitly understood. Similarly, in the second equation, we identify the tensor products $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$. We also write $a \cdot b := \mu(a, b)$.

2. *A morphism of algebras $(A, \mu, \eta) \to (A', \mu', \eta')$ is a $\mathbb{K}$-linear map*
$$\varphi : A \to A' ,$$
such that
$$\varphi \circ \mu = \mu' \circ (\varphi \otimes \varphi) \quad \text{and} \quad \varphi \circ \eta = \eta' .$$

3. *Consider again the flip map*
$$\begin{array}{rcl} \tau_{A,A} : A \otimes A & \to & A \otimes A \\ u \otimes v & \mapsto & v \otimes u \end{array}$$
The opposite algebra A^{opp} is the triple $(A, \mu^{\mathrm{opp}} = \mu \circ \tau_{A,A}, \eta)$. Thus $a \cdot_{\mathrm{opp}} b = b \cdot a$.

4. *An algebra is called commutative, if $\mu^{\mathrm{opp}} = \mu$ holds, i.e. if $a \cdot b = b \cdot a$ for all $a, b \in A$.*

Examples 2.1.2.

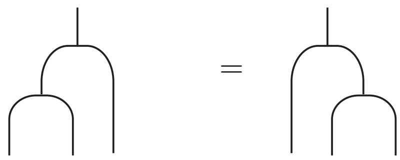
1. The unit η is unique, if it exists.

2. The ground field $\mathbb{K}$ itself is a commutative $\mathbb{K}$-algebra. The polynomial algebra $\mathbb{K}[X]$ is a commutative $\mathbb{K}$-algebra.

3. For any $\mathbb{K}$-vector space M, the vector space $\mathrm{End}_{\mathbb{K}}(M)$ of $\mathbb{K}$-linear endomorphisms of M is a $\mathbb{K}$-algebra. The product is composition of linear maps. For $\dim M > 1$, it is not commutative.

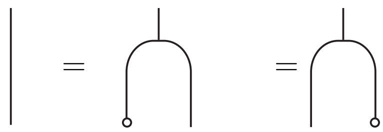
4. Let $\mathbb{K}$ be a field and G a group. Denote by $\mathbb{K}[G]$ the vector space freely generated by G. It has a basis labelled by elements of G which we denote by a slight abuse of notation by $(g)_{g \in G}$. The multiplication on basis elements $g \cdot h = gh$ is inherited from the multiplication of G. It is thus associative, and the neutral element $e \in G$ of the group G provides a unit for the group algebra $\mathbb{K}[G]$.

We introduce a graphical calculus in which associativity reads



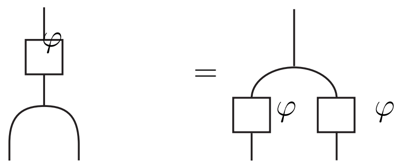
Our convention is to read such a diagram from below to above. Lines here represent the algebra A, trivalent vertices with two ingoing and one outgoing line the multiplication morphism μ. The diagram is progressive, i.e. lines are not allowed to "go back" downwards. The juxtaposition of lines represents the tensor product. We have identified again the tensor products $(A \otimes A) \otimes A \cong A \otimes (A \otimes A)$.

Similarly, we represent unitality by

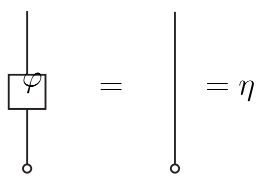


where we identified again the tensor products $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$. Invisible lines denote the ground field $\mathbb{K}$. Note that we have required that the unit element $1_A := \eta(1_{\mathbb{K}}) \in A$ is both a left and a right unit element. If it exists, such an element is unique.

A morphism φ of unital algebras obeys



and



Alternatively, we can characterize associativity by the following commutative diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"'] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

while unitality reads

$$\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d] & A \otimes A \arrow[d, "\mu"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

Examples 2.1.3.

1. We give another important example of a $\mathbb{K}$-algebra: let V be a $\mathbb{K}$-vector space. The tensor algebra over V is the associative unital $\mathbb{K}$-algebra

$$T(V) = \bigoplus_{r\geq 0} V^{\otimes r}.$$

with the tensor product as multiplication:

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_r) \cdot (w_1 \otimes \cdots \otimes w_t) := v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_t \,.$$

The tensor algebra is a $\mathbb{Z}_+$-graded algebra: with the homogeneous component $T^{(r)} := V^{\otimes r}$ we have

$$T^{(r)} \cdot T^{(s)} \subset T^{(r+s)} \,.$$

The tensor algebra is infinite-dimensional, even if V is finite-dimensional. In this case, obviously

$$\dim T^{(r)} = \dim V^{\otimes r} = (\dim V)^r \,.$$

On the homogenous subspace $V^{\otimes r}$, it carries an action of the symmetric group S_r.

2. Denote by $I_+(V)$ the two-sided ideal of $T(V)$ that is generated by all elements of the form $x \otimes y - y \otimes x$ with $x, y \in V$. The quotient

$$S(V) := T(V)/I_+(V)$$

with its natural algebra structure is called the symmetric algebra over V. Since the two-sided ideal $I_+(V)$ is a graded ideal, the symmetric algebra is a $\mathbb{Z}_+$-graded algebra, as well. It is infinite-dimensional, even if V is finite-dimensional. Note that in $S(V)$ the multiplciation is commutative.

3. Similarly, denote by $I_-(V)$ the graded two-sided ideal of $T(V)$ that is generated by all elements of the form $x \otimes x$ with $x \in V$. The quotient

$$\Lambda(V) := T(V)/I_-(V)$$

with its natural algebra structure is called the alternating algebra or exterior algebra over V. The alternating algebra is a $\mathbb{Z}_+$-graded algebra, as well. If V is finite-dimensional, $n := \dim V$, it is finite-dimensional. The dimension of the homogeneous component is

$$\dim \Lambda^r(V) = \begin{pmatrix} n \\ r \end{pmatrix}$$

The notion of a module is central for these lectures:

Definition 2.1.4
Let A be a (unital) $\mathbb{K}$ algebra. A left A-module is a pair (M, ρ), consisting of a $\mathbb{K}$-vector space M and a (unital) morphism of $\mathbb{K}$-algebras

$$\rho : A \to \mathrm{End}_{\mathbb{K}}(M) \,.$$

Remark 2.1.5.

1. We also write
$$a.m := \rho(a)\, m \quad \text{for all} \quad a \in A \text{ and } m \in M$$
and thus obtain a $\mathbb{K}$-linear map which by abuse of notation we also denote by ρ:
$$\begin{array}{rcl} \rho : A \otimes M & \to & M \\ a \otimes m & \mapsto & a.m \end{array}$$
such that for all $a, b \in A$ and $m, n \in M$ and $\lambda, \mu \in \mathbb{K}$ the following identities hold:
$$\begin{array}{rcl} a.(\lambda m + \mu n) & = & \lambda(a.m) + \mu(a.n) \\ (\lambda a + \mu b).m & = & \lambda(a.m) + \mu(b.m) \\ (a \cdot b).m & = & a.(b.m) \\ 1.m & = & m \end{array}$$
(The first two lines just express that ρ is $\mathbb{K}$-bilinear.) For the properties of this map, one can again use a graphical representation and write down the two commuting diagrams:
$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu\otimes\mathrm{id}_M"] \arrow[d, "\mathrm{id}\otimes\rho"'] & A \otimes M \arrow[d, "\rho"'] \\ A \otimes M \arrow[r, "\rho"'] & A \end{tikzcd}$$
while unitality reads
$$\begin{tikzcd} \mathbb{K} \otimes M \arrow[r, "\eta\otimes\mathrm{id}_M"] \arrow[d] & A \otimes M \arrow[d, "\rho"] \\ M \arrow[r, equal] & M \end{tikzcd}$$
2. A right A-module is a left A^{opp}-module (M, ρ) with $\rho : A^{\mathrm{opp}} \to \mathrm{End}(M)$. We write $m.a := \rho(a)m$ and find the relations:
$$\begin{array}{rcl} (\lambda m + \mu n).a & = & \lambda(m.a) + \mu(n.a) \\ m.(\lambda a + \mu b) & = & \lambda(m.a + \mu(m.b) \\ m.(a \cdot b) & = & (m.a).b \\ m.1 & = & m \end{array}$$
for all $a, b \in A$ and $\lambda, \mu \in K$ and $m, n \in M$. This explains the word "right module". This also becomes evident in the graphical notation.
3. Multiplication endows any algebra with the structure of a module over itself, $a.b := a \cdot b$. The corresponding module is called the left regular module. Similarly, a right regular module can be defined. Notice that for a general $\mathbb{K}$-algebra, the ground field $\mathbb{K}$ cannot be necessarily endowed with the structure of an A-module.
4. To give a module
$$\rho : \mathbb{K}[G] \to \mathrm{End}(M)$$
over a group algebra $\mathbb{K}[G]$, it is sufficient to specify the algebra morphism ρ on the distinguished basis $(g)_{g \in G}$ of $\mathbb{K}[G]$. This amounts to giving a group homomorphism into the group of invertible $\mathbb{K}$-linear endomorphisms:
$$\rho_G : G \to \mathrm{GL}(M) := \{\varphi \in \mathrm{End}_{\mathbb{K}}(M), \varphi \text{ invertible }\}\,.$$
The pair (M, ρ_G) is called a representation of the group G.

Remarks 2.1.6.

1. Any $\mathbb{K}$-vector space V carries a representation of its automorphism group $\mathrm{GL}(V)$ by $\rho = \mathrm{id}_{\mathrm{GL}(V)}$. This representation is called the defining representation of the general linear group $\mathrm{GL}(V)$.

2. Any vector space M becomes a representation of any group G by the trivial operation $\rho(g) = \mathrm{id}_M$ for all $g \in G$.

3. To specify a representation (M, ρ) of the free abelian group $\mathbb{Z}$ amounts to specifying an automorphism $A \in \mathrm{GL}(M)$, namely $A = \rho(1)$. Then $\rho(n) = A^n$ for all $n \in \mathbb{Z}$.

4. To specify a module of the polynomial algebra $\mathbb{K}[X]$ amounts to specifying a $\mathbb{K}$-vector space M and an endomorphism $\varphi : M \to M$. By the universal property of the polynomial algebra, this uniquely specifies a morphism of algebras $\mathbb{K}[X] \to \mathrm{End}(M)$ and thus a representaton of $\mathbb{K}[X]$.

5. A representation of the cyclic group $\mathbb{Z}/2\mathbb{Z}$ on a $\mathbb{K}$-vector space V amounts to an automorphism $A : V \to V$ such that $A^2 = \mathrm{id}_V$.

   If $\mathrm{char}\mathbb{K} \neq 2$, V is the direct sum of eigenspaces of A to the eigenvalues ± 1,
   $$V = V^+ \oplus V^- ,$$
   since any vector $v \in V$ can be decomposed as
   $$v = \frac{1}{2}(v + Av) + \frac{1}{2}(v - Av) \ .$$
   Since
   $$A\frac{1}{2}(v \pm Av) = \frac{1}{2}(Av \pm A^2 v) = \pm\frac{1}{2}(v \pm Av) \quad ,$$
   these are eigenvectors of A to the eigenvalues ± 1. This decomposition into eigenspaces is unique.

   If $\mathrm{char}\mathbb{K} = 2$, the only possible eigenvalue is $+1$. Because of $A^2 = \mathrm{id}_V$, the minimal polynomial of A has to divide $X^2 - 1 = (X - 1)^2$. It has to be a power of the prime polynomial $X - 1$ so that a Jordan block decomposition exists. The Jordan blocks of the automorphism A have size 1 or 2. Indeed, we find for a Jordan block of size 2:
   $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Definition 2.1.7

Let A be a $\mathbb{K}$-algebra and M, N be A-modules. A $\mathbb{K}$-linear map $\varphi : M \to N$ is called a morphism of A-modules or, equivalently, an A-linear map, if
$$\varphi(a.m) = a.\varphi(m) \quad \textit{for all} \quad m \in M, a \in A \ .$$

As a diagram, this reads
$$\begin{tikzcd} A \otimes M \arrow[r, "\mathrm{id}_A \otimes \varphi"] \arrow[d, "\rho_M"'] & A \otimes N \arrow[d, "\rho_N"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd} .$$

If A is a group algebra, A-linear maps are also called intertwiners of G-representations.

One goal of this lecture is to obtain insights on representations of groups and to generalize them to a class of algebraic structures beyond groups. To this end, it is convenient to have more terminology available to talk about *all* modules over a given algebra A at once: they form a category.

Definition 2.1.8

1. *A category $\mathcal{C}$ consists*

   (a) of a class of objects $\mathrm{Obj}(\mathcal{C})$*, whose entries are called the objects of the category.*

   (b) a class $\mathrm{Hom}(\mathcal{C})$*, whose entries are called morphisms of the category*

   (c) Maps

$$\begin{aligned} \mathrm{id} &: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}\ (\mathcal{C}) \\ s, t &: \mathrm{Hom}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}) \\ o &: \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}\ (\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}) \end{aligned}$$

   such that

   (a) $s(\mathrm{id}_V) = t(\mathrm{id}_V) = V$ *for all* $V \in \mathrm{Obj}(\mathcal{C})$

   (b) $\mathrm{id}_{t(f)} \circ f = f \circ \mathrm{id}_{s(f)} = f$ *for all* $f \in \mathrm{Hom}(\mathcal{C})$

   (c) for all $f, g, h \in \mathrm{Hom}(\mathcal{C})$ *with* $t(f) = s(g)$ *and* $t(g) = s(h)$ *the associativity identity* $(h \circ g) \circ f = h \circ (g \circ f)$ *holds.*

2. *We write for* $V, W \in \mathrm{Obj}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}}(V, W) = \{f \in \mathrm{Hom}(\mathcal{C}) \mid s(f) = V, t(f) = W\}$$

   and $\mathrm{End}_{\mathcal{C}}(V)$ *for* $\mathrm{Hom}_{\mathcal{C}}(V, V)$*. For any pair* V, W*, we require* $\mathrm{Hom}_{\mathcal{C}}(V, W)$ *to be a set. Elements of* $\mathrm{End}_{\mathcal{C}}(V)$ *are called endomorphisms of* V*.*

3. *A morphism* $f \in \mathrm{Hom}(V, W)$ *which we also write* $V \xrightarrow{f} W$ *or in the form* $f : V \to W$ *is called an isomorphism, if there exists a morphism* $g : W \to V$*, such that*

$$g \circ f = \mathrm{id}_V \quad \text{and } f \circ g = \mathrm{id}_W \ .$$

   Two objects V, W *of a category are called isomorphic, if there is an isomorphism* $V \to W$*. Being isomorphic is an equivalence relation; the equivalence classes of the category* $\mathcal{C}$ *are denoted by* $\pi_0(\mathcal{C})$*.*

Remarks 2.1.9.

1. Never require two objects of a category to be *equal* - rather require them to be *isomorphic*. The isomorphism is then an interesting piece of data. For example, any finite-dimensional vector space is isomorphic to its dual vector space, but there is no distinguished such isomorphism (for example, one has to chose a basis to exhibit such an isomorphism).

   As a more subtle example, consider finite-dimensional representations of the compact Lie group $\mathrm{SU}(n)$. We should not ask whether the defining n-dimensional complex representation equals it dual, but rather whether it is isomorphic to its dual; then we can ask refined questions about the isomorphism, leading e.g. to the distinction of real and pseudoreal representations.

2. We explain why in the definition of a category we talk about *sets* and *classes*: for applying category in practice one would like to have a notion of a "category of all sets" and, for constructing interesting categories, for a given a property $\varphi(x)$ of a set x, also a category "$\{x \mid \varphi(x)\}$" of all sets having the property φ. Famously, this leads to contradictions, such as the one of the category of all sets that are not elements of themselves.

   A solution to this problem is to restrict the application of φ to be allowed only for sets that are elements in some specific set $\mathfrak{U}$ (where it is supposed that the notion of a set is defined, e.g. by working with Zermelo-Fraenkel axioms.) Further, such a set $\mathfrak{U}$ must be sufficiently nice – technically speaking, it must be a *universe* (for details see [McL, Sect. I.6]). All mathematical constructions are then carried out inside the universe $\mathfrak{U}$. A set that is an element of $\mathfrak{U}$ is called *small* (relative to $\mathfrak{U}$). It should be appreciated that, with this definition, sets that are small in terms of cardinality are not necessarily $\mathfrak{U}$-small; for example, the one-element set $\{\mathfrak{U}\}$ is *not* $\mathfrak{U}$-small. Functions between small sets relative to $\mathfrak{U}$ can be constructed inside $\mathfrak{U}$. This yields for each universe $\mathfrak{U}$ a category of $\mathfrak{U}$-small sets.

   A category $\mathcal{C}$ is now called $\mathfrak{U}$-*small* if the set $\mathrm{Obj}(\mathcal{C})$ of objects is in $\mathfrak{U}$. The category of $\mathfrak{U}$-small categories is *not* $\mathfrak{U}$-small, because this would imply $\mathfrak{U} \in \mathfrak{U}$, thus violating the axioms of a universe. A *class* C (relative to a universe $\mathfrak{U}$) can then be defined as an arbitrary subset $C \subseteq \mathfrak{U}$. It follows that every $\mathfrak{U}$-small set is a $\mathfrak{U}$-class, but the converse is not true. Using classes, we can now talk about the category of $\mathfrak{U}$-small categories.

   The choice of $\mathfrak{U}$ is usually supressed in the notation. It is common to enlarge the axioms of set theory by requiring that for any set X there is a universe $\mathfrak{U}$ such that $X \in \mathfrak{U}$, which in particular ensures the existence of universes.

3. Let $\mathbb{K}$ be a field. A <u>$\mathbb{K}$-linear category</u> is a category for which all hom-sets have the additional structure of a $\mathbb{K}$-vector space and for which the composition operation has the property of being $\mathbb{K}$-bilinear.

Examples 2.1.10.

1. Any set X can be endowed with a trivial structure of a category $\underline{X}$ in which the only morphisms are the identity morphisms. This category is called the discrete category.

2. The category $\mathrm{Cob}_{1,0}$ has as objects sets of finitely many oriented points and as morphisms arrows (or, rather, oriented one-dimensional manifolds up to diffeomorphism). This category (or rather its higher-dimensional analogues) is central for topological field theory. They contain much information on the collection of all manifolds.

3. Vector spaces over a field $\mathbb{K}$, together with linear maps, form a category $\mathrm{vect}(\mathbb{K})$. It is a particular feature of this category that its Hom-sets are not only sets, but $\mathbb{K}$-vector spaces, and that composition is $\mathbb{K}$-bilinear. Hence, the category of $\mathbb{K}$-vector spaces is $\mathbb{K}$-linear. We say that the category $\mathrm{vect}(\mathbb{K})$ is <u>enriched</u> over the category $\mathrm{vect}(\mathbb{K})$.

4. More generally, left modules over a ring R form a category R-mod. Complex representations of a given group G, together with intertwiners, form an $\mathbb{R}$-linear or a $\mathbb{C}$-linear category, respectively.

5. Consider a category with a single object $*$; this category is completely described by the set $\mathrm{End}(*)$ which has the structure of an (associative, unital) monoid. The category is called the delooping of the monoid.

6. A category in which all morphisms are isomorphisms is a called a groupoid. A groupoid with single object $*$ is completely described by the monoid $G := \mathrm{End}(*)$ which is a group. We write $*//G$ for this groupoid.

   More generally, we can consider for any associative unital $\mathbb{K}$-algebra A the category $*//A$ with a single object and morphisms given by A. This category is $\mathbb{K}$-linear.

7. An important example of a groupoid is the fundamental groupoid $\Pi_1(M)$ of a topological space M: its objects are the points of the space M, a morphism from $p \in M$ to $q \in M$ is a homotopy class of paths from p to q. For this groupoid $\mathrm{End}(p) =: \pi_1(M,p)$ is the fundamental group for the base point $p \in M$. The isomorphism classes of $\Pi_1(M)$ are the path-connected components of M. See [B06] for a textbook on topology that uses fundamental groupoids.

8. Let G be a group and X a set, together with an action

$$\begin{array}{rrcl} \rho: & G \times X & \to & X \\ & (g,x) & \mapsto & g.x \end{array}$$

   of G on X, i.e. $(gh).x = g.(h.x)$. Define a category, the action groupoid, $X//G$ whose objects are elements $x \in X$ and which has a morphism $x \to g.x$ for every pair $(g,x) \in G \times X$. (We use the somewhat counterintuitive notation $X//G$ for a *left* action.) The isomorphism classes of objects are the G-orbits, thus $\pi_0(X//G) = X/G$ with X/G the orbit set, the set-theoretic quotient.

9. The category Man has as objects smooth finite-dimensional manifolds and as morphisms smooth maps of manifolds. All manifolds in this lecture will be smooth manifolds.

For the next observation, we need the following notion:

Proposition 2.1.11.
Let (A, μ_A, η_A) and (B, μ_B, η_B) be unital associative $\mathbb{K}$-algebras. Then the tensor product $A \otimes B$ has a natural structure of an associative unital algebra determined by

$$(a \otimes b) \cdot (a' \otimes b') := aa' \otimes b \cdot b' \quad \text{for all} \quad a, a' \in A, b, b' \in B$$

and $\eta_{A\otimes B} := \eta_A \otimes \eta_B$.

Put differently, the multiplication $\mu_{A\otimes B}$ is the map

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B \ ,$$

with τ the flip map $\tau : a \otimes b \mapsto b \otimes a$ from Example 1.1.3 (i).

Observation 2.1.12.
The category of modules over a group algebra has more structure than just the structure of a $\mathbb{K}$-linear category:

- Let V, W be $\mathbb{K}[G]$-modules. Then the ground field $\mathbb{K}$, the tensor product $V \otimes_{\mathbb{K}} W$ and the dual vector space $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ can be turned into $\mathbb{K}[G]$-modules as well by

$$\begin{array}{rcll} g.1 & := & 1 & \text{for all } g \in G \\ g.(v \otimes w) & := & g.v \otimes g.w & \text{for all } g \in G, v \in V \text{ and } w \in W \\ (g.\phi)(v) & := & \phi(g^{-1}.v) & \text{for all } g \in G, v \in V \text{ and } \phi \in V^* \ . \end{array}$$

  (In physics, the representation on the ground field $\mathbb{K}$ is used to describe invariant states, and the representation on $V \otimes W$ corresponds to "coupling systems" for symmetries leading to multiplicative quantum numbers.)

- We want to encode this information in additional algebraic structure on the group algebra $\mathbb{K}[G]$. To this end, we note the following simple fact:

  Let $\varphi : A \to A'$ be a morphism of $\mathbb{K}$-algebras and M an A'-module described by $\rho' : A' \to \mathrm{End}(M)$. Then

  $$A \xrightarrow{\varphi} A' \xrightarrow{\rho'} \mathrm{End}(M)$$

  is an A-module, denoted by $\varphi^* M$. The A-action on M is

  $$a.m := \varphi(a).m \quad \text{for all} \quad a \in A, m \in M \ .$$

  The operation is called <u>restriction of scalars</u>, even if A is not a subalgebra of A'. One also calls the A-module $\varphi^* M$ the <u>pullback</u> of M along the algebra morphism φ.

- Now suppose that (M, ρ) and (M', ρ') are two A'-modules and $M \xrightarrow{f} M'$ is a morphism of A'-modules. Then the linear map f is also a morphism $\varphi^* M \to \varphi^* M'$ of A-modules which we denote by $\varphi^* f$.

- In the case of the tensor product $V \otimes W$, we naturally get a morphism of algebras

  $$\begin{array}{lcl} \mathbb{K}[G] \otimes \mathbb{K}[G] & \xrightarrow{\rho_V \otimes \rho_W} & \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W) \\ g_1 \otimes g_2 & \mapsto & \rho_V(g_1) \otimes \rho_W(g_2) \end{array} .$$

  If we identify the algebras $\mathbb{K}[G] \otimes \mathbb{K}[G] \cong \mathbb{K}[G \times G]$, we get a representation of the group $G \times G$, but not of the group G itself. The remedy is to take the additional datum of the morphism of algebras

  $$\begin{array}{rcl} \Delta : \mathbb{K}[G] & \to & \mathbb{K}[G] \otimes \mathbb{K}[G] \\ g & \mapsto & g \otimes g \qquad \text{for all} \ \ g \in G \ . \end{array}$$

  The $\mathbb{K}[G]$-module structure on $V \otimes W$ is then obtained by pullback

  $$\begin{array}{lclcl} \mathbb{K}[G] & \xrightarrow{\Delta} & \mathbb{K}[G] \otimes \mathbb{K}[G] & \xrightarrow{\rho_V \otimes \rho_W} & \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W) \\ g & \mapsto & g \otimes g & \mapsto & \rho_V(g) \otimes \rho_W(g) \end{array} .$$

  We thus get the $\mathbb{K}[G]$-module structure on $V \otimes W$ as the pullback along Δ of the natural $\mathbb{K}[G] \otimes \mathbb{K}[G]$-module structure on $V \otimes W$.

  For the case of the ground field, consider the algebra morphism

  $$\begin{array}{rcl} \epsilon : \mathbb{K}[G] & \to & \mathbb{K} \\ g & \mapsto & 1 \quad \text{for all} \ \ g \in G \end{array}$$

  The $\mathbb{K}[G]$-module structure on $\mathbb{K}$ is then obtained from

  $$\mathbb{K}[G] \xrightarrow{\epsilon} \mathbb{K} \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}) \ .$$

  Finally, for the dual vector space, consider the algebra morphism

  $$\begin{array}{rcl} S : \mathbb{K}[G] & \to & \mathbb{K}[G]^{\mathrm{opp}} \\ g & \mapsto & g^{-1} \quad \text{for all} \ \ g \in G \end{array}$$

  The $\mathbb{K}[G]$-module structure on V^* is then obtained via the transpose from

  $$\begin{array}{lclcl} \mathbb{K}[G] & \xrightarrow{S} & \mathbb{K}[G]^{\mathrm{opp}} & \xrightarrow{\rho^t} & \mathrm{End}(V^*) \\ g & \mapsto & g^{-1} & \mapsto & (\varphi \mapsto \varphi \circ \rho_V(g^{-1})) \ . \end{array}$$

The same type of algebraic structure is present on another class of associative algebras. To this end, we first introduce Lie algebras:

Definition 2.1.13

1. *A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space, $\mathfrak{g}$ together with a bilinear map, called the Lie bracket,*
$$\begin{array}{rccc} [-,-]: & \mathfrak{g}\otimes\mathfrak{g} & \to & \mathfrak{g} \\ & x\otimes y & \mapsto & [x,y] \end{array}$$
which is alternating, i.e. $[x,x]=0$ for all $x\in\mathfrak{g}$, and for which the Jacobi identity
$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$
holds for all $x,y,z\in\mathfrak{g}$.

2. *A morphism of Lie algebras $\varphi:\mathfrak{g}\to\mathfrak{g}'$ is a $\mathbb{K}$-linear map which preserves the Lie bracket,*
$$\varphi([x,y])=[\varphi(x),\varphi(y)] \quad \textit{for all} \quad x,y\in\mathfrak{g} \ .$$

3. *Given a Lie algebra $\mathfrak{g}$, we define the opposed Lie algebra $\mathfrak{g}^{\mathrm{opp}}$ as the Lie algebra with the same underlying vector space and Lie bracket*
$$[x,y]_{\mathrm{opp}}:=-[x,y]=[y,x] \quad \textit{for all} \quad x,y\in\mathfrak{g} \ .$$

Examples 2.1.14.

1. For any $\mathbb{K}$-vector space V, the vector space $\mathrm{End}_{\mathbb{K}}(V)$ is endowed with the structure of a Lie algebra by the commutator
$$[f,g]:=f\circ g-g\circ f \ .$$
We denote this Lie algebra by $\mathrm{gl}(V)$.

2. More generally, any associative $\mathbb{K}$-algebra A inherits a structure of a Lie algebra by using the commutator:
$$[a,b]:=a\cdot b-b\cdot a \quad \text{for all } a,b\in A \ .$$
The reader should check that the associativity of the product of A implies that Jacobi identity for the commutator.

3. Let V be a finite-dimensional $\mathbb{K}$-vector space. The subspace $\mathrm{sl}(V)$ of endomorphisms with vanishing trace is a Lie subalgebra of $\mathrm{gl}(V)$.

4. Consider the algebra $\mathrm{End}_{\mathbb{K}}(A)$ of $\mathbb{K}$-linear endomorphisms of a $\mathbb{K}$-algebra A. A linear endomorphism $\varphi:A\to A$ is called a derivation, if it obeys the Leibniz rule:
$$\varphi(a\cdot b)=\varphi(a)\cdot b+a\cdot\varphi(b) \quad \text{for all } a,b\in A \ .$$
Denote by $\mathrm{Der}(A)\subset\mathrm{End}_{\mathbb{K}}(A)$ the subspace of derivations. It is a Lie subalgebra of $\mathrm{End}_{\mathbb{K}}(A)$:
$$\begin{aligned} [\varphi,\psi](a\cdot b) &= \varphi(a\psi(b)+\psi(a)b)-\psi(\varphi(a)b+a\varphi(b)) \\ &= \varphi\psi(a)b+a\varphi\psi(b)-\psi\varphi(a)b-a\psi\varphi(b) \\ &= [\varphi,\psi](a)\cdot b+a\cdot[\varphi,\psi](b) \end{aligned}$$

5. Examples of Lie algebras are abundant. In particular, the smooth vector fields on a smooth manifold form a Lie algebra.

Remarks 2.1.15.

- To any Lie algebra $\mathfrak{g}$, one can associate a unital associative algebra, the <u>universal enveloping algebra</u>. It is constructed as a quotient of the tensor algebra

$$T(\mathfrak{g}) := \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$$

by the two-sided ideal $I(\mathfrak{g})$ that is generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \quad \text{with } x, y \in \mathfrak{g}$$

i.e.

$$\mathrm{U}(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}) \, .$$

Since the ideal $I(\mathfrak{g})$ is not homogeneous, we only have a filtration: define $\mathrm{U}^r(\mathfrak{g})$ as the image of

$$\mathrm{U}^r(\mathfrak{g}) := \pi(\oplus_{i=0}^r T^i(\mathfrak{g})) \subset \mathrm{U}(\mathfrak{g}) \ .$$

Then we have an increasing series of subspaces

$$\mathbb{K} \subset \mathrm{U}^1(\mathfrak{g}) \subset \mathrm{U}^2(\mathfrak{g}) \subset \ldots \subset \mathrm{U}^r(\mathfrak{g}) \subset \mathrm{U}^{r+1}(\mathfrak{g}) \subset \ldots$$

with $\cup_{i=1}^\infty \mathrm{U}^i(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})$ which is is compatible with the multiplication:

$$\mathrm{U}^r(\mathfrak{g}) \cdot \mathrm{U}^s(\mathfrak{g}) \subset \mathrm{U}^{r+s}(\mathfrak{g}) \ .$$

- As an example, take V to be any vector space. It is turned into a Lie algebra by $[v, w] = 0$ for all $v, w \in V$. Such a Lie algebra is called <u>abelian</u>. In this case, the universal enveloping algebra is just the symmetric algebra $S(V)$ which is not only filtered, but even graded.

- If the Lie algebra $\mathfrak{g}$ has a totally ordered basis (x_i), the <u>Poincaré-Birkhoff-Witt theorem</u> gives a $\mathbb{K}$-basis of $\mathrm{U}(\mathfrak{g})$.

  Consider the map

$$\iota_\mathfrak{g} : \mathfrak{g} \to T(\mathfrak{g}) \xrightarrow{\pi} T(\mathfrak{g})/I(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})$$

  which is a morphism of Lie algebras. Then the $\mathbb{K}$-basis of $\mathrm{U}(\mathfrak{g})$ consists of the elements $\iota(x_{i_1}) \cdot \iota(x_{i_2}) \ldots \iota(x_{i_k})$ with $k = 0, 1, \ldots$ and $i_1 \leq i_2 \leq \ldots$.

  In particular, the elements $(\iota(x_i))$ generate $\mathrm{U}(\mathfrak{g})$ as an associative algebra. As a consequence of the Poincaré-Birkhoff-Witt theorem, [2] the map $\iota_\mathfrak{g} : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is an *injective* map of Lie algebras.

- For later purposes, we note that for two $\mathbb{K}$-Lie algebras $\mathfrak{g}, \mathfrak{h}$, we have

$$\mathrm{U}(\mathfrak{g} \oplus \mathfrak{h}) \cong \mathrm{U}(\mathfrak{g}) \otimes_\mathbb{K} \mathrm{U}(\mathfrak{h}) \ .$$

[2] Alsen 1911 - Hamburg 1991, Professor in Hamburg 1938-1991

- The universal enveloping algebras $\mathrm{U}(\mathfrak{g})$ has the following universal property: for any associative $\mathbb{K}$-algebra A and any $\mathbb{K}$-linear map

  $$\varphi : \mathfrak{g} \to A\,,$$

  that is a morphism of Lie algebras,

  $$\varphi([x,y]) = [\varphi(x), \varphi(y)] \quad \text{for all } x, y \in \mathfrak{g}$$

  with the Lie algebra structure on A from example 2.1.14.2, there is a *unique* morphism of associative algebras $\widetilde{\varphi} : \mathrm{U}(\mathfrak{g}) \to A$ such that the diagram

  $$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\iota_{\mathfrak{g}}"] \arrow[dr, "\varphi"'] & & \mathrm{U}(\mathfrak{g}) \arrow[dl, "\exists!\tilde{\varphi}"] \\ & A & \end{tikzcd}$$

  of morphisms of Lie algebras commutes. Explicitly, we have

  $$\tilde{\varphi}(x_1 \cdot \cdots \cdot x_n) = \varphi(x_1) \cdots \varphi(x_n)\ .$$

  The diagram implies that any morphism $\varphi : \mathfrak{g} \to A$ of Lie algebras can be uniquely extended to a morphism $\tilde{\varphi} : \mathrm{U}(\mathfrak{g}) \to A$ of associative algebras. As a consequence, it is possible to construct algebra morphisms out of the universal enveloping $\mathrm{U}(\mathfrak{g})$ algebra into an associative algebra by giving a morphism $\mathfrak{g} \to A$ of Lie algebras.

  For example, the linear map underlying the morphism $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ of Lie algebras can also be seen as a morphism of Lie algebra $\mathfrak{g}^{\mathrm{opp}} \to \mathrm{U}(\mathfrak{g})^{\mathrm{opp}}$, where on the codomain we take the opposed algebra structure. It extends to a map of algebras $\mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) \to \mathrm{U}(\mathfrak{g})^{\mathrm{opp}}$ which can be shown to be an isomorphism using the Poincaré-Birkhoff-Witt theorem.

Lie algebras have representations as well:

Definition 2.1.16
Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. A <u>representation</u> *of $\mathfrak{g}$ is a pair (M, ρ), consisting of a $\mathbb{K}$-vector space M and morphism of Lie algebras*

$$\rho : \mathfrak{g} \to \mathrm{gl}(M)\ .$$

Remark 2.1.17.
We also write

$$x.m := \rho(x)\, m \quad \text{for all} \quad x \in \mathfrak{g} \text{ and } m \in M$$

and thus obtain a $\mathbb{K}$-linear map

$$\begin{array}{rcl} \mathfrak{g} \otimes M & \to & M \\ x \otimes m & \mapsto & x.m \end{array}$$

such that for all $x, y \in \mathfrak{g}$ and $m, n \in M$ the following identities hold:

$$\begin{array}{rcl} x.(\lambda m + \mu n) & = & \lambda(x.m) + \mu(x.n) \\ (\lambda x + \mu y).m & = & (\lambda x.m) + (\mu x.m) \\ ([x,y]).m & = & x.(y.m) - y.(x.m)\ . \end{array}$$

Again, the first two lines express that we have a $\mathbb{K}$-bilinear map.

Definition 2.1.18
Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and let M, N be representations of $\mathfrak{g}$. A $\mathbb{K}$-linear map $\varphi : M \to N$ is called a <u>*morphism*</u> *of representations of $\mathfrak{g}$, if*

$$\varphi(x.m) = x.\varphi(m) \quad \text{for all} \quad m \in M \;\text{ and }\; x \in \mathfrak{g} \ .$$

This defines the category $\mathfrak{g}-\mathrm{rep}$ *of representations of* $\mathfrak{g}$.

Using the universal property of the universal enveloping algebra, every representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(M)$ of a Lie algebra $\mathfrak{g}$ extends uniquely to a representation $\tilde{\rho} : \mathrm{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{K}}(M)$ of the universal enveloping algebra:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\iota_{\mathfrak{g}}"] \arrow[dr, "\rho"'] & & \mathrm{U}(\mathfrak{g}) \arrow[dl, dashed, "\exists! \tilde{\rho}"] \\ & \mathrm{End}_{\mathbb{K}}(M) & \end{tikzcd}$$

We have thus proven:

Proposition 2.1.19.
There is a canonical bijection between representations of the Lie algebra $\mathfrak{g}$ and modules over its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. One can show that morphisms of representations of $\mathfrak{g}$ are in bijection to $\mathrm{U}(\mathfrak{g})$-module morphisms.

These bijections are, however, not an appropriate language to compare the categories $\mathrm{U}(\mathfrak{g})-\mathrm{mod}$ and $\mathfrak{g}-\mathrm{rep}$ which are bilayered structures consisting of objects and morphisms, the intertwiners.

Definition 2.1.20
Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A <u>*functor*</u> *$F : \mathcal{C} \to \mathcal{C}'$ consists of two maps:*

$$\begin{array}{lll} F & : & \mathrm{Obj}(\mathcal{C}) \;\to \mathrm{Obj}(\mathcal{C}') \\ F & : & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}'), \end{array}$$

which obey the following conditions:

(a) $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$ *for all objects* $V \in \mathrm{Obj}(\mathcal{C})$

(b) $s(F(f)) = Fs(f)$ *and* $t(F(f)) = Ft(f)$ *for all morphisms* $f \in \mathrm{Hom}\ (\mathcal{C})$

(c) For any pair f, g of composable morphisms, we have

$$F(g \circ f) = F(g) \circ F(f) \ .$$

Two functors

$$\begin{array}{lllll} F & : & \mathcal{C} & \to & \mathcal{C}' \\ G & : & \mathcal{C}' & \to & \mathcal{C}'' \end{array}$$

can be concatenated to a functor $G \circ F : \mathcal{C} \to \mathcal{C}''$.

We have already encountered examples of functors:

Examples 2.1.21.

1. A functor $*//G \to \mathrm{vect}(\mathbb{K})$ assigns to the single object $*$ a $\mathbb{K}$-vector space M and to any group element $g \in G$ an endomorphism $\rho(g)$ of M. Since functors preserve composition, the map ρ defines a representation of the group G. Thus $\mathbb{K}$-linear representations of G are just functors $*//G \to \mathrm{vect}(\mathbb{K})$.

2. Associating to a vector space V its dual space provides a functor

$$\begin{array}{rcl} \mathrm{vect}(\mathbb{K}) & \to & \mathrm{vect}(\mathbb{K})^{\mathrm{opp}} \\ V & \mapsto & V^* \ . \end{array}$$

   Here we have introduced the opposed category $\mathcal{C}^{\mathrm{opp}}$ of a category $\mathcal{C}$. It has the same objects as $\mathcal{C}$, but $\mathrm{Hom}^{\mathrm{opp}}(U,V) := \mathrm{Hom}(V,U)$. The composition is defined in a compatible way. It thus implements the idea of "reversing arrows".

   The bidual provides a functor

$$\begin{array}{rcl} \mathrm{Bi} : \mathrm{vect}(\mathbb{K}) & \to & \mathrm{vect}(\mathbb{K}) \\ V & \mapsto & V^{**} \ . \end{array}$$

3. Let $\varphi : A \to A'$ be a morphism of algebras. As in observation 2.1.12, we consider for any A'-module M, $\rho : A' \to \mathrm{End}(M)$ the A-module $\varphi^* M$ that is defined on the same $\mathbb{K}$-vector space M. The $\mathbb{K}$-linear map underlying a morphism $f :: M \to M'$ of A'-modules is also a morphism of modules $\varphi^* M \to \varphi^* M'$. We thus obtain a functor

$$\varphi^* = \mathrm{Res}_A^{A'} \ : A'\mathrm{-mod} \to A\mathrm{-mod}$$

   that is called, by abuse of language, a <u>restriction functor</u> or <u>pullback functor</u>.

4. We have learned that any associative algebra is endowed, by the commutator, with the structure of a Lie algebra. This provides a functor

$$\mathrm{Alg}_{\mathbb{K}} \to \mathrm{Lie}_{\mathbb{K}} \ .$$

5. The universal enveloping algebra provides a functor from the category of Lie algebras to the category of associative algebras,

$$\begin{array}{rcl} \mathrm{U} : \mathrm{Lie}_{\mathbb{K}} & \to & \mathrm{Alg}_{\mathbb{K}} \\ \mathfrak{g} & \mapsto & \mathrm{U}(\mathfrak{g}) \ . \end{array}$$

6. In proposition 2.1.19, we have constructed a functor $\mathfrak{g}\mathrm{-rep} \to \mathrm{U}(\mathfrak{g})\mathrm{-mod}$.

It is important to compare two functors $F, G : \mathcal{C} \to \mathcal{C}'$ between the same categories. We give two motivations:

- We have seen in example 2.1.21.1 that for G a group, a functor $F_\rho : *//G \to \mathrm{vect}(\mathbb{K})$ corresponds to a $\mathbb{K}$-linear representation of the group G. From definition 2.1.7 we know that there are intertwiners between different representations. Given two functors $F_\rho, F_{\rho'} : *//G \to \mathrm{vect}(\mathbb{K})$, we thus need the analogue of an intertwiner.

- To get an idea on how to relate functors, we remark that any vector space V can be embedded into its bidual vector space. This means that for every V there is a linear map

$$\begin{array}{rcl} \iota_V : \mathrm{id}(V) = V & \to & V^{**} = \mathrm{Bi}(V) \\ v & \mapsto & (\beta \mapsto \beta(v)) \end{array}$$

  that relates the two functors $\mathrm{id}, \mathrm{Bi} : \mathrm{vect}(\mathbb{K}) \to \mathrm{vect}(\mathbb{K})$.

We formalize this as follows:

Definition 2.1.22

1. *Let $F, G : \mathcal{C} \to \mathcal{C}'$ be functors. A* <u>*natural transformation*</u>
$$\eta : F \to G$$
is a family of morphisms
$$\eta_V : F(V) \to G(V)$$
in $\mathcal{C}'$, indexed by objects $V \in \mathrm{Obj}(\mathcal{C})$ in the source category such that for any morphism
$$f : V \to W$$
in the source category $\mathcal{C}$ the diagram in $\mathcal{C}'$
$$\begin{tikzcd} F(V) \arrow[r, "\eta_V"] \arrow[d, "F(f)"'] & G(V) \arrow[d, "G(f)"] \\ F(W) \arrow[r, "\eta_W"] & G(W) \end{tikzcd}$$
commutes.

2. *If for each object $V \in \mathrm{Obj}(\mathcal{C})$ the morphism $\eta_V : F(V) \to G(V)$ is an isomorphism in $\mathcal{C}'$, then $\eta : F \to G$ is called a* <u>*natural isomorphism*</u>*.*

3. *A functor*
$$F : \mathcal{C} \to \mathcal{D}$$
is called an <u>*equivalence of categories*</u>*, if there is a functor*
$$G : \mathcal{D} \to \mathcal{C}$$
and natural isomorphisms
$$\begin{aligned} \eta :&\ \mathrm{id}_{\mathcal{D}} \to FG \\ \theta :&\ GF \to \mathrm{id}_{\mathcal{C}}\ . \end{aligned}$$

Remarks 2.1.23.

1. Let G be a finite group, $\mathbb{K}$ a field and consider two functors $F_\rho, F_{\rho'} : *//G \to \mathrm{vect}(\mathbb{K})$. A natural transformation $\eta : F_\rho \to F_{\rho'}$ is a $\mathbb{K}$-linear map $\eta_* : F_\rho(*) \to F_{\rho'}(*)$ which by the commuting diagram in 2.1.22.1 is an intertwiner of G-representations.

2. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors. Two natural transformations $\eta : F \to G$ and $\eta' : G \to H$ can be composed. Indeed, consider for $V \in \mathcal{C}$ the morphism:
$$(\eta' \circ \eta)_V : \quad F(V) \stackrel{\eta_V}{\to} G(V) \stackrel{\eta'_V}{\to} H(V)\ .$$
Since for any morphism $V \stackrel{f}{\to} W$ in $\mathcal{C}$ the two squares
$$\begin{tikzcd} F(V) \arrow[r, "\eta_V"] \arrow[d, "F(f)"'] & G(V) \arrow[r, "\eta'_V"] \arrow[d, "G(f)"'] & H(V) \arrow[d, "H(f)"'] \\ F(W) \arrow[r, "\eta_W"] & G(W) \arrow[r, "\eta'_W"] & H(W) \end{tikzcd}$$
commute, also the outer square commutes so that $(\eta' \circ \eta)_V : F \to H$ defines a natural transformation. The composition of natural transformations is associative and has the identity natural transformation $\eta : F \to F$ with $\eta_V = \mathrm{id}_V$ as a unit.

3. If the class $\mathrm{Obj}(\mathcal{C})$ is a set, then there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ whose objects are functors $F, G : \mathcal{C} \to \mathcal{C}$ and whose morphisms natural transformations $\eta : F \to G$.

The following lemma is useful to find equivalences of categories:

Lemma 2.1.24.
A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, if and only if

(a) The functor F is essentially surjective, i.e. for any $W \in \mathrm{Obj}(\mathcal{D})$ there is $V \in \mathrm{Obj}(\mathcal{C})$ such that $F(V) \cong W$ in $\mathcal{D}$.

(b) The functor F is fully faithful: for any pair V, V' of objects in $\mathcal{C}$, the map

$$F : \mathrm{Hom}_{\mathcal{C}}(V, V') \to \ \mathrm{Hom}_{\mathcal{D}}(F(V), F(V'))$$

on Hom-spaces is bijective.

Proof: see [Kassel, p. 278] and the exercises. The proof uses the axiom of choice twice.
An example for an equivalence of categories is the functor $\mathfrak{g}$-rep $\to$ $\mathrm{U}(\mathfrak{g})$-mod constructed in proposition 2.1.19.

We finally present some structure on universal enveloping algebras that should be compared to the structure found in observation 2.1.12 for group algebras. As a further consequence of the universal property of the enveloping algebra $\mathrm{U}(\mathfrak{g})$, we get from maps of Lie algebras maps of unital associative algebras:

$$\begin{array}{rcllrcl}
\mathfrak{g} & \to & \mathbb{K} & \text{gives} & \epsilon : \mathrm{U}(\mathfrak{g}) & \to & \mathbb{K} \\
x & \mapsto & 0 & & & & \\
\mathfrak{g} & \to & \mathfrak{g} \oplus \mathfrak{g} \subset \mathrm{U}(\mathfrak{g} \oplus \mathfrak{g}) & \text{gives} & \Delta : \mathrm{U}(\mathfrak{g}) & \to & \mathrm{U}(\mathfrak{g} \oplus \mathfrak{g}) \cong \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \\
x & \mapsto & (x, x) & & & & \\
\mathfrak{g} & \to & \mathfrak{g}^{\mathrm{opp}} \subset \mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) & \text{gives} & S : \mathrm{U}(\mathfrak{g}) & \to & \mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) \cong \mathrm{U}(\mathfrak{g})^{\mathrm{opp}} \\
x & \mapsto & -x & & & &
\end{array}$$

These morphisms of algebras are explicitly given by the following expressions on the generators $x \in \mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$

$$\epsilon(x) = 0$$
$$\Delta(x) = 1 \otimes x + x \otimes 1$$
$$S(x) = -x$$

These maps allow us to endow tensor products of representations of $\mathfrak{g}$, the dual of a vector space underlying a representation of $\mathfrak{g}$ and the ground field $\mathbb{K}$ with the structure of $\mathfrak{g}$-representations.

Observation 2.1.25.

- Let V, W be representations of $\mathfrak{g}$. The $\mathrm{U}(\mathfrak{g})$-module structure on the tensor product $V \otimes W$ is then obtained from

$$\mathrm{U}(\mathfrak{g}) \xrightarrow{\Delta} \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \xrightarrow{\rho_V \otimes \rho_W} \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W) \ .$$

The $\mathrm{U}(\mathfrak{g})$-module structure is uniquely determined by the condition

$$(*) \qquad x.(v \otimes w) = x.v \otimes w + v \otimes x.w \quad \text{for all } v \in V, w \in W \text{ and } x \in \mathfrak{g} \ .$$

- The $\mathrm{U}(\mathfrak{g})$-module structure on the ground field $\mathbb{K}$ is obtained from the unital algebra morphism
$$\mathrm{U}(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K} \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}) \ .$$
This is uniquely determined by the condition $x.v = 0$ for all $x \in \mathfrak{g}$ and $v \in \mathbb{K}$.

- The $\mathrm{U}(\mathfrak{g})$-module structure on V^* is then obtained via the transpose from
$$\mathrm{U}(\mathfrak{g}) \xrightarrow{S} \mathrm{U}(\mathfrak{g})^{\mathrm{opp}} \xrightarrow{\rho^t} \mathrm{End}(V^*) \ .$$

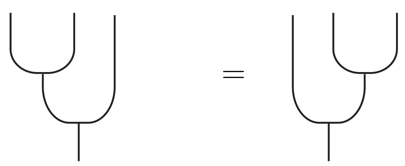
- Again, in physics, the representation on $\mathbb{K}$ is used to introduce the notion of an invariant state, and the representation on $V \otimes W$ corresponds to the "coupling of two systems" for symmetries leading by condition $(*)$ to additive quantum numbers.

2.2 Coalgebras and comodules

The maps (Δ, ϵ) in the two examples of a group algebra $\mathbb{K}[G]$ of a group G and the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ have properties that are best understood by reversing arrows in the definition of an algebra.

Definition 2.2.1

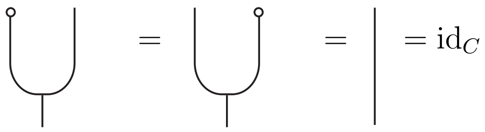
1. *A coassociative coalgebra over a field $\mathbb{K}$ is a pair (C, Δ), consisting of a $\mathbb{K}$-vector space C and a $\mathbb{K}$-linear map*
$$\Delta : \quad C \to C \otimes C \ ,$$
called the coproduct, such that the coassociativity condition $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$ holds. As a picture, we have



In terms of commuting diagrams, we have
$$\begin{tikzcd} C \otimes C \otimes C & & C \otimes C \arrow[ll, "\Delta\otimes\mathrm{id}_C"'] \\ C \otimes C \arrow[u, "\mathrm{id}_C\otimes\Delta"] & & C \arrow[ll, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

2. *A coassociative coalgebra is called counital, if there is a $\mathbb{K}$-linear map*
$$\epsilon : C \to \mathbb{K} \ ,$$

called the <u>*counit*</u>*, such that* $(\epsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C$ *holds. As a picture, we have*



In terms of commuting diagrams, we have

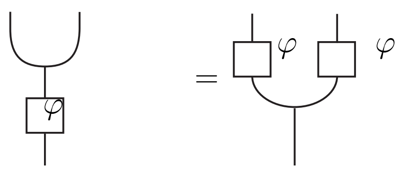
$$\begin{tikzcd} \mathbb{K} \otimes C & C \otimes C \arrow[l, "\epsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\epsilon"] & C \otimes \mathbb{K} \\ C \arrow[u] \arrow[r, equal] & C \arrow[u, "\Delta"] \arrow[r, equal] & C \arrow[u] \end{tikzcd}$$

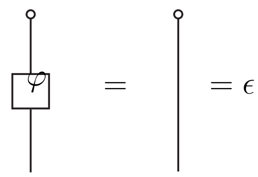
3. *Given a coalgebra* (C, Δ, ϵ)*, the* <u>*coopposed coalgebra*</u> C^{copp} *is given by* $(C, \Delta^{\mathrm{copp}} := \tau_{C,C} \circ \Delta, \epsilon)$.
   A coalgebra is called <u>*cocommutative*</u> *, if the identity* $\Delta^{\mathrm{copp}} = \Delta$ *holds. Here* τ *is again the map flipping the two tensor factors.*

4. *A* <u>*coalgebra map*</u> *is a linear map* $\varphi : C \to C'$*, such that the equation*

$$\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$$

holds. It is called counital, if also the equation $\epsilon' \circ \varphi = \epsilon$ *holds. Pictorially,*





Examples 2.2.2.

1. Let S be any set and $C = \mathbb{K}[S]$ the free $\mathbb{K}$-vector space with distinguished basis S. Then C becomes a coassociative counital coalgebra with coproduct given on the distinguished basis of $\mathbb{K}[S]$ by the diagonal map $\Delta(s) = s \otimes s$ and $\epsilon(s) = 1$ for all $s \in S$. It is cocommutative. It should be noted that for a general element $v \in \mathbb{K}[S]$, we do not have $\Delta(v) = v \otimes v$ and that the distinguished basis enters explicitly. Hence, it is somewhat misleading to call Δ a diagonal.

2. In particular, the group algebra $\mathbb{K}[G]$ for any group G with the maps Δ, ϵ discussed in observation 2.1.12 is a cocommutative coalgebra.

3. The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of any Lie algebra with the maps Δ, ϵ discussed before observation 2.1.25 will be shown to be a coalgebra which is cocommutative. (This is easier to do once we have stated compatibility conditions between product and coproduct.)

Remarks 2.2.3.

1. The counit ϵ is uniquely determined, if it exists.

2. The following notation is due to Heyneman and Sweedler and frequently called Sweedler notation: let (C, Δ, ϵ) be a coalgebra. For any $x \in C$, we can find finitely many elements $x_i' \in C$ and $x_i'' \in C$ such that
$$\Delta(x) = \sum_i x_i' \otimes x_i'' \ .$$
Dropping the summation indices, this is written as
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \ .$$
It is common to even omit the sum and write
$$\Delta(x) = x_{(1)} \otimes x_{(2)} \ .$$
In this notation, counitality reads
$$\epsilon(x_{(1)})x_{(2)} = x_{(1)}\epsilon(x_{(2)}) = x \quad \text{for all } x \in C,$$
and cocommutativity
$$x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)} \quad \text{for all } x \in C \ .$$
Finally, coassociativity reads
$$(x_{(1)})_{(1)} \otimes (x_{(1)})_{(2)} \otimes x_{(2)} = x_{(1)} \otimes (x_{(2)})_{(1)} \otimes (x_{(2)})_{(2)} \, .$$
For the sake of a compact notation, we denote this element also by $x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

Lemma 2.2.4.

1. If C is a coalgebra, then the dual vector space C^* is an algebra, with multiplication from $m = \Delta^*|_{C^* \otimes C^*}$ and unit $\eta = \epsilon^*$.

   Explicitly,
$$m(f \otimes g)(c) = \Delta^*(f \otimes g)(c) = (f \otimes g)\Delta(c) \in \mathbb{K} \quad \text{for all} \quad f, g \in C^* \text{ and } c \in C \ .$$

2. If the coalgebra C is cocommutative, then the algebra C^* is commutative.

Proof.
This is shown by dualizing diagrams, together with one additional observation: the dual of the copoduct $\Delta : C \to C \otimes C$ is a map $(C \otimes C)^* \to C^*$. Using the canonical injection $C^* \otimes C^* \subset (C \otimes C)^*$, we can restrict Δ^* to the subspace $C^* \otimes C^*$ to get the multiplication on C^*. Details will be in an exercise. □

Remarks 2.2.5.

1. Let S be a set. The algebra $\mathbb{K}[S]^*$ dual to the coalgebra $\mathbb{K}[S]$ in example 2.2.2 has the product
$$\varphi \cdot \varphi'(s) = \varphi \otimes \varphi'(\Delta(s)) = \varphi \otimes \varphi'(s \otimes s) = \varphi(s)\varphi'(s) \ ,$$
which shows that $\mathbb{K}[S]^*$ is the algebra of functions on S.

2. Warning: algebras cannot be simply dualized to coalgebras: the dual of the multiplication is a map $m^* : A^* \to (A \otimes A)^*$, but we need a map $A^* \to A^* \otimes A^*$. If A is finite-dimensional, we have $A^* \otimes A^* = (A \otimes A)^*$ and A^* is a coalgebra. In general, $A^* \otimes A^*$ is a proper subspace, $A^* \otimes A^* \subsetneq (A \otimes A)^*$.

3. For this reason, we denote by A^o the <u>finite dual</u> of A:

$$A^o := \{f \in A^* \,|\, f(I) = 0 \quad \text{for some ideal } I \subset A \text{ of finite codimension, } \dim A/I < \infty\} \ .$$

If A is an algebra, then the finite dual A^o can be shown to be a coalgebra, with coproduct $\Delta = m^*$ and counit η^*. If A is commutative, then A^o is cocommutative.

We dualize the notion of an ideal to get coalgebra structures on certain quotients:

Definition 2.2.6
Let C be a coalgebra.

1. *A subspace $I \subset C$ is a <u>left coideal</u>, if $\Delta I \subset C \otimes I$.*

2. *A subspace $I \subset C$ is a <u>right coideal</u>, if $\Delta I \subset I \otimes C$.*

3. *A subspace $I \subset C$ is a <u>two-sided coideal</u>, if*

$$\Delta I \subset I \otimes C + C \otimes I \qquad \text{and } \epsilon(I) = 0 \ .$$

Any two-sided ideal of an algebra is, in particular, a left ideal and a right ideal. For coideals of a coalgebra, however, an left or right coideal is a two-sided coideal, provided $\epsilon(I) = 0$ holds. It is easy to check that a subspace $I \subset C$ is a two-sided coideal, if and only if C/I is a coalgebra with comultiplication induced by Δ.

This raises the question what the algebraic structure on the quotient of C/I with I a left or right ideal is. To this end, we also dualize the notion of a module:

Definition 2.2.7
Let $\mathbb{K}$ be a field and (C, Δ, ϵ) be a $\mathbb{K}$-coalgebra.

1. *A <u>right C-comodule</u> is a pair (M, Δ_M), consisting of a $\mathbb{K}$-vector space M and a $\mathbb{K}$-linear map*

$$\Delta_M : M \to M \otimes C \ ,$$

called the <u>coaction</u> such that the two diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\Delta_M"'] & M \otimes C \arrow[d, "\Delta_M \otimes \mathrm{id}_C"] \\ M \otimes C \arrow[r, "\mathrm{id}_M \otimes \Delta"'] & M \otimes C \otimes C \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\cong"] & M \otimes C \arrow[d, "\mathrm{id}_M \otimes \epsilon"] \\ M \otimes \mathbb{K} \arrow[r, equal] & M \otimes \mathbb{K} \ . \end{tikzcd}$$

2. *A $\mathbb{K}$-linear map $\varphi : M \to N$ between right C-comodules M, N is said to be a* <u>*comodule map*</u>*, if the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\Delta_M"'] & N \arrow[d, "\Delta_N"] \\ M \otimes C \arrow[r, "\varphi\otimes\mathrm{id}_C"'] & N \otimes C \end{tikzcd} .$$

3. *We denote the category of right C-comodules by* comod-C.

4. *Left comodules and morphisms of left comodules are defined analogously. They form a category, denoted by* $C-$comod.

Again, the reader should draw pictures in a graphical notation.

Examples 2.2.8.

1. A left coideal I of a coalgebra is a subspace that is also, by restriction of the coproduct of C a left comodule. Similarly, a right coideal $I \subset C$ is a subspace that is, by restriction of the coproduct of C a right comodule.

2. Let C be a coalgebra. A subspace $I \subset C$ is a left coideal, if and only if C/I with the natural map
$$\overline{\Delta} : C/I \to C \otimes C/I$$
inherited from the coproduct of C is a left comodule. There is an analogous statement for right coideals. A subspace $I \subset C$ is a two-sided ideal, if and only if the quotient C/I with the inherited map
$$\overline{\Delta} : C/I \to C/I \otimes C/I$$
is a coalgebra. All statements will be exercises.

3. Let C be a coalgebra and M be a right C-comodule with coaction
$$\Delta_M(m) = \sum m_0 \otimes m_1 \quad \text{with } m_0 \in M \text{ and } m_1 \in C \ .$$
Here we have adapted Sweedler notation to comodules. The coassociativity of the coaction is then encoded in the notion
$$(\mathrm{id}_M \otimes \Delta_C) \circ \Delta_M(m) = (\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M(m) = m_0 \otimes m_1 \otimes m_2$$
with $m_0 \in M$ and $m_1, m_2 \in C$. By lemma 2.2.4, then C^* is an algebra and M is a left C^*-module, where the action of $f \in C^*$ is defined by
$$f.m = \sum \langle f, m_1 \rangle m_0 \ ,$$
where $\langle f, m_1 \rangle$ denotes the evaluation of $f \in C^*$ on $m_1 \in C$. Warning: in this way, we do not get all C^*-modules, but only the so-called *rational* C^*-modules.

4. Let S be a set and $C := \mathbb{K}[S]$ the coalgebra described in example 2.2.2.1. Then a $\mathbb{K}$-vector space M has the structure of a C-comodule, if and only if it is S-graded, i.e. if it can be written as a direct sum of subspaces $M_s \subset M$ for $s \in S$:
$$M = \oplus_{s \in S} M_s \ .$$

Given an S-graded vector space M, set $\Delta_M(m) := m \otimes s$ for a homogeneous element $m \in M_s$. One directly checks that this is a coassociative counital coaction of the coalgebra C. Conversely, given a C-comodule M, write $\Delta_M(m) = \sum_{s\in S} m_s \otimes s$, using the distinguished basis of C. We find

$$(\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M(m) = \sum_{s,t\in S} (m_s)_t \otimes t \otimes s$$

which by coassociativity of the coaction has to be equal to

$$(\mathrm{id}_M \otimes \Delta) \circ \Delta_M(m) = \sum_{s\in S} m_s \otimes s \otimes s \ .$$

Thus $(m_s)_t = m_s \delta_{s,t}$, which implies $\Delta_M(m_s) = m_s \otimes s$. We introduce the subspaces

$$M_s := \{m_s \,|\, m \in M\} \ .$$

The sum of the subspaces $\oplus M_s$ is direct: $m \in M_s \cap M_t$ for $s \neq t$ implies $m = m'_s = m''_t$ for some $m', m'' \in M$. Then the comparison of

$$\Delta(m) = \Delta(m'_s) = m'_s \otimes s = m \otimes s$$

with the same relation for t shows that $m \otimes s = m \otimes t$ and thus $m = 0$. Moreover, counitality of the coaction implies

$$m = \mathrm{id}_M(m) = (\mathrm{id}_M \otimes \epsilon) \circ \Delta_M(m) = \sum_{s\in S} m_s \epsilon(s) = \sum_{s\in S} m_s \ ,$$

so that $M = \oplus_{s\in S} M_s$.

2.3 Bialgebras

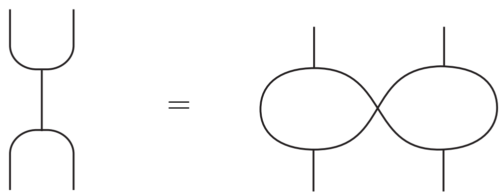
Definition 2.3.1

1. *A triple (A, μ, Δ) is called a bialgebra, if*

   - *(A, μ) is an associative algebra, having a unit $\eta : \mathbb{K} \to A$.*
   - *(A, Δ) is a coassociative coalgebra, having a counit $\epsilon : A \to \mathbb{K}$.*
   - *The coproduct $\Delta : A \to A \otimes A$ is a map of unital algebras, where $A \otimes A$ has the algebra structure described in proposition 2.1.11:*

   $$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \quad \textit{for all } a, b \in A$$

   in pictures

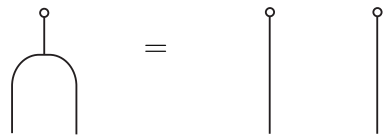
   

   or in Sweedler notation

   $$\sum_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} = \sum_{(a)(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \ .$$

   and $\Delta(1) = 1 \otimes 1$.

- *The counit $\epsilon : A \to \mathbb{K}$ is a map of unital algebras: $\epsilon(a \cdot b) = \epsilon(a) \cdot \epsilon(b)$. In pictures*



and $\epsilon(1) = 1$.

2. *A $\mathbb{K}$-linear map is said to be a* <u>*bialgebra map*</u>*, if it is both an algebra and a coalgebra map.*

Examples 2.3.2.

1. To endow the tensor algebra $T(V)$ with the structure of a bialgebra, it is enough to specify the unital algebra morphisms Δ and ϵ on the generators $v \in T^{(1)}V = V$. We set

$$\Delta(v) = v \otimes 1 + 1 \otimes v \quad \text{and } \epsilon(v) = 0 \qquad \text{for all } v \in V \ .$$

Since ϵ is required to be a morphism of algebras, one has

$$\epsilon(v_1 \cdot \dots \cdot v_n) = \epsilon(v_1) \cdot \ldots \cdot \epsilon(v_n) = 0 \ .$$

Together with unitality, $\epsilon(1) = 1$, this fixes the counit uniquely. Inductively, one uses the property that Δ is a morphism of algebras to show

$$\Delta(v_1 \dots v_n) = 1 \otimes (v_1 \dots v_n) + \sum_{p=1}^{n-1} \sum_{\sigma} (v_{\sigma(1)} \dots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \dots v_{\sigma(n)}) + (v_1 \dots v_n) \otimes 1 \,.$$

where the sum is over all $(p, n-p)$-shuffle permutations, i.e. over all permutations $\sigma \in S_n$ for which $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(n)$.

Counitality of Δ is now a direct consequence of the explicit formulae for coproduct and counit. Similarly, coassociativity can be derived. Finally, cocommutativity comes from the explicit formula for the coproduct, together with the observation that $(p, n-p)$-shuffles are in bijection to $(n-p, p)$-shuffles via the cyclic permutation in S_n that acts as $(1, 2, \dots, n) \mapsto (p+1, p+2, \dots, n, 1, \dots p)$.

2. A direct calculation shows that the group algebra $\mathbb{K}[G]$ of a group G is a bialgebra. Note that here we do not make use of the inverses in the group G, hence monoid algebras are bialgebras as well. The algebra of functions on a finite group is a bialgebra as well.

3. The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a bialgebra. In particular, any symmetric algebra over a vector space has a natural bialgebra structure. Since the arguments are similar to the case of the tensor algebra, we do not repeat them.

Remarks 2.3.3.

1. Since the counit $\epsilon : B \to \mathbb{K}$ of a bialgebra is a morphism of algebras, the ground field $\mathbb{K}$ can be endowed with the structure of a B-module by $b.\lambda = \epsilon(b) \cdot \lambda$ for $b \in B$ and $\lambda \in \mathbb{K}$. A bialgebra thus has a distinguished module, the <u>trivial module</u>. Similarly, the ground field $\mathbb{K}$ has also the structure of a comodule by $\Delta(\lambda) = \lambda \otimes 1_B$ for $\lambda \in \mathbb{K}$. This gives the <u>trivial comodule</u>.

2. If C and D are coalgebras, the tensor product $C \otimes D$ can be endowed with a natural structure of a coalgebra with comultiplication

$$C \otimes D \stackrel{\Delta_C \otimes \Delta_D}{\longrightarrow} C \otimes C \otimes D \otimes D \stackrel{\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_D}{\longrightarrow} C \otimes D \otimes C \otimes D$$

and counit

$$C \otimes D \stackrel{\epsilon_C \otimes \epsilon_D}{\longrightarrow} \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} \; .$$

This is just the anaologue of proposition 2.1.11.

3. In the definition of a bialgebra, the last two axioms of coproduct Δ and counit ϵ being morphisms of unital algebras can be replaced by the equivalent condition of the product μ and and the unit η being morphisms of counital coalgebras.

4. Since the counit $\epsilon : A \to \mathbb{K}$ is a morphism of algebras, the kernel $A^+ := \ker \epsilon$ is a two-sided ideal of codimension 1, called the augmentation ideal.

5. There is a weakening of the axioms of a bialgebra: one drops the condition of unitality for the coproduct and the counit and replaces them the following identities

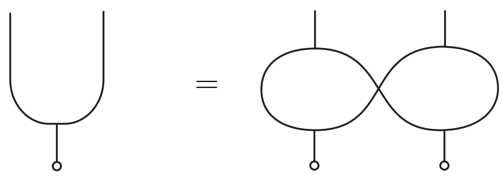
$$\begin{aligned} (\Delta \otimes \mathrm{id}_A) \cdot \Delta(1) &= (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1)) \\ &= (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1) \end{aligned} \qquad (1)$$

and

$$\begin{aligned} \epsilon(fgh) &= \epsilon(fg_{(1)})\epsilon(g_{(2)}h) \\ &= \epsilon(fg_{(2)})\epsilon(g_{(1)}h) \quad \text{for all } f, g, h \in A \, . \end{aligned} \qquad (2)$$

This defines the notion of a weak bialgebra. In a weak bialgebra, we only have the relation

$$\Delta(1) = \Delta(1 \cdot 1) = \Delta(1)\Delta(1) \, ,$$



i.e. $\Delta(1)$ is an idempotent in $A \otimes A$.

Remark 2.3.4.
A subspace $I \subset B$ of a bialgebra B is called a biideal, if it is both an ideal and a coideal. In this case, B/I is again a bialgebra.

We again discuss duals:

Lemma 2.3.5.
Let $(A, \mu, \eta, \Delta, \epsilon)$ be a *finite-dimensional* (weak) bialgebra and $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ its linear dual. Then the dual maps

$$\begin{aligned} \Delta^* &: (A \otimes A)^* \cong A^* \otimes A^* \to A^* \\ \epsilon^* &: \mathbb{K} \to A^* \\ \mu^* &: A^* \to (A \otimes A)^* = A^* \otimes A^* \\ \eta^* &: A^* \to \mathbb{K} \end{aligned}$$

define the structure of a (weak) bialgebra $(A^*, \Delta^*, \epsilon^*, \mu^*, \eta^*)$.

Remark 2.3.6.
For any (weak) bialgebra $(A, \mu, \eta, \Delta, \epsilon)$, we have three more (weak) bialgebras:

$$\begin{array}{rcl} A^{\text{opp}} & = & (A, \mu^{\text{opp}}, \eta, \Delta, \epsilon) \qquad A^{\text{opp,copp}} = (A, \mu^{\text{opp}}, \eta, \Delta^{\text{copp}}, \epsilon) \\ A^{\text{copp}} & = & (A, \mu, \eta, \Delta^{\text{copp}}, \epsilon) \end{array}$$

2.4 Tensor categories

We wish to understand the additional structure that is present on the categories of modules over bialgebras. Given two modules V, W over an algebra A, the tensor product has the structure of an $A \otimes A$-module by

$$A \otimes A \stackrel{\rho_V \otimes \rho_W}{\longrightarrow} \mathrm{End}(V) \otimes \mathrm{End}(W) \hookrightarrow \mathrm{End}(V \otimes W) \ ,$$

i.e. as in the case of Lie algebras and group algebras, cf. observation 2.1.12, will use for a bialgebra the pullback along the group homomorphism $\Delta : A \to A \otimes A$ to endow the tensor product $V \otimes W$ with the structure of an A-module. This turns a pair of objects (V, W) of the category $A-$mod into an object $V \otimes W$, and a pair of morphisms (f, g) into a morphism $f \otimes g$. We formalize this structure:

Definition 2.4.1
The Cartesian product of two categories $\mathcal{C}, \mathcal{D}$ is defined as the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs $(V, W) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and whose morphism sets are given by the Cartesian product of sets:

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((V, W), (V', W')) = \mathrm{Hom}_{\mathcal{C}}(V, V') \times \mathrm{Hom}_{\mathcal{D}}(W, W') \ .$$

We are now ready to discuss the structure induced by the tensor product of modules:

Definition 2.4.2

1. *Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor, called a tensor product.*

   Note that this associates to any pair (V, W) of objects an object $V \otimes W$ and to any pair of morphisms (f, g) a morphism $f \otimes g$ with source and target given by the tensor products of the source and target objects. In particular, $\mathrm{id}_{V\otimes W} = \mathrm{id}_V \otimes \mathrm{id}_W$ and for composable morphisms

   $$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g) \ .$$

2. *A monoidal category or tensor category consists of a category $(\mathcal{C}, \otimes)$ with tensor product, an object $\mathbb{I} \in \mathcal{C}$, called the tensor unit, and a natural isomorphism, called the associator,*

   $$a : \otimes(\otimes \times \ \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes) \, .$$

   of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

   $$r : \mathrm{id} \otimes \mathbb{I} \to \mathrm{id} \quad \textit{and} \quad l : \mathbb{I} \otimes \mathrm{id} \to \mathrm{id}$$

   called unit constraints such that the following axioms hold:

- *The* <u>pentagon axiom</u>*: for all quadruples of objects* $U, V, W, X \in \mathrm{Obj}(\mathcal{C})$ *the following diagram commutes*

$$\begin{tikzcd}
& (U \otimes V) \otimes (W \otimes X) \arrow[dr, "a_{U,V,W\otimes X}"] & \\
((U \otimes V) \otimes W) \otimes X \arrow[ur, "a_{U\otimes V,W,X}"] \arrow[d, "a_{U,V,W}\otimes \mathrm{id}_X"'] & & U \otimes (V \otimes (W \otimes X)) \\
(U \otimes (V \otimes W)) \otimes X \arrow[rr, "a_{U,V\otimes W,X}"'] & & U \otimes ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id}_U \otimes a_{V,W,X}"']
\end{tikzcd}$$

- *The* <u>triangle axiom</u>*: for all pairs of objects* $V, W \in \mathrm{Obj}(\mathcal{C})$ *the following diagram commutes*

$$\begin{tikzcd}
(V \otimes \mathbb{I}) \otimes W \arrow[rr, "a_{V,\mathbb{I},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbb{I} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd}$$

Remarks 2.4.3.

1. A monoidal category can be considered as a higher analogue of an associative, unital monoid, hence the name. The <u>associator</u> a is, however, a *structure*, not a *property.* A property is imposed at the level of natural transformations in the form of the pentagon axiom. For a given category $\mathcal{C}$ and a given tensor product $\otimes$, inequivalent associators can exist. Any associator a gives for any triple U, V, W of objects an isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

such that all diagrams of the form

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "(f\otimes g)\otimes h"'] & U \otimes (V \otimes W) \arrow[d, "f\otimes(g\otimes h)"] \\
(U' \otimes V') \otimes W' \arrow[r, "a_{U',V',W'}"'] & U \otimes (V \otimes W)
\end{tikzcd}$$

commute.

2. The pentagon axiom can be shown to guarantee the following property: suppose we are given finitely many objects $U_1, U_2, \ldots, U_n \in \mathcal{C}$. Any bracketing of this string determines an object in $\mathcal{C}$. For example, for $n = 4$, we get the five different objects at the vertices of the pentagon diagram. The pentagon diagram illustrates that the associator can be applied repeatedly in different ways to get an isomorphism between two objects arising from different bracketings. In the case of the pentagon, we have two isomorphisms $((U \otimes V) \otimes W) \otimes X \to U \otimes (V \otimes (W \otimes X))$. The pentagon axioms ensures that for arbitrary n all such isomorphisms coincide. This is known as Mac Lane's coherence theorem. We refer to [Kassel, XI.5] for details.

3. A tensor category is called <u>strict</u> , if the natural transformations a, l and r are the identity. One can show that any tensor category is equivalent, as a tensor category, to a strict tensor category.

4. Let $(\mathcal{C}, \otimes, a, l, r)$ be a tensor category. From the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we can get the functor $\otimes^{\mathrm{opp}} = \otimes \circ \tau$ with

$$V \otimes^{\mathrm{opp}} W := W \otimes V \quad \text{and } f \otimes^{\mathrm{opp}} g := g \otimes f \ .$$

It defines a tensor product: given an associator a for $\otimes$, one verifies that $a^{\mathrm{opp}}_{U,V,W} := a^{-1}_{W,V,U}$ is an associator for the tensor product $\otimes^{\mathrm{opp}}$. Similarly, one obtains left and right unit constraints.xxxxx

Examples 2.4.4.

1. The category of vector spaces over a fixed field $\mathbb{K}$ is a tensor category which is not strict. (See the appendix for information about this tensor category.) Tacitly, it is frequently replaced by an equivalent strict tensor category.

2. Let G be a group and $\mathrm{vect}_G(\mathbb{K})$ be the category of G-graded $\mathbb{K}$-vector spaces, i.e. of $\mathbb{K}$-vector spaces with a direct sum decomposition $V = \oplus_{g\in G} V_g$. Then the tensor product $V \otimes W$ is bigraded, $V \otimes W = \oplus_{g,h\in G} V_g \otimes W_h$ and becomes G-graded by the total degree

$$V \otimes W = \oplus_{g\in G} \left(\oplus_{h\in G} V_h \otimes W_{h^{-1}g}\right) \ .$$

Together with the associativity constraints inherited from $\mathrm{vect}(\mathbb{K})$ and with $\mathbb{K}_e$, i.e. the ground field $\mathbb{K}$ in homogeneous degree $e \in G$, as the tensor unit, this is a monoidal category. For these considerations, inverses in G are not needed and we could consider the monoidal category of vector spaces graded by any unital associative monoid.

3. Let $\mathcal{C}$ be a small category. The endofunctors

$$F \,:\, \mathcal{C} \to \mathcal{C}$$

are the objects of a tensor category $\mathrm{End}(\mathcal{C})$. The morphisms in this category are natural transformations, the tensor product is composition of functors. This tensor category is strict.

4. Let $(G_n)_{n\in\mathbb{N}_0}$ be a family of groups such that $G_0 = \{1\}$.

Define a category $\mathcal{G}$ whose objects are the natural numbers and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{G}}(m,n) = \left\{ \begin{array}{lll} \emptyset & m & \neq & n \\ G_n & m & = & n \end{array} \right.$$

Composition is the product in the group, the identity is the neutral element, $\mathrm{id}_n = e \in G_n$.

Suppose that we are given as further data a group homomorphism for any pair (m, n)

$$\rho_{m,n} : G_m \times G_n \to G_{m+n}$$

such that for all $m, n, p \in \mathbb{N}$, we have

$$\rho_{m+n,p} \circ (\rho_{m,n} \times \mathrm{id}_{G_p}) = \rho_{m,n+p} \circ (\mathrm{id}_{G_m} \times \rho_{n,p}) \ .$$

We define a functor

$$\otimes \,:\, \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

on objects by $m \otimes n = m + n$ and on morphisms by

$$\begin{array}{rcl} G_m \times G_n & \to & G_{m,n} \\ (f,g) & \mapsto & f \otimes g := \rho_{m,n}(f,g) \ . \end{array}$$

This turns $\mathcal{G}$ into a strict tensor category.

Such a structure is provided in particular by the collection $(S_n)_{n\in\mathbb{N}_0}$ of symmetric groups and the collection $(B_n)_{n\in\mathbb{N}}$ of braid groups. Define

$$\begin{array}{rcccl}\rho_{m,n} & : & B_m \times B_n & \to & B_{m+n} \\ & & (\sigma_i, \sigma_j) & \mapsto & \sigma_i \sigma_{j+m}\,, \end{array}$$

as the juxtaposition of a braid from B_m to a braid B_n.

Remarks 2.4.5.

1. In any monoidal category, we have a notion of an associative unital algebra (A, μ, η): this is a triple, consisting of an object $A \in \mathcal{C}$, multiplication $\mu : A \otimes A \to A$ and a unit morphism $\eta : \mathbb{I} \to A$ such that associativity identity
$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu) \circ a_{A,A,A}$$
for the morphisms $(A \otimes A) \otimes A \to A$ and the unit property
$$\mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A \circ r_A \quad \text{and} \quad \mu \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A \circ l_A$$
hold. Note that the associator enters. For a general monoidal category, we do not have a notion of a commutative algebra.

2. Similarly, we introduce the notion of a coalgebra (C, Δ, ϵ) in any monoidal category. For a general monoidal category, we do not have a notion of a cocommutative coalgebra.

3. Similarly, one can define modules and comodules in any monoidal category.

4. A coalgebra in $\mathcal{C}$ gives an algebra in $\mathcal{C}^{\mathrm{opp}}$ and vice versa.

The graphical notation for algebras, coalgebras, modules and comodules in a (strict) monoidal category is introduced in the obvious way.

If $\mathcal{C}$ is any category, the category of endofunctors $\mathrm{End}(\mathcal{C})$ has a composition. (If the category is small, it is even a strict monoidal category with composition as the tensor product.) This composition allows us to define:

Definition 2.4.6
Let $\mathcal{C}$ be a category. An associative unital algebra in the monoidal category $\mathrm{End}(\mathcal{C})$ *of endofunctors is called a* <u>*monad*</u> *on $\mathcal{C}$. Concretely, a monad is an endofunctor $Z : \mathcal{C} \to \mathcal{C}$, together with two natural transformations*

$$\mu : Z \circ Z \to Z \quad \text{and} \quad \eta : \mathrm{id}_{\mathcal{C}} \to Z$$

such that the two diagrams

$$\begin{tikzcd} Z^3(c) \arrow[r, "\mu_{Z(c)}"] \arrow[d, "Z(\mu_c)"'] & Z^2(c) \arrow[d, "\mu_c"] \\ Z^2(c) \arrow[r, "\mu_c"'] & Z(c) \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} Z(c) \arrow[r, "\eta_{Z(c)}"] \arrow[d, "Z(\eta_c)"'] \arrow[dr, "\mathrm{id}"] & Z^2(c) \arrow[d, "\mu_c"] \\ Z^2(c) \arrow[r, "\mu_c"'] & Z(c) \end{tikzcd}$$

expressing associativity and unitality commute for all $c \in \mathcal{C}$.

Considering Z as a monoid in $\mathrm{End}(\mathcal{C})$, one could define a module as as an object m in the monidal category $\mathrm{End}(\mathcal{C})$ with a morphism $Z \circ m \to m$. The following notion of a Z-module is different and more useful:

Definition 2.4.7

1. *Let $\mathcal{C}$ be a category and $Z : \mathcal{C} \to \mathcal{C}$ a monad on $\mathcal{C}$. A* <u>*Z-module*</u> *is a pair (m, ρ), consisting of an object $m \in \mathcal{C}$ and a morphism $\rho : Z(m) \to m$ such that the following two diagrams*

$$\begin{tikzcd} Z^2(m) \arrow[r, "\mu_m"] \arrow[d, "Z(\rho)"'] & Z(m) \arrow[d, "\rho"] \\ Z(m) \arrow[r, "\rho"'] & m \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} m \arrow[r, "\eta_m"] \arrow[dr, "id_m"'] & Z(m) \arrow[d, "\rho"] \\ & m \end{tikzcd}$$

   expressing associativity and unitality of the action commute.

2. *Given two Z-modules (m, ρ) and (m, ρ'), the set of Z-module morphisms is*

$$\mathrm{Hom}_Z((m, \rho), (m', \rho')) := \{f : m \to m' \mid \rho' \circ Z(f) = f \circ \rho\} .$$

   We denote by $Z-mod$ the category of Z-modules.

There exists also the notion of a comonad which the reader should work out as an exercise.

Tensor categories are categories with some additional structure. It should not come as a surprise that we need also a class of functors and natural transformations that is adapted to this extra structure.

Definition 2.4.8

1. *Let $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{I}_\mathcal{C}, a_\mathcal{C}, l_\mathcal{C}, r_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{I}_\mathcal{D}, a_\mathcal{D}, l_\mathcal{D}, r_\mathcal{D})$ be tensor categories. (We will sometimes suppress indices indicating the category to which the data belong.) A* <u>*tensor functor*</u> *or* <u>*monoidal functor*</u> *from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ consisting of*

a functor	$F : \mathcal{C}$	$\to$	$\mathcal{D}$
an isomorphism	$\varphi_0 : \mathbb{I}_\mathcal{D}$	$\to$	$F(\mathbb{I}_\mathcal{C})$ *in the category* $\mathcal{D}$
a natural isomorphism	$\varphi_2 : \otimes_\mathcal{D} \circ (F \times F)$	$\to$	$F \circ \otimes_\mathcal{C}$

   of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$. This includes in particular an isomorphism for any pair of objects $U, V \in \mathcal{C}$

$$\varphi_2(U, V) : F(U) \otimes_\mathcal{D} F(V) \xrightarrow{\sim} F(U \otimes_\mathcal{C} V) .$$

   These data have to obey a series of constraints expressed by commuting diagrams:

   - *Compatibility with the associativity constraint:*

$$\begin{tikzcd} (F(U) \otimes F(V)) \otimes F(W) \arrow[r, "a_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}"'] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d, "\mathrm{id}_{F(U)}\otimes\varphi_2(V,W)"] \\ F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_2(U\otimes V,W)"'] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_2(U,V\otimes W)"] \\ F((U \otimes V) \otimes W) \arrow[r, "F(a_{U,V,W})"'] & F(U \otimes (V \otimes W)) \end{tikzcd}$$

   - *Compatibility with the left unit constraint:*

$$\begin{tikzcd} \mathbb{I}_\mathcal{D} \otimes F(U) \arrow[r, "l_{F(U)}"] \arrow[d, "\varphi_0\otimes\mathrm{id}_{F(U)}"'] & F(U) \\ F(\mathbb{I}_\mathcal{C}) \otimes F(U) \arrow[r, "\varphi_2(\mathbb{I}_\mathcal{C},U)"'] & F(\mathbb{I}_\mathcal{C} \otimes U) \arrow[u, "F(l_U)"'] \end{tikzcd}$$

- *Compatibility with the right unit constraint:*

$$\begin{tikzcd}
F(U) \otimes \mathbb{I}_{\mathcal{D}} \arrow[r, "r_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)} \otimes \varphi_0"'] & F(U) \\
F(U) \otimes F(\mathbb{I}_{\mathcal{C}}) \arrow[r, "\varphi_2(U, \mathbb{I}_{\mathcal{C}})"'] & F(U \otimes \mathbb{I}_{\mathcal{C}}) \arrow[u, "F(r_U)"']
\end{tikzcd}$$

2. *A tensor functor is called* <u>*strict*</u>*, if the isomorphism φ_0 and the natural transformation φ_2 are identities in $\mathcal{D}$. In general, the isomorphism and the natural isomorphism is additional* structure.

3. *A* <u>*monoidal natural transformation*</u> *between tensor functors*

$$\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$$

is a natural transformation $\eta : F \to F'$ with the following two properties: such that diagram involving the tensor unit

$$\begin{tikzcd}
 & F(\mathbb{I}_{\mathcal{C}}) \arrow[dd, "\eta_{\mathbb{I}}"] \\
\mathbb{I}_{\mathcal{D}} \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\
 & F'(\mathbb{I}_{\mathcal{C}})
\end{tikzcd}$$

commutes, and for all pairs (U, V) of objects the diagram

$$\begin{tikzcd}
F(U) \otimes F(V) \arrow[r, "\varphi_2(U,V)"] \arrow[d, "\eta_U \otimes \eta_V"'] & F(U \otimes V) \arrow[d, "\eta_{U \otimes V}"] \\
F'(U) \otimes F'(V) \arrow[r, "\varphi_2'(U,V)"'] & F'(U \otimes V)
\end{tikzcd}$$

commutes.

4. *One then defines* <u>*monoidal natural isomorphisms*</u> *as invertible monoidal natural transformations. An* <u>*equivalence*</u> *of tensor categories $\mathcal{C}$,$\mathcal{D}$ is given by a pair of tensor functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural* monoidal *isomorphisms*

$$\eta : \quad \mathrm{id}_{\mathcal{D}} \to FG \quad \text{and} \quad \theta : \quad GF \to \mathrm{id}_{\mathcal{C}} \ .$$

Remarks 2.4.9.

1. Suppose that a tensor functor $(F, \varphi_0, \varphi_2)$ has the property that the underlying functor F is an equivalence of categories. One then then show that then there exists a tensor functor G such that (F, G) is an equivalence of tensor categories [DM, Proposition 1.11] which refers to [Saa, Proposition 4.4.2]

2. The strictification result for tensor categories can now be stated more precisely: any tensor category $\mathcal{C}$ is monoidally equivalent to a strict tensor category $\hat{\mathcal{C}}$. The strict tensor category $\hat{\mathcal{C}}$ equivalence $F : \mathcal{C} \to \hat{\mathcal{C}}$ can even be chosen such that F is a strict monoidal functor, see [JS, Corollary 1.4].

3. There is also a strictification result for tensor functors: Proposition 1.5 in [JS] explains how tensor functors can be replaced by strict tensor functors.

We can now characterize algebras whose representation categories are monoidal categories.

Proposition 2.4.10.
Let (A, μ) be a unital associative algebra. Suppose we are given unital algebra maps

$$\Delta : A \to A \otimes A \quad \text{and} \quad \epsilon : A \to \mathbb{K} \ .$$

Use the pullback along the morphism of algebras $\epsilon : A \to \mathbb{K} \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K})$ to endow the ground field $\mathbb{K}$ with the structure of an A-module $(\mathbb{K}, \epsilon)$, i.e. $a.\lambda := \epsilon(a) \cdot \lambda$ for $a \in A$ and $\lambda \in \mathbb{K}$. Let

$$\otimes : A\mathrm{-mod} \times A\mathrm{-mod} \to A\mathrm{-mod}$$

be the functor which associates to a pair M, N of A-modules their tensor product $M \otimes_{\mathbb{K}} N$ as vector spaces with the A-module structure given by the morphism of algebras

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho_M \otimes \rho_N} \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N) \ .$$

Then $(A\mathrm{-mod}, \otimes, (\mathbb{K}, \epsilon))$, together with the canonical associativity and unit constraints of the category $\mathrm{vect}(\mathbb{K})$ of $\mathbb{K}$-vector spaces is a monoidal category, if and only if (A, μ, Δ) is a bialgebra with counit ϵ, i.e. if and only if (A, Δ, ϵ) is a coalgebra.

Proof.

- Suppose that (A, μ, Δ) is a bialgebra. We have to show that the canonical isomorphisms of vector spaces

  $$\begin{array}{ccc} (U \otimes V) \otimes W & \to & U \otimes (V \otimes W) \\ (u \otimes v) \otimes w & \mapsto & u \otimes (v \otimes w) \end{array}$$

  are morphisms of A-modules. Using Sweedler notation, the element $a \in A$ acts on the left hand side by

  $$a.(u \otimes v) \otimes w = a_{(1)}.(u \otimes v) \otimes a_{(2)}.w = ((a_{(1)})_{(1)}.u \otimes (a_{(1)})_{(2)}.v) \otimes a_{(2)}.w \qquad (*)$$

  and on the right hand side

  $$a.u \otimes (v \otimes w) = a_{(1)}.u \otimes a_{(2)}.(v \otimes w) = a_{(1)}.u \otimes ((a_{(2)})_{(1)}.v \otimes (a_{(2)})_{(2)}.w) \qquad (**)$$

  Coassociativity of A implies that the right hand side of the first equation is mapped to the right hand side of the second equation after rebracketing.

  Since the standard associativity constraints in $\mathrm{vect}(\mathbb{K})$ obey the pentagon relation, this relation holds in $A\mathrm{-mod}$, as well. Similarly, we have to show that the two unit constraints

  $$\begin{array}{ccc} V \otimes \mathbb{K} & \to & V \\ v \otimes \lambda & \mapsto & \lambda v \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{K} \otimes V & \to & V \\ \lambda \otimes v & \mapsto & \lambda v \end{array}$$

  are morphisms of A-modules. For the second isomorphism, note that

  $$a.(\lambda \otimes v) = \epsilon(a_{(1)})\lambda \otimes a_{(2)}.v \mapsto \epsilon(a_{(1)})a_{(2)}.\lambda v = a.\lambda v$$

  where in the last step we used one defining property of the counit. The other unit constraint is dealt with in complete analogy.

- Conversely, suppose that $(A\text{−mod}, \otimes, \mathbb{K})$ is a monoidal category. We have to extract from this categorical structure structure and relations on the algebra. This is usually done using the following observation: the algebra A itself, with the action by left multiplication, is a left A-module, the left regular A-module ${}_AA$. In the specific case $U = V = W =_A A$, the associator provides an isomorphism

  $$\begin{array}{rcl} (A \otimes A) \otimes A & \to & A \otimes (A \otimes A) \\ (u \otimes v) \otimes w & \mapsto & u \otimes (v \otimes w) \end{array}$$

  of A-modules. Taking the associator in the category of vector spaces to be the identity, the fact that the identity intertwines the action of A leads to the equation

  $$(a_{(1)})_{(1)}u \otimes (a_{(1)})_{(2)}v \otimes a_{(2)}w = a_{(1)}u \otimes (a_{(2)})_{(1)}v \otimes (a_{(2)})_{(2)}w$$

  for all $u, v, w \in_A A$ and all $a \in A$. Choosing $u = v = w = 1_A$, implies coassociativity,

  $$(a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} = a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}$$

  for all $a \in A$.

  Similarly, we conclude from the fact that the canonical maps $\mathbb{K} \otimes A \to A$ and $A \otimes \mathbb{K} \to A$ are isomorphisms of A-modules that ϵ is a counit.

□

Remark 2.4.11.

1. Let (A, μ, Δ) again be a bialgebra. Then the category comod-A of right A-comodules is a tensor category as well. Given two comodules (M, Δ_M) and (N, Δ_N), the coaction on the tensor product $M \otimes N$ is defined using the multiplication:

   $$\Delta_{M\otimes N}: \quad M \otimes N \xrightarrow{\Delta_M \otimes \Delta_N} M \otimes A \otimes N \otimes A \xrightarrow{\mathrm{id}_M \otimes \tau \otimes \mathrm{id}_A} M \otimes N \otimes A \otimes A \xrightarrow{\mathrm{id}_{M\otimes N} \otimes \mu} M \otimes N \otimes A\,.$$

   It is straightforward to dualize all statements we made earlier.

   In particular, the tensor unit is the trivial comodule which is the ground field $\mathbb{K}$ with a coaction that is given by the unit $\eta : \mathbb{K} \to A$:

   $$\mathbb{K} \xrightarrow{\eta} A \cong \mathbb{K} \otimes A\,.$$

   Again, the associativity and unit constraints of comodules are inherited from the constraints for vector spaces:

   $$\begin{array}{rcl} (M \otimes N) \otimes P & \cong & M \otimes (N \otimes P) \\ \mathbb{K} \otimes M \cong & M & \cong M \otimes \mathbb{K}\,. \end{array}$$

2. A bialgebra thus gives rise to four monoidal categories, left and right modules and comodules. These categories are, in general, rather different. For example, for a group algebra $\mathbb{K}[G]$, the monoidal category of comodules is the category vect_G of G-graded vector spaces which has only one-dimensional simple modules while the category of left modules is the category rep$G$ of G-representations.

2.5 Hopf algebras

Observation 2.5.1.
Let (A, μ) be a unital algebra and (C, Δ) a counital coalgebra over the same field $\mathbb{K}$. We then define on the $\mathbb{K}$-vector space of $\mathbb{K}$-linear maps $\mathrm{Hom}(C, A)$ a product, called convolution. The product $f * g$ of $f, g \in \mathrm{Hom}(C, A)$ is the $\mathbb{K}$-linear map

$$f * g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A\,.$$

This product is $\mathbb{K}$-bilinear and associative. In Sweedler notation

$$(f * g)(x) = f(x_{(1)}) \cdot g(x_{(2)})\,.$$

The linear map

$$C \xrightarrow{\epsilon} \mathbb{K} \xrightarrow{\eta} A$$

is a unit for this product.

This endows in particular the space $\mathrm{End}_{\mathbb{K}}(A)$ of endomorphisms of a bialgebra A with the structure of a unital associative $\mathbb{K}$-algebra. Its unit is *not* the identity $\mathrm{id}_A \in \mathrm{End}_{\mathbb{K}}(A)$. It is, however, not clear whether in this case the identity id_A has the property of being an invertible element of the convolution algebra.

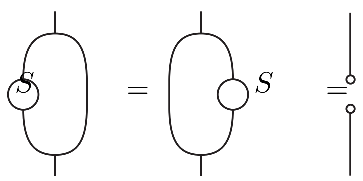
Definition 2.5.2
We say that a bialgebra (H, μ, Δ) is a Hopf algebra, if the identity id_H has a two-sided inverse $S : H \to H$ under the convolution product. This inverse is then called the antipode of the Hopf algebra.

Remarks 2.5.3.

1. The defining identity of the antipode

$$S * \mathrm{id}_H = \mathrm{id}_H * S = \eta\epsilon$$

   reads in graphical notation



   and in Sweedler notation

$$x_{(1)} \cdot S(x_{(2)}) = \epsilon(x) \cdot 1 = S(x_{(1)}) \cdot x_{(2)}\,.$$

2. If an antipode exists, it is, as a two-sided inverse for an associative product, uniquely determined:

$$\begin{aligned} S &= S * (\eta\epsilon) = S * (\mathrm{id}_H * S') = (S * \mathrm{id}_H) * S' \\ &= \eta\epsilon * S' = S'\ . \end{aligned}$$

   Thus, for a bialgebra, being a Hopf algebra is a property rather than a structure.

3. If $H = (A, \mu, \eta, \Delta, \epsilon, S)$ is a finite-dimensional Hopf algebra, its dual $H^* = (A^*, \Delta^*, \epsilon^*, \mu^*, \eta^*, S^*)$ is a Hopf algebra as well.

4. We will see in corollary 2.5.10 that any morphism $f : H \to K$ of bialgebras between Hopf algebras respects the antipode, $f(S_H h) = S_K f(h)$ for all $h \in H$. It is thus a morphism of Hopf algebras.

5. The antipode is not necessarily invertible as a linear map $S : H \to H$. For counter examples, see [T].

6. A subspace $I \subset H$ of a Hopf algebra H is a Hopf ideal, if it is a biideal, cf. remark 2.3.4 and if $S(I) \subset I$. In this case, H/I with the structure induced from H is a Hopf algebra.

Example 2.5.4.
If G is a group, the group algebra $\mathbb{K}[G]$ is a Hopf algebra with antipode

$$S(g) = g^{-1} \quad \text{for all} \quad g \in G \ .$$

Indeed, we have for $g \in G$:

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(g) = \mu \circ (S \otimes \mathrm{id})(g \otimes g) = g^{-1} \cdot g = \epsilon(g)1 \ .$$

Before giving more examples, we need a fundamental property of the antipode. If (A, μ_A) and (B, μ_B) are algebras, a map $f : A \to B$ is called an antialgebra map, if it is a map of unital algebras $f : A \to B^{\mathrm{opp}}$, i.e. if $f(a \cdot a') = f(a') \cdot f(a)$ for all $a, a' \in A$ and $f(1_A) = 1_B$.

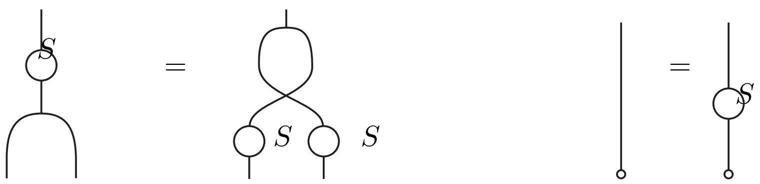
Similarly, if (C, Δ_C) and (D, Δ_D) are coalgebras, a map $g : C \to D$ is called an anticoalgebra map, if it is a counital coalgebra map $g : C \to C^{\mathrm{copp}}$, i.e. if $\epsilon_D \circ g = \epsilon_C$ and

$$g(c)_{(2)} \otimes g(c)_{(1)} = g(c_{(1)}) \otimes g(c_{(2)}) \ .$$

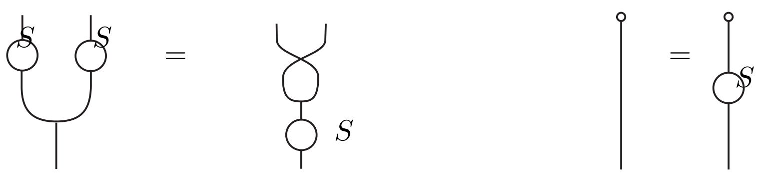
Proposition 2.5.5.
Let H be a Hopf algebra. Then the antipode S is a morphism of bialgebras $S : H \to H^{\mathrm{opp,copp}}$, i.e. an antialgebra and anticoalgebra map: we have for all $x, y \in H$

$$\begin{array}{rrcll} & S(xy) & = & S(y)S(x) & \qquad S(1) = 1 \\ \text{and} & (S \otimes S) \circ \Delta & = & \Delta^{\mathrm{copp}} \circ S & \qquad \epsilon \circ S = \epsilon \, . \end{array}$$

Graphically,



and



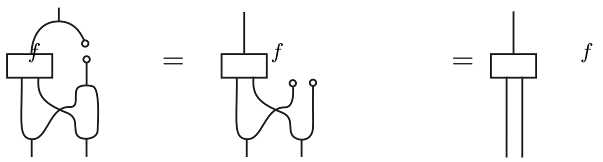
Proof.
Since $H \otimes H$ is in particular a coalgebra and H an algebra, we can endow the vector space $B :=$ Hom $(H \otimes H, H)$ with bilinear product given by the convolution product: the product $\nu * \rho$ of $\nu, \rho \in$ Hom $(H \otimes H, H)$ is by definition

$$\nu * \rho : H \otimes H \stackrel{(id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)}{\longrightarrow} H^{\otimes 4} \stackrel{\nu \otimes \rho}{\longrightarrow} H^{\otimes 2} \stackrel{\mu}{\longrightarrow} H\,.$$

As any convolution product involving an associative algebra and a coassociative coalgebra, this product is associative. The unit is

$$1_B := \eta \circ \epsilon \circ \mu \quad : \quad H \otimes H \stackrel{\mu}{\longrightarrow} H \stackrel{\epsilon}{\longrightarrow} \mathbb{K} \stackrel{\eta}{\longrightarrow} H$$

as can be seen graphically: for any $f \in \mathrm{Hom}_{\mathbb{K}}(H \otimes H, H)$, we have



Here we used that the counit ϵ of a bialgebra is a morphism of algebras and then we used the counit property twice. Recall that, as for any associative product, two-sided inverses are unique: given $\mu \in B$, for any $\rho, \nu \in B$, the relation

$$\rho * \mu = \mu * \nu = 1_B$$

implies

$$\nu = 1 * \nu = (\rho * \mu) * \nu = \rho * (\mu * \nu) = \rho * 1 = \rho\ .$$

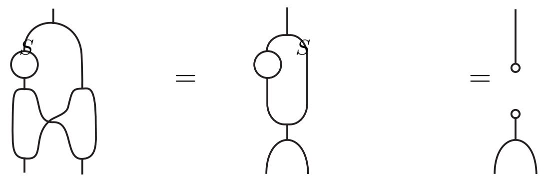
We apply this to the two elements in the algebra B

$$\begin{array}{rccl} & H \otimes H & \to & H \\ \nu : & x \otimes y & \mapsto & S(y) \cdot S(x) \\ \rho : & x \otimes y & \mapsto & S(x \cdot y) \end{array}$$

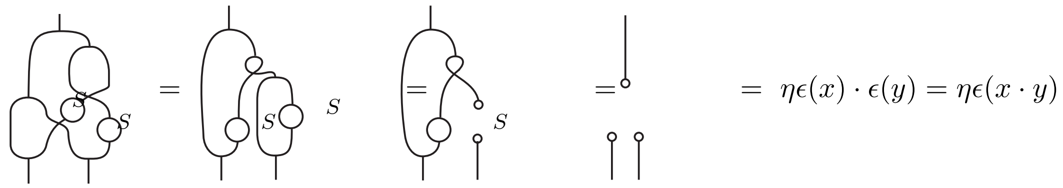
We compute for $x, y \in H$:

$$\begin{aligned}
(\rho * \mu)(x \otimes y) &= \sum_{x \otimes y} \rho((x \otimes y)_{(1)}) \cdot \mu((x \otimes y)_{(2)}) \quad \text{[defn. of the convolution $*$]} \\
&= \sum \rho(x_{(1)} \otimes y_{(1)}) \mu(x_{(2)} \otimes y_{(2)}) \quad \text{[defn. of the coproduct of $H \otimes H$]} \\
&= \sum S(x_{(1)} y_{(1)}) x_{(2)} y_{(2)} \qquad \text{[defn. of ρ and μ]} \\
&= \sum_{(xy)} S((xy)_{(1)})(xy)_{(2)} \qquad \text{[Δ is a morphism of algebras]} \\
&= \eta\epsilon(xy) \qquad \text{[defn. of the antipode]} \\
&= 1_B(x \otimes y)
\end{aligned}$$

It is instructive to do such a calculation graphically:



The first equality is the multiplicativity of the coproduct in a bialgebra, the second is the definition of the antipode.
On the other hand, we compute $\mu * \nu$:



where in the first step we used associativity twice and in last step we used that the counit ϵ is a map of algebras.

Finally, the equality defining the antipode

$$\mathrm{id} \; * S = \eta\epsilon \,,$$

can be applied to 1_H and then yields

$$1_H \cdot S(1_H) = \mathrm{id} * S(1_H) = \eta\epsilon(1_H) = 1_H \,,$$

where the first equality is unitality of the coproduct Δ and the last identity is the unitality of the counit ϵ. This identity in H implies $S(1_H) = 1_H$. The assertions about the coproduct are proven in an analogous way by showing the equivalent identity

$$\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{copp}} \quad \text{in} \quad \mathrm{Hom}(H, H \otimes H) \ .$$

Finally, apply ϵ to the equality

$$\epsilon(x)1 = S(x_{(1)}) \cdot x_{(2)}$$

to get

$$\epsilon(x) = \epsilon(x)\epsilon(1) = \epsilon(S(x_{(1)})\epsilon(x_{(2)}) = \epsilon \circ S(x) \quad \text{for all } x \in H \ .$$

□

We now present another class of examples of Hopf algebras

Example 2.5.6.
The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra with antipode

$$S(x) = -x \quad \text{for all} \quad x \in \mathfrak{g} \ .$$

Indeed, we have for $x \in \mathfrak{g}$:

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \mu(-x \otimes 1 + 1 \otimes x) = -x + x = 0 = 1\epsilon(x) \ .$$

We extend this to all of $\mathrm{U}(\mathfrak{g})$ by the following observation: let H be a bialgebra that is generated, as an algebra, by a subset $X \subset H$. Suppose that the defining relation for an antipode holds for all $x \in X$, i.e.

$$S * \mathrm{id}_H(x) = \mathrm{id}_H * S(x) = \eta\epsilon(x) \quad \text{for all } x \in X \;.$$

Then S is an antipode for H. In fact, it is enought to check that the relation holds for products xy with $x, y \in X$. Then

$$\begin{array}{rcl} (xy)_{(1)}S((xy)_{(2)}) & = & x_{(1)}y_{(1)}S(x_{(2)}y_{2)}) \quad \text{[bialgebra]} \\ & = & x_{(1)}y_{(1)}S(y_{(2)})S(x_{2)}) \quad \text{[antialgebra morphism]} \\ & = & \epsilon(x)\epsilon(y) = \epsilon(xy) \quad \text{[relation for generators } x, y \text{ and } \epsilon \text{ algebra morphism.]} \end{array}$$

The other relation follows analogously.

In particular, the symmetric algebra over a vector space V is a Hopf algebra, since it is the universal enveloping algebra of the abelian Lie algebra on the vector space V. Similarly, the tensor algebra TV over a vector space V is a Hopf algebra, since it can be considered as the enveloping algebra of the free Lie algebra on V.

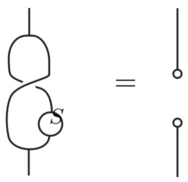
Proposition 2.5.7.
Let H be a Hopf algebra. Then the following identities are equivalent:

(a) $S^2 = \mathrm{id}_H$

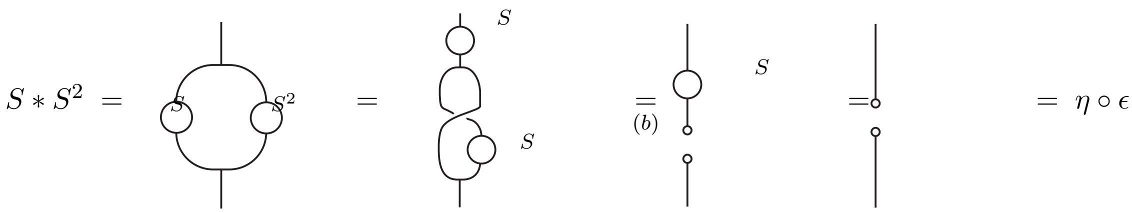
(b) $\sum_x S(x_{(2)})x_{(1)} = \epsilon(x)1_H$ for all $x \in H$.

(c) $\sum_x x_{(2)}S(x_{(1)}) = \epsilon(x)1_H$ for all $x \in H$.

Proof.
We show (b) $\Rightarrow$ (a) by first showing from (b) that $S * S^2$ is the unit $\eta \circ \epsilon$ of the convolution product. In graphical notation, (b) reads



Thus

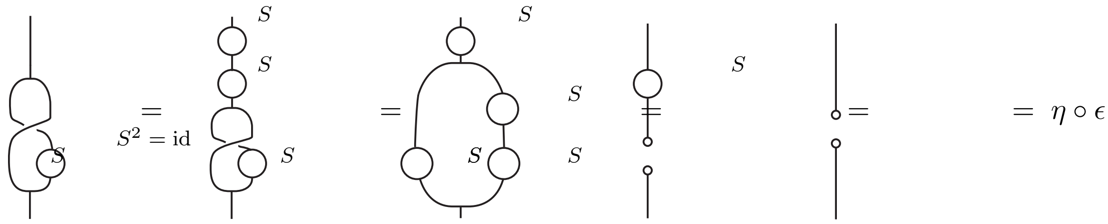


For comparison, we also compute in equations:

$$\begin{array}{rcl} S * S^2(x) & = & \sum_{(x)} S(x_{(1)})S^2(x_{(2)}) = S\Big(\sum_{(x)} S(x_{(2)})x_{(1)}\Big) \\ & \overset{(b)}{=} & S(\epsilon(x)1) = \epsilon(x)S(1) = \epsilon(x)1\,. \end{array}$$

Multiplying from the left with id yields $\mathrm{id} = \mathrm{id} * (S * S^2) = (\mathrm{id} * S) * S^2 = S^2$.

Conversely, assume $S^2 = \mathrm{id}_H$



where we used $S^2 = \mathrm{id}$, the fact that S is an anticoalgebra map, again $S^2 = \mathrm{id}$ and then the defining property of the antipode S. The equivalence of (c) and (a) is proven in complete analogy. $\square$

The following simple lemma will be useful in many places:

Lemma 2.5.8.
Let H be a Hopf algebra with invertible antipode. Then

$$S^{-1}(a_{(2)}) \cdot a_{(1)} = a_{(2)} \cdot S^{-1}(a_{(1)}) = 1_H \epsilon(a) \quad \text{for all } a \in H\ .$$

Proof.
The following calculation shows the claim:

$$\begin{aligned} S^{-1}(a_{(2)}) \cdot a_{(1)} &= S^{-1} \circ S \left(S^{-1}(a_{(2)}) \cdot a_{(1)} \right) \\ &= S^{-1} \left(S(a_{(1)}) \cdot a_{(2)} \right) \qquad [\ S \text{ is antialgebra morphism}] \\ &= S^{-1}(1_H)\epsilon(a) = 1_H \epsilon(a) \end{aligned}$$

The other identity is proven analogously. $\square$

Remark 2.5.9.
Let H be a bialgebra. An endomorphism $\tilde{S} : H \to H$ such that

$$\sum_x \tilde{S}(x_{(2)}) x_{(1)} = \sum_x x_{(2)} \tilde{S}(x_{(1)}) = \epsilon(x) 1_H \quad \text{for all} \quad x \in H$$

is also called a skew antipode. For any invertible antipode, S^{-1} is a skew-antipode. Conversely, a bialgebra with an antipode and a skew-antipode has an invertible antipode. As we will see, a theorem of Larson and Sweedler asserts that for any finite-dimensional Hopf algebra the antipode is invertible. Hence, finite-dimensional Hopf algebras also have a skew antipode.

Corollary 2.5.10.

1. If H is either commutative or cocommutative, then the identity $S^2 = \mathrm{id}_H$ holds.

2. If H and K are Hopf algebras with antipodes S_H and S_K, respectively, then any (unital and counital) bialgebra map $\varphi : H \to K$ is a Hopf algebra map, i.e. $\varphi \circ S_H = S_K \circ \varphi$.

Proof.

1. If H is commutative, then

$$x_{(2)} \cdot S(x_{(1)}) = S(x_{(1)}) \cdot x_{(2)} \stackrel{\text{defn. of } S}{=} \epsilon(x) 1_H \ .$$

From proposition 2.5.7, we conclude that $S^2 = \mathrm{id}_H$. If H is cocommutative, then

$$x_{(2)} \cdot S(x_{(1)}) = x_{(1)} \cdot S(x_{(2)}) \stackrel{\text{defn. of } S}{=} \epsilon(x) 1_H \ .$$

Again we conclude that $S^2 = \mathrm{id}_H$.

2. Use again a convolution product to endow $B := \mathrm{Hom}(H, K)$ with the structure of an associative unital algebra. Then compute

$$(\varphi \circ S_H) * \varphi = \mu_K \circ (\varphi \otimes \varphi) \circ (S_H \otimes \mathrm{id}_H) \circ \Delta_H = \varphi \circ \mu_H (S_H \otimes \mathrm{id}_H) \circ \Delta_H = 1_K \epsilon_H$$

and

$$\varphi * (S_K \circ \varphi) = \mu_K \circ (\mathrm{id} \otimes S_K) \circ \Delta_K \circ \varphi = 1_K \epsilon_K \circ \varphi = 1_K \epsilon_H$$

The uniqueness of the inverse of φ for the convolution product shows the claim.

□

We use the antipode to endow the category of left modules over a Hopf algebra H with a structure that generalizes contragredient or dual representations of groups. We first state a more general fact:

Proposition 2.5.11.

1. Let A be a $\mathbb{K}$-algebra and U, V objects in $A-$mod. Then the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(U, V)$ is an $A \otimes A^{\mathrm{opp}}$-module by

$$\Big[(a \otimes a').f\Big](u) := a.f(a'.u) \ .$$

2. If H is a Hopf algebra, then $\mathrm{Hom}_{\mathbb{K}}(U, V)$ is an H-module by

$$(af)(u) = \sum_{(a)} a_{(1)} f(S(a_{(2)})u) \, .$$

In the special case of the trivial module, $V = \mathbb{K}$, the dual vector space $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ becomes an H-module by

$$(af)u = f(S(a)u) \ .$$

3. Similarly, if H is a Hopf algebra and if the antipode S of H is an invertible endomorphism of H (or if a skew antipode exists), then the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(U, V)$ is also an H-module by

$$(af)(u) = \sum_{(a)} a_{(1)} f(S^{-1}(a_{(2)})u) \, .$$

In the special case $V = \mathbb{K}$, the dual vector space $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ becomes an H-module by

$$(af)u = f(S^{-1}(a)u) \ .$$

Proof.
We compute with $a, b \in A$ and $a', b' \in A^{\mathrm{opp}}$:

$$\begin{aligned}
\big((a\otimes a')(b\otimes b')\big)f(u) &= (ab\otimes b'a')f(u)\\
&= abf(b'a'u)\\
&= a\Big((b\otimes b')f\Big)(a'u)\\
&= (a\otimes a')\big((b\otimes b')f(u)\big)
\end{aligned}$$

For the second assertion, note that

$$A \xrightarrow{\Delta} A\otimes A \xrightarrow{\mathrm{id}_A\otimes S} A\otimes A^{\mathrm{opp}}$$

and, if S is invertible, also

$$A \xrightarrow{\Delta} A\otimes A \xrightarrow{\mathrm{id}_A\otimes S^{-1}} A\otimes A^{\mathrm{opp}}$$

are morphisms of algebras.

In the specific case of the trivial module, $V = \mathbb{K}$, we find

$$(af)(u) = \sum_{(a)} \epsilon(a_{(1)})f(S(a_{(2)})u) = \sum_{(a)} f\Big(S(\epsilon(a_{(1)})a_{(2)})u\Big) = f(S(a)u)$$

where the second equality holds since f is $\mathbb{K}$-linear and the last equality holds by counitality.
□

We recall the following maps relating a $\mathbb{K}$-vector space X and its dual $X^* = \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K})$: we have two evaluation maps

$$\begin{aligned}
d_X : X^*\otimes X &\to \mathbb{K}\\
\beta\otimes x &\mapsto \beta(x)\\
\tilde{d}_X : X\otimes X^* &\to \mathbb{K}\\
x\otimes\beta &\mapsto \beta(x)
\end{aligned}$$

We call d_X a right evaluation and $\tilde{d}_X$ a left evaluation. If the $\mathbb{K}$-vector space X is *finite-dimensional*, consider a basis $\{x_i\}_{i\in I}$ of X and a dual basis $\{x^i\}_{i\in I}$ of X^*. We then have two coevaluation maps:

$$\begin{aligned}
b_X : \mathbb{K} &\to X\otimes X^*\\
\lambda &\mapsto \lambda\textstyle\sum_{i\in I} x_i\otimes x^i\\
\tilde{b}_X : \mathbb{K} &\to X^*\otimes X\\
\lambda &\mapsto \lambda\textstyle\sum_{i\in I} x^i\otimes x_i
\end{aligned}$$

The two maps b_X and $\tilde{b}_X$ are in fact independent of the choice of basis. For example,

$$\begin{aligned}
b_X : \mathbb{K} &\to \mathrm{End}_{\mathbb{K}}(X) \cong X\otimes X^*\\
\lambda &\mapsto \lambda\mathrm{id}_X
\end{aligned}$$

We call b_X a right coevaluation and $\tilde{b}_X$ a left coevaluation.

Definition 2.5.12

1. *Let $\mathcal{C}$ be a tensor category. An object V of $\mathcal{C}$ is called right dualizable, if there exists an object $V^\vee \in \mathcal{C}$ and morphisms*

$$b_V : \mathbb{I} \to V \otimes V^\vee \quad \text{and} \quad d_V : V^\vee \otimes V \to \mathbb{I}$$

such that

$$\begin{aligned} r_V \circ (\mathrm{id}_V \otimes d_V) \circ a_{V,V^\vee,V} \circ (b_V \otimes \mathrm{id}_V) \circ l_V^{-1} &= \mathrm{id}_V \\ l_{V^\vee} \circ (d_V \otimes \mathrm{id}_{V^\vee}) \circ a^{-1}_{V^\vee,V,V^\vee} \circ (\mathrm{id}_{V^\vee} \otimes b_V) \circ r^{-1}_{V^\vee} &= \mathrm{id}_{V^\vee} \end{aligned}$$

Such an object $V^\vee$ is called a right dual to V.

The morphism d_V is called an evaluation, the morphism b_V a coevaluation.

2. *A monoidal category is called right-rigid or right-autonomous, if every object has a right dual.*

3. *A left dual to V is an object ${}^\vee V$ of $\mathcal{C}$, together with two morphisms*

$$\tilde{b}_V : \mathbb{I} \to {}^\vee V \otimes V \quad \text{and} \quad \tilde{d}_V : V \otimes {}^\vee V \to \mathbb{I}$$

such that analogous equations hold. A left-rigid or left autonomous category is a monoidal category in which every object has a left dual.

4. *A monoidal category is rigid or autonomous, if it is both left and right rigid or autonomous.*

Lemma 2.5.13.
A $\mathbb{K}$-vector space V has a right dual, if and only if it is finite-dimensional.

Proof.
Consider the element

$$b_V(1) = \sum_{i=1}^{N} b_i \otimes \beta_i \in V \otimes V^* \quad \text{with} \quad b_i \in V \quad \text{and} \quad \beta_i \in V^*$$

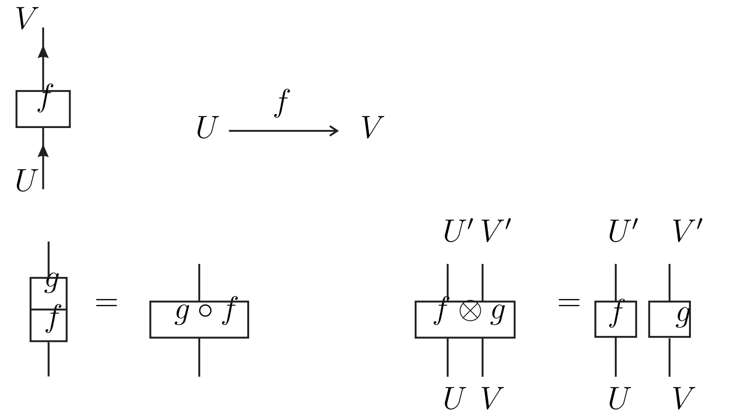
which is necessarily a finite linear combination. Then by the axioms of a duality

$$v = (\mathrm{id}_V \otimes d_V)(b_V(1) \otimes \mathrm{id}_V)(v) = \sum_{i=1}^{N} b_i \beta_i(v) \ .$$

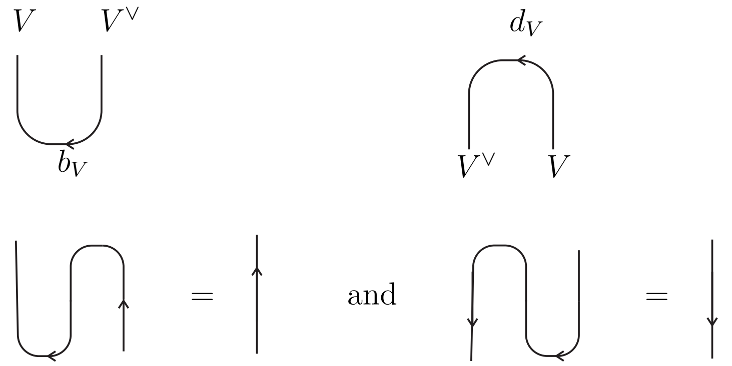
This shows that the vectors $(b_i)_{i=1,\ldots N}$ are a finite set of generators for V and thus that V is finite-dimensional. The converse is obvious. □

Remarks 2.5.14.

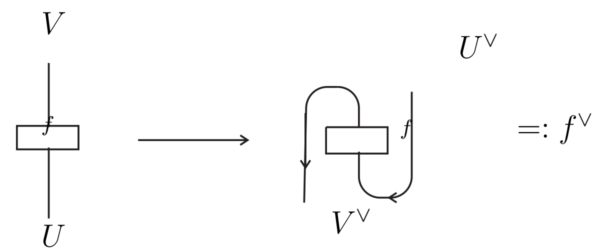
1. In any *strict* tensor category, we have a graphical calculus. Morphisms are to be read from below to above. Composition of morphisms is by joining vertically superposed boxes. The tensor product of morphisms is described by horizontally juxtaposed boxes.



We represent coevaluation and evaluation of a right duality and their defining properties as follows.



2. By definition, a right duality in a rigid tensor category associates to every object V another object $V^\vee$. We also define its action on morphisms:



   One checks graphically that this gives a functor $?^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$, i.e. a contravariant functor. Similarly, we get from the left duality a functor ${}^\vee? : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$. There is no reason, in general, for these functors to be isomorphic. The reader should check that as functors

$$ {}^\vee? : \mathcal{C} \to \mathcal{C}^{\mathrm{opp,mopp}} $$

   where we denote by $\mathcal{C}^{\mathrm{mopp}}$ the monoidal category with the opposed tensor product, are monoidal equivalences, cf. Remark 2.4.3.4.

Rigid duals are a property, not a structure as the following Lemma shows:

Lemma 2.5.15.
Let V be an object in a tensor category. Let $(V^\vee, d_V, b_V)$ and $(\tilde{V}^\vee, \tilde{d}_V, \tilde{b}_V)$ be two right duals of V. Then $V^\vee$ and $\tilde{V}^\vee$ are canonically isomorphic: there is a unique isomorphism $\varphi : V^\vee \to \tilde{V}^\vee$, such that the two diagrams

$$\begin{tikzcd} V^\vee \otimes V \arrow[rr, "\varphi\otimes \mathrm{id}_V"] \arrow[dr, "d_V"'] & & \tilde{V}^\vee \otimes V \arrow[dl, "\tilde{d}_V"] \\ & \mathbb{I} & \end{tikzcd} \qquad \begin{tikzcd} V \otimes V^\vee \arrow[rr, "\mathrm{id}_V\otimes\varphi"] & & V \otimes \tilde{V}^\vee \\ & \mathbb{I} \arrow[ul, "b_V"] \arrow[ur, "\tilde{b}_V"'] & \end{tikzcd}$$

commute.

Proof.
For simplicity, we assume that the tensor category is strict. The axioms of a duality imply that

$$\alpha : V^\vee \stackrel{\mathrm{id}_{V^\vee}\otimes\tilde{b}_V}{\longrightarrow} V^\vee \otimes V \otimes \tilde{V}^\vee \stackrel{d_V\otimes\mathrm{id}_{\tilde{V}^\vee}}{\longrightarrow} \tilde{V}^\vee$$

and

$$\beta : \tilde{V}^\vee \stackrel{\mathrm{id}_{\tilde{V}^\vee}\otimes b_V}{\longrightarrow} \tilde{V}^\vee \otimes V \otimes V^\vee \stackrel{\tilde{d}_V\otimes\mathrm{id}_{V^\vee}}{\longrightarrow} V^\vee$$

are inverse to each other. Uniqueness is easy to see. □

Proposition 2.5.16.
Let H be a Hopf algebra. Let V be an H-module. We denote by $V^\vee$ the H-module defined on the dual vector space $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ with the action given by pullback of the transpose along S. If the antipode has an inverse $S^{-1} \in \mathrm{End}(H)$ or if a skew-antipode exists, then denote by ${}^\vee V$ the H-module defined on the same vector space $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ with the action given by pullback of the transpose along S^{-1}.

1. The right evaluation
$$\begin{array}{rcl} d_V : V^\vee \otimes V & \to & \mathbb{K} \\ \alpha \otimes v & \mapsto & \alpha(v) \end{array}$$
is a map of H-modules.

2. If the antipode S of H is invertible, the left evaluation
$$\begin{array}{rcl} \tilde{d}_V : V \otimes {}^\vee V & \to & \mathbb{K} \\ v \otimes \alpha & \mapsto & \alpha(v) \end{array}$$
is a map of H-modules.

3. If V is finite-dimensional, then the right coevaluation
$$b_V : \mathbb{K} \to V \otimes V^\vee$$
is a map of H-modules.

4. If V is finite-dimensional and if the antipode S of H is invertible, then the left coevaluation is a map of H-modules.

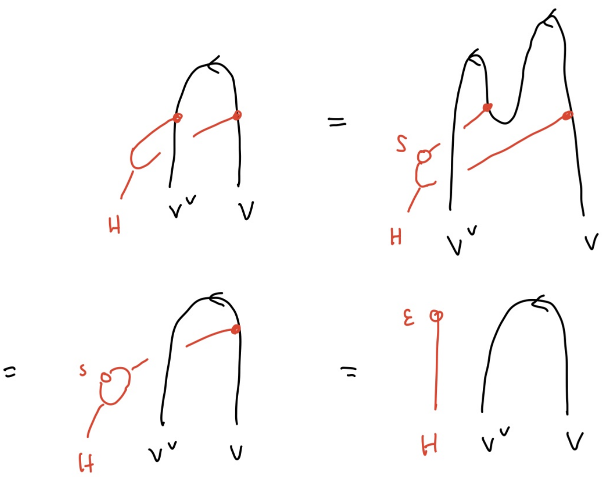
Proof.

1. Let $a \in H, v \in V$ and $\alpha \in V^*$. Then we compute

$$\begin{aligned}
d_V(a.(\alpha \otimes v)) &= \sum_{(a)} d_V(a_{(1)}.\alpha \otimes a_{(2)}.v) \\
&= \sum_{(a)} (a_{(1)}.\alpha)(a_{(2)}.v) \quad \text{[defn. of } d_V\text{]} \\
&= \alpha\Big(\sum_{(a)} S(a_{(1)})a_{(2)}.v\Big) \quad \text{[defn. of action for } V^\vee\text{]} \\
&= \alpha(\epsilon(a)v) = \epsilon(a)\alpha(v) = a.d_V(\alpha \otimes v)\,.
\end{aligned}$$

In the last line, we used the defining property of the antipode, linearity of α and the definition of the H-action on the trivial module $\mathbb{K}$.

Here is also a graphical proof:



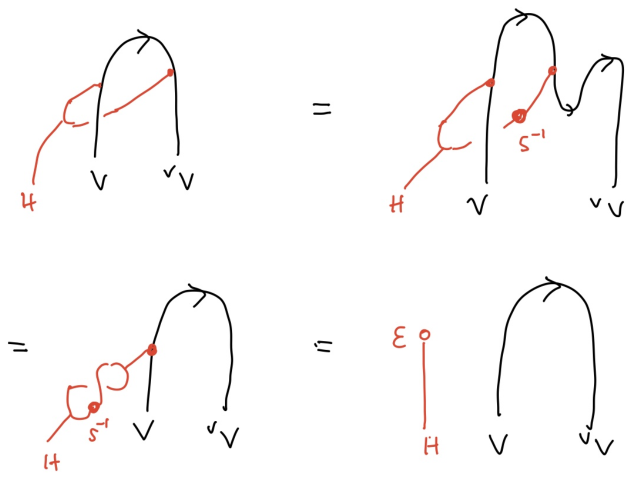
2. Similarly, we use the identity

$$S^{-1}(a_{(2)}) \cdot a_{(1)} = 1_H \epsilon(a)$$

from lemma 2.5.8 to compute for $v \in V$ and $\alpha \in V^*$

$$\begin{aligned}
\tilde{d}_V(a.(v \otimes \alpha) &= \tilde{d}_V(a_{(1)}.v \otimes a_{(2)}.\alpha) \\
&= \alpha(S^{-1}(a_{(2)})a_{(1)}.v) \\
&= \alpha(\epsilon(a)v) = a.\alpha(v)
\end{aligned}$$

Again, we present a graphical proof:



3. As a final example, we discuss the left coevaluation. We have to compare linear maps $\mathbb{K} \to V^* \otimes V \cong \mathrm{End}_{\mathbb{K}}(V)$. We compute for $\lambda \in K$ and $v \in V$

$$\begin{aligned} a.\tilde{b}_V(\lambda)v &= \lambda \textstyle\sum_i x^i(S^{-1}(a_{(1)}).v) \otimes a_{(2)}.x_i \\ &= \lambda a_{(2)} \left((\textstyle\sum_i x^i \otimes x_i)(S^{-1}(a_{(1)}).v) \right) \\ &= \lambda \left(a_{(2)} \cdot S^{-1}(a_{(1)}) \right).v = \epsilon(a)\lambda v = \tilde{b}_V(a.\lambda)v \end{aligned}$$

□

We conclude:

Corollary 2.5.17.
The category H-mod$_{\mathrm{fd}}$ of finite-dimensional modules over any Hopf algebra is right rigid. If the antipode S of H is a (composition-)invertible element of $\mathrm{End}_{\mathbb{K}}(H)$, the category H-mod$_{\mathrm{fd}}$ is rigid.

We construct another example of a monoidal category.

Definition 2.5.18

1. *Let n be any positive integer. We define a category* $\mathrm{Cob}(n)$ *of n-dimensional* <u>*cobordisms*</u> *as follows:*

   (a) *An object of* $\mathrm{Cob}(n)$ *is a closed oriented $(n-1)$-dimensional smooth oriented manifold. The empty set $\emptyset$ is considered as an $(n-1)$-dimensional manifold and thus an object of* $\mathrm{Cob}(n)$.

   (b) *Given a pair of objects $M, N \in \mathrm{Cob}(n)$, a morphism $M \to N$ is a class of bordisms from M to N. A bordism is an oriented, n-dimensional smooth manifold B with boundary, together with an orientation preserving diffeomorphism*

   $$\phi_B : \quad \overline{M} \coprod N \xrightarrow{\sim} \partial B \; .$$

   Here $\overline{M}$ denotes the same manifold with opposite orientation. (Since the empty set is an object in $\mathrm{Cob}(n)$, *every closed oriented n-dimensional smooth manifold B defines a morphism $\emptyset \to \emptyset$.)*

Two bordisms B, B' give the same morphism, if there is an orientation-preserving diffeomorphism $\phi : B \to B'$ which restricts to the evident diffeomorphism

$$\partial B \xrightarrow{\phi_{B'}^{-1}} \overline{M} \coprod N \xrightarrow{\phi_B} \partial B' \ ,$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
B \arrow[rr, "\phi"] & & B' \\
& \overline{M} \coprod N \arrow[ul, "\phi_B"] \arrow[ur, "\phi'_B"'] &
\end{tikzcd}$$

(c) *For any object $M \in \mathrm{Cob}(n)$, the identity map is represented by the product bordism $B = M \times [0, 1]$, i.e. the so-called cylinder over M.*

(d) *Composition of morphisms in $\mathrm{Cob}(n)$ is given by gluing bordisms together: given objects $M, M', M'' \in \mathrm{Cob}(n)$, and bordisms $B : M \to M'$ and $B' : M' \to M''$, the composition is defined to be the morphism represented by the manifold $B \coprod_{M'} B'$. (To get a smooth structure on this manifold, choices like collars are necessary. They lead to diffeomorphic glued bordisms, however.)*

2. *For each n, the category $\mathrm{Cob}(n)$ can be endowed with the structure of a tensor category. The tensor product*

$$\otimes : \mathrm{Cob}(n) \times \mathrm{Cob}(n) \to \mathrm{Cob}(n)$$

is given by disjoint union. The unit object of $\mathrm{Cob}(n)$ is the empty set, regarded as a smooth manifold of dimension $n - 1$.

Example 2.5.19.
The objects of $\mathrm{Cob}(1)$ are finitely many oriented points. Thus objects are finite unions of $(\bullet, +)$ and $(\bullet, -)$.

The morphisms are oriented one-dimensional manifolds, possibly with boundary, i.e. unions of intervals and circles.

An isomorphism class of objects is characterized by the numbers (n_+, n_-) of points with positive and negative orientation. Sometimes, one also considers another equivalence relation on objects: two $d-1$-dimensional closed manifolds M and N are called cobordant, if there exists a cobordism $B : M \to N$. Since there is a cobordism $(\bullet, +) \coprod (\bullet, -) \to \emptyset$, the objects (n_+, n_-) and (n'_+, n'_-) are cobordant, if and only if $n_+ - n_- = n'_+ - n'_-$.

One can also define a category of unoriented cobordisms. In this case, objects are finite disjoint unions of points, isomorphism classes are in bijection to the number of points. Since a pair of points can annihilate, there are only two cobordism classes, consisting of the set with an even and and odd number of points, respectively.

We next comment on the rigidity of the category $\mathrm{Cob}(n)$:

Observation 2.5.20.
Let M be a closed oriented $n-1$-dimensional smooth manifold. Then the oriented n-dimensional manifold $B := M \times [0, 1]$, the cylinder over M, has boundary $M \coprod \overline{M}$. The manifold B can be considered as a cobordism in six different ways, corresponding to decomposition of its boundary:

- As a bordism $M \to M$. This represents the identity on M.

- As a bordism $\overline{M} \to \overline{M}$. This represents the identity on $\overline{M}$.
- As a morphism $d_M : \overline{M} \coprod M \to \emptyset$ or, alternatively, as a morphism $\tilde{d}_M : M \coprod \overline{M} \to \emptyset$.
- As a morphism $\tilde{b}_M : \emptyset \to \overline{M} \coprod M$ or, alternatively, as a morphism $b_M : \emptyset \to M \coprod \overline{M}$.

One checks that the axioms of a left and a right duality hold. We conclude that the category $\mathrm{Cob}(n)$ is rigid. (It has in fact more structure.)

We discuss a final example.

Example 2.5.21.
We have seen that for any small category $\mathcal{C}$, the endofunctors of $\mathcal{C}$, together with natural transformations, form a monoidal category. In this case, a left dual of an object, i.e. of a functor, is also called its left adjoint functor. Indeed, the following generalization beyond endofunctors is natural and a central notion of category theory.

Definition 2.5.22

1. *Let $\mathcal{C}$ and $\mathcal{D}$ be any categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$, if for any two objects c in $\mathcal{C}$ and d in $\mathcal{D}$ there is an isomorphism of Hom-spaces*

$$\Phi_{c,d} : \mathrm{Hom}_{\mathcal{C}}(c, Gd) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Fc, d)$$

with the following naturality property:

For any homomorphism $c' \xrightarrow{f} c$ in $\mathcal{C}$ and $d \xrightarrow{g} d'$ in $\mathcal{D}$ consider for $\varphi \in \mathrm{Hom}_{\mathcal{D}}(Fc, d)$ the morphism

$$\mathrm{Hom}(Ff, g)(\varphi) := Fc' \xrightarrow{Ff} Fc \xrightarrow{\varphi} d \xrightarrow{g} d' \in \mathrm{Hom}_{\mathcal{D}}(Fc', d')$$

and for $\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ the morphism

$$\mathrm{Hom}(f, Gg)(\varphi) := c' \xrightarrow{f} c \xrightarrow{\varphi} Gd \xrightarrow{Gg} Gd' \in \mathrm{Hom}_{\mathcal{C}}(c', Gd') \ .$$

The naturality requirement for the family $(\Phi_{c,d})$ of isomorphisms is then the requirement that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[r, "\mathrm{Hom}(f,Gg)"'] \arrow[d, "\Phi_{c,d}"] & \mathrm{Hom}_{\mathcal{C}}(c', Gd') \arrow[d, "\Phi_{c',d'}"'] \\
\mathrm{Hom}_{\mathcal{D}}(Fc, d) \arrow[r, "\mathrm{Hom}(Ff,g)"'] & \mathrm{Hom}_{\mathcal{D}}(Fc', d')
\end{tikzcd}$$

commutes for all morphisms f, g.

2. *We write $F \dashv G$ and also say that the functor G is a right adjoint to F.*

Examples 2.5.23.

1. In general, the existence of a left adjoint functor does not imply the existence of a right adjoint functor. Even if both adjoints exist, they need not coincide. Also, the isomorphisms $\Phi_{c,d}$ are in general not unique.

2. As explained in the appendix, the forgetful functor

$$U : \text{ vect}(\mathbb{K}) \to \text{Set} \ ,$$

which assigns to any $\mathbb{K}$-vector space the underlying set has as a left adjoint, the free vector space on a set:

$$F : \text{ Set} \to \text{vect}(\mathbb{K}) \ ,$$

Indeed, we have for any set M and any $\mathbb{K}$-vector space V an isomorphism

$$\begin{array}{rlcl} \Phi_{M,V} : & \mathrm{Hom}_{\mathrm{Set}}(M, U(V)) & \to & \mathrm{Hom}_K(F(M), V) \\ & \varphi & \mapsto & \Phi_{M,V}(\varphi) \end{array}$$

where $\Phi_{M,V}(\varphi)$ is the $\mathbb{K}$-linear map defined by prescribing values in V on the distinguished basis of $F(M)$ using φ and extending linearly:

$$\Phi_{M,V}(\varphi)(\sum_{m\in M} \lambda_m m) := \sum_{m\in M} \lambda_m \varphi(m) \ .$$

In particular, we find the isomorphism of sets $\mathrm{Hom}_{\mathrm{Set}}(\emptyset, U(V)) \cong \mathrm{Hom}_{\mathbb{K}}(F(\emptyset), V)$ for all $\mathbb{K}$-vector spaces V. Thus $\mathrm{Hom}_{\mathbb{K}}(F(\emptyset), V)$ has exactly one element for any vector space V. This shows $F(\emptyset) = \{0\}$, i.e. the vector space freely generated by the empty set is the zero-dimensional vector space.

3. In general, freely generated objects are obtained as images under left adjoints of forgetful functors. It is, however, not true that any forgetful functor has a left adjoint. As a counterexample, take the forgetful functor U from the category of all fields to sets. Suppose a left adjoint exists and study the image K of the empty set under it. Then K is a field such that for any other field L, we have a bijection

$$\mathrm{Hom}_{Field}(K, L) \cong \mathrm{Hom}_{Set}(\emptyset, U(L)) \cong \star \ .$$

Since morphisms of fields are injective, such a field K would be a subfield of any field L. Such a field does not exist.

To make contact with the notion of duality, the following reformulation of the notion of a pair $F \dashv G$ of adjoint functors is needed:

Observation 2.5.24.

1. Let $F \dashv G$ be adjoint functors. From the definition, we get isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(G(d), G(d)) \cong \mathrm{Hom}_{\mathcal{D}}(F(G(d)), d)$$

and

$$\mathrm{Hom}_{\mathcal{D}}(F(c), F(c)) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(F(c))) \ .$$

The images of the identity on $G(d)$ and $F(c)$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ respectively form two natural transformations

$$\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}} \quad \text{ and } \quad \eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \ .$$

Note the different order of the functors F, G in the composition and compare to the definition 2.5.12 of a pair of dual objects. The natural transformation η is called the <u>unit</u>, ϵ is called the <u>counit</u> of the adjunction. The natural transformations η and ϵ are not unique, since the isomorphisms $\Phi_{c,d}$ are not unique. In particular, for a $\mathbb{K}$- linear category, given any $\lambda \in K^\times$, we can replace η by $\lambda\eta$ and ϵ by $\lambda^{-1}\epsilon$ to get another pair of morphisms.

These natural transformations have the property that for all objects c in $\mathcal{C}$ and d in $\mathcal{D}$ the morphisms

$$G(d) \xrightarrow{\eta_{G(d)}} (GF)\,G(d) = \; G(FG)(d) \xrightarrow{G(\epsilon_d)} G(d)$$

and

$$F(c) \;\xrightarrow{F(\eta_c)} F(GF)(c) = (FG)F(c) \xrightarrow{\epsilon_{F(c)}} F(c)$$

are identities. Again compare with the properties of a pair of dual objects. In particular, the left adjoint of an endofunctor is its left dual in the monoidal category of endofunctors with monoidal product $F \otimes G = G \circ F$. For proofs, we refer to [McL, Chapter IV]

2. Conversely, we can recover the adjunction isomorphisms $\Phi_{c,d}$ from the natural transformations ϵ and η by

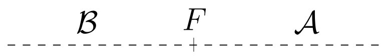
$$\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(c), F(G(d)) \xrightarrow{(\epsilon_d)_*} \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

and their inverses by

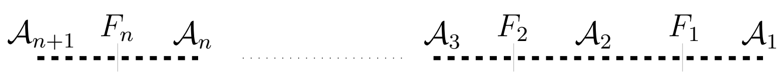
$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(F(c)), G(d)) \xrightarrow{\eta_c^*} \mathrm{Hom}_{\mathcal{C}}(c, G(d))\,.$$

3. Note that a pair of adjoint functors $F \dashv G$ is an equivalence of categories, if and only if ϵ and η are natural isomorphisms of functors. In this case, one has an adjoint equivalence. Any equivalence of categories can be improved to an adjoint equivalence, cf. [McL, IV.4, Thm. 1].

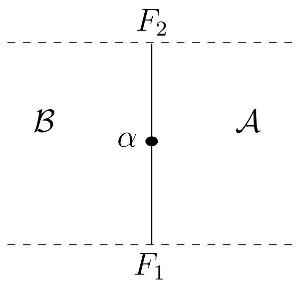
Remark 2.5.25.
Adjoint functors can be understood in a planar diagrammatics, cf. e.g. [Kh, Section 1]. Consider one-dimensional diagrams, with one-dimensional segments describing categories and zero-dimensional parts indicating functors. In our convention, such diagrams are drawn horizontally and are to be read from right to left. Thus for $\mathcal{A}$ and $\mathcal{B}$ categories and a functor $F\colon \mathcal{A} \to \mathcal{B}$, we draw the diagram



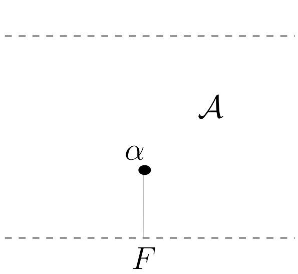
The composition $F_n \cdots F_1 \equiv F_n \circ \cdots \circ F_1\colon \mathcal{A}_1 \to \mathcal{A}_n$ of functors $F_i\colon \mathcal{A}_i \to \mathcal{A}_{i+1}$ is represented by horizontal concatenation



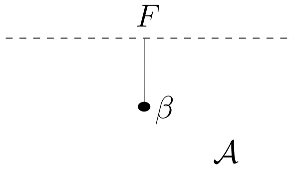
To accommodate also natural transformations, a second dimension is needed. Categories are now represented by two-dimensional regions and functors by one-dimensional vertical segments, while zero-dimensional parts indicate natural transformations. In our convention, the vertical direction is to be read from bottom to top. Thus a natural transformation $\alpha\colon F_1 \Rightarrow F_2$ between the functors F_1, F_2 from objects $\mathcal{A}$ to $\mathcal{B}$ is depicted by the diagram



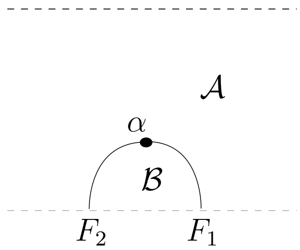
For the moment, we require that the strands always go from bottom to top and do not allow 'U-turns'. Such diagramms are called progressive in [JS]. For the identity natural transformation $\alpha = \mathrm{id}_F$ we omit the blob in the diagram. For the identity functor $\mathrm{id}_\mathcal{A}$ we omit any label except for the one referring to the category $\mathcal{A}$. With these conventions, natural transformations $\alpha\colon F \Rightarrow \mathrm{id}_\mathcal{A}$ and $\beta\colon \mathrm{id}_\mathcal{A} \Rightarrow F$ with F an endofunctor of the category $\mathcal{A}$ are drawn as



and

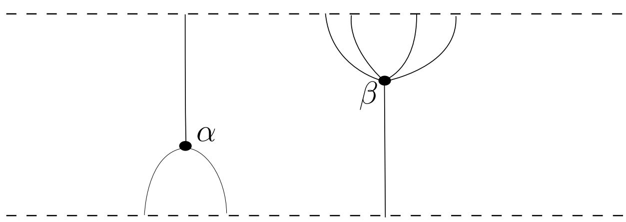


respectively, while a natural transformation $F_2F_1 \Rightarrow \mathrm{id}_\mathcal{A}$ is represented by



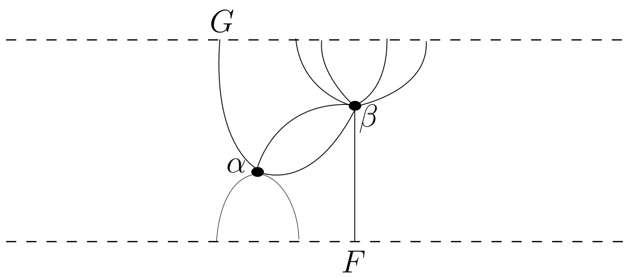
Natural transformations can be composed horizontally and vertically. Horizontal composition is depicted as juxtaposition, as in

$\alpha \otimes \beta =$



Vertical composition is represented as vertical concatenation of diagrams; thus e.g.

$(\mathrm{id}_G \otimes \beta) \circ (\alpha \otimes \mathrm{id}_F) =$



By observation 2.5.24, for a pair of adjoint functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, with $F \dashv G$, we have natural transformations

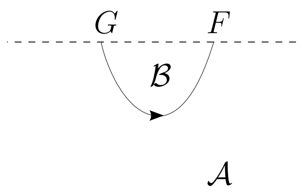
$$\eta: \quad \mathrm{id}_\mathcal{A} \Rightarrow GF \quad \text{and} \quad \epsilon: \quad FG \Rightarrow \mathrm{Id}_\mathcal{B}$$

satisfying

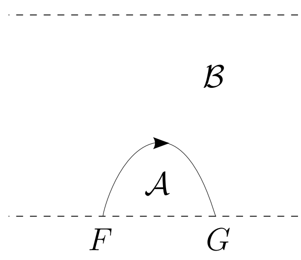
$$(\mathrm{id}_F \otimes \eta) \circ (\epsilon \otimes \mathrm{id}_F) = \mathrm{id}_F \quad \text{and} \quad (\eta \otimes \mathrm{id}_G) \circ (\mathrm{id}_G \otimes \epsilon) = \mathrm{id}_G\,.$$

In the diagrammatic description, special notation is introduced for the unit and counit of an adjoint pair of functors: we depict them as

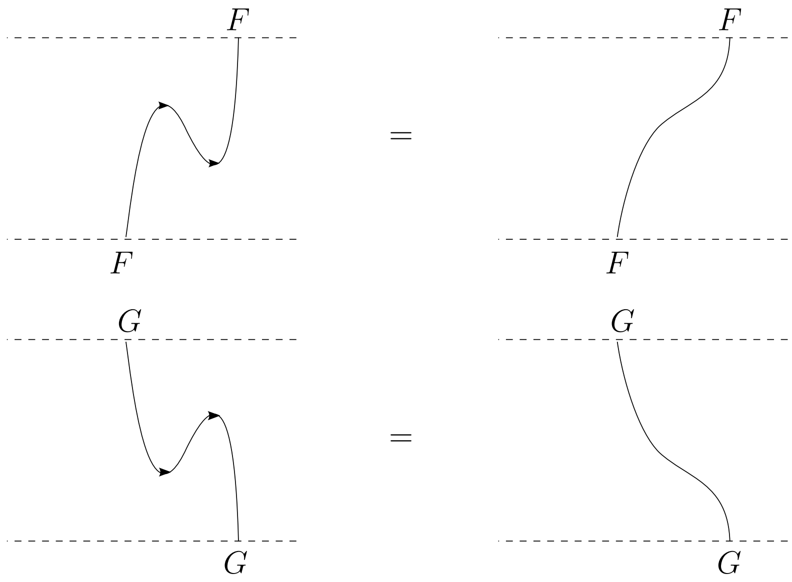
$\eta =$



and $\epsilon =$



The equalities (2.5.25) amount to the identifications



of diagrams.

Example 2.5.26.
Let $\mathcal{C}$ be a rigid tensor category. Then for any triple U, V, W of objects of $\mathcal{C}$, we have natural bijections

$$\begin{array}{ccc}
\mathrm{Hom}(U \otimes V, W) & \cong & \mathrm{Hom}(U, W \otimes V^\vee) \\
\lambda & \mapsto & (\lambda \otimes \mathrm{id}_{V^\vee}) \circ (\mathrm{id}_U \otimes b_V) \\
\mathrm{Hom}(U \otimes V, W) & \cong & \mathrm{Hom}(V, {}^\vee U \otimes W) \\
\lambda & \mapsto & (\mathrm{id}_{{}^\vee U} \otimes \lambda) \circ (\tilde{b}_U \otimes \mathrm{id}_V)
\end{array}$$

We have thus the following adjunctions of functors:

$$(? \otimes V) \dashv (? \otimes V^\vee) \quad \text{and} \quad (U \otimes ?) \dashv ({}^\vee U \otimes ?) \ .$$

We will see in Lemma 3.2.18 that this implies that for $\mathcal{C}$ a rigid abelian category, the functors $- \otimes V$ and $V \otimes -$ are exact.

We now relate adjunctions and monads.

Proposition 2.5.27.

1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors, i.e. $F \dashv G$. Then the endofunctor $T := G \circ F : \mathcal{C} \to \mathcal{C}$, together with the natural transformation

$$\mu : \quad T \circ T = G \circ F \circ G \circ F \xrightarrow{G\epsilon F} G \circ F$$

which on objects reads

$$\mu_c : \quad GFGF(c) \stackrel{G\epsilon_{F(c)}}{\longrightarrow} GF(c)$$

together with the natural transformation $\mathrm{id}_\mathcal{C} \to G \circ F$ in Observation 2.5.24 is a monad.

2. Given a monad $T : \mathcal{C} \to \mathcal{C}$, consider the forgetful functor $U : T-\mathrm{mod} \to \mathcal{C}$ which sends the module $(m, \rho) \in T-\mathrm{mod}$ to $m \in \mathcal{C}$ and the induction functor $I : \mathcal{C} \to T-\mathrm{mod}$ sending $c \in \mathcal{C}$ to the free module $T(c)$ with action $\mu_c : T(T(c)) \to T(c)$. These functors are adjoints, $I \dashv U$.

Proof.
The proof of 1. will be an exercise. For 2., we remark that the adjunction

$$\mathrm{Hom}_T(Ic, m) \cong \mathrm{Hom}_\mathcal{C}(c, Um)$$

with $m = (d, \rho : Td \to d)$ is given by the map that sends the morphism $c \stackrel{f}{\to} d$ in $\mathcal{C}$ to the morphism

$$Tc \stackrel{Tf}{\to} Td \stackrel{\rho}{\to} d$$

which is a morphism of T-modules. Its inverse sends a morphism of T-modules $Tc \stackrel{f}{\to} m$ to the morphism

$$c \stackrel{\eta_c}{\to} Tc \stackrel{f}{\to} m\ .$$

One should notice that $U \circ I = T$. Hence, different adjunctions can give rise to the same monad. □

Remarks 2.5.28.

1. In the situation of the preceding proposition, we define a comparison functor

$$K : \mathcal{D} \to T-\mathrm{mod}$$

which sends $d \in \mathcal{D}$ to the object $Gd \in \mathcal{C}$ with module structure given by

$$GFGd \stackrel{G\epsilon_d}{\to} Gd$$

2. An adjunction is called <u>monadic</u>, if K is an equivalence of categories. Beck's monadicity theorem [McL, Chapter VI.7] gives criteria on on the functor G ensuring this.

Example 2.5.29.
Let $\mathcal{C} = \mathrm{Set}$ and $\mathcal{D} = Grp$ be the category of groups. The forgetful functor $U : Grp \to \mathrm{Set}$ has as a left adjoint the functor $\mathcal{F} : \mathrm{Set} \to Grp$ that assigns to a set X the free group $\mathcal{F}(X)$ on this set. This adjunction is monadic.

The monad $T : \mathrm{Set} \to \mathrm{Set}$ assigns to a set X the set $\dot{\mathcal{F}}(X)$ underlying the free group on X. These are words in an alphabet $(x^+, x^-)_{x \in X}$ modulo the relation identifying x_+x_- and x_-x_+ with the empty word. The statement that the adjunction is monad means that we can describe a group G in terms of a map of sets from the free group $\mathcal{F}(G) \to G$ which encodes the relations for the set of generators G that characterize the group. For more information on adjunctions, we refer to [Riehl, Chapter 4,5] for a helpful exposition where it is also explained how categories can be seen as modules over a monad on a category of graphs.

2.6 Examples of Hopf algebras

We will now consider several examples of Hopf algebras that are neither group algebras nor universal enveloping algebras.

Observation 2.6.1.
The following example is due to Taft. Let $\mathbb{K}$ be a field and $N \geq 2$ a natural number. Assume that there exists a primitive N-th root of unity ζ in $\mathbb{K}$. Consider the algebra $H = H_N$ generated over $\mathbb{K}$ by two elements g and x, subject to the relations

$$g^N = 1, \quad x^N = 0, \quad xg = \zeta gx \ .$$

We say that the elements x and g ζ-commute. The algebra H_N can be shown to have finite dimension N^2 and a basis $g^i x^j$ with $0 \leq i, j \leq N-1$.

We claim that there are algebra maps

$$\Delta : H \to H \otimes H, \quad S : H \to H^{\mathrm{opp}} \quad \text{and } \epsilon : H \to \mathbb{K}$$

uniquely determined on the generators g, x by

$$\begin{gathered} \Delta(g) = g \otimes g \text{ and } \Delta(x) = 1 \otimes x + x \otimes g \\ \epsilon(x) = 0 \text{ and } \epsilon(g) = 1 \\ S(g) = g^{-1} \text{ and } S(x) = -xg^{-1} \end{gathered}$$

and that these maps endow the algebra H_N with the structure of a Hopf algebra. The special case $\zeta = -1$, i.e. $N = 2$, is also known as Sweedler's Hopf algebra.

We work out the coproduct Δ in detail and leave the discussion of the counit ϵ and the antipode S to the reader. We have to show that the map Δ extends to a well-defined algbera morphism, i.e. that it is compatible with the three defining relations of H_N. To check compatibility with the relation $g^N = 1$, we compute for $n \in \mathbb{N}$

$$\Delta(g^n) = \Delta(g)^n = (g \otimes g)^n = g^n \otimes g^n \ ,$$

where we use that Δ has to be a morphism of algebras, the definition of $\Delta(g)$ and the product in $H_N \otimes H_N$. To be compatible with the relation $g^N = 1$, the expression $\Delta(g^N)$ has to be equal to $\Delta(1) = 1 \otimes 1$, which indeed follows from the relation $g^N = 1$. To show compatibility with the relation $xg = \zeta gx$, compare

$$\Delta(xg) = \Delta(x) \cdot \Delta(g) = (1 \otimes x + x \otimes g) \cdot (g \otimes g) = g \otimes xg + xg \otimes g^2$$

and

$$\Delta(\zeta gx) = \zeta\Delta(g) \cdot \Delta(x) = \zeta(g \otimes g) \cdot (1 \otimes x + x \otimes g) = \zeta g \otimes gx + \zeta gx \otimes g^2$$

which implies $\Delta(xg) = \Delta(\zeta gx)$.

For the remaining relation $x^N = 0$, we need a few more relations:

Observation 2.6.2.

1. We denote for $n \in \mathbb{N} \setminus \{0\}$ the following element in the polynomial ring $\mathbb{Z}[\mathbf{q}]$

$$(n)_{\mathbf{q}} := 1 + \mathbf{q} + \ldots + \mathbf{q}^{n-1} \in \mathbb{Z}[\mathbf{q}]$$

and

$$(n)!_{\mathbf{q}} := (n)_{\mathbf{q}} \cdots (2)_{\mathbf{q}}(1)_{\mathbf{q}} \in \mathbb{Z}[\mathbf{q}] \ .$$

Finally, define for $0 \leq i \leq n$ in the field of fractions of $\mathbb{Z}[\mathbf{q}]$

$$\begin{pmatrix} n \\ i \end{pmatrix}_{\mathbf{q}} := \frac{(n)!_{\mathbf{q}}}{(n-i)!_{\mathbf{q}}(i)!_{\mathbf{q}}} \ .$$

2. We note the identity in the polynomial ring $\mathbb{Z}[\mathbf{q}]$:

$$\mathbf{q}^k (n+1-k)_{\mathbf{q}} + (k)_{\mathbf{q}} = (n+1)_{\mathbf{q}}$$

and thus deduce

$$\begin{aligned} \mathbf{q}^k \begin{pmatrix} n \\ k \end{pmatrix}_{\mathbf{q}} + \begin{pmatrix} n \\ k-1 \end{pmatrix}_{\mathbf{q}} &= \tfrac{(n)!_{\mathbf{q}}}{(n+1-k)!_{\mathbf{q}}(k)!_{\mathbf{q}}} \cdot \left(\mathbf{q}^k (n+1-k)_{\mathbf{q}} + (k)_{\mathbf{q}} \right) \\ &= \begin{pmatrix} n+1 \\ k \end{pmatrix}_{\mathbf{q}} \end{aligned}$$

from which we conclude by induction on n that $\begin{pmatrix} n \\ k \end{pmatrix}_{\mathbf{q}} \in \mathbb{Z}[\mathbf{q}]$.

Given a field $\mathbb{K}$, we can then specialize for $q \in \mathbb{K}$ the values of $(n)_{\mathbf{q}}$, $(n)!_{\mathbf{q}}$ and the of the $\mathbf{q}$-binomials and to obtain elements $(n)_q, (n)!_q \in \mathbb{K}$ and $\begin{pmatrix} n \\ k \end{pmatrix}_q \in \mathbb{K}$. Note that $(n)_1 = n$. If $q \in \mathbb{K}$ is an N-th root of unity different from 1, then

$$(N)_q = 1 + q + \ldots + q^{N-1} = \frac{1-q^N}{1-q} = 0 \ .$$

In a field of characteristic $p > 0$, the quantity $N := 1 + \ldots + 1$ with p summands also vanishes. There are indeed similarities between q-deformed situations and situations in fields of prime characteristic. As a further consequence, for q an N-th root of unity,

$$\begin{pmatrix} N \\ k \end{pmatrix}_q = 0 \quad \text{for all} \quad 0 < k < N \ .$$

Lemma 2.6.3.
Let A be an associative algebra over a field $\mathbb{K}$ and $q \in \mathbb{K}$. Let $x, y \in A$ be two elements that q-commute, i.e. $xy = qyx$. Then the quantum binomial formula holds for all $n \in \mathbb{N}$:

$$(x+y)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix}_q y^i x^{n-i} \ .$$

Proof.
By induction on n, using the relation we proven in Observation 2.6.2.2. □

We then conclude, since for the Taft-Hopf algebra $1 \otimes x$ and $x \otimes g$ ζ-commute, we have

$$\begin{aligned} \Delta(x^N) = \Delta(x)^N &= (1 \otimes x + x \otimes g)^N = \textstyle\sum_{i=0}^{N} \begin{pmatrix} N \\ i \end{pmatrix}_\zeta (x \otimes g)^i (1 \otimes x)^{N-i} \\ &= (x \otimes g)^N + (1 \otimes x)^N = x^N \otimes g^N + 1 \otimes x^N = 0 \end{aligned}$$

In the second identity, we used that the binomial coefficients vanish, except for $i = 0, N$. This shows that the coproduct of the Taft Hopf algebra is well-defined.

We remark that for the square of the antipode, we have

$$S^2(g) = S(g^{-1}) = g \quad \text{ and } S^2(x) = S(-xg^{-1}) = -S(g^{-1})S(x) = gxg^{-1}$$

which is a so-called inner automorphism of order N. Thus there exist finite-dimensional Hopf algebras with antipode S of any even order.

Note that the Taft algebra is, in general, not cocommutative. Indeed, one can show that over an algebraically closed field $\mathbb{K}$ of characteristic zero, all finite-dimensional cocommutative Hopf algebras are group algebras of some finite group. More precisely, the Cartier-Kostant-Milnor-Moore theorem [Sweedler, Theorem 8.1.5] asserts that over an algebraically closed field $\mathbb{K}$ of characteristic zero, any cocommutative Hopf algebra can be written as $U(\mathfrak{g}) \rtimes \mathbb{K}[G]$, where G is a group acting on a Lie algebra $\mathfrak{g}$.

This is not true in finite characteristic. To provide a counterexample, we need a class of Lie algebras with extra structure: restricted Lie algebras.

Observation 2.6.4.
Let $\mathbb{K}$ be a field of prime characteristic, $\mathrm{char}\mathbb{K} = p$. Let A be any $\mathbb{K}$-algebra. The algebra A might even be non-associative. Then the derivations $\mathrm{Der}(A)$ form a Lie subalgebra of the Lie algebra $\mathrm{End}_{\mathbb{K}}(A)$. Moreover, if $D : A \to A$ is a derivation, then because of

$$D^p(a \cdot b) = \sum_{i=0}^{p} \binom{p}{i} D^i(a) \cdot D^{p-i}(b) = D^p(a) \cdot b + a \cdot D^p(b)$$

the p-th power of D, i.e. $D^p : A \to A$ is a derivation as well. Thus the Lie algebra $\mathrm{Der}(A)$ has more structure: the structure of a restricted Lie algebra.

Definition 2.6.5

1. *Let $\mathbb{K}$ be a field of characteristic $p > 0$. A* <u>*restricted Lie algebra*</u> *L over $\mathbb{K}$ is a Lie algebra, together with a map*
$$\begin{array}{ccc} L & \to & L \\ a & \mapsto & a^{[p]} \end{array}$$
such that for all $a, b \in L$ and $\lambda \in \mathbb{K}$
$$\begin{array}{rcl} (\lambda a)^{[p]} & = & \lambda^p a^{[p]} \\ \mathrm{ad}(b^{[p]}) & = & (\mathrm{ad} b)^p \\ (a+b)^{[p]} & = & a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b) \end{array}$$
Here $\mathrm{ad}(a) : L \to L$ denotes the adjoint representation of L on L with $\mathrm{ad}(a)(b) = [a, b]$. Moreover, $i \cdot s_i(a, b)$ is the coefficient of λ^{i-1} in $\mathrm{ad}(\lambda a + b)^{p-1}(a)$.

2. *A morphism of restricted Lie algebras $f : L \to L'$ is a morphism of Lie algebras such that $f(a^{[p]}) = f(a)^{[p]}$ for all $a \in L$.*

Example 2.6.6.
If A is an associative $\mathbb{K}$-algebra with $\mathbb{K}$ a field of prime characteristic, $\mathrm{char}(\mathbb{K}) = p$, then the commutator and the map $a \mapsto a^p$ turns it into a restricted Lie algebra.

Observation 2.6.7.

1. Let L be a restricted Lie algebra, U its universal enveloping algebra. Denote by B the two-sided ideal in U generated by $a^p - a^{[p]}$ for all $a \in L$. Denote by $\mathcal{U}$ the quotient algebra $\mathcal{U} := U/B$. It is a restricted Lie algebra with $a^{[p]}$ given by the p-th power.

2. Then the canonical quotient map $\pi : L \to \mathcal{U}$ is a morphism of restricted Lie algebras. It is universal in the following sense: if A is any associative algebra over $\mathbb{K}$ and $f : L \to A$ a

morphism of restricted Lie algebras, then there exists a unique algebra map $F : \mathcal{U} \to A$ such that $f = F \circ \pi$:

$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathcal{U} \arrow[d, dashed, "\exists! F"] \\ & A \end{tikzcd}$$

3. By the universal property, the restricted morphisms

$$\begin{array}{rcl} L & \to & \mathbb{K} \\ a & \mapsto & 0 \\ L & \to & L \times L \\ a & \mapsto & (a, a) \\ L & \to & L^{\mathrm{opp}} \\ a & \mapsto & -a \end{array}$$

define algebra maps

$$\epsilon : \mathcal{U} \to \mathbb{K} \ , \quad \Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad S : \mathcal{U} \to \mathcal{U}^{\mathrm{opp}}$$

that are uniquely determined by

$$\begin{array}{rcl} \epsilon(\pi(a)) & = & 0 \\ \Delta(\pi(a)) & = & 1 \otimes \pi(a) + \pi(a) \otimes 1 \\ S(\pi(a)) & = & -\pi(a) \end{array}$$

for $a \in L$ that turn $\mathcal{U}$ into a cocommutative Hopf algebra. It is called the u-algebra of the restricted Lie algebra L.

4. One has the following variant of the Poincaré-Birkhoff-Witt theorem: the natural map $\iota_L : L \to \mathcal{U}$ is injective. If $(u_i)_{i \in I}$ is an ordered basis for L, then

$$u_{i_1}^{k_1} \cdot u_{i_2}^{k_2} \ldots u_{i_r}^{k_r} \quad \text{with} \quad i_1 \leq i_2 \leq \ldots i_r \quad \text{and} \quad 0 \leq k_j \leq p-1$$

is a basis of $\mathcal{U}$.

5. Thus if L has finite-dimension, $\dim_{\mathbb{K}} L = n$, then $\mathcal{U}$ is finite-dimensional of dimension $\dim \mathcal{U} = p^n$. Thus $\mathcal{U}$ is a cocommutative finite-dimensional Hopf algebra.

To show that a restricted Lie algebra is not isomorphic to the group algebra of any finite group, we need some notions which are of independent interest:

Definition 2.6.8

1. *An element $h \in H \setminus \{0\}$ of a Hopf algebra H is called group-like , if $\Delta(h) = h \otimes h$. The set of group-like elements of a Hopf algebra H is denoted by $G(H)$.*

2. *An element $h \in H$ of a bialgebra H is called a primitive element, if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of a bialgebra H is denoted by $P(H)$.*

3. *More generally, if $g_1, g_2 \in G(H)$ are group-like elements, an element $h \in H$ is called g_1, g_2-primitive, if $\Delta(h) = g_1 \otimes h + h \otimes g_2$.*

Remark 2.6.9.

1. Consider group-like elements in the dual $\mathbb{K}$-linear Hopf algebra H^*. These are $\mathbb{K}$-linear maps $\beta : H \to \mathbb{K}$ such that for $h_1, h_2 \in H$

$$\beta(h_1 \cdot h_2) = \Delta(\beta)(h_1 \otimes h_2) = (\beta \otimes \beta)(h_1 \otimes h_2) = \beta(h_1) \cdot \beta(h_2) \ .$$

   Thus the group-like elements in the dual Hopf algebra H^* are the algebra maps $H \to \mathbb{K}$ which are also called <u>characters</u> of H.

2. The primitive elements in the dual Hopf algebra H^* are linear maps $D : H \to \mathbb{K}$ such that for all $h_1, h_2 \in H$

$$D(h_1 \cdot h_2) = (1 \otimes D + D \otimes 1)(h_1 \otimes h_2) = \epsilon(h_1) \cdot D(h_2) + D(h_1) \cdot \epsilon(h_2) \ .$$

   Thus, they are the $\mathbb{K}$-valued derivations of H.

We need the following Lemma which is also important in Galois theory:

Lemma 2.6.10 (Artin)**.**
Let M be an associative monoid. Let $\chi_1, \dots \chi_n$ be pairwise different characters $\chi_i : M \to \mathbb{K}^\times$, i.e. group homomorphisms of the monoid M with values in the multiplicative group $\mathbb{K}^\times$ of a field $\mathbb{K}$. Then these characters are linearly independent as $\mathbb{K}$-valued functions on M.

Proof.
By induction on n. The assertion holds for $n = 1$, since for a character $\chi(M) \subseteq \mathbb{K}^\times$ so that a single character is linearly independent.

Thus assume $n > 1$ and consider a non-trivial relation

$$a_1\chi_1 + \cdots + a_m\chi_m = 0 \qquad\qquad (*)$$

of minimal length m in which all coefficients are non-zero, $a_i \neq 0$ for all $i = 1, \dots m$. Thus $2 \leq m \leq n$.

From $\chi_1 \neq \chi_2$ we deduce that there is $z \in M$ such that $\chi_1(z) \neq \chi_2(z)$. Using the multiplicativity of characters, we find for all $x \in M$:

$$\begin{aligned} 0 &= a_1\chi_1(zx) + \cdots + a_m\chi_m(zx) \\ &= a_1\chi_1(z)\chi_1(x) + \cdots + a_m\chi_m(z)\chi_m(x). \end{aligned}$$

and thus a different non-trivial linear relation of the characters:

$$\sum_{i=1}^m a_i\chi_i(z)\,\chi_i = 0\,.$$

Dividing this relation by $\chi_1(z) \neq 0$ and subtracting it from$(*)$, we find

$$a_2\left(\underbrace{\frac{\chi_2(z)}{\chi_1(z)} - 1}_{\neq 0}\right)\chi_2 + \cdots + a_m\left(\frac{\chi_m(z)}{\chi_1(z)} - 1\right)\chi_m = 0\,.$$

and thus a shorter non-trivial relation. □

Proposition 2.6.11.
Let H be a Hopf algebra over a field $\mathbb{K}$.

1. We have $\epsilon(x) = 1$ for any group-like element $x \in H$.
2. The set of group-like elements $G(H)$ is a subgroup of the set of units of H. The inverse of $x \in G(H)$ is $S(x)$.
3. Distinct group-like elements are linearly independent. In particular, the set of group-like elements of a group algebra $\mathbb{K}[G]$ is precisely G.

Proof.

1. We note that by definition of the counit ϵ,
$$x = (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = \epsilon(x)x \ .$$
Since by definition for a group-like element x, we have $x \neq 0$, this implies over a field $\epsilon(x) = 1$.
2. Using the fact that S is a coalgebra antihomomorphism, we find for a group-like element $x \in H$
$$\Delta(S(x)) = (S \otimes S) \circ \Delta^{\mathrm{copp}}(x) = (S \otimes S)(x \otimes x) = S(x) \otimes S(x)$$
so that $S(x)$ is group-like, provided that $S(x) \neq 0$. The defining identity of the antipode, applied to a group-like element x shows
$$xS(x) = (\mathrm{id} * S)(x) = 1\epsilon(x) \overset{1.}{=} 1$$
so that $S(x)$ is the multiplicative inverse of x in the algebra underlying H. In particular, it follows that $S(x) \neq 0$ for all grouplike elements $x \in H$.
3. Using the embedding $H \hookrightarrow H^{**}$, group-like elements of H are characters on the monoid H^* with values in the field $\mathbb{K}$. By Artin's lemma 2.6.10, characters are linearly independent.

□

Proposition 2.6.12.

1. For any primitive element x in a bialgebra H, we have $\epsilon(x) = 0$.
2. The commutator
$$[x, y] = xy - yx$$
of two primitive elements x, y of a bialgebra H is again primitive.

Proof.

1. The equation
$$x = (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = (\epsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = \epsilon(x)1 + \epsilon(1)x$$
for x primitive implies $\epsilon(x) = 0$.
2. We compute for primitive elements $x, y \in H$
$$\begin{aligned} \Delta(x \cdot y) &= \Delta(x)\Delta(y) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1 \end{aligned}$$
Subtracting the corresponding identity for $\Delta(yx)$, we find
$$\Delta([x, y]) = 1 \otimes [x, y] + [x, y] \otimes 1 \,.$$

□

The following proposition applies in particular to universal enveloping algebras of Lie algebras and u-algebras of restricted Lie algebras.

Lemma 2.6.13.
Let $\mathbb{K}$ be a field. If H is a Hopf algebra over $\mathbb{K}$ which is generated as an algebra by primitive elements, then the group of group-like elements of H is trivial, $G(H) = \{1_H\}$.

Proof.
Let $\{x_i\}_{i\in I}$ denote the family of non-zero primitive elements of H. Let $A_0 = \mathbb{K}1_A$. For $n > 0$, denote by A_n the linear span in H of elements of the form $x_{i_1}^{k_1} \dots x_{i_m}^{k_m}$ with $k_j \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 + \dots k_m \leq n$. Then

- $A_n \subset A_{n+1}$.
- Since H is generated, as an algebra, by primitive elements, we have $\cup_{n\geq 0} A_n = H$.
- By multiplicativity of the coproduct, $\Delta(A_n) \subset \sum_{i=0}^{n} A_i \otimes A_{n-i}$

Let $g \neq 1$ be group-like. Then $g \in A_m$ for some m. Choose m to be minimal. Since g is non-trivial, $g \notin \mathbb{K}1_A = A_0$. Then find $f \in H^*$ such that $f(A_0) = 0$ and $f(g) = 1$.

Now $g \in A_m$ implies

$$\Delta(g) = \sum_{i=0}^{m} a_i \otimes a'_{m-i}$$

for some $a_j, a'_j \in A_j$ which in turn implies

$$g = \langle \mathrm{id} \otimes f, g \otimes g \rangle = \langle \mathrm{id} \otimes f, \Delta(g) \rangle = \sum_{i=0}^{m-1} a_i f(a'_{m-i}) \in A_{m-1} \quad ,$$

where the second equality follows from the fact that g is grouplike, contradicting the minimality of m.

□

The lemma implies that the u-algebra of a non-trivial restricted Lie algebra cannot be isomorphic, as a Hopf algebra, to a group algebra, since it contains no non-trivial group-like elements. It cannot be isomorphic to a universal enveloping algebra either, since it is finite-dimensional. This shows that the Milnor-Moore theorem does not hold in finite characteristic.

We remark that over fields of characteristic zero, we can recover a Lie algebra from the primitive elements in its universal enveloping algebra:

Proposition 2.6.14.
Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ of characteristic zero with an ordered basis and $\iota_\mathfrak{g} : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ its universal enveloping algebra. Then the primitive elements of $\mathrm{U}(\mathfrak{g})$ are given by the image of $\mathfrak{g}$,

$$P(\mathrm{U}(\mathfrak{g})) = \iota_\mathfrak{g}(\mathfrak{g}) \,.$$

If $\mathrm{char}(\mathbb{K}) = p$, then the subspace of primitive elements of $\mathrm{U}(\mathfrak{g})$ is the span of all x^{p^k} with $x \in \mathfrak{g}$ and $k \geq 0$. It is a restricted Lie algebra.

Proof.
Define
$$\mathrm{U}^n(\mathfrak{g}) := \mathrm{span}_{\mathbb{K}}\{x^n | x \in \mathfrak{g}\}$$
and consider the subspace of $\mathrm{U}(\mathfrak{g})$ given by the direct sum:
$$\mathrm{U}(\mathfrak{g}) \supset \bigoplus_{n=0}^{\infty} \mathrm{U}^n(\mathfrak{g})\,. \qquad (*)$$
Since $x \in \mathfrak{g}$ is primitive in the Hopf algebra $\mathrm{U}(\mathfrak{g})$, we find
$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} \ .$$
Thus the subspace in $(*)$ is a subcoalgebra of $\mathrm{U}(\mathfrak{g})$ and the coproduct
$$\Delta : U(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$$
preserves the degree where the right hand side is endowed with the total degree. One checks inductively using the Poincaré-Birkhoff-Witt theorem, that the direct sum is closed under multiplication as well (the multiplication is not homogeneous, though). Since the elements $x \in \mathfrak{g}$ generate $\mathrm{U}(\mathfrak{g})$ as an algebra, we conclude $\mathrm{U}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathrm{U}^n(\mathfrak{g})$.

Since Δ preserves the grade, we can restrict to homogenous elements
$$x = \sum_{j=1}^{l} \lambda_j (x_j)^n \in U^n(\mathfrak{g})$$
with $n \geq 2$ to find a primitive element x. We obtain for the coproduct:
$$\Delta(x) = \sum_{j=1}^{l} \lambda_j \sum_{k=0}^{n} \binom{n}{k} (x_j)^k \otimes (x_j)^{n-k}$$
Then x is primitive, if and only if all components with bigrade $(k, n-k)$ and $1 \leq k \leq n-1$ vanish. Applying multiplication to the sum of these terms, we find
$$\left[\sum_{k=1}^{n-1} \binom{n}{k}\right] \cdot \sum_{j=1}^{l} \lambda_j (x_j)^n = 0 \quad \text{for all} \quad k = 1, \ldots, n-1\,.$$
Over a field of characteristic zero, this implies $x = 0$. In finite characteristic p, n can be a power of p. $\square$

3 Finite-dimensional Hopf algebras

3.1 Hopf modules and integrals

The goal of this subsection is to introduce the notion of an integral on a Hopf algebra that is fundamental for representation theory and some applications to topological field theory. Hopf modules are an essential tool to show the existence of integrals.

Definition 3.1.1

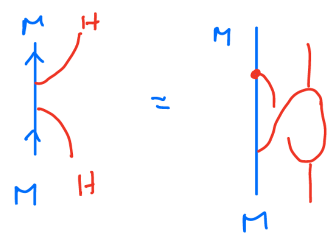
1. *Let H be a $\mathbb{K}$-Hopf algebra. A $\mathbb{K}$-vector space V is called a right Hopf module, if*
   - *It has the structure of a right (unital) H-module.*
   - *It has the structure of a right (counital) H-comodule with right coaction $\Delta_V : V \to V \otimes H$.*
   - *Δ_V is a morphism of right H-modules.*
2. *If V and W are Hopf modules, a $\mathbb{K}$-linear map $f : V \to W$ is a map of Hopf modules, if it is both a module and a comodule map.*
3. *We denote by $\mathcal{M}_H^H$ the category of right Hopf modules. The categories ${}_H\mathcal{M}^H$, ${}^H\mathcal{M}_H$ and ${}_H^H\mathcal{M}$ are defined analogously.*

Remarks 3.1.2.

1. We have in Sweedler notation for the right coaction with $\Delta_V(v) = v_{(0)} \otimes v_{(1)}$ where $v_{(0)} \in V$ and $v_{(1)} \in H$

   $$\Delta_V(v.x) = v_{(0)}.x_{(1)} \otimes v_{(1)} \cdot x_{(2)} \quad \text{for all} \quad x \in H, v \in V \ .$$

   In the graphical calculus, the condition reads:

2. Any Hopf algebra H is a Hopf module over itself with action given by multiplication and coaction given by the coproduct.
3. More generally, let $K \subset H$ be a Hopf subalgebra. We may consider the restriction of the right action to K, but the coaction of all of H to get the category of right (H, K)-Hopf modules $\mathcal{M}_K^H$.
4. Given any H-module M, the tensor product $M \otimes H$ is a right H-module, where H is seen as a regular right H-module. Using $\Delta_{M\otimes H} := \mathrm{id}_M \otimes \Delta$ as a coaction, one checks that it becomes a Hopf module.

5. Let M be a $\mathbb{K}$-vector space; then $M \otimes H$ becomes an H-Hopf module by

$$(m \otimes h).\tilde{h} := m \otimes (h \cdot \tilde{h}) \quad \text{and } \Delta_{M\otimes H}(m \otimes h) := m \otimes h_{(1)} \otimes h_{(2)}$$

We call such Hopf module a <u>trivial</u> Hopf module.

Formally, this can be reduced to a special case of the examples given in 4.: let M be a left H-module on which H acts tivially in the sense of remark 2.3.3.1, i.e. $m.h = \epsilon(h) \cdot m$ for all $h \in H$ and $m \in M$. Then, the right action on the tensor product $M \otimes H$ is $(m \otimes k).h = m \otimes k \cdot h$. (Note that this H-action is *not* trivial. In particular, both action and coaction of a trivial Hopf module are not given by the counit and the unit!)

We also need the notion of invariants and coinvariants:

Definition 3.1.3
Let H be a Hopf algebra.

1. *Let M be a left H-module. The <u>invariants</u> of H on M are defined as the $\mathbb{K}$-vector subspace*

$$M^H := \{m \in M \mid h.m = \epsilon(h)m \quad \text{for all } h \in H\}$$

of M. This defines a functor $H-\mathrm{mod} \to \mathrm{vect}(\mathbb{K})$.

2. *Let (M, Δ_M) be a right H-comodule. The <u>coinvariants</u> of H on M are defined as the $\mathbb{K}$-vector space*

$$M^{\mathrm{co}H} := \{m \in M \mid \Delta_M(m) = m \otimes 1\} \ .$$

Remarks 3.1.4.

1. For invariants of left modules, the notation ${}^H M$ would be more logical, but is not common. One can also define invariants for right H-modules.
2. The invariants, M^H, endowed with the trivial action $h.v = \epsilon(h)v$, form a submodule of M. The coinvariants $M^{\mathrm{co}H}$ form a subcomodule.

Examples 3.1.5.

1. If M is a right H-comodule, it can be considered as a left H^*-module. Then

$$M^{H^*} = \{m \in M \ \beta.m = \beta(1)m \quad \forall \beta \in H^*\} = \{m \in M \ m_{(0)}\beta(m_{(1)}) = \beta(1)m \quad \forall \beta \in H^*\} = M^{\mathrm{co}H} \ .$$

2. Consider a group algebra, $H = \mathbb{K}[G]$. For a left $\mathbb{K}[G]$-module

$$M^{\mathbb{K}[G]} = \{m \in M \mid g.m = m \quad \text{for all } g \in G\} \ .$$

For a $\mathbb{K}[G]$-comodule the coinvariants

$$M^{\mathrm{co}\mathbb{K}[G]} = M_e$$

are the identity component of the G-graded vector space underlying according to example 2.2.8.3 the comodule.

3. For a module M over the universal enveloping algebra $H = \mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$,

$$M^{\mathrm{U}(\mathfrak{g})} = \{m \in M \mid x.m = 0 \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

The category of Hopf modules in itself is not of particular interest, but the equivalence to be stated next provides a powerful tool:

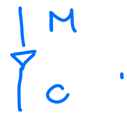
Theorem 3.1.6.
Let M be a right H-Hopf module. Then the multiplication map:

$$\begin{array}{rrcl} \rho: & M^{\mathrm{co}H} \otimes H & \to & M \\ & m \otimes h & \mapsto & m.h \end{array}$$

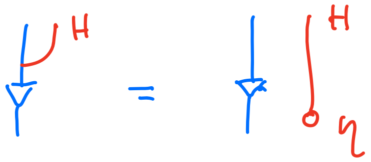
is an isomorphism of Hopf modules, where the left hand side has the structure of a trivial Hopf module, cf. Remark 3.1.2.4.

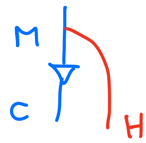
In particular, any Hopf module M is equivalent to a trivial Hopf module and thus a free right H-module of rank $\dim_{\mathbb{K}} M^{\mathrm{co}H}$.

Proof.
We perform the proof graphically.

Denote the embedding $C := M^{coH} \hookrightarrow M$ by  .

We have

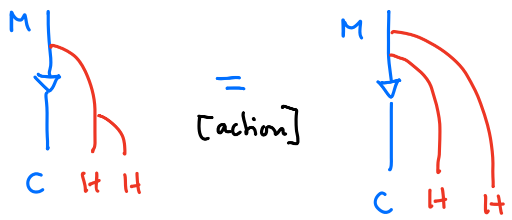


<u>Thm</u> : $\rho : C \otimes H \longrightarrow M$ 

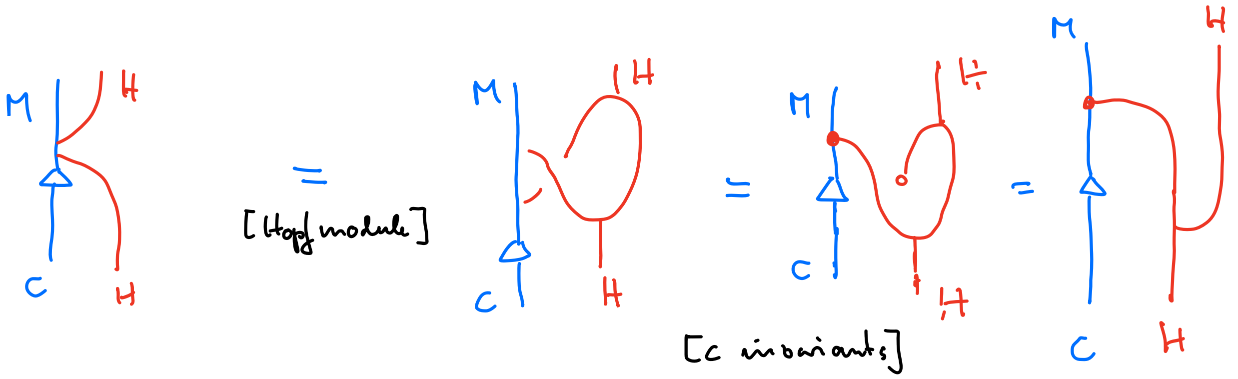
is an isomorphism of Hopf modules.

<u>Proof</u>

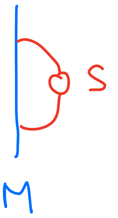
1. ρ is a morphism of H-modules:



2. ρ is a morphism of H-comodules

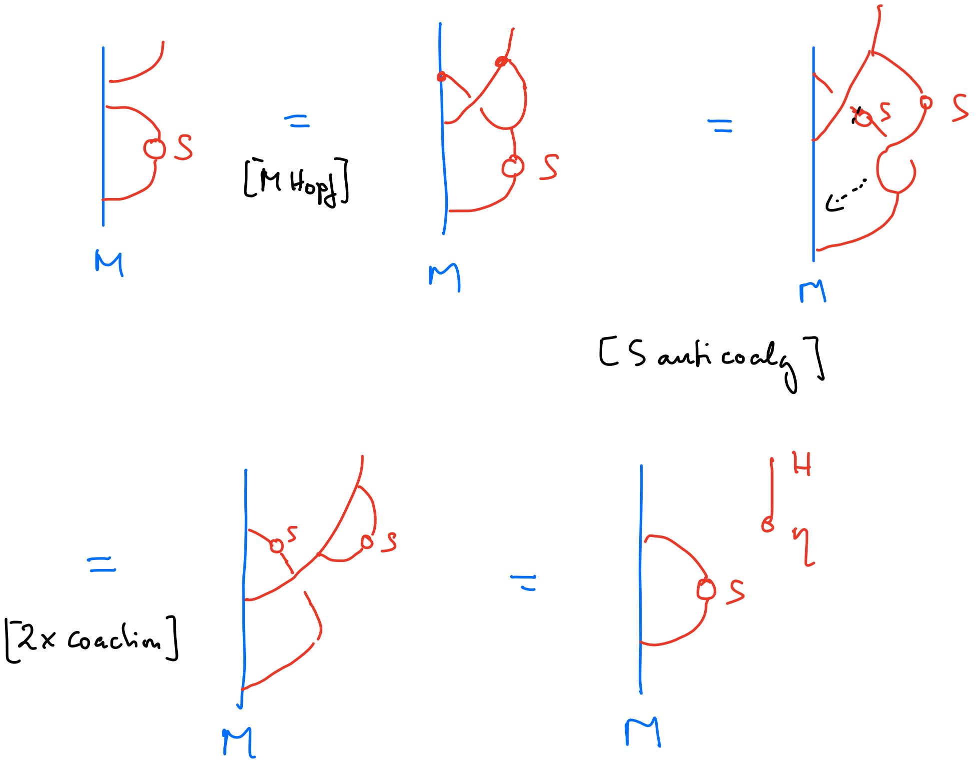


3. Define $\phi =$

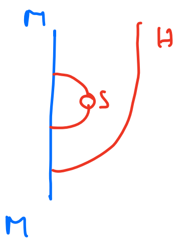


$\in \mathrm{End}_K(M)$

obeys $\mathrm{im}\,\phi \subset C$.

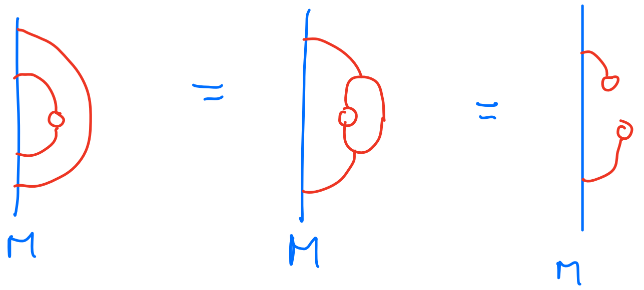


4. $\psi :=$

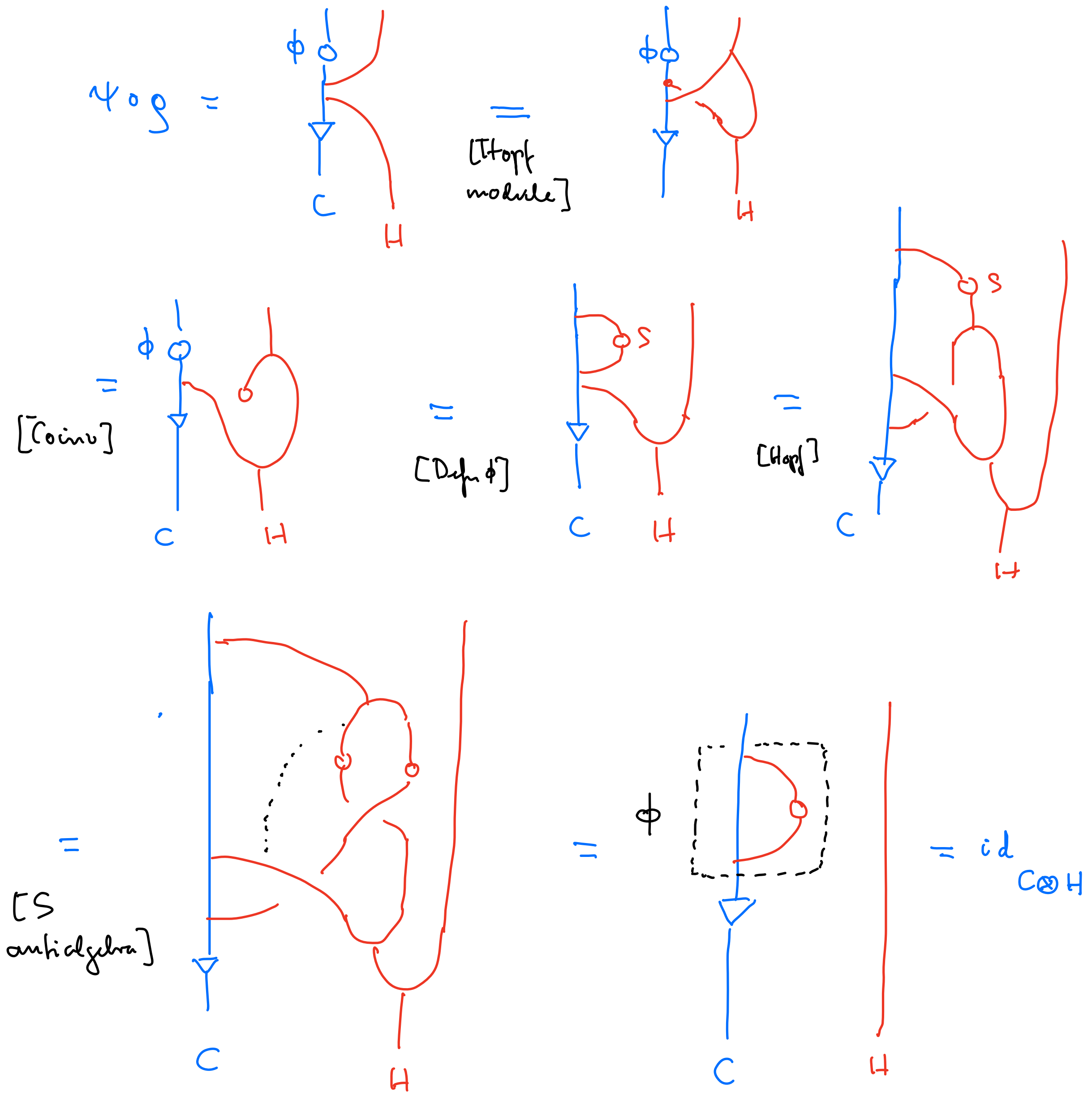


is a linear inverse of ρ and thus morphism of Hopf modules.

$\rho \circ \psi =$



$= \mathrm{id}_M$



$\Psi \circ g =$
[Hopf module]
[Coinv]
[Def. ϕ]
[Hopf]
[S antialgebra]
$= id_{C \otimes H}$
ϕ
S
C
H

□

Example 3.1.7.
Consider a Hopf module M over a group algebra $\mathbb{K}[G]$. Since it is a comodule, M has by example 2.2.8.4 the structure of a G-graded vector space

$$M = \oplus_{g \in G} M_g$$

with coaction $\Delta_M(m_g) = m_g \otimes g$ for $m_g \in M_g$. Moreover, G acts on M. Since we have a Hopf module, G acts such that $\Delta_M(m.h) = \Delta_M(m).h$. Thus for $m_g \in M_g$ and $h \in G$, we have $\Delta_M(m_g.h) = m_g.h \otimes gh$. Thus $M_g.h \subset M_{gh}$. Using the action of h^{-1}, we find a canonical identification of the subspaces, $M_g.h \cong M_{gh}$. Thus the G-action permutes the homogeneous components and

$$M_g = M_1.g = M^{\mathrm{co}\mathbb{K}[G]}.g \ .$$

This is exactly the statement of the fundamental theorem: $M \cong M^{\mathrm{co}\mathbb{K}[G]} \otimes \mathbb{K}[G]$.

We discuss a first simple application to finite-dimensional Hopf algebras:

Corollary 3.1.8.
Let H be a finite-dimensional Hopf algebra. If $I \subset H$ is a right ideal and a right coideal, then $I = H$ or $I = (0)$.

Proof.
As a right ideal, I is a right submodule of H. Similarly, as a right coideal, it is a right H-subcomodule. The condition of a Hopf module is inherited, so I is a Hopf submodule. The fundamental theorem 3.1.6 for Hopf modules implies

$$I \cong I^{\mathrm{co}H} \otimes_{\mathbb{K}} H \ .$$

Taking dimensions, we find

$$\dim_{\mathbb{K}} H \cdot \dim_{\mathbb{K}} I^{\mathrm{co}H} = \dim_{\mathbb{K}} I \leq \dim_{\mathbb{K}} H$$

where the inequality comes from the fact that I is a vector subspace of H. This only leaves the two possibilities $\dim_{\mathbb{K}} I^{\mathrm{co}H} = 0, 1$ and thus $I = (0)$ or $I = H$. □

Definition 3.1.9

1. *Let H be a Hopf algebra. The $\mathbb{K}$-linear subspace*

$$\mathcal{I}_l(H) := \{x \in H \,|\, h \cdot x = \epsilon(h)x \quad \textit{for all } h \in H\}$$

is called the space of <u>*left integrals*</u> *of the Hopf algebra H. Similarly,*

$$\mathcal{I}_r(H) := \{x \in H \,|\, x \cdot h = \epsilon(h)x \quad \textit{for all } h \in H\}$$

is called the space of <u>*right integrals*</u> *of H.*

2. *Similarly, the subspace of the linear dual H^**

$$C\mathcal{I}_l(H) := \{\phi \in H^* \,|\, (\mathrm{id}_H \otimes \phi) \circ \Delta_H(h) = 1_H \phi(h) \quad \textit{for all } h \in H\}$$

is called the space of <u>*left cointegrals*</u>. *Right cointegrals are defined analogously.*

3. *A Hopf algebra is called* <u>unimodular</u>*, if* $\mathcal{I}_l(H) = \mathcal{I}_r(H)$*.*

Remarks 3.1.10.

1. The space of left integrals is the space of left invariants for the left action of H on itself by multiplication. Alternatively, consider the canonical isomorphism $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, H) \cong H$ that sends $\phi \mapsto \phi(1)$ and $H \ni \Lambda \to (\lambda \to \lambda\Lambda)$. Under this identification, an integral $h \in \mathcal{I}_l(H) \subset H$ corresponds to a morphism of left H-modules in $\mathrm{Hom}_H(\mathbb{K}, H) \subset \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, H)$ from the trivial H-module to the left regular H-module. A similar statement holds for right integrals.

2. Even if a Hopf algebra H is cocommutative, it can be not unimodular. For an example, see [Montgomery, p. 17].

3. Let H be finite-dimensional. Then, by definition, $\phi \in H^*$ is a left integral for the dual Hopf algebra H^*, if and only if
$$\mu^*(\beta, \phi) = \epsilon^*(\beta)\phi \quad \text{for all} \quad \beta \in H^* \ .$$
(Our convention needs an explanation: μ^* is here the multiplication of H^*, not the linear map dual to the multiplication on H. Similarly, ϵ^* is the counit of H^*.) Applying this identity in H^* to $h \in H$ and using the definition of the bialgebra structure on H^*, we obtain the equality
$$\beta(h_{(1)}) \cdot \phi(h_{(2)}) = \beta(1_H) \cdot \phi(h) \ \text{ for all } \quad \beta \in H^* \ \text{ and } \ h \in H \ .$$
Thus ϕ is a left integral of H^*, if and only if
$$h_{(1)}\langle \phi, h_{(2)} \rangle = \langle \phi, h \rangle 1_H \ \text{ for all } \quad h \in H \ ,$$
i.e. if and only if ϕ is a left cointegral for H. In this way, (co-)integrals are compatible with duality.

4. Let G be a *finite* group. Then the group algebra $\mathbb{K}[G]$ is a unimodular Hopf algebra, with integrals
$$\mathcal{I}_l = \mathcal{I}_r = \mathbb{K} \sum_{g \in G} g \ .$$
Indeed, for $I := \sum_{h \in G} h$ we have $g.I = \sum_{h \in G} gh = I = \epsilon(g)I$ for all $g \in G$, and it is enough to check this relation on the distinguished basis of $\mathbb{K}[G]$.

5. The dual $\mathbb{K}G$ of the group algebra $\mathbb{K}[G]$ is a commutative Hopf algebra. Suppose that G is a finite group; then it can be identified with the commutative algebra of $\mathbb{K}$-valued functions on G. In this case, a right integral $\lambda \in \mathbb{K}[G]$ can be considered as an element in the bidual, $\lambda \in \mathbb{K}G^*$, i.e. a linear form $\tilde{\lambda} : \phi \mapsto \phi(\lambda)$ on functions on G. The form $\tilde{\lambda}$ is called a measure.

   On the space of functions on a group G, we have a left action of G by <u>translations</u>:
$$L_g : \quad \mathbb{K}G \to \mathbb{K}G$$
defined by $(L_g\phi)(h) = \phi(hg)$. We compute, using that λ is a right integral:
$$\lambda(L_g\phi) = (L_g\phi)(\lambda) = \phi(\lambda \cdot g) = \phi(\lambda) = \lambda(\phi) \ .$$
Thus the measure on $\mathbb{K}G$ given by a right integral is invariant under left translations.

6. The spaces of integrals for the Taft algebra are

$$\mathcal{I}_l = \mathbb{K} \sum_{j=0}^{N-1} g^j x^{N-1}$$

and

$$\mathcal{I}_r = \mathbb{K} \sum_{j=0}^{N-1} \zeta^j g^j x^{N-1} \ .$$

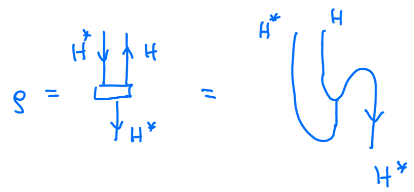
This is an expansion as a linear combination of basis elements, hence the integrals differ and the Taft algebra is thus not unimodular.

We need some actions and coactions of the Hopf algebra H on the dual vector space H^*. Since we will use dualities, we assume H to be finite-dimensional.

Observation 3.1.11.

1. We consider H^* as a right H-comodule

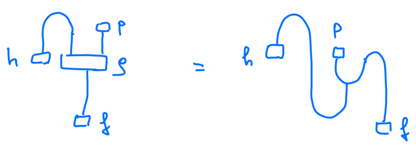
$$\begin{array}{rcl} \rho : H^* & \to & H^* \otimes H \\ f & \mapsto & f_{(0)} \otimes f_{(1)} \end{array}$$



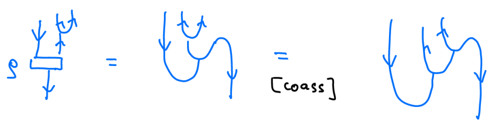
with coaction derived from the coproduct in H:

$$\langle f_{(0)}, h \rangle \cdot \langle p, f_{(1)} \rangle = \langle p, h_{(1)} \rangle \cdot \langle f, h_{(2)} \rangle \quad \text{for all} \quad p \in H^*, h \in H \ .$$

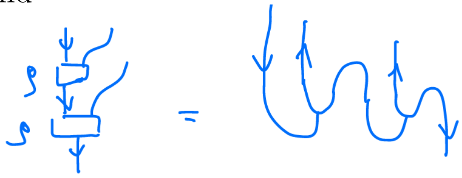
Graphically, this definition is simpler to understand



and the proof that ρ is a coaction follows from comparing



and



2. Consider for $x \in H$ the $\mathbb{K}$-linear endomorphism given by right multiplication with x

$$\begin{array}{rcl} m_x : H & \to & H \\ h & \mapsto & h \cdot x \end{array}$$

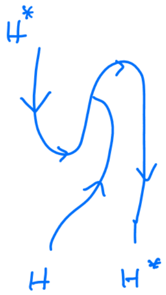
It is a morphism of left modules. The transpose is a map $m_x^* : H^* \to H^*$, for each $x \in H$. One checks graphically that these maps define the structure of a left H-module on H^*. We write

$$h \rightharpoonup h^* \quad \in H^*$$

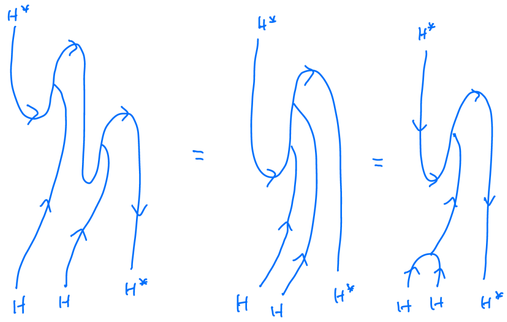
for the image of $h^* \in H^*$ under the left action of $h \in H$. Thus

$$\langle h \rightharpoonup h^*, g\rangle = \langle h^*, gh\rangle \quad \text{for all} \quad g \in H \ .$$

One can perform this construction quite generally for an algebra in a rigid monoidal category. In this case, one has to take the *left* dual for this construction to work to avoid crossings of lines.



This is again immediately obvious from the graphical proof:



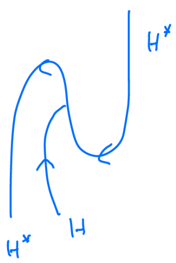
3. In the same vein, the transpose of left multiplication defines a right action of H on H^*. We write

$$h^* \leftharpoonup h \quad \in H^*$$

for the image of $h^* \in H^*$ under the right action of $h \in H$. Thus

$$\langle h^* \leftharpoonup h, g\rangle = \langle h^*, hg\rangle \quad \text{for all} \quad g \in H \ .$$

One can perform this construction quite generally for an algebra in a rigid monoidal category. In this case, one has to take the *right* dual for this construction to work:

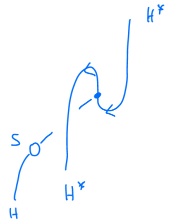


This is again immediately obvious from the graphical proof.

4. Since the antipode is an antialgebra morphism, we can use it to turn left actions into right actions and vice versa.

   In this way, we get a left action of H on H^* by

   $(h \rightharpoondown h^*) := (h^* \leftharpoonup S(h))$ which is graphically

   It obeys
   $$\langle h \rightharpoondown h^*, g\rangle = \langle h^*, S(h)g\rangle \quad \text{for all} \quad g \in H \ .$$
   Similarly, we get a right action of H on H^* by

   $(h^* \leftharpoondown h) := (S(h) \rightharpoonup h^*)$ which is graphically

   with
   $$\langle h^* \leftharpoondown h, g\rangle = \langle h^*, gS(h)\rangle \quad \text{for all} \quad g \in H \ .$$

The following Lemma will be needed to show the existence of integrals:

Lemma 3.1.12.
Let H be a finite-dimensional Hopf algebra. Then H^* with right H action $\leftharpoondown$ and right coaction ρ from observation 3.1.11 is a Hopf module.

Proof.
The condition in $H^* \otimes H$ to have a Hopf module is

$$\rho(f \leftharpoondown h) = (f_{(0)} \leftharpoondown h_{(1)}) \otimes (f_{(1)} \cdot h_{(2)})$$

for all $f \in H^*$ and $h \in H$. The right coaction ρ appears on the right hand side in the form of the Sweedler-like notation $f_{(0)} \otimes f_{(1)}$. By the definition of the coaction ρ, this amounts to showing for all $p \in H^*$ and $x \in H$:

$$\langle p, x_{(1)}\rangle\langle f \leftharpoondown h, x_{(2)}\rangle = \langle f_{(0)} \leftharpoondown h_{(1)}, x\rangle\langle p, f_{(1)} \cdot h_{(2)}\rangle \ .$$

We start with the right hand side:

$$\begin{aligned}
\langle f_{(0)} \leftharpoondown h_{(1)}, x\rangle\langle p, f_{(1)}h_{(2)}\rangle &= \langle f_{(0)}, xS(h_{(1)})\rangle\langle h_{(2)} \rightharpoonup p, f_{(1)}\rangle \quad [\text{defn. of} \leftharpoondown \text{and} \rightharpoonup]\\
&= \langle h_{(3)} \rightharpoonup p, x_{(1)}S(h_{(2)})\rangle \cdot \langle f, x_{(2)}S(h_{(1)})\rangle \quad [\text{defn. of } \rho]\\
&= \langle p, x_{(1)}\langle \epsilon, h_{(2)}\rangle\rangle \cdot \langle f, x_{(2)}S(h_{(1)})\rangle \quad [\text{defn. of} \rightharpoonup \text{and antipode}]\\
&= \langle p, x_{(1)}\rangle\langle f, x_{(2)}S(h)\rangle \quad [\text{counit}]\\
&= \langle p, x_{(1)}\rangle\langle f \leftharpoondown h, x_{(2)}\rangle \quad [\text{defn. of} \leftharpoondown]
\end{aligned}$$

□

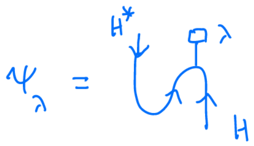
Lemma 3.1.13.
Let H be a finite-dimensional Hopf algebra. Consider H^* as a right comodule with the H-coaction ρ. Then
$$(H^*)^{\mathrm{co}H} = \mathcal{I}_l(H^*) \ .$$

Proof.
We recall from remark 3.1.10 that elements $\beta \in \mathcal{I}_l(H^*)$ are left cointegrals for H: they are elements such that
$$\mu^*(h^*, \beta) = \epsilon^*(h^*)\beta \quad \text{for all} \quad h^* \in H^* \ .$$
This means that we have $\beta \in \mathcal{I}_l(H^*)$, if and only if for all $h \in H$ and $h^* \in H^*$, we have
$$h^*(h_{(1)}) \cdot \beta(h_{(2)}) = \mu^*(h^*, \beta)(h) = \epsilon^*(h^*)\beta(h) = h^*(1)\beta(h) \ .$$
On the other hand, we have for coinvariants $\beta \in H^*$ under the coaction ρ
$$\rho(\beta) = \beta \otimes 1_H$$
and thus by definition of ρ
$$\langle h^*, h_{(1)} \rangle \cdot \langle \beta, h_{(2)} \rangle = \langle h^*, 1 \rangle \cdot \langle \beta, h \rangle$$
for all $h^* \in H$ and $h \in H$. $\square$

Theorem 3.1.14.
Let H be a *finite-dimensional* Hopf algebra over a field $\mathbb{K}$.

1. Then $\dim \mathcal{I}_l(H) = \dim \mathcal{I}_r(H) = 1$.
2. The antipode S is bijective and $S(\mathcal{I}_l) = S^{-1}(\mathcal{I}_l) = \mathcal{I}_r$. In particular, a skew antipode exists.
3. For any non-zero left cointegral $\lambda \in \mathcal{I}_l(H^*) \setminus \{0\}$, the so-called Frobenius map
$$\begin{array}{rcl} \Psi_\lambda : H & \to & H^* \\ h & \mapsto & (S(h) \rightharpoonup \lambda) = (\lambda \leftharpoonup h) \end{array}$$
is an isomorphism of right H-modules, where H is endowed with the regular right action, i.e. by multiplication, and H^* with the action $h^* \leftharpoonup h$.

We see that 

and thus 

Proof.
1. Consider H^* with the Hopf module structure described in lemma 3.1.12. By the fundamental theorem on Hopf modules,
$$H^* \cong (H^*)^{\mathrm{co}H} \otimes H \ .$$
Since H is finite-dimensional, we can take dimensions and find $\dim(H^*)^{\mathrm{co}H} = 1$. By lemma 3.1.13, we have $\dim \mathcal{I}_l(H^*) = \dim(H^*)^{\mathrm{co}H} = 1$. Thus the Hopf algebra H^* has a one-dimensional space of left integrals. Since any finite-dimensional Hopf algebra can be written as the dual of a Hopf algebra, we get the first equality. The second equality follows analogously or from the assertion in 2.

2. Again by the fundamental theorem 3.1.6 on Hopf modules, the map

$$\begin{array}{rcl} \mathcal{I}_l(H^*) \otimes H & \to & H^* \\ \lambda \otimes h & \mapsto & (\lambda \leftharpoondown h) \end{array} \tag{3}$$

is an isomorphism of Hopf-modules. In particular, it is a morphism of right H-modules. The compatibility with the right action is also shown graphically. Keeping $\lambda \in \mathcal{I}_l(H^*)\setminus\{0\}$ fixed, we deduce the third assertion.

3. Fix a non-zero left integral $\lambda \in \mathcal{I}_l(H^*) \setminus \{0\}$ and suppose that there is $h \in H$ such that $S(h) = 0$. Then

$$0 = (S(h) \rightharpoonup \lambda) \stackrel{\text{def}}{=} (\lambda \leftharpoondown h)$$

and thus by injectivity of the map (3), we have $\lambda \otimes h = 0$. This implies over a field that $h = 0$. Thus the antipode S is injective and, as an endomorphism of a finite-dimensional vector space, bijective.

If $\Lambda \in \mathcal{I}_l(H)$, we have $h \cdot \Lambda = \epsilon(h)\Lambda$ for all $h \in H$. Applying the antipode, and its inverse respectively, which are antialgebra morphisms and preserve the counit, we find

$$S(\Lambda) \cdot S(h) = \epsilon(h)S(\Lambda) = \epsilon(S(h))S(\Lambda) \quad \text{for all} \quad h \in H$$

and

$$S^{-1}(\Lambda) \cdot S^{-1}(h) = \epsilon(h)S^{-1}(\Lambda) = \epsilon(S^{-1}(h))S^{-1}(\Lambda) \quad \text{for all} \quad h \in H \ .$$

Since S is bijective, this implies that $S(\Lambda)$ is a right integral. We have now also proven the second assertion.

□

Corollary 3.1.15.
Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$. Then the monoidal category $H-\mathrm{mod}_{fd}$ of finite-dimensional H-modules is rigid.

Proof.
Given the existence of a skew antipode, this follows immediately from from Lemma 2.5.17. □

We find a different relation between the left and right integrals on a finite-dimensional Hopf algebra in the following

Observation 3.1.16.

1. Let $t \in \mathcal{I}_l(H)$ be a left integral. Then for any $h \in H$, the element $th \in H$ is a left integral as well: we have for all $h' \in H$

$$h'(th) = (h't)h = \epsilon(h')th \ .$$

Since the subspace of left integrals is one-dimensional, we have $t \cdot h = t\alpha(h)$ with some linear form $\alpha \in H^*$.

2. One directly checks that $\alpha : H \to \mathbb{K}$ is a morphism of algebras and thus a group-like element of H^*.

3. Let now be $t' \in \mathcal{I}_r(H)$ a non-vanishing right integral. Then by theorem 3.1.14.2, the element St' is a left integral and thus for all $h \in H$

$$S(ht') = St' \cdot Sh = \alpha(Sh)\, St' \ .$$

The invertibility of the antipode implies $ht' = \alpha(Sh)t' = (S^*\alpha)(h)t'$ for all $h \in H$. Here S^* is the antipode of the dual Hopf algebra H^*. Thus the inverse $\alpha^{-1} = S^*\alpha \in G(H^*)$ plays a similar role for right integrals.

4. Since $\alpha : H \to \mathbb{K}$ is a morphism of algebras, it endows the ground field with the structure of an H-module by $h.\lambda := \alpha(h) \cdot \lambda$ for $h \in H$ and $\lambda \in \mathbb{K}$. We conclude that the category $H{-}\mathrm{mod}_{fd}$ of finite-dimensional modules over a finite-dimensional Hopf algebra contains, apart from the monoidal unit, another distinguished object, given by α. It is called the distinguished invertible object D, since we have for the object D^{-1} given by α^{-1} isomorphisms $D \otimes D^{-1} \cong \mathbb{I} \cong D^{-1} \otimes D$.

Definition 3.1.17
Let H be a finite-dimensional Hopf algebra. The element $\alpha \in G(H^*)$ constructed in observation 3.1.16 is called the distinguished group-like element or the modular element of H^*.

Corollary 3.1.18.
A finite-dimensional Hopf algebra is unimodular, if and only if the distinguished group-like element α equals the counit, $\alpha = \epsilon$. In this case, the distinguished invertible element D is isomorphic to the monoidal unit $\mathbb{I}$.

Proof.
Let $t \in \mathcal{I}_l(H) \setminus \{0\}$. If $\alpha = \epsilon$, then $t \cdot h = t\alpha(h) = t\epsilon(h)$ for all $h \in H$ so that t is a right integral as well. The converse is obvious. □

The third assertion of theorem 3.1.14 about the bijectivity of the Frobenius map allows us to identify additional algebraic structure on any finite-dimensional Hopf algebra.

Definition 3.1.19
Let (A, μ, η) be a unital associative algebra in a monoidal category $\mathcal{C}$.

1. *A (Δ, ϵ)-Frobenius structure on A is the structure of a coassociative, counital coalgebra (Δ, ϵ) such that $\Delta : A \to A \otimes A$ is a morphism of A-bimodules.*

2. *Assume now that the monoidal category $\mathcal{C}$ is rigid. A κ-Frobenius structure on A is a pairing $\kappa \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, \mathbb{I})$ that is invariant (or associative) i.e. satisfies*

$$\kappa \circ (\mu \otimes \mathrm{id}_A) = \kappa \circ (\mathrm{id}_A \otimes \mu) \,,$$

and that is non-degenerate in the sense that

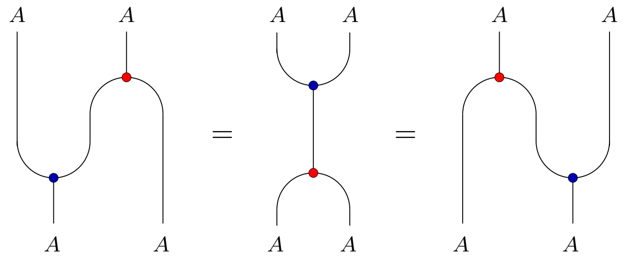
$$\Phi_\kappa := (\mathrm{id}_{^\vee A} \otimes \kappa) \circ (\tilde{b}_A \otimes \mathrm{id}_A) \ \in \mathrm{Hom}(A, {}^\vee A)$$

is an isomorphism.

3. *Assume again that the monoidal category $\mathcal{C}$ is rigid. A Φ_ρ-Frobenius structure on A is a left-module isomorphism $\Phi_\rho \in \mathrm{Hom}_{A-\mathrm{mod}}(A, {}^\vee A)$ between the left regular A-module (A, μ) and left A-module ${}^\vee A$ with the left dual action.*

Remarks 3.1.20.

1. Graphically, the condition in the (Δ, ϵ)-Frobenius structure that Δ is a morphism of bimodules reads



2. Note that, unlike in the case of bialgebras (which needs the swap of two tensorands and hence cannot be defined in any monoidal category), neither the coproduct Δ nor the counit ϵ is an algebra morphism.

3. Concerning the Φ_ρ-Frobenius structure, we remark that if $\Phi_\rho \in \mathrm{Hom}(A, {}^\vee A)$ is an isomorphism between the left regular A-module (A, μ) and left A-module ${}^\vee A$, then the right dual morphism
$$\Phi_\rho^\vee \in \mathrm{Hom}(({}^\vee A)^\vee, A^\vee) = \mathrm{Hom}(A, A^\vee)$$
is an isomorphism between the right regular A-module (A, μ) and the right A-module $A^\vee$ with the right dual action. This is shown graphically on the next page.

It turns out that the three concepts are equivalent:

Proposition 3.1.21.
In a rigid monoidal category $\mathcal{C}$ the notions of a (Δ, ϵ)-Frobenius structure and of a κ-Frobenius structure on an algebra (A, μ, η) are equivalent.

More concretely:

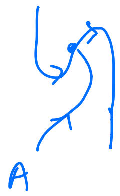
1. If $(A, \mu, \eta, \Delta, \epsilon)$ is an algebra with a (Δ, ϵ)-Frobenius structure, then $(A, \mu, \eta, \kappa_\epsilon)$ with
$$\kappa_\epsilon := \epsilon \circ \mu$$
is an algebra with κ-Frobenius structure.

2. If (A, μ, η, κ) is an algebra with κ-Frobenius structure, then $(A, \mu, \eta, \Delta_\kappa, \epsilon_\kappa)$ with
$$\Delta_\kappa := (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes \Phi_\kappa^{-1} \otimes \mathrm{id}_A) \circ (b_A \otimes \mathrm{id}_A) \qquad \text{and} \qquad \epsilon_\kappa := \kappa \circ (\mathrm{id}_A \otimes \eta)$$
with $\Phi_\kappa \in \mathrm{Hom}(A, A^\vee)$ the morphism that exists by the assumption that κ is non-degenerate is an algebra with (Δ, ϵ)-Frobenius structure.

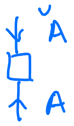
Proof.
We present the proof that a (Δ, ϵ)-Frobenius structure gives a κ-Frobenius structure graphically. The converse statement is relegated to an exercise. □

Remark 3.1.18.3

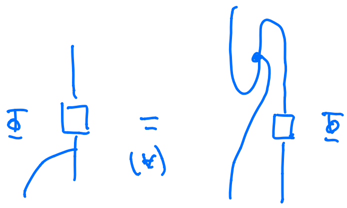
Recall: left module on ${}^\vee A$



right module on $A^\vee$

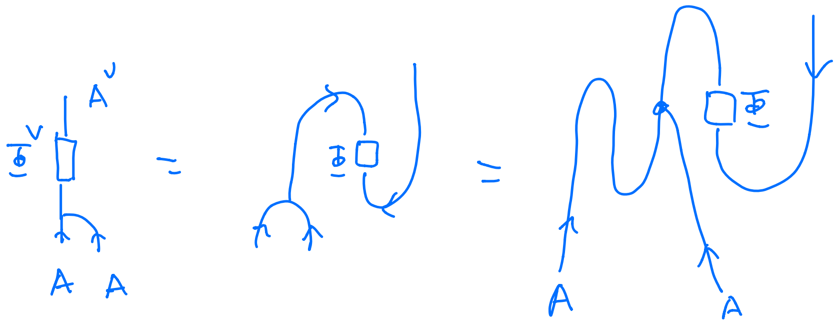
ϕ = 

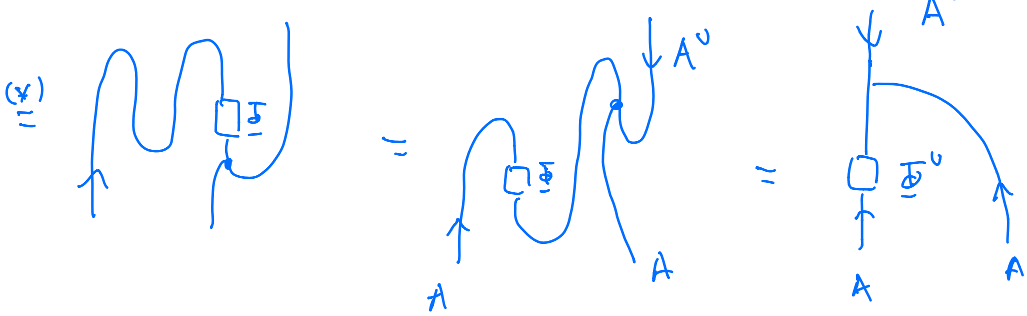
intertwines left action:



Then $\phi^\vee$ = 

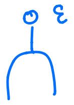
intertwines right action:



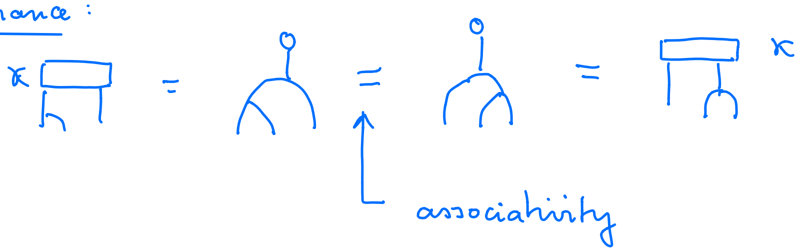


Proposition 3.1.19

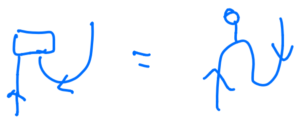
Suppose A is (Δ, ε) Frobenius. Then define

$\kappa :=$  $\in \mathrm{Hom}(A \otimes A, \mathbb{1})$

Invariance:

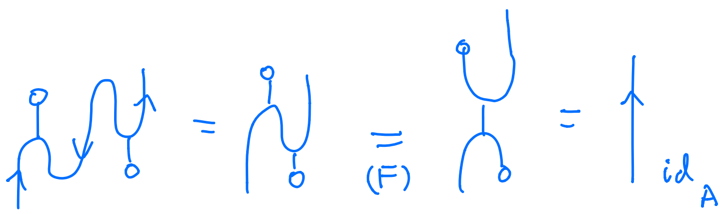


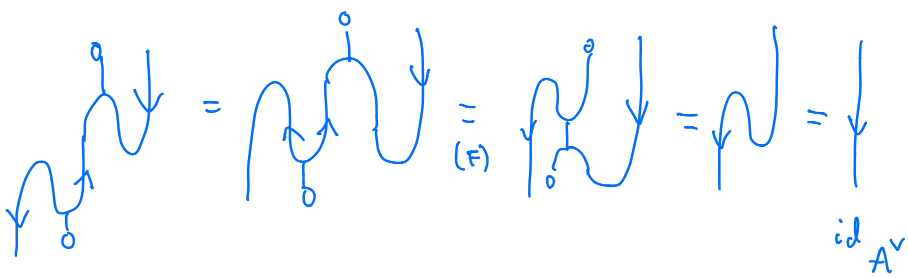
Non-degenerate:

Inverse for $\Phi :=$  is

$\Psi =$ 

Indeed:

$\Psi \circ \Phi =$ 

$\Phi \circ \Psi =$ 

Proposition 3.1.22.
In a rigid monoidal category $\mathcal{C}$ the notions of a κ-Frobenius structure and of a Φ_ρ-Frobenius structure on an algebra (A, μ, η) are equivalent.

More specifically, for any algebra A in $\mathcal{C}$ the following holds:

1. There exists a non-degenerate pairing on A, if and only if A is isomorphic to ${}^\vee A$ as an object of $\mathcal{C}$.

2. There exists an invariant pairing on A, if and only if there exists a morphism from A to ${}^\vee A$ that is a morphism of left A-modules.

Proof.
Given a morphism $\Phi \in \mathrm{Hom}_\mathcal{C}(A, {}^\vee A)$, we define a pairing on A by

$$\kappa_\Phi := \tilde{d}_A \circ (\mathrm{id}_A \otimes \Phi)\,.$$

Conversely, using the dualities, we find for any non-degenerate pairing an isomorphism $\psi \in \mathrm{Hom}(A, {}^\vee A)$ such that the operations are inverse.

A pairing is obviously non-degenerate, if and only if the morphism Φ is an isomorphism. Similarly, invariance of the pairing amounts to the fact that Φ is a morphism of left modules. This can be seen graphically and is relegated to an exercise. $\square$

Definition 3.1.23
A Frobenius algebra in a rigid monoidal category $\mathcal{C}$ is an associative unital algebra A in $\mathcal{C}$ together with the choice of one of the following three equivalent structures:

1. *A (Δ, ϵ)-Frobenius structure on A.*

2. *A κ-Frobenius structure on A.*

3. *A Φ_ρ-Frobenius structure on A.*

Example 3.1.24.
It is instructive to write down explicitly a distinguished Frobenius algebra structure on the group algebra $\mathbb{K}[G]$ of a finite group.

1. The invariant bilinear form is defined on the distinguished basis by

$$\kappa(g, h) = \delta_{gh,e} \quad \text{for all } g, h \in G$$

and hence the evaluation of the product on the component of the neutral element e. This form is obviously non-degenerate and invariant, $\kappa(gh, l) = \delta_{ghl,e} = \kappa(g, hl)$ for all $g, h, l \in G$.

2. The corresponding Φ_ρ-Frobenius structure is the morphism

$$\begin{array}{rccl} \Phi_\rho : & \mathbb{K}[G] & \to & \mathbb{K}(G) = \mathbb{K}[G]^* \\ & g & \mapsto & \delta_{g^{-1}} \end{array}$$

To show that this is indeed a morphism of left modules, we have to show $\Phi_\rho(hg) = h \rightharpoonup \Phi_\rho(g)$. Indeed, evaluating this on $x \in G$, we find

$$(h \rightharpoonup \delta_{g^{-1}})(x) = \delta_{g^{-1}}(xh) = \delta_{g^{-1}h^{-1}}(x) = \delta_{(hg)^{-1}}(x) \quad \text{for all } x \in G\,.$$

3. We can finally deduce the (Δ_F, ϵ_F)-Frobenius structure, where we added an index F to the Frobenius coproduct and counit to distinguish them from the Hopf coproduct and counit. We find
$$\epsilon_F(g) = \delta_{g,e} \quad \text{and} \quad \Delta_F(g) = \sum_{h\in G} gh^{-1} \otimes h$$
which is indeed different from the coproduct and counit giving the Hopf algebra structure on $\mathbb{K}[G]$ which were only using the structure of the set underlying G. Note that here, in contrast to the Hopf coproduct, the product in the group enters and the elements $g \in G$ are not group-like.

We can now state:

Theorem 3.1.25.
Let H be a finite-dimensional Hopf-algebra with left non-zero integral $\lambda \in H^*$. Then H has the structure of a Frobenius algebra with bilinear pairing
$$\kappa(h, h') := \lambda(h \cdot h') \quad \text{for } h, h' \in H \ .$$

Proof.
From the associativity and bilinearity of the product of the algebra H, it is obvious that the form is bilinear and invariant. To show non-degeneracy, assume that there exists $a \in H$ such that
$$0 = \kappa(a, h) = \lambda(ah) = \langle h \rightharpoonup \lambda, a\rangle \quad \text{for all } \ h \in H \ .$$
But $(H \rightharpoonup \lambda) = H^*$ by equation (3) in the proof of theorem 3.1.14, and the pairing between the vector space H and its dual H^* is non-degenerate. $\square$

Example 3.1.26.
Consider the case of a group algebra $H = \mathbb{K}[G]$ for a finite group G. Then the cointegral $\lambda \in H^*$ is the projection to the component of the neutral element: $\lambda(g) = \delta_{g,e}$ for all $g \in G$. Indeed,
$$(\mathrm{id}_H \otimes \lambda) \circ \Delta(g) = g\lambda(g) = e\delta_{g,e} = 1_H\lambda(g) \quad \text{for all } g \in G \ .$$
This yields the Frobenius structure on $\mathbb{K}[G]$ discussed in example 3.1.24.

Proposition 3.1.27.
Let H be a finite-dimensional Hopf algebra. Recall from theorem 3.1.14 that for a non-zero $\lambda \in \mathcal{I}_l(H^*)$
$$\begin{array}{rcl} \Psi_\lambda : H & \to & H^* \\ h & \mapsto & (S(h) \rightharpoonup \lambda) \end{array}$$
is an isomorphism of right H-modules. As a consequence, also the map $H \to H^*$ with $h \mapsto (\lambda \leftharpoonup h)$ is a linear isomorphism $H \to H^*$.

1. Let λ be a left integral in H^*. We can find $\Lambda \in H$ such $\lambda \leftharpoonup \Lambda = \epsilon$ equals the counit ϵ. Then Λ is a right integral.

2. Conversely, if $I \in H$ is a right integral, then $\langle \lambda, I\rangle \neq 0$. If we normalize $I \in H$ such that $\langle \lambda, I\rangle = 1$, we have $\lambda \leftharpoonup I = \epsilon$.

Proof.

1. We first show 2. and assume that I is a non-zero right integral. Then for all $h \in H$

$$\langle \lambda \leftharpoonup I, h \rangle = \langle \lambda, I \cdot h \rangle = \langle \lambda, I \rangle \epsilon(h)$$

   and thus $\lambda \leftharpoonup I = \langle \lambda, I \rangle \epsilon$. By injectivity, since $I \neq 0$, we conclude $\langle \lambda, I \rangle \neq 0$. Normalizing I, we find the identity $\lambda \leftharpoonup I = \epsilon$.

2. Conversely, suppose that we have $\Lambda \in H$ such that $\lambda \leftharpoonup \Lambda = \epsilon$. Then

$$\langle \lambda \leftharpoonup \Lambda, h \rangle = \langle \lambda, \Lambda h \rangle = \epsilon(h) \quad \text{for all} \quad h \in H \ .$$

   Applying this to $h = 1_H \in H$, we find

$$\langle \lambda, \Lambda \rangle = \langle \lambda, \Lambda 1_H \rangle = \epsilon(1_H) = 1 \ .$$

   Thus

$$\langle \lambda, \Lambda h \rangle = \epsilon(h) = \epsilon(h) \langle \lambda, \Lambda \rangle = \langle \lambda, \epsilon(h) \Lambda \rangle \ .$$

   By the injectivity of the map $h \mapsto (\lambda \leftharpoonup h)$, we conclude $\Lambda h = \epsilon(h)\Lambda$ for all $h \in H$. Thus Λ is a right integral.

□

3.2 Integrals and semisimplicity

We now need the important notion of semi-simplicity.

Definition 3.2.1

1. *A module M over a $\mathbb{K}$-algebra A is called simple, if it has no non-trivial submodules, i.e. the only submodules of M are (0) and M itself.*

2. *A module M over a $\mathbb{K}$-algebra A is called semisimple, if every submodule $U \subset M$ has a* complement *D, i.e. if we can find for any submodule U a submodule D such that $D \oplus U = M$.*

3. *An algebra is called semisimple , if it is semisimple as a left module over itself.*

We also have the corresponding categorical definitions they are formulated in the framework of abelian categories. In appendix **??**, we develop the necessary background knowledge.

Definition 3.2.2

1. *Let $\mathcal{C}$ be an abelian category. A morphism $f : X \to Y$ is said to be a monomorphism if* $\mathrm{Ker}\,(f) = 0$*. It is said to be an epimorphism if* $\mathrm{Coker}(f) = 0$.

2. *A subobject of an object Y is an object X, together with a monomorphism $\iota : X \to Y$. We write $X \subset Y$*

3. *A quotient object of Y is an object Z with an epimorphism $p : Y \to Z$. For a subobject $X \subset Y$, define the quotient object $Z = Y/X$ to be the cokernel of the monomorphism $f : X \to Y$.*

4. *A nonzero object $X \in \mathcal{C}$ is called* <u>*simple*</u>*, if* 0 *and X are its only subobjects. An object $X \in \mathcal{C}$ is called* <u>*semisimple*</u>*, if it is a direct sum of simple objects,*

Remarks 3.2.3.

1. A $\mathbb{K}$-vector space is a semisimple module over the $\mathbb{K}$-algebra $\mathbb{K}$. A $\mathbb{K}$-vector space V together with an endomorphism $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$ is a module over the polynomial algebra $\mathbb{K}[X]$. Suppose that V is finite-dimensional and that φ can be described by a matrix that consists of a single Jordan block of size $n \geq 2$. Then the $\mathbb{K}[X]$-module contains a one-dimensional submodule, the only eigenspace of φ. This submodule does not have a complement.

2. One has a similar notion of semisimplicity for right modules. It follows from the structure theory of semisimple algebras that an algebra is semisimple as a right module over itself, if and only if it is semisimple as a left module over itself [JS, Satz VII.2.10].

Proposition 3.2.4.
Let A be a $\mathbb{K}$-algebra and M an A-module. Then the following assertions are equivalent:

(i) M is a direct sum of simple submodules.

(ii) M is a (not necessarily direct) sum of simple submodules.

(iii) M is semisimple, i.e. any submodule $U \subset M$ has a complement D.

For the proof, we refer to the lecture notes on advanced algebra.

Corollary 3.2.5.
Any quotient and any submodule of a semisimple module is semisimple.

Proof.
Suppose we are given a submodule $U \subset M$ of a semisimple module M. Consider the canonical surjection $M \to M/U$. The image of a simple submodule of M is then either zero or simple. Thus the quotient module is a sum of simple modules and thus semisimple by proposition 3.2.4.

Next, find a complement D of U. Then the submodule U is isomorphic to the quotient $U \cong M/D$ and by the result just obtained semisimple. □

We next need the important notion of a projective module. We recall that a collection of morphisms of A-modules

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

is called a short exact sequence, if f is injective, g is surjective and $\mathrm{Im}\, f = \ker g$. The injectivity of f means that $\ker f \subset N'$ is the image of the left most morphisms; the surjectivity of g means that $\mathrm{Im}\,(g) \subset N''$ is the kernel of the right most morphism. Hence we have at all objects that the image of the preceeding morphism is the kernel of the subsequent morphism. This property can be formulated in an arbitrary abelian category $\mathcal{C}$.

Definition 3.2.6
A functor $F : \mathcal{C} \to \mathcal{D}$ to another abelian category $\mathcal{D}$ does not necessarily preserve this property. If it preserves it, it is called <u>*exact*</u>*; if the sequence*

$$0 \to F(N') \xrightarrow{F(f)} F(N) \xrightarrow{F(g)} F(N'')$$

is exact, the functor is called <u>*left exact*</u>*; if the sequence*

$$F(N') \stackrel{F(f)}{\to} F(N) \stackrel{F(g)}{\to} F(N'') \to 0$$

is exact, F is said to be <u>*right exact*</u>*. The functor* $\mathrm{Hom}_{\mathcal{C}}(c,-) : \mathcal{C} \to \mathrm{vect}_{\mathbb{K}}$ *for a* $\mathbb{K}$*-linear category* $\mathcal{C}$ *is left exact.*

Proposition 3.2.7.
Let A be a $\mathbb{K}$-algebra. Then the following assertions about an A-module M are equivalent:

1. For every diagram with A-modules N_1, N_2

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d] & \\ N_1 \arrow[r] & N_2 \arrow[r] & 0 \end{tikzcd}$$

   with exact line, there is a lift such that the diagram commutes. (The lift is indicated by the dotted arrow. The lift is, in general, not unique.)

2. There is an A-module N such that $M \oplus N$ is a free A-module.

3. Any short exact sequence of the form

$$0 \to N' \to N \stackrel{f}{\to} M \to 0$$

   splits, i.e. there is a morphism $s : M \to N$ such that $f \circ s = \mathrm{id}_M$. Then $N \cong N' \oplus s(M)$.

4. For any short exact sequence of modules

$$0 \to T' \to T \to T'' \to 0$$

   the sequence of $\mathbb{K}$-vector spaces with morphisms given by postcomposition

$$0 \to \mathrm{Hom}_A(M,T') \to \mathrm{Hom}_A(M,T) \to \mathrm{Hom}_A(M,T'') \to 0$$

   is exact. (Note that the sequence

$$0 \to \mathrm{Hom}_A(M,T') \to \mathrm{Hom}_A(M,T) \to \mathrm{Hom}_A(M,T'')$$

   is exact for any module M.)

Proof.

1⇒ 3 The split is given by the lift in the specific diagram

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d, "\mathrm{id}"] & \\ N \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

3⇒ 2 Any A-module M is a quotient of a free module, e.g. by the surjection

$$\begin{array}{rcl} \oplus_{m\in M} A & \to & M \\ a_m & \mapsto & a_m.m \end{array}$$

Take a surjection $N \to M$ with kernel N' and N a free module. Since the short exact sequence $0 \to N' \to N \to M \to 0$ splits, we have $N \cong M \oplus N'$, where N is a free module.

2⇒ 4 We first note that assertion 4 holds in the case when M is a free module: then $\mathrm{Hom}_A(M,T) \cong \mathrm{Hom}_A(\oplus_{i\in I}A,T) \cong \prod_{i\in I} T$ for any module T, where the index set I labels a basis of M. The maps are simply in each component the given maps.

In particular, if N is a complement of M to a free module, the sequence

$$0 \to \mathrm{Hom}_A(M\oplus N,T') \to \mathrm{Hom}_A(M\oplus N,T) \to \mathrm{Hom}_A(M\oplus N,T'') \to 0$$

is exact. Using the universal property of the direct sum, this amounts to

$$\begin{aligned}&0 \to \mathrm{Hom}_A(M,T') \times \mathrm{Hom}_A(N,T') \to \mathrm{Hom}_A(M,T) \times \mathrm{Hom}_A(N,T)\\ &\to \mathrm{Hom}_A(M,T'') \times \mathrm{Hom}_A(N,T'') \to 0\ .\end{aligned}$$

The kernel of a Cartesian product of maps is the product of kernels; the image of the Cartesian product of maps is the Cartesian product of the images. This implies the exactness of the sequence in 4.

4⇒ 1 From the surjectivity of the horizontal line, we get a short exact sequence

$$0 \to \ker((N_1 \to N_2)) \to N_1 \xrightarrow{f} N_2 \to 0$$

By 4, we get a short exact sequence

$$0 \to \mathrm{Hom}_A(M,\ker(N_1 \to N_2)) \to \mathrm{Hom}_A(M,N_1) \to \mathrm{Hom}_A(M,N_2) \to 0$$

where the last arrow is

$$\begin{array}{rccc} f_* : & \mathrm{Hom}_A(M,N_1) & \to & \mathrm{Hom}_A(M,N_2)\\ & \varphi & \mapsto & f\circ\varphi =: f_*(\varphi)\end{array}$$

The surjectivity of this morphism amounts to property 1.

□

Definition 3.2.8
An A-module with one of the four equivalent properties from proposition 3.2.7 is called a projective module.

We also have the corresponding notion in a general abelian category:

Definition 3.2.9

1. *An object P in an abelian category $\mathcal{C}$ is called projective, if the functor* $\mathrm{Hom}_{\mathcal{C}}(P,-)$ *is exact.*

2. *Dually, an object $I \in \mathcal{C}$ is called injective, if the functor* $\mathrm{Hom}_{\mathcal{C}}(-,I)$ *is exact.*

We can now define

Definition 3.2.10
Let $\mathbb{K}$ be a field. A $\mathbb{K}$-linear category $\mathcal{C}$ is called a finite category,

1. *$\mathcal{C}$ has finite-dimensional spaces of morphisms.*

2. *Every object of $\mathcal{C}$ has finite length, i.e. for any object $c \in \mathcal{C}$ there exists a finite filtration*

$$0 = c_0 \subset c_1 \subset c_1 \subset \ldots \subset c_n = c$$

   by subobjects such that the quotient object c_i/c_{i-1} is a simple object.

3. *$\mathcal{C}$ has enough projectives, i.e. every simple object has a projective cover. (A projective cover of an object $c \in \mathcal{C}$ is a projective object $p(c) \in \mathcal{C}$, together with an epimorphism $\pi : p(c) \to c$ such that if $g : p' \to c$ is an epimorphism from a projective object p' to c, then there exists an epimorphism $h : p' \to p(c)$ such that $\pi \circ h = g$.*

4. *There are finitely many isomorphism classes of simple objects.*

Remark 3.2.11.
A $\mathbb{K}$-linear category is finite, if and only if it is equivalent to the category A-mod of finite-dimensional A-modules over a finite-dimensional $\mathbb{K}$-algebra. A detailled proof can be found in [DSPS2, Proposition 1.4]. The algebra can be given explicitly: chose for any isomorphism class of simple module with representative S_i a projective cover $P_i \to S_i$. Then A can be chosen to be the endomorphism algebra $\mathrm{End}(\oplus_{i \in I} P_i)$, where the sum goes over a system of representatives for the isomorphism classes of simple objects.

Definition 3.2.12
A finite tensor category is a finite rigid monoidal linear category.

Remarks 3.2.13.

1. The category $H-\mathrm{mod}_{fd}$ of finite-dimensional modules over a finite-dimensional Hopf algebra is a finite tensor category. It is, however, not true that every finite tensor category is monoidally equivalent to the category of finite-dimensional modules over some Hopf algebra.

2. It can be shown though that every finite tensor category is monoidally equivalent to the category of finite-dimensional modules over a Hopf algebroid, see [BLV, Theorem 7.6]; for the definition of a Hopf algebroid see the beginning of [BLV, Section 7].

Proposition 3.2.14.
Let A be a $\mathbb{K}$-algebra. Then the following assertions are equivalent:

1. The algebra A is semisimple, i.e. seen as a left module over itself, it is a direct sum of simple submodules.

2. Any A-module is semisimple, i.e. direct sum of simple submodules.

3. The category $A-\mathrm{mod}$ is semisimple, i.e. all A-modules are projective.

As a consequence of this result, we need to understand only *simple* modules to understand the representation category of a semisimple algebra. The morphisms between simple modules are controlled by Schur's lemma; there are no extensions, so the homological algebra of such categories is trivial.

Proof.

3.⇒ 2. Suppose that the category $A-$mod is semisimple. Let M be an A-module. Any submodule $U \subset M$ yields a short exact sequence

$$0 \to U \to M \to M/U \to 0 \ ,$$

which splits, since the module M/U is projective. Then $M \cong U \oplus M/U$ and the submodule U has a complement in M. Thus, the module M is semisimple.

2.⇒ 1. Trivial, since 1. is a special case of 2.

1.⇒ 3. We have to show that every module is projective, i.e. direct summand of a free module. Since every module is a homomorphic image of a free module F, we have a short exact sequence:

$$0 \to \ker \pi \to F \xrightarrow{\pi} M \to 0$$

A being semisimple by assumption, implies that also the direct sum F of copies of A is semisimple. Thus the submodule $\ker \pi$ has a complement which is isomorphic to M, $F \cong M \oplus \ker \pi$. Thus M is projective by property 2 of a projective module.

□

Lemma 3.2.15.
Let $\mathcal{C}$ be an abelian category. Then a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{C}$, if for any object $X \in \mathcal{C}$ the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{C}}(X, C)$$

of abelian groups is exact.

Proof.
Let $X = A$ and find from the exact Hom-sequence $\beta \circ \alpha = \beta_* \circ \alpha_*(\mathrm{id}_A) = 0$. Thus $\mathrm{Im}\,\alpha \subset \ker \beta$.

Next consider $X = \ker \beta$ with inclusion map $\iota : \ker \beta \to B$. Since ι is the embedding of the kernel of β, we have $\beta_*(\iota) = \beta \circ \iota = 0$. By exactness of the Hom sequence, there exists $\varphi \in \mathrm{Hom}_{\mathcal{C}}(\ker \beta, A)$ such that $\alpha \circ \varphi = \alpha_*(\varphi) = \iota$. Thus $\ker \beta \subset \mathrm{Im}\,\alpha$. The converse statement follows since the Hom-functor is left exact. □

Lemma 3.2.16.
Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ be an additive functor left adjoint to $G : \mathcal{D} \to \mathcal{C}$, i.e. $F \dashv G$. Then F is a right exact functor and G is a left exact functor.

Proof.
Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{D}$ and let $X \in \mathcal{C}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(F(X), A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(F(X), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(F(X), C) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(X, G(A)) \arrow[r] & \mathrm{Hom}(X, G(B)) \arrow[r] & \mathrm{Hom}(X, G(C))
\end{tikzcd}$$

The vertical arrows are the adjunction isomorphisms and isomorphisms of abelian groups. The top row is exact since the Hom-functor is left exact, thus the bottom row is exact as well. By lemma 3.2.15, this implies that $0 \to G(A) \to G(B) \to G(C)$ is exact. Thus any right adjoint functor is left exact.

To see that F is right exact, notice that $F^{\text{opp}} : \mathcal{C}^{\text{opp}} \to \mathcal{D}^{\text{opp}}$ is a right adjoint of G^{opp} and thus by the previous argument left exact. But this amounts to the statement that F is right exact. $\square$

A certain converse is provided by so-called adjoint functor theorems: they state that under certain conditions a functor that preserves limits (which generalizes left exactness) is a right adjoint, and that a functor that preserves colimits (which preserves right exatness) is a left adjoint. In our setting, the following proposition can be shown [DSPS2, Proposition 1.7]:

Proposition 3.2.17.
Let $\mathcal{C}$ and $\mathcal{D}$ be finite linear categories and let $G : \mathcal{C} \to \mathcal{D}$ be a linear functor. Then G is left exact if and only if G admits a linear left adjoint, and G is right exact if and only if G admits a linear right adjoint.

Lemma 3.2.18.
Let $\mathcal{C}$ be an abelian monoidal category. Suppose that the object X is rigid. Then the functor $-\otimes X$ of tensoring with X is exact, i.e. if

$$0 \to U \to V \to W \to 0$$

is an exact sequence in $\mathcal{C}$, then

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

is exact in $\mathcal{C}$ as well. In particular, the tensor product of a finite tensor category $\mathcal{C}$ provides exact functors $c \otimes -$ and $- \otimes c$ for every object $c \in \mathcal{C}$.

Proof.
This follows from lemma 3.2.16, since the functor of tensoring with a rigid object has a left and a right adjoint by example 2.5.26. $\square$

For the following propositions, the reader might wish to keep the category $\mathcal{C} = H\text{–mod}_{fd}$ of finite-dimensional modules over a finite-dimensional Hopf algebra in mind.

Lemma 3.2.19.
Let $\mathcal{C}$ be an abelian tensor category. Let P be a projective object and let M be an object that has a right dual. Then the object $P \otimes M$ is projective.

Proof.
By rigidity, we have adjunction isomorphisms

$$\text{Hom}(P \otimes M, N) \cong \text{Hom}(P, N \otimes M^\vee) \ .$$

Thus the functor $\text{Hom}(P \otimes M, -)$ is isomorphic to the concatenation of the functor $- \otimes M^\vee$ (which is exact by lemma 3.2.18) with the functor $\text{Hom}(P, -)$ which is exact by property 4 of the projective object P. $\square$

Corollary 3.2.20.
A finite tensor category $\mathcal{C}$ is semi-simple, if and only if the monoidal unit is projective. In particular, a $\mathbb{K}$-Hopf algebra is semi-simple, if and only if the trivial module $(\mathbb{K}, \epsilon)$ is projective.

Proof.
If the tensor unit $\mathbb{I}$ is semisimple, then by lemma 3.2.19 any object $M \cong M \otimes \mathbb{I}$ is projective. The converse is trivial. In the case of $\mathcal{C} = H-\mathrm{mod}_{fd}$, the trivial module $(\mathbb{K}, \epsilon)$ is the tensor unit in $H-$mod. □

Definition 3.2.21
A semisimple finite tensor category is called a <u>*fusion category*</u>.

Theorem 3.2.22 (Maschke)**.**
Let H be a finite-dimensional Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.

2. The counit takes non-zero values on the one-dimensional space of left integrals, $\epsilon(\mathcal{I}_l(H)) \neq 0$.

Proof.

1. Suppose that H is semisimple. Then any module is projective, in particular the trivial module $(\mathbb{K}, \epsilon)$. Thus the exact sequence of left H-modules given by the counit

$$(0) \to \ker \epsilon \to \ {}_H H \xrightarrow{\epsilon} \mathbb{K} \to (0) \qquad (*)$$

splits. Here, H is considered with the structure of the left regular module; note that ϵ is a morphism of modules: $\epsilon(h.x) = \epsilon(h \cdot x) = \epsilon(h)\epsilon(x) = h.\epsilon x$. Thus, we have a direct sum decomposition of H-left modules $H = \ker \epsilon \oplus I$ with I a left ideal of H.

We first note that for any $h \in H$, we have $h - \epsilon(h)1 \in \ker \epsilon$. Moreover, since I is a left ideal, we have $h \cdot z \in I$ for any $h \in H$ and $z \in I$. Thus, in the equality

$$h \cdot z = (h - \epsilon(h)1) \cdot z + \epsilon(h) z$$

the left hand side is in I, while on the right hand side the first summand is in $\ker \epsilon$ and the second summand is in I. Because of the direct sum decomposition $H = \ker \epsilon \oplus I$, the first summand has to vanish and we find $h \cdot z = \epsilon(h) z$ for all $h \in H$ and all $z \in I$. This is the defining equation for a left integral, so any $z \in \mathcal{I}_l(H)$ is a left integral in H.

Since $\dim_K I = 1$, we may choose $z \neq 0$. Then $z \notin \ker \epsilon$ and thus $\epsilon(\mathcal{I}_l(H)) \neq 0$.

2. Conversely, let Λ be a left integral and assume that $\epsilon(\Lambda) \neq 0$. Replacing Λ by a non-zero scalar multiple, we can assume that $\epsilon(\Lambda) = 1$. Then

$$\begin{array}{rccc} s: & \mathbb{K} & \to & H \\ & \mu & \mapsto & \mu\Lambda \end{array}$$

obeys $\epsilon \circ s(\lambda) = \lambda\epsilon(\Lambda) = \lambda$ and is a morphism of left H modules, since Λ is a left integral, so that the exact sequence $(*)$ splits. Thus the trivial module is projective and the claim follows from corollary 3.2.20.

□

Example 3.2.23.
Consider the group algebra $\mathbb{K}[G]$ of a finite group G with two-sided integral $\Lambda = \sum_{g\in G} g$. Then

$$\epsilon(\Lambda) = \sum_{g\in G} \epsilon(g) = |G| \in \mathbb{K} \ .$$

Thus the group algebra $\mathbb{K}[G]$ is semisimple, if and only if $\mathrm{char}(\mathbb{K}) \not| \ |G|$. In this case, the category $G-\mathrm{rep}_{fd}$ of finite-dimensional G-representations is a fusion category.

The category vect_G of finite-dimensional G-graded vector spaces is a fusion category for any field $\mathbb{K}$.

Corollary 3.2.24.
A finite-dimensional semisimple Hopf algebra is unimodular.

Proof.
Since H is semisimple, we can choose by Maschke's theorem 3.2.22 a left integral $t \in H$ such that $\epsilon(t) \neq 0$. Then for any $h \in H$, we have

$$\alpha(h)\epsilon(t)t = \alpha(h)t^2 = (th)t = t(ht) = \epsilon(h)t^2 = \epsilon(h)\epsilon(t)t \ ,$$

where we used the definition of a left integral and of the distinguished group-like element α of H^*. Since $\epsilon(t) \neq 0$, we have $\alpha(h) = \epsilon(h)$ for all $h \in H$ which implies unimodularity by corollary 3.1.18. □

Remarks 3.2.25.

1. We can immediately conclude from Remark 3.1.10 that the Taft algebra is not semisimple, since it is not unimodular.

2. A distinguished invertible object D can also be introduced for a general finite tensor category $\mathcal{C}$, e.g. as the socle of the projective cover of the monoidal unit [EGNO, Section 6.4]. The finite tensor category is called unimodular, if D is isomorphic to the monoidal unit $\mathbb{I}$. Every semisimple finite tensor category is automatically unimodular [EGNO, Remark 6.5.9].

We recall the notion of a separable algebra over a field $\mathbb{K}$. To this end, let A be an associative unital $\mathbb{K}$-algebra. The algebra $A^e := A \otimes A^{\mathrm{opp}}$ is called the enveloping algebra of A. If B is an A-bimodule, it is a left module over A^e by

$$(a_1 \otimes a_2).b := (a_1.b).a_2 \ .$$

Conversely, any A^e-left module M carries a canonical structure of an A-bimodule with left action $a.m := (a \otimes 1).m$ and right action $m.a := (1 \otimes a).m$. Thus the categories of A^e-left modules and A-bimodules are canonically isomorphic as $\mathbb{K}$-linear abelian categories. (The category of bimodules has on top of this a monoidal structure with A as a bimodule as the monoidal unit.)

Proposition 3.2.26.
Let $\mathbb{K}$ be a field and A be an associative unital $\mathbb{K}$-algebra. Then the following properties are equivalent:

1. A is projective as an A^e-module.

2. The short exact sequence of A^e-modules

$$0 \to \ker \mu \to A^e \xrightarrow{\mu} A \to 0$$

splits. Put differently, the multiplication epimorphism

$$\mu : A \otimes_{\mathbb{K}} A \to A$$

has a right inverse as a morphism of bimodules:

$$\varphi : A \to A \otimes_{\mathbb{K}} A$$

with $\mu \circ \varphi = \mathrm{id}_A$ and $\varphi(abc) = a \cdot \varphi(b) \cdot c$ for all $a, b, c \in A$.

3. Given any extension of fields $\mathbb{K} \subset E$, the E-algebra $A \otimes_{\mathbb{K}} E$ induced by extension of scalars is semisimple.

For the proof of this statement, we refer to [Pierce, Chapter 10]. Note that over a field that is not perfect there are semisimple algebras that are not separable. For example, for any prime p, consider the field $\mathbb{K} := \mathbb{F}_p(t)$ of rational functions over the field $\mathbb{F}_p$ with p elements. Then the algebra $\mathbb{K}[X]/(X^p - t)$ is known to be semisimple but not separable.

Definition 3.2.27
A $\mathbb{K}$-algebra A that has one of the properties of proposition 3.2.26 is called separable.

Remarks 3.2.28.

1. The choice of a right inverse φ of the multiplication $\mu : A \otimes_{\mathbb{K}} A \to A$ is called the choice of a separability structure of A.

2. We can describe φ by the element $e := \varphi(1_A) \in A^e$. Indeed, since φ is a morphism of A-bimodules, $\varphi(a) = a.e = (a \otimes 1_A)e$. Since s is a section of the multiplication, we have $\mu(e) = \mu(s(1_A)) = 1_A$. Finally, we have

$$(a \otimes 1_A)e = a.e = \varphi(a \cdot 1) = \varphi(1 \cdot a) = e.a = e(1_A \otimes a)$$

for all $a \in A$. An element $C \in B$ in an A-bimodule B that obeys $a.C = C.a$ is called a Casimir element.

With the multiplication in A^e, we have $e^2 = e$, see [Pierce, p. 182].. The element $e \in A^e$ is therefore called a separability idempotent.

3. Separable algebras over fields are finite-dimensional and semisimple.

More precisely, a $\mathbb{K}$-algebra A is separable, if and only if

$$A \cong A_1 \oplus \cdots \oplus A_r$$

is a direct sum of finite-dimensional simple $\mathbb{K}$-algebras where all $Z(A_i)/\mathbb{K}$ are separable extensions of fields.

4. We present an example: for any field $\mathbb{K}$, the full matrix algebra $\mathrm{Mat}_n(\mathbb{K})$ is a separable $\mathbb{K}$-algebra.

Introduce matrix units ϵ_{ij}, i.e. ϵ_{ij} is the matrix with zero entries, except for one in the i-th line and j-th column. Fix some index $1 \leq j \leq n$; then

$$e^{(j)} := \sum_{i=1}^{n} \epsilon_{ij} \otimes \epsilon_{ji} \in \mathrm{Mat}_n(\mathbb{K}) \otimes \mathrm{Mat}_n(\mathbb{K})^{\mathrm{opp}}$$

obeys

$$\mu(e^{(j)}) = \sum_{i=1}^{n} \epsilon_{ij}\epsilon_{ji} = \sum_{i=1}^{n} \epsilon_{ii} = 1 \in \mathrm{Mat}_n(\mathbb{K})$$

and for all $k, l = 1, \ldots n$

$$\sum_{i=1}^{n} \epsilon_{ij} \otimes \epsilon_{ji} \cdot \epsilon_{kl} = \epsilon_{kj} \otimes \epsilon_{jl} = \sum_{i=1}^{n} \epsilon_{kl}\epsilon_{ij} \otimes \epsilon_{ji}$$

so that all elements $e^{(j)}$ are separability idempotents. In particular, the separability idempotent is not unique.

Proposition 3.2.29.
Let H be a finite-dimensional *semisimple* $\mathbb{K}$-Hopf algebra.

1. H is a separable $\mathbb{K}$-algebra.

2. Any Hopf subalgebra $U \subset H$ such that H is free over U is semisimple as well.

Proof.

1. We have to show that for any field extension $\mathbb{K} \subset E$, the algebra $H \otimes_{\mathbb{K}} E$ is semisimple as well. Note that $H \otimes E$ is an E-Hopf algebra with morphisms

$$\begin{aligned} \Delta(h \otimes \alpha) &:= \Delta(h) \otimes \alpha \in H \otimes H \otimes E \cong (H \otimes E) \otimes_E (H \otimes E) \\ \epsilon(h \otimes \alpha) &:= \epsilon(h) \otimes \alpha \\ S(h \otimes \alpha) &:= S(h) \otimes \alpha \end{aligned}$$

for all $h \in H$ and all $\alpha \in E$. This implies that the ideal of left integrals is obtained by extension of scalars as well,

$$\mathcal{I}_l(H \otimes_{\mathbb{K}} E) = \mathcal{I}_l(H) \otimes_{\mathbb{K}} E$$

and thus that the counit ϵ is non-zero on $\mathcal{I}_l(H \otimes_{\mathbb{K}} E)$. Now Maschke's theorem 3.2.22 implies that the Hopf algebra $H \otimes_{\mathbb{K}} E$ is semisimple.

2. Since H is semisimple, find $t \in \mathcal{I}_l(H)$ with $\epsilon(t) \neq 0$. Since H is free as a U-module, find a U-basis $\{h_i\}$ of ${}_U H$ and write $t = \sum_{i \in I} u_i h_i$ with coefficients $u_i \in U$. Then for all $u \in U$, we have

$$\sum_{i \in I} (u u_i) h_i = ut = \epsilon(u) t = \sum_{i \in I} (\epsilon(u) u_i) h_i \; .$$

Comparison of coefficients with respect to the basis $\{h_i\}$ shows $u u_i = \epsilon(u) u_i$ for all $i \in I$ and all $k \in K$. Thus all coefficients $u_i \in U$ of t are integrals of U, i.e. $u_i \in \mathcal{I}_l(U)$.

Now $0 \neq \epsilon(t) = \sum_{i \in I} \epsilon(u_i)\epsilon(h_i)$ implies that $\epsilon(u_i) \neq 0$ for some $i \in I$. Thus, by Maschke's theorem 3.2.22, the Hopf subalgebra U is semisimple.

□

Observation 3.2.30.

1. We comment on the results in a language using bases. Let A be a Frobenius algebra. It is finite-dimensional and let $(l_i)_{i=1,\dots N}$ be any $\mathbb{K}$-basis of A. Since the Frobenius form κ is non-degenerate, we can find another basis $(r_i)_{i=1,\dots,N}$ such that
$$\kappa(l_i, r_j) = \delta_{ij} \ .$$
Such a pair of bases is called a pair (r_i, l_i) of <u>dual bases</u> for the Frobenius form κ.

2. Since $(l_i)_{i=1,\dots N}$ is a basis, we can write any $x \in A$ as a linear combination, $x = \sum_{i=1}^N x_i l_i$. Since
$$\kappa(x, r_j) = \sum_{i=1}^N x_i \kappa(l_i, r_j) = x_j$$
we have explicit expressions for the coefficients $x_j \in \mathbb{K}$ and find for the expansion in the basis $(l_i)_{i=1,\dots N}$
$$x = \sum_{i=1}^N \kappa(x, r_i) l_i \quad \text{for all } x \in A \ ; \qquad (*)$$
similarly, we find for the expansion in the basis $(r_i)_{i=1,\dots N}$
$$x = \sum_{i=1}^N \kappa(l_i, x) r_i \quad \text{for all } x \in A \ . \qquad (**)$$

3. Conversely, given a pair of bases (r_i, l_i) such that equation $(*)$ holds for all $x \in A$, we find with $x = l_j$ by comparing coefficients that $\kappa(l_i, r_i) = \delta_{ij}$ holds. We conclude that $(**)$ holds for all $x \in A$.

4. Consider for any pair of dual bases of a Frobenius algebra A the element
$$C := \sum_{i=1}^N r_i \otimes l_i \in A \otimes A \ .$$
We claim that it is a Casimir element, i.e. $x.C = C.x$ for all $x \in A$. Indeed, by $(*)$
$$l_i x = \sum_{i=1}^N \kappa(l_i x, r_i) l_i$$
which implies
$$C.x = \sum_{i=1}^N r_i \otimes l_i x = \sum_{i,j=1}^N \kappa(l_i x, r_j) r_i \otimes l_i \ .$$
Similarly, we find with $(**)$
$$x.C = \sum_{i=1}^N x r_i \otimes l_i = \sum_{i,j=1}^N \kappa(l_i, x r_j) r_i \otimes l_i \ .$$
The invariance of the Frobenius form κ now implies the Casimir relation $xC = Cx$.

Remark 3.2.31.
We can give explicitly a separability idempotent of a finite-dimensional semisimple Hopf algebra.

1. Let $\lambda \in H^*$ be a non-zero left integral and let $\Lambda \in H$ be a right integral such that $\lambda(\Lambda) = 1$, cf. proposition 3.1.27. Then we have for all $x \in H$

$$\begin{aligned} S(\Lambda_{(1)})\langle\lambda, \Lambda_{(2)}x\rangle &= S(\Lambda_{(1)})\Lambda_{(2)}x_{(1)}\langle\lambda, \Lambda_{(3)}x_{(2)}\rangle \quad [\lambda \text{ left integral for } H^*] \\ &= x_{(1)}\ \langle\lambda, \Lambda x_{(2)}\rangle \quad [S \text{ antipode}] \\ &= x\langle\lambda, \Lambda\rangle = x \quad [\Lambda \text{ right integral, normalization}] \end{aligned}$$

It follows that the components $\Lambda^i_{(1)}$ of any representation of $\Delta(\Lambda)$ as a sum

$$\Delta(\Lambda) = \sum_i \Lambda^i_{(1)} \otimes \Lambda^i_{(2)} \in H \otimes H$$

form a generating system of H. We can thus express $\Delta(\Lambda)$ as a sum such that the components $(\Lambda^i_{(1)})$ form a basis. Thus $(S(\Lambda_{(1)}), \Lambda_{(2)})$ form a pair of dual bases for the standard Frobenius structure on on the Hopf algebra H given by λ, cf. Theorem 3.1.25.

2. Assume that H is moreover semisimple. By Maschke's theorem $\kappa := \epsilon(\Lambda) \neq 0$. Then

$$e := \kappa^{-1} \cdot S(\Lambda_{(1)}) \otimes \Lambda_{(2)} \in H \otimes H$$

is a separability idempotent. Indeed,

$$\mu(e) := \kappa^{-1}S(\Lambda_{(1)}) \cdot \Lambda_{(2)} = \kappa^{-1}1_H\epsilon(\Lambda) = 1_H$$

by the defining property of the antipode. The Casimir property of a separability idempotent follows directly from observation 3.2.30.4, since it is built from a pair of dual bases. The integral thus provides a separability idempotent for a semisimple Hopf algebra.

3.3 Powers of the antipode

Observation 3.3.1.
Let V be a finite-dimensional $\mathbb{K}$-vector space. Using the canonical identification

$$\begin{array}{rcl} V^* \otimes V & \to & \mathrm{End}_{\mathbb{K}}(V) \\ \beta \otimes v & \mapsto & (w \mapsto \beta(w)v) \end{array}$$

we can associate to $\beta \otimes v$ the value of the trace. We have $\mathrm{Tr}(\beta \otimes v) = \beta(v)$. Indeed, consider dual bases $(e_i)_{i\in I}$ of V and $(e^i)_{i\in I}$ of V^* and write $\beta = \sum_i \beta_i e^i$ and $v = \sum_i v^i e_i$. The linear endomorphism corresponding to $\beta \otimes v$ maps the basis vector e_i to

$$e_i \mapsto \beta(e_i)\sum_j v^j e_j = \sum_j \beta_i v^j e_j$$

and thus has trace $\sum_j \beta_j v^j = \beta(v)$.

Lemma 3.3.2.
Let H be a finite-dimensional Hopf algebra with $\lambda \in \mathcal{I}_l(H^*)$ and a right integral $\Lambda \in H$ such that $\lambda(\Lambda) = 1$. Let F be a linear endomorphism of H. Then

$$\mathrm{Tr}(F) = \langle\lambda, F(\Lambda_{(2)})S(\Lambda_{(1)})\rangle\ .$$

Proof.
We know by remark 3.2.31.1 that for all $x \in H$, we have

$$F(x) = \langle \lambda, F(x)S(\Lambda_{(1)})\rangle \Lambda_{(2)} \ .$$

Thus under the identification $H^* \otimes H \cong \mathrm{End}(H)$, the endomorphism F corresponds to

$$\langle \lambda, F(-)S(\Lambda_{(1)})\rangle \otimes \Lambda_{(2)} \ ;$$

thus by observation 3.3.1

$$\mathrm{Tr}(F) = \langle \lambda, F(\Lambda_{(2)})S(\Lambda_{(1)})\rangle \ .$$

□

We need to understand the powers of the antipode. We first need another structure: for any element $h \in H$, left multiplication yields a $\mathbb{K}$-linear endomorphism

$$\begin{array}{rccc} L_h : & H & \to & H \\ & x & \mapsto & hx \end{array}$$

We thus define a linear form

$$\begin{array}{rccc} \mathrm{Tr}_H : & H & \to & \mathbb{K} \\ & h & \mapsto & \mathrm{Tr}(L_h) \ . \end{array}$$

Proposition 3.3.3.
Let H be a finite-dimensional Hopf algebra with $\lambda \in \mathcal{I}_l(H^*)$ and a right integral $\Lambda \in H$ such that $\lambda(\Lambda) = 1$.

1. We have for the trace of the endomorphism $S^2 : H \to H$

$$\mathrm{Tr} S^2 = \langle \epsilon, \Lambda\rangle \langle \lambda, 1\rangle \ .$$

2. If $S^2 = \mathrm{id}_H$, then $\mathrm{Tr}_H = \langle \epsilon, \Lambda\rangle\lambda$.

Proof.

1. Taking S^2 in lemma 3.3.2, we find

$$\mathrm{Tr}(S^2) = \langle \lambda, S^2(\Lambda_{(2)})S(\Lambda_{(1)})\rangle = \langle \lambda, S(\Lambda_{(1)} \cdot S(\Lambda_{(2)}))\rangle = \langle \epsilon, \Lambda\rangle \cdot \langle \lambda, 1\rangle \ .$$

2. The identity $S^2 = \mathrm{id}_H$ implies that S is also a skew antipode and thus by proposition 2.5.7

$$h_{(2)}S(h_{(1)}) = \langle \epsilon, h\rangle 1 \quad \text{for all } h \in H \ . \qquad (*)$$

Taking $F = L_h$, we find

$$\mathrm{Tr}_H(h) \stackrel{\text{def}}{=} \mathrm{Tr}(L_h) \stackrel{3.3.2}{=} \langle \lambda, h\Lambda_{(2)}S(\Lambda_{(1)})\rangle \stackrel{(*)}{=} \langle \epsilon, \Lambda\rangle \cdot \langle \lambda, h\rangle \ ,$$

where we used in the last step equation $(*)$ for $h = \Lambda$.

□

Corollary 3.3.4.

1. H and H^* are semisimple, if and only if $\mathrm{Tr} S^2 \neq 0$.
2. If $S^2 = \mathrm{id}_H$ and $\mathrm{char}\mathbb{K}$ does not divide $\dim H$, then H and H^* are semisimple.

Indeed, for the Taft algebra $S^2 \neq \mathrm{id}$, and the Taft algebra over any field is not semisimple.

Proof.

1. By Maschke's theorem 3.2.22, H is semisimple, if and only if $\langle \epsilon, \Lambda \rangle \neq 0$. Similarly, again by Maschke's theorem, H^* is semisimple, if and only if $\langle \epsilon^*, \Lambda^* \rangle = \langle \lambda, 1 \rangle \neq 0$. Together with proposition 3.3.3.1, this implies the assertion.
2. If $S^2 = \mathrm{id}_H$, then by proposition 3.3.3.1
$$\dim H = \mathrm{Tr} S^2 = \langle \epsilon, \Lambda \rangle \langle \lambda, 1 \rangle$$
which is non-zero by the assumption on the characteristic of $\mathbb{K}$. Now Maschke's theorem 3.2.22 implies the assertion.

□

Remark 3.3.5.

1. Assume that H is a cocommutative Hopf algebra. We have seen in corollary 2.5.10.1 that $S^2 = \mathrm{id}_H$ for a cocommutative Hopf algebra. Thus by corollary 3.3.4 a cocommutative finite-dimensional Hopf algebra over a field $\mathbb{K}$ of characteristic zero is always semisimple and cosemisimple.
2. Let us assume that the field $\mathbb{K}$ of characteristic 0 is moreover algebraically closed. Since H^* is semisimple, it is, as an algebra, isomorphic to $H^* \cong \mathbb{K} \times \mathbb{K} \times \ldots \times \mathbb{K}$ by the Artin-Wedderburn theorem. The projection p_i to the i-th factor is a morphism of algebras or, put differently, a grouplike element in $H^{**} \cong H$. All projections give a basis of H consisting of grouplike elements. Thus a cocommutatitve finite-dimensional Hopf algebra H over an algebraically closed field is a group algebra of a finite group. We will therefore have to go beyond cocommutativity.

Observation 3.3.6.

1. Consider a Frobenius algebra A over a field $\mathbb{K}$ with bilinear form κ. Since κ is non-degenerate, it provides a bijection
$$\begin{array}{rcl} A & \to & A^* \\ h & \mapsto & \kappa(-, h) \ . \end{array}$$
Consider for fixed $x \in A$ the linear form on A
$$y \mapsto \kappa(x, y) \ .$$
Using the bijection $A \to A^*$ above, we find $\rho(x) \in A$ such that
$$\kappa(x, y) = \kappa(y, \rho(x)) \quad \text{for all } y \in A \ .$$
The map $\rho : A \to A$ is obviously $\mathbb{K}$-linear and a bijection.

2. The map $\rho : A \to A$ is a morphism of algebras. Indeed, using the definition of ρ and the invariance (I) of κ, we find for all $x, y, z \in A$

$$\begin{aligned} \kappa(z, \rho(xy)) &= \kappa(xy, z) \stackrel{(I)}{=} \kappa(x, yz) = \kappa(yz, \rho(x)) \stackrel{(I)}{=} \kappa(y, z\rho(x)) \\ &= \kappa(z\rho(x), \rho(y)) \stackrel{(I)}{=} \kappa(z, \rho(x)\rho(y)) \end{aligned}$$

Since the Frobenius form κ is non-degenerate, this implies $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in A$.

Definition 3.3.7
Let A be a Frobenius algebra over a field $\mathbb{K}$.

1. *The automorphism ρ is called the Nakayama automorphism of the Frobenius algebra A with respect to the Frobenius structure κ.*

2. *A Frobenius algebra is called symmetric, if the Nakayama automorphism equals the identity, i.e. if $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in A$.*

Remark 3.3.8.

1. If the Frobenius algebra A is commutative, the Nakayama involution is the identity and the Frobenius algebra is symmetric.

2. Group algebras over a field are examples of symmetric Frobenius algebras that are not necessarily commutative.

3. The notion of a symmetric Frobenius algebra can not be defined for an Frobenius algebra A in a general tensor category $\mathcal{C}$, since it involves left and right duals of A which can be different.

4. Denote by $\nu : A \to A$ the inverse of the Nakayama automorphism and define an endofunctor

$$N : A\text{−mod} \to A\text{−mod}$$

which sends a module $(M, \rho_M : A \to \mathrm{End}_{\mathbb{K}}(M))$ to the module with twisted action $\rho_M \circ \nu : A \to \mathrm{End}_{\mathbb{K}}(M)$. This so-called Nakayama functor is an autoequivalence of A−mod.

We compute

$$\kappa(\nu(z) \cdot x, y) \stackrel{(I)}{=} \kappa(\nu(z), xy) = \kappa(xy, z) \stackrel{(I)}{=} \kappa(x, yz) \qquad (*)$$

Endowing A^* with the standard $\rightharpoonup$-action of A and A with the A-action twisted by μ, we see that the morphism

$$\begin{array}{rccc} \Phi : & A & \to & A^* \\ & x & \mapsto & \kappa(x, -) \end{array}$$

intertwines the two A-actions:

$$\Phi(z.x)(y) = \kappa(z.x, y) = \kappa(\nu(z)x, y) \stackrel{(*)}{=} \kappa(x, yz) = (z \rightharpoonup \Phi(x))(y) \ .$$

The Eilenberg-Watts theorem then shows that the Nakayama functor is equivalent to $A^* \otimes_A -$.

5. The Nakayama functor can be defined for any finite $\mathbb{K}$-linear category [FSS]; it is not necessary an equivalence. If $\mathcal{C} \cong A-$mod, the Nakayama functor is $A^* \otimes_A -$. It is right exact and has a right adjoint.

Our strategy will now be to compute the Nakayama automorphism for the Frobenius algebra structure of a finite-dimensional Hopf algebra given by the right cointegral, cf. theorem 3.1.25 in two different ways. (One can show that the Nakayama automorphism has, in this case, always finite order.) We need two lemmas. Denote by S^{-1} the composition inverse of the antipode, $S \circ S^{-1} = S^{-1} \circ S = \mathrm{id}_H$. It exists by theorem 3.1.14.

Lemma 3.3.9.
Let H be a finite-dimensional Hopf algebra. Let $\gamma \in H^*$ a non-zero right integral and let $\Gamma \in H$ be the left integral such that $\langle \gamma, \Gamma \rangle = 1$. Such a left integral exists by proposition 3.1.27.2. By theorem 3.1.14, $t := S(\Gamma) \in H$ is then a non-zero right integral. Denote by $\alpha \in H^*$ the distinguished group like element which is an algebra morphism $H \to \mathbb{K}$.

1. Then $(S^{-1}(t_{(2)}), t_{(1)})$ is a pair of dual bases for γ.

2. We have for the Nakayama automorphism for the Frobenius structure given by the right integral γ:
$$\rho(h) = \langle \alpha, h_{(1)} \rangle \; S^{-2}(h_{(2)}) \; .$$

Proof.

- We already know from remark 3.2.31 that for the Frobenius form given by a left integral λ for H^* we have a dual basis $(S\Lambda_{(1)}, \Lambda_{(2)})$. Applying this to the Hopf algebra H^{copp} which has antipode S^{-1}, we find the assertion.

  We remark that for all $x \in H$, we have by the general facts about dual bases
$$x = \langle \gamma, t_{(1)} x \rangle S^{-1}(t_{(2)})$$
  Let $x = 1$ and apply the counit ϵ; then
$$1 = \epsilon(1) = \langle \gamma, t_{(1)} \rangle \epsilon(S^{-1}(t_{(2)})) = \langle \gamma, t_{(1)} \rangle \epsilon((t_{(2)})) = \langle \gamma, t \rangle \qquad (*)$$

- We also have have, using dual bases,
$$\rho(h) = S^{-1}(t_{(2)}) \; \langle \gamma, t_{(1)} \rho(h) \rangle = S^{-1}(t_{(2)}) \; \langle \gamma, h t_{(1)} \rangle$$
  where in the second step applied the definition of the Nakayama automorphism ρ of H with Frobenius structure given by γ. Applying S^2, we find
$$\begin{aligned} S^2 \rho(h) &= \langle \gamma, h t_{(1)} \rangle S(t_{(2)}) \\ &= \langle \gamma, h_{(1)} t_{(1)} \rangle h_{(2)} t_{(2)} S(t_{(3)}) \quad [\gamma \text{ right integral of } H^*] \\ &= \langle \gamma, h_{(1)} t \rangle h_{(2)} = \langle \gamma, \alpha(h_{(1)}) t \rangle h_{(2)} \quad [\text{antipode, } \alpha \text{ distinguished element}] \\ &= \langle \alpha, h_{(1)} \rangle \, h_{(2)} \quad [\text{normalization in } (*)] \end{aligned}$$
  Applying S^{-2} yields the claim.

□

Similarly, we have

Lemma 3.3.10.
Let $a \in G(H)$ be the distinguished group-like element of H and t, γ as before in lemma 3.3.9.

1. Then $(S(t_{(1)})a, t_{(2)})$ is a pair of dual bases for γ.
2. We have for the Nakayama automorphism for the Frobenius structure given by the right integral γ:
$$\rho(h) = a^{-1}S^2(h_{(1)})\langle \alpha, h_{(2)}\rangle a \ .$$

Proof.

- Using the definitions, we find for all $h \in H$:
$$\begin{aligned} S(t_{(1)})a\langle \gamma, t_{(2)}h\rangle &= S(t_{(1)})t_{(2)}h_{(1)}\langle \gamma, t_{(3)}h_{(2)}\rangle \quad [\gamma \text{ right integral of } H^*] \\ &= h_{(1)}\langle \gamma, th_{(2)}\rangle = h_{(1)}\epsilon(h_{(2)}) = h \end{aligned}$$
so that we have dual bases.
- By the fact that we have dual bases, we can write
$$\rho(h) = S(t_{(1)})a\langle \gamma, t_{(2)}\rho(h)\rangle = S(t_{(1)})a\langle \gamma, ht_{(2)}\rangle \ ,$$
where in the second identity, we applied the definition of the Nakayama automorphism ρ. Applying S^{-2} and conjugating with a, we find
$$\begin{aligned} aS^{-2}(\rho(h))a^{-1} &= a\langle \gamma, ht_{(2)}\rangle S^{-1}(t_{(1)}) \\ &= h_{(1)}t_{(2)}\langle \gamma, h_{(2)}t_{(3)}\rangle S^{-1}(t_{(1)}) \quad [\gamma \text{ right cointegral}] \\ &= h_{(1)}\langle \gamma, h_{(2)}t\rangle = h_{(1)}\langle \alpha, h_{(2)}\rangle \ . \end{aligned}$$
Conjugating with a^{-1} and then applying S^2 yields the claim.

□

Observation 3.3.11.
If H is a finite-dimensional Hopf algebra, then H^* is a finite-dimensional Hopf algebra as well. We then have the structure of a left and right H^*-module on H by

$$h^* \rightharpoonup h := h_{(1)}\langle h^*, h_{(2)}\rangle \qquad \text{and} \qquad h \leftharpoonup h^* := \langle h^*, h_{(1)}\rangle h_{(2)} \ .$$

This follows by identifying $H \cong H^{**}$ from

$$(h^* \rightharpoonup h).\beta = h(\beta \cdot h^*) = \beta \cdot h^*(h) = \beta(h_{(1)}) \cdot h^*(h_{(2)})$$

for all $h^*, \beta \in H^*$ and $h \in H$. This is indeed a left H^*-action:

$$g^* \rightharpoonup (h^* \rightharpoonup h) = g^* \rightharpoonup (h_{(1)}\langle h^*, h_{(2)}\rangle) = h_{(1)}\langle g^*, h_{(2)}\rangle\langle h^*, h_{(3)}\rangle = h_{(1)}\langle g^* \cdot h^*, h_{(2)}\rangle = g^* \cdot h^* \rightharpoonup h$$

Theorem 3.3.12 (Radford,1976)**.**
Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$. Let $a \in G(H)$ and $\alpha \in G(H^*)$ be the distinguished grouplike elements. Then the following identity holds:

$$S^4(h) = a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha$$

Proof.

- We first show the second identity

$$a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha$$

  We transform the left hand side by using the definition of the H^*-actions for observation 3.3.11:

$$a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \langle \alpha, h_{(1)} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, h_{(3)} \rangle \ . \qquad (*)$$

  We transform the right hand side, using the definition of the H^* actions from observation 3.3.11 and the fact that a is group-like:

$$\begin{aligned} \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha &= \langle \alpha, a h_{(1)} a^{-1} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, a h_{(3)} a^{-1} \rangle \\ &= \langle \alpha, h_{(1)} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, h_{(3)} \rangle \ , \end{aligned}$$

  where in the last identity we used that α as a group-like element of H^* is a morphism of algebras $H \to \mathbb{K}$, cf. remark 2.6.9.1.

- The two lemmata 3.3.9 and 3.3.10 for the Nakayama automorphism ρ for the right cointegral γ imply the identity

$$\langle \alpha, h_{(1)} \rangle \ S^{-2}(h_{(2)}) \overset{3.3.9}{=} \rho(h) \overset{3.3.10}{=} a^{-1} S^2(h_{(1)}) \langle \alpha, h_{(2)} \rangle a \ .$$

  Applying S^2 to this equation and conjugating with $a \in G(H)$, we get

$$a \cdot \langle \alpha, h_{(1)} \rangle \ h_{(2)} \cdot a^{-1} = S^4(h_{(1)}) \langle \alpha, h_{(2)} \rangle \ . \qquad (**)$$

  We now compute $\langle \alpha^{-1}, h_{(3)} \rangle$, we find

$$\begin{aligned} a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} &\overset{(*)}{=} a \cdot \langle \alpha, h_{(1)} \rangle \ h_{(2)} \langle \alpha^{-1}, h_{(3)} \rangle \cdot a^{-1} \\ &\overset{(**)}{=} S^4(h_{(1)}) \langle \alpha, h_{(2)} \rangle \langle \alpha^{-1}, h_{(3)} \rangle \\ &= S^4(h_{(1)}) \langle \alpha \cdot \alpha^{-1}, h_{(2)} \rangle \\ &= S^4(h) \end{aligned}$$

□

Corollary 3.3.13.
Let H be a finite-dimensional Hopf algebra.

1. The order of the antipode S of H is finite.
2. If H is unimodular, then S^4 coincides with the inner automorphism of H induced by a grouplike element. In particular, the order of the antipode is at most $4 \cdot \dim H$.
3. If both H and H^* are unimodular, then $S^4 = \mathrm{id}_H$.

Proof.

1. Since H and H^* are finite-dimensional and since distinct powers of a group-like element are linearly independent by proposition 2.6.11, every group-like element in H or H^* has finite order. By Radford's formula 3.3.12, S^4 has finite order and thus S has finite order.

2. By corollary 3.1.18, the Hopf algebra H is unimodular, if and only if the distinguished group-like element α equals the counit. The action of the counit on H is trivial, thus for unimodular Hopf algebras, Radford's formula reads $S^4(h) = aha^{-1}$. The last assertion follows by applying the same reasoning to the Hopf algebra H^* as well.

□

Remark 3.3.14.
For a finite tensor category $\mathcal{C}$, the Nakayama functor comes with coherent isomorphisms [FSS]

$$N(a \otimes m \otimes b) \cong {}^{\vee\vee}a \otimes N(m) \otimes b^{\vee\vee} \ .$$

Using that $N(1) = D^{-1}$ is the inverse of the distinguished invertible object, we conclude

$$N(a) = N(1 \otimes a) \cong N(1) \otimes a^{\vee\vee} \cong D^{-1} \otimes a^{\vee\vee}$$

and

$$N(a) = N(a \otimes 1) \cong {}^{\vee\vee}a \otimes N(1) \cong {}^{\vee\vee}a \otimes D^{-1}$$

which implies the following categorical variant of Radford's S^4-theorem:

$$a^{\vee\vee\vee\vee} \cong D \otimes a \otimes D^{-1} \ .$$

We finally derive a result relating the order of the antipode S of H to the semisimplicity of the Hopf algebra H.

Lemma 3.3.15.
Let A be a Frobenius algebra with bilinear form κ and dual bases (r_i, l_i) as in observation 3.2.30. Suppose that $e \in A$ has the property that $e^2 = \alpha e$ with some $\alpha \in \mathbb{K}$. Consider an $\mathbb{K}$-linear endomorphism f of the subspace $eA := \{ea \,|\, a \in A\}$ of A. Then

1. $\alpha \mathrm{Tr}(f) = \sum_i \kappa(f(el_i), r_i)$.
2. $\alpha \mathrm{Tr}(f) = \sum_i \kappa(l_i, f(er_i))$.

Proof.
Using the defining property of dual bases, we find

$$\alpha ex = e^2 x = e \left(\sum_i \kappa(ex, r_i) l_i \right) = \sum_i \kappa(ex, r_i) el_i \ .$$

Thus, since f is linear,

$$\alpha f(ex) = \sum_i \kappa(ex, r_i) f(el_i)$$

so that under the isomorphism

$$(eA)^* \otimes (eA) \to \mathrm{End}_{\mathbb{K}}(eA)$$

we have

$$\sum_i \kappa(-, r_i) \otimes f(el_i) \mapsto \alpha f \ .$$

Combined with lemma 3.3.2 on the computation of traces, this shows the first formula. The second formula is shown analogously. □

Definition 3.3.16
Let A be a unital associative $\mathbb{K}$-algebra. Let V be a finite-dimensional left A-module given by the algebra map $\rho : A \to \mathrm{End}_{\mathbb{K}}(V)$. Then the linear form

$$\chi_V : \begin{array}{rcl} A & \to & \mathbb{K} \\ a & \mapsto & \mathrm{Tr}\rho(a) \end{array}$$

is called the <u>*character*</u> *of the module V.*

Remarks 3.3.17.
The following properties are easy to check:

1. $\chi_V(1_A) = \dim_{\mathbb{K}} V$ for any A-module V.
2. Let V, W be A-modules. Any isomorphism of modules $V \cong W$ implies identity of characters, $\chi_V = \chi_W$. The converse is, in general, wrong. As an example, consider the group algebra $K[C_2]$ of the cyclic group of order two over a field of characteristic two which was discussed in remark 2.1.6.5. The direct sum of two copies of the one-dimensional simple representation and the regular representation have the same character (which identically vanishes).
3. Let V, W be A-modules. Then we have $\chi_{V\oplus W} = \chi_V + \chi_W$, as a consequence of the behaviour of the trace on direct sums of vector spaces.
4. Suppose that we consider modules over a Hopf algebra H. Then $\chi_{V\otimes W} = \chi_V \cdot \chi_W$ with the product in H^*. Indeed,
$$\chi_{V\otimes W}(h) = \mathrm{Tr}_{V\otimes W}\rho_V(h_{(1)}) \otimes \rho_W(h_{(2)}) = \chi_V(h_{(1)}) \cdot \chi_W(h_{(2)}) = (\chi_V \cdot \chi_W)(h) \ .$$
5. Suppose again that we consider modules over a Hopf algebra H. Then for a trivial module $T = (V, \epsilon \otimes \mathrm{id}_V)$, we have $\chi_T(h) = \epsilon(h) \dim V$.
6. For the character of the right dual module $V^\vee$, we have
$$\chi_{V^\vee}(h) = \mathrm{Tr}_{V^*}\rho_V(S(h))^t = \mathrm{Tr}_V\rho_V(S(h)) = \chi_V(S(h)) \ .$$
For the character of the left dual module, we have
$$\chi_{{}^\vee V}(h) = \mathrm{Tr}_{V^*}\rho_V(S^{-1}(h))^t = \mathrm{Tr}_V\rho_V(S^{-1}(h)) = \chi_V(S^{-1}(h)) \ .$$

Lemma 3.3.18.
Let H be a Hopf algebra. Then we have for the character of the left regular module

1. $\chi_H^2 = \dim H \cdot \chi_H$.
2. $S^2\chi_H = \chi_H$, where we use S for the antipode of H^* as well,

as identities of elements in the Hopf algebra H^*.

Proof.

1. Let V be any H-module and V_ϵ the trivial H-module structure on the vector space underlying V. Then the Hopf algebra property of H implies that the linear map

$$\begin{array}{rcl} H \otimes V_\epsilon & \to & H \otimes V \\ h \otimes v & \mapsto & h_{(1)} \otimes h_{(2)}.v \end{array}$$

defines an isomorphism of H-modules with inverse

$$\begin{array}{rcl} H \otimes V & \to & H \otimes V_\epsilon \\ h \otimes v & \mapsto & h_{(1)} \otimes S(h_{(2)}).v \end{array}$$

This implies

$$\chi_H \chi_V = \chi_H \chi_{V_\epsilon} = \chi_H \dim V \ ,$$

where all products are products in H^* and where we used that the counit ϵ is the unit of H^*. Then specialize to $V = H$.

2. Let $h \in H$. Then

$$\langle S^2(\chi_H), h\rangle = \langle \chi_H, S^2 h\rangle = \mathrm{Tr}_H(L_{S^2(h)}) \ .$$

Since S^2 is an algebra automorphism, we have

$$\mathrm{Tr}_H(L_{S^2(h)}) = \mathrm{Tr}_H(L_h) = \langle \chi_H, h\rangle \ .$$

□

Since $S^2\chi_H = \chi_H$ by lemma 3.3.18.2 and since S^2 is a an algebra auto morphism of H^*, we have for any $\beta \in H^*$

$$S^2(\chi_H \beta) = S^2(\chi_H) \cdot S^2(\beta) = \chi_H \cdot S^2(\beta)$$

so that S^2 restricts to an endomorphism of the linear subspace $\chi_H H^* \subset H^*$.

Lemma 3.3.19.
Let H be a Hopf algebra. Let $\gamma \in H^*$ be a nonzero right integral and $\Gamma \in H$ be a left integral, normalized such that $\langle \gamma, \Gamma\rangle = 1$. Then

$$\mathrm{Tr}_{H^*}(S^2) = \langle \epsilon, \Gamma\rangle \langle \gamma, 1\rangle = (\dim H)\mathrm{Tr}\ (S^2|_{\chi_H H^*}) \ .$$

Proof.
By applying proposition 3.3.3 to $H^{\mathrm{opp,copp}}$, we find

$$\mathrm{Tr}_H S^2 = \langle \gamma, 1\rangle \cdot \langle \epsilon, \Gamma\rangle \ .$$

We denote by $\tilde\Gamma \in H^{**}$ the image of $\Gamma \in H$ in the bidual of H. Now γ is a Frobenius form with dual bases $(\Gamma_{(1)}, S(\Gamma_{(2)}))$ which implies that $\tilde\Gamma$ is a Frobenius form for H^* with dual bases $(S(\gamma_{(1)}), \gamma_{(2)})$.

Now lemma 3.3.15 applies to $e := \chi_H$ with $\alpha = \dim H$, thus yielding

$$\begin{array}{rcll} \dim H \cdot \mathrm{Tr}(S^2)|_{\chi_H H^*} & = & \langle \tilde\Gamma, S^2(\chi_H \gamma_{(2)}) S(\gamma_{(1)})\rangle & \\ & = & \langle \tilde\Gamma, S^2(\chi_H) S^2(\gamma_{(2)}) S(\gamma_{(1)})\rangle & [S^2 \text{ algebra morphism}] \\ & = & \langle \tilde\Gamma, \chi_H S(\gamma_{(1)} \cdot S(\gamma_{(2)})\rangle & [S^2\chi_H = \chi_H] \\ & = & \langle \gamma, 1\rangle \cdot \langle \chi_H, \Gamma\rangle & \end{array}$$

By the same lemma 3.3.15, taking $f = L_\Gamma$ and $e = 1$ with $\alpha = 1$, we have for the second factor

$$\begin{aligned}\chi_H(\Gamma) &= \langle \gamma, S(\Gamma_{(2)})\Gamma\Gamma_{(1)}\rangle \\ &= \langle \gamma, \epsilon(\Gamma_{(2)})\Gamma\Gamma_{(1)}\rangle \quad [\Gamma \text{ is a left integral of } H] \\ &= \langle \gamma, \Gamma\Gamma\rangle = \epsilon(\Gamma)\langle \gamma, \Gamma\rangle = \epsilon(\Gamma)\end{aligned}$$

Combining the two results yields

$$\dim H \cdot \mathrm{Tr}(S^2)|_{\chi_H H^*} = \langle \gamma, 1\rangle \cdot \langle \chi_H, \Gamma\rangle = \langle \gamma, 1\rangle \cdot \langle \epsilon, \Gamma\rangle$$

which completes the proof of the lemma. □

Theorem 3.3.20 (Larson-Radford, 1988)**.**
Let $\mathbb{K}$ be a field of characteristic zero. Let H be a finite-dimensional $\mathbb{K}$-Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.
2. H^* is semisimple.
3. $S^2 = \mathrm{id}_H$.

It has been shown [EG, Theorem 3.1] that over a field of any characteristic, the following holds: if H *and* H^* are semisimple, then $S^2 = \mathrm{id}_H$.

Proof.
We have already seen in corollary 3.3.4.2 that 3. implies 1. and 2. One can show [LR, Theorem 3.3] that 2 implies 1. Here we only show that 1. and 2. together imply 3. Suppose that H and H^* are both semisimple and thus, by corollary 3.2.24, unimodular. By corollary 3.3.13.3, then $S^4 = (S^2)^2 = \mathrm{id}$.

Hence the eigenvalues of S^2 on H and of $S^2|_{\chi_H H^*}$ are all ± 1 and S^2 can be diagonalized, cf. remark 2.1.6.5. Call the eigenvalues $(\mu_j)_{1\leq j\leq n}$ with $n := \dim H$ and $(\eta_i)_{1\leq i\leq m}$ with $m := \dim \chi_H H^*$. Thus

$$\mathrm{Tr}_{H^*}(S^2) = \sum_{j=1}^n \mu_j \quad \text{and} \quad \mathrm{Tr}(S^2|_{\chi_H H^*}) = \sum_{i=1}^m \eta_i\ ,$$

By lemma 3.3.19,

$$\sum_{j=1}^n \mu_j = n \sum_{i=1}^m \eta_i\ . \qquad (*)$$

This implies

$$n \cdot |\sum_{i=1}^m \eta_i| = \left|\sum_{j=1}^n \mu_j\right| \leq \sum_{j=1}^n |\mu_j| = n\ .$$

For a semisimple Hopf algebra, we have seen in corollary 3.3.4 that $0 \neq \mathrm{Tr}S^2 = \sum_{j=1}^n \mu_j$ and thus by the equality $(*)$ we find $\sum_{i=1}^m \eta_i \neq 0$. This implies $\sum_{i=1}^m \eta_i = \pm 1$ and, as a further consequence of equation $(*)$ we have $\sum_{j=1}^n \mu_j = \pm n$. Since $S^2(1_H) = 1_H$, we have at least one eigenvalue $+1$. Thus all eigenvalues of S^2 on H have to be $+1$ which amounts to $S^2 = \mathrm{id}_H$. □

There are some important results we do not cover in these lectures. The following theorem is proven in [Schneider]:

Theorem 3.3.21 (Nichols-Zoeller, 1989).
Let H be a finite-dimensional Hopf algebra, and let $R \subset H$ be a Hopf subalgebra. Then H is a free R-module.

Corollary 3.3.22 ("Langrange's theorem for Hopf algebras").
If $R \subset H$ are finite-dimensional Hopf algebras, then the order of R divides the order of H. (The order of a Hopf algebra is, by definition, its dimension.)

We finally refer to chapter 4 of Schneider's lecture notes [Schneider] for a character theory for finite-dimensional semisimple Hopf algebras that closely parallels the character theory for finite groups.

4 Quasi-triangular Hopf algebras and braided categories

4.1 Braidings and topological field theory

In this subsection, we introduce the notion of a topological field theory and investigate low-dimensional topological field theories. To this end, we need more structure on monoidal categories.

We recall from Remark 2.4.3.4 that, given a tensor category $(\mathcal{C}, \otimes, a, l, r)$ we obtain from the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a functor $\otimes^{\mathrm{opp}} = \otimes \circ \tau$ with

$$V \otimes^{\mathrm{opp}} W := W \otimes V \quad \text{and } f \otimes^{\mathrm{opp}} g := g \otimes f \ .$$

which admits the associator $a^{\mathrm{opp}}_{U,V,W} := a^{-1}_{W,V,U}$.

Definition 4.1.1

1. *A* <u>*commutativity constraint*</u> *for a tensor category* $(\mathcal{C}, \otimes)$ *is a natural isomorphism*

$$c : \otimes \to \otimes^{\mathrm{opp}}$$

of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$*. Explicitly, we have for any pair* (V, W) *of objects of* $\mathcal{C}$ *an isomorphism*

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$

such that for all morphisms $V \xrightarrow{f} V'$ *and* $W \xrightarrow{g} W'$ *the diagrams*

$$\begin{tikzcd} V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"'] & W \otimes V \arrow[d, "g \otimes f"] \\ V' \otimes W' \arrow[r, "c_{V',W'}"] & W' \otimes V' \end{tikzcd}$$

commute.

2. *Let* $\mathcal{C}$ *be, for simplicity, a strict tensor category. A* <u>*braiding*</u> *is a commutatitivity constraint such that for all objects* U, V, W *the compatibility relations with the tensor product*

$$\begin{aligned} c_{U\otimes V,W} &= (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \\ c_{U,V\otimes W} &= (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) \end{aligned}$$

hold.

If the category is not strict, the following two <u>*hexagon axioms*</u> *involving also the associators have to hold:*

$$\begin{tikzcd} & U \otimes (V \otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[dr, "a_{V,W,U}"] & \\ (U \otimes V) \otimes W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V \otimes (W \otimes U) \\ & (V \otimes U) \otimes W \arrow[r, "a_{V,U,W}"] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{id}_V \otimes c_{U,W}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[r, "c_{U\otimes V,W}"] & W \otimes (U \otimes V) \arrow[dr, "a^{-1}_{W,U,V}"] & \\
U \otimes (V \otimes W) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U \otimes c_{V,W}"'] & & & (W \otimes U) \otimes V \\
 & U \otimes (W \otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U \otimes W) \otimes V \arrow[ur, "c_{U,W} \otimes \mathrm{id}_V"'] &
\end{tikzcd}$$

3. *A braided tensor category is a tensor category together with the structure of a braiding.*
4. *With c_{UV}, also c^{-1}_{VU} is a braiding. If the identity $c_{U,V} = c^{-1}_{V,U}$ holds, the braided tensor category is called symmetric.*

Remarks 4.1.2.

1. Graphically, we represent the braiding by overcrossings and its inverse by undercrossings. Overcrossings and undercrossings have to be distinguished, unless the category is symmetric.
2. It is not necessary to impose the correct behaviour of the monoidal unit as an axiom, see [JS, Proposition 2.1].
3. The flip map
$$\begin{array}{rccc} \tau : & V \otimes W & \to & W \otimes V \\ & v \otimes w & \mapsto & w \otimes v \end{array}$$
defines a symmetric braiding on the monoidal category $\mathrm{vect}(\mathbb{K})$ of $\mathbb{K}$-vector spaces. It also induces a symmetric braiding on the category $\mathbb{K}[G]$-mod of $\mathbb{K}$-linear representations of a group. More generally, flip maps give a symmetric braiding on the category $H-$mod for any cocommutative Hopf algebra. Since the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is cocommutative, the category of $\mathbb{K}$-linear representations of $\mathfrak{g}$ has the structure of a symmetric tensor category, as well.
4. There are tensor categories that do not admit a braiding. For example, for G a non-abelian group, the category $\mathrm{vect}(G)$ of G-graded vector spaces does not admit a braiding since
$$\mathbb{K}_g \otimes \mathbb{K}_h \cong \mathbb{K}_{gh} \quad \text{and} \quad \mathbb{K}_h \otimes \mathbb{K}_g \cong \mathbb{K}_{hg}$$
are not isomorphic, if $gh \neq hg$.
5. The category $\mathrm{vect}(G)$ admits the flip as a braiding, if the group G is abelian. In the case of $G = \mathbb{Z}_2$, objects are $\mathbb{Z}_2$-graded vector spaces $V_0 \oplus V_1$. We can introduce another symmetric braiding c: on homogeneous components, it is the flip up to signs:
$$\begin{array}{rcr} c : V_i \otimes W_j & \to & W_j \otimes V_i \\ v_i \otimes w_j & \mapsto & (-1)^{ij} w_j \otimes v_i \end{array}$$
This category is the symmetric category underlying the spherical category of super vector spaces. For more details, see remark 5.1.12. In particular, a tensor category can admit inequivalent braidings.

6. Recall from definition 2.5.18 that the monoidal category of cobordisms has disjoint union as the tensor product, $\mathbb{S}_1 \otimes \mathbb{S}_2 = \mathbb{S}_1 \sqcup \mathbb{S}_2$. It admits symmetric braiding given by the morphism represented by the bordism of two cylinders

$$(\mathbb{S}_1 \sqcup \mathbb{S}_2) \times I = \mathbb{S}_1 \times I \sqcup \mathbb{S}_2 \times I$$

with maps

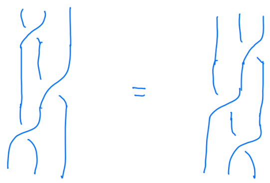
$$\begin{array}{rcl} \Psi_i : \mathbb{S}_1 \sqcup \mathbb{S}_2 & \to & (\mathbb{S}_1 \sqcup \mathbb{S}_2) \times I \\ (s_1, s_2) & \mapsto & (s_1, s_2, 0) \end{array} \quad \begin{array}{c} \text{and} \\ \text{and} \end{array} \quad \begin{array}{rcl} \mathbb{S}_2 \sqcup \mathbb{S}_1 & \to & (\mathbb{S}_1 \sqcup \mathbb{S}_2) \times I \\ (s_2, s_1) & \mapsto & (s_1, s_2, 1) \ . \end{array}$$

We note that in a braided category, a version of the Yang-Baxter equation holds:

Proposition 4.1.3.
Let U, V, W be objects in a strict braided tensor category. Then the following identity of morphisms $U \otimes V \otimes W \to W \otimes V \otimes U$ holds:

$$\begin{aligned} &(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) \\ &\quad = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \, . \end{aligned}$$

Expressed using the graphical calculus, we find



If the braided category is not strict, this amounts to a commuting diagram with 12 corners, a dodecagon. The reader should draw the graphical representation of this identity.

In particular $c_{V,V} \in \mathrm{Aut}\ (V \otimes V)$ is a solution of the Yang-Baxter equation, cf. remark 1.1.2. Thus any object V of a braided tensor category provides a group homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$, cf. proposition 1.2.4.

Proof.
The equality is a direct consequence of the hexagon axiom from definition 4.1.1 and the functoriality of the braiding:

$$\begin{array}{rll} & (c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) & \\ = & (c_{V,W} \otimes \mathrm{id}_U) \circ c_{U,V\otimes W} & [\text{Hexagon axiom}] \\ = & c_{U,W\otimes V} \circ (\mathrm{id}_U \otimes c_{V,W}) & [\text{Naturality of } c_{U,V\otimes W}] \\ = & (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) & [\text{Hexagon axiom}] \end{array}$$

□

Remarks 4.1.4.

1. As in any tensor category, we can consider algebras and coalgebras in a braided tensor category $(\mathcal{C}, \otimes, c)$. Now, we have the notion of a *commutative* associative unital algebra (A, μ, η): here the product is required to obey $\mu \circ c_{A,A} = \mu$. We also have the opposed algebra with multiplication $\mu^{\mathrm{opp}} := \mu \circ c_{A,A}$. Similarly, we have the notion of the coopposed coalgebra with coproduct $\Delta^{\mathrm{copp}} := c_{C,C} \circ \Delta$, and the notion of a cocommative coassociative counital coalgebra with coproduct obeying $c_{C,C} \circ \Delta = \Delta$.

2. Another construction that uses the braiding is the following: Suppose that we have two associative unital algebras (A, μ, η) and (A', μ', η') in a braided tensor category $\mathcal{C}$. Then the tensor product $A \otimes A'$ can be endowed with the structure of an associative algebra with product

$$A \otimes A' \otimes A \otimes A' \xrightarrow{\mathrm{id}_A \otimes c_{A,A'} \otimes \mathrm{id}_{A'}} A \otimes A \otimes A' \otimes A' \xrightarrow{\mu \otimes \mu'} A \otimes A' .$$

   A unit is then $\eta \otimes \eta'$.

   Dually, also the tensor product of two counital coassociative coalgebras can be endowed with the structure of a coalgebra.

3. Hence, in braided tensor categories, it makes sense to consider an object H which has both the structure of an associative unital algebra and of a coassociative counital coalgebra such that the coproduct $\Delta : H \to H \otimes H$ is a morphism of algebras. We are thus able to introduce the notion of a bialgebra and, moreover, of a Hopf algebra, in a braided category. Such Hopf algebras play an important role in the construction of modular functors and of invariants of three-manifolds. For a detailled exposition of the modular functor which goes back to the work of Lyubashenko we refer to [LMSS].

4. We will see in an exercise that the exterior algebra is a Hopf algebra in the symmetric tensor category of super vector spaces. A generalization are Nichols algebras [H].

We again need functors and natural transformations with appropriate compatibilities:

Definition 4.1.5

1. *A tensor functor $(F, \varphi_0, \varphi_2)$ from a braided tensor category $\mathcal{C}$ to a braided tensor category $\mathcal{D}$ is called a* <u>*braided tensor functor*</u>*, if for any pair of objects (V, V') of $\mathcal{C}$, the square*

$$\begin{tikzcd} F(V) \otimes F(V') \arrow[r, "\varphi_2"] \arrow[d, "c_{F(V),F(V')}"'] & F(V \otimes V') \arrow[d, "F(c_{V,V'})"] \\ F(V') \otimes F(V) \arrow[r, "\varphi_2"] & F(V' \otimes V) \end{tikzcd}$$

   commutes. If the braided tensor category has the property of being symmetric, a braided tensor functor is also called a symmetric tensor functor.

2. *As braided monoidal natural transformations, we take all monoidal natural transformations.*

Example 4.1.6.
We discuss an important class of braided monoidal categories. Let G be a finite abelian group and vect_G the category of (finite-dimensional) G-graded vector spaces. For simplicity, we assume that the category of finite-dimensional vector spaces has been replaced by an equivalent strict monoidal category. An associator is then determined on simple objects by

$$\alpha_{g_1,g_2,g_3} = \omega(g_1, g_2, g_3)\mathrm{id} : \quad (\mathbb{K}_{g_1} \otimes \mathbb{K}_{g_2}) \otimes \mathbb{K}_{g_3} \to \mathbb{K}_{g_1} \otimes (\mathbb{K}_{g_2} \otimes \mathbb{K}_{g_3})$$

and the braiding by

$$c_{g_1,g_2} = \beta(g_1, g_2)\mathrm{id} : \quad \mathbb{K}_{g_1} \otimes \mathbb{K}_{g_2} \cong \mathbb{K}_{g_1 g_2} \to \mathbb{K}_{g_1 g_2} \cong \mathbb{K}_{g_2} \otimes \mathbb{K}_{g_1}$$

with functions

$$\omega: \ G\times G\times G\to \mathbb{K}^\times \quad \text{and} \quad \beta: \ G\times G\to \mathbb{K}^\times$$

obeying by the pentagon and the two hexagon equations

$$\begin{aligned}
\omega(g_1g_2,g_3,g_4)\omega(g_1,g_2,g_3g_4) &= \omega(g_1,g_2,g_3)\omega(g_1,g_2g_3,g_4)\omega(g_2,g_3.g_4)\\
\omega(g_2,g_3,g_1)\beta(g_1,g_2g_3)\omega(g_1,g_2,g_3) &= \beta(g_1,g_3)\omega(g_2,g_1,g_3)\beta(g_1,g_2)\\
\omega(g_3,g_1,g_2)^{-1}\beta(g_1g_2,g_3)\omega(g_1,g_2,g_3)^{-1} &= \beta(g_1,g_3)^{-1}\omega(g_1,g_3,g_2)^{-1}\beta(g_2,g_3)
\end{aligned}$$

for all $g_1,g_2,g_3,g_4\in G$.

Using these equations, one shows [EGNO, Theorem 8.4.9] that the function

$$\begin{array}{rccl} q: & G & \to & \mathbb{K}^\times \\ & g & \mapsto & \beta(g,g) \end{array}$$

is a quadratic form on G, i.e. it obeys $q(g)=q(g^{-1})$ and

$$b(g,h):=\frac{q(gh)}{q(g)q(h)}$$

is a symmetric bicharacter, $b(g_1g_2,h) = b(g_1,h)b(g_2,h)$. Such quadratic forms even classify structures of a braided monoidal category on vect_G up to braided equivalence. This makes this class of braided monoidal categories accessible by tools from (abelian) group cohomology. For more information, see [EGNO, Section 8.4].

Definition 4.1.7 *[Atiyah]*
Let $\mathbb{K}$ be a field. A topological field theory of dimension n is a symmetric monoidal functor

$$Z:\mathrm{Cob}(n)\to \mathrm{vect}(\mathbb{K}) \ .$$

Remarks 4.1.8.

1. The category $\mathrm{vect}(\mathbb{K})$ can be replaced by any symmetric monoidal category. (Interesting examples include e.g. categories of complexes of vector spaces.) Also variants of cobordism categories are in use: spin cobordisms, manifolds with principal bundle, unoriented manifolds, ... One also considers categories with less morphisms, e.g. admitting only cobordisms $\mathbb{S}\to\mathbb{S}$ that are cylinders, $M=\mathbb{S}\times I$ with two diffoemorphisms $\psi_{i,o}:\mathbb{S}\to\mathbb{S}\otimes I$.

2. Without loss of generality, one can suppose that the symmetric monoidal functor Z is strict.

3. Recall for G a finite group from example 2.1.21 the functor category $[*//G,\mathrm{vect}(\mathbb{K})]$ which is the category of $\mathbb{K}$-linear representations of G. Topological field theories can be seen as representations of cobordism categories.

4. We deduce from the definition that a topological field theory Z of dimension n is given by the following data:

   (a) For every oriented closed manifold M of dimension $(n-1)$, a $\mathbb{K}$-vector space $Z(M)$.
   (b) For every oriented bordism B from an $(n-1)$-manifold M to another $(n-1)$-manifold N, a $\mathbb{K}$-linear map $Z(B):Z(M)\to Z(N)$.

(c) A collection of coherent isomorphisms

$$Z(\emptyset) \cong \mathbb{K} \qquad Z(M \coprod N) \cong Z(M) \otimes Z(N).$$

Functoriality implies that we can glue cobordisms and get the composition of linear maps. Moreover, these data are required to satisfy a number of natural coherence properties which we will not make explicit.

5. A closed oriented manifold M of dimension n can be regarded as a bordism from the empty $(n-1)$-manifold to itself, $M : \emptyset \to \emptyset$. Thus

$$Z(M) : \mathbb{K} \cong Z(\emptyset) \to Z(\emptyset) \cong \mathbb{K}$$

and thus $Z(M) \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$ is a number: an invariant assigned to every closed oriented manifold of dimension n.

Observation 4.1.9.
Let Z be an n-dimensional topological field theory. For any closed oriented $(n-1)$-dimensional manifold M, $Z(M)$ is a vector space. The cylinder on M gives a bordism $d_M : \overline{M} \coprod M \to \emptyset$ which is a right evaluation. Similarly, we get a right coevaluation $b_M : \emptyset \to M \coprod \overline{M}$.

Applying the functor Z, we get a vector space $Z(M)$ together with another vector space $Z(\overline{M})$ that is a right dual.

We can now use Lemma 2.5.15 and Lemma 2.5.13 which state the uniqueness of duals and the fact that the finite-dimensional vector spaces are precisely the ones that admit duals.

Corollary 4.1.10.
Let Z be a topological field theory of dimension n. Then for every closed $(n-1)$-manifold M, the vector space $Z(M)$ is finite-dimensional, and the pairing $Z(\overline{M}) \otimes Z(M) \to \mathbb{K}$ is perfect: that is, it induces an isomorphism α from $Z(\overline{M})$ to the dual space of $Z(M)$.

In low dimensions, it is possible to describe topological field theories very explicitly.

Example 4.1.11 (Topological field theories in dimension 1)**.**

- Let Z be a 1-dimensional topological field theory. Then Z assigns a finite-dimensional vector space $Z(M)$ to every closed oriented 0-manifold M, i.e. to a finite set of oriented points. Since the functor Z is monoidal, it suffices to know its values $Z(\bullet, +)$ and $Z(\bullet, -)$ on the positively and negatively oriented point which are finite-dimensional vector spaces dual to each other. Thus

$$Z(M) \cong (\bigotimes_{x \in M_+} V) \otimes (\bigotimes_{y \in M_-} V^\vee)$$

with $V := Z(\bullet, +)$.

- To fully determine Z, we must also specify Z on 1-manifolds B with boundary. Since Z is a symmetric monoidal functor, it suffices to specify $Z(B)$ when B is *connected*. In this case, the 1-manifold B is diffeomorphic either to a closed interval $[0, 1]$ or to a circle S^1.

- There are five cases to consider, depending on how we interpret the one-dimensional oriented manifold B with boundary, the interval, as cobordism:

  (a) Suppose that $B = [0, 1]$, regarded as a bordism from $(\bullet, +)$ to itself. Then $Z(B)$ is the identity map $\mathrm{id}_V : V \to V$.

(b) Suppose that $B = [0,1]$, regarded as a bordism from $(\bullet, -)$ to itself. Then $Z(B)$ is the identity map $\mathrm{id} : V^\vee \to V^\vee$.

(c) Suppose that $B = [0,1]$, regarded as a bordism from $(\bullet,+)\coprod(\bullet,-)$ to the empty set. Then $Z(B)$ is a linear map from $V \otimes V^\vee$ into the ground field $\mathbb{K}$: the evaluation map $(v,\lambda) \mapsto \lambda(v)$. Since the order matters, we also consider the related bordism from $(\bullet,-)\coprod(\bullet,+)$ to the empty set. Then $Z(B)$ is a linear map from $V^\vee \otimes V$ into the ground field $\mathbb{K}$: the evaluation map $(\lambda, v) \mapsto \lambda(v)$.

(d) Suppose that $B = [0,1]$, regarded as a bordism from the empty set to $(\bullet,+)\coprod(\bullet,-)$. Then $Z(B)$ is a linear map from $\mathbb{K}$ to $Z((\bullet,+)\coprod(\bullet,-)) \cong V \otimes V^\vee$. Under the canonical isomorphism $V \otimes V^\vee \cong \mathrm{End}(V)$, this linear map is given by the coevalution $x \mapsto x\mathrm{id}_V$. Again, we can exchange the order of the objects.

(e) Suppose that $B = S^1$, regarded as a bordism from the empty set to itself. Then $Z(B)$ is a linear map from $\mathbb{K}$ to itself, which we can identify with an element of $\mathbb{K}$. To compute this element, decompose the circle $S^1 \cong \{z \in \mathbb{C} : |z| = 1\}$ into two intervals

$$S^1_- = \{z \in \mathbb{C} : (|z| = 1) \wedge \mathrm{Im}\,(z) \le 0\} \qquad S^1_+ = \{z \in \mathbb{C} : (|z| = 1) \wedge \mathrm{Im}\,(z) \ge 0\},$$

with intersection

$$S^1_- \cap S^1_+ = \{\pm 1\} \subseteq S^1.$$

It follows that $Z(S^1)$ is given as the composition of the linear maps

$$\mathbb{K} \simeq Z(\emptyset) \xrightarrow{Z(S^1_-)} Z(\pm 1) \xrightarrow{Z(S^1_+)} Z(\emptyset) \simeq \mathbb{K} \ .$$

These maps were described by (c) and (d) above. We thus get a map

$$\begin{array}{rcccl} \mathbb{K} \cong Z(\emptyset) & \to & Z(\pm 1) \cong V \otimes V^\vee & \to & Z(\emptyset) \cong \mathbb{K} \\ \lambda & \mapsto & \lambda \sum v^i \otimes v_i & \mapsto & \lambda \sum_i v^i(v_i) = \lambda \cdot \dim V \end{array}$$

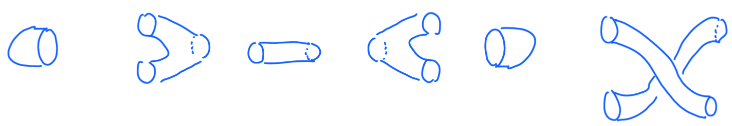
where we have chosen a basis $(v_i)_{i\in I}$ of V and a dual basis $(v^i)_{i\in I}$ of V^*. Consequently, $Z(S^1)$ is given by the dimension of V.

In physical language, we have a quantum mechanical system which has only ground states and thus trivial Hamiltonian. Then the only invariant of the system is the degeneracy $\dim V$ of the space of ground states.

Example 4.1.12 (Topological field theories in dimension 2)**.**

- A two-dimensional topological field theory assigns a vector space $Z(M)$ to every closed, oriented 1-manifold M. Such a manifold is diffeomorphic to a disjoint union of circles, $M \cong (S^1)^{\amalg n}$ for some $n \ge 0$. Since Z is monoidal, $Z(M) \cong A^{\otimes n}$ with $A := Z(S^1)$ by Lemma 2.5.13 a finite-dimensional vector space.

- One can show (see e.g. [Kock, Proposition 1.4.13]) that the morphisms of monoidal category Cob(2) are generated under composition and disjoint union by six cobordisms: cap or disc, trinion, also called pair of pants, the cylinder, the trinion with two outoing circles, a disc with one ingoing circle and two exchanging cylinders. This a proposition about the structure of oriented 2-manifolds!



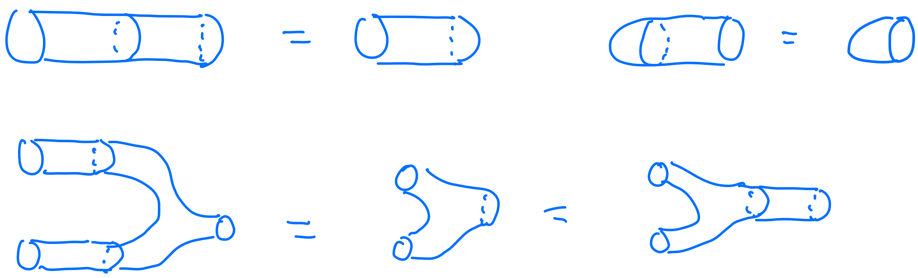
Cap $\emptyset \to S^1$, trinion or pair-of-pants $S^1 \sqcup S^1 \to S^1$, cylinder $S^1 \to S^1$, their mirror images and commutativity constraint.

Applying the functor Z to these cobordisms, we get the following linear maps:

$$\begin{array}{rl} \text{cap} & \eta : \mathbb{K} \to A \\ \text{trinion} & \mu : A \otimes A \to A \\ \text{cylinder} & I_A : A \to A \\ \text{opposite trinion} & \Delta : A \to A \otimes A \\ \text{opposite cap} & \epsilon : A \to \mathbb{K} \\ \text{exchanging cylinder} & \tau : A \otimes A \to A \otimes A \end{array}$$

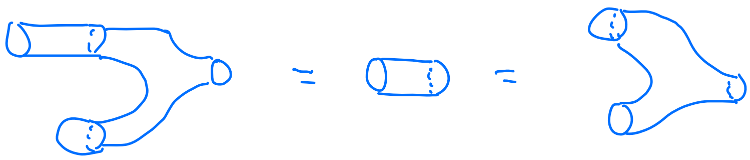
- One can also classify all relations between the generators, see [Kock, 1.4.24-1.4.28].

Identity relations



and mirror images

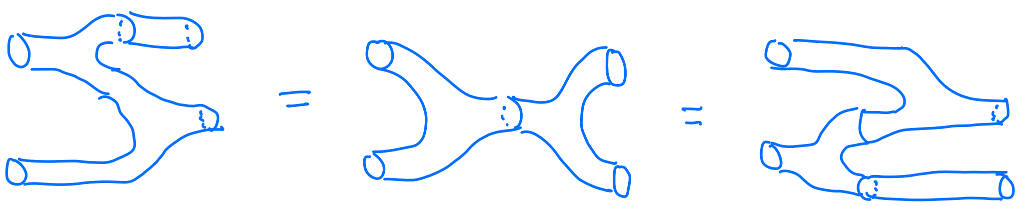
Unitality relations



Associativity + coassociativity

Commutativity and cocommutativity

Frobenius :



The relations can be summarized that category Cob(2) is the free symmetric monoidal category on a commutative Frobenius object [Kock, Theorem 3.6.19]. The relations imply that A has the structure of a commutative (Δ, ϵ)-Frobenius algebra.

- The converse is true as well: given a commutative Frobenius algebra A, one can construct a 2-dimensional topological field theory Z such that $A = Z(S^1)$ and the multiplication and Frobenius form on A are given by evaluating Z on a pair of pants and a disk, respectively.

- In a categorical language, we thus arrive at the following classification result for two-dimensional topological field theories: the topological field theories are described by the category $[\mathrm{Cob}(2), \mathrm{vect}(\mathbb{K})]_{\text{symm. monoidal}}$ of symmetric monoidal functors. This category which is actually a groupoid [Kock, Lemma 2.4.5] is equivalent to the category of commutative $\mathbb{K}$-Frobenius algebras.

This example can be generalized:

Example 4.1.13 (Open/closed topological field Theories in dimension 2).

- We define a larger category $\mathrm{Cob}(2)^{o/cl}$ of open-closed cobordisms:

  - Objects are compact oriented 1-manifolds which are allowed to have boundaries. These are finite disjoint unions of oriented intervals and oriented circles.
  - As a bordism $B : M \to N$, we consider a smooth oriented two-dimensional manifold B, together with an orientation preserving smooth map

    $$\phi_B : \quad \overline{M} \coprod N \to \partial B$$

    which is a diffeomorphism to its image. The map is not required to be surjective. In particular, we have parametrized and unparametrized intervals on the boundary circles of M. The unparametrized intervals are called <u>free boundaries</u> and constitute physical boundaries of two-manifolds. The other boundaries are cut-and-paste boundaries and implement (aspects of) locality of the topological field theory.

    Two bordisms B, B' give the same morphism, if there is an orientation-preserving diffeomorphism $\phi : B \to B'$ such that the following diagram commutes:

    $$\begin{tikzcd} B \arrow[rr, "\phi"] & & B' \\ & \overline{M}\coprod N \arrow[ul, "\phi_B"] \arrow[ur, "\phi'_B"'] & \end{tikzcd}$$

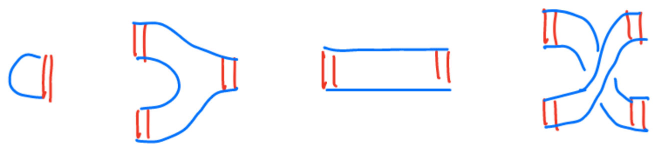
    Thus the diffeomorphism respects parametrizations of intervals on boundary circles and parametrizations of whole boundary circles.
  - For any object M, the identity morphism id_M is represented by the cylinder over M.
  - Composition is again by gluing.

- Again, disjoint union endows $\mathrm{Cob}(2)^{o/cl}$ with the structure of a symmetric monodial category with the empty set as the tensor unit.

- An <u>open-closed TFT</u> is defined as a symmetric monoidal functor

  $$Z : \mathrm{Cob}(2)^{o/cl} \to \mathrm{vect}(\mathbb{K}) \ .$$

Again $C := Z(S^1)$ is a commutative Frobenius algebra. One can again write generators and relations for the cobordism category. Generators for morphisms are the generators for morphisms of the closed TFT, together with the additional generators:

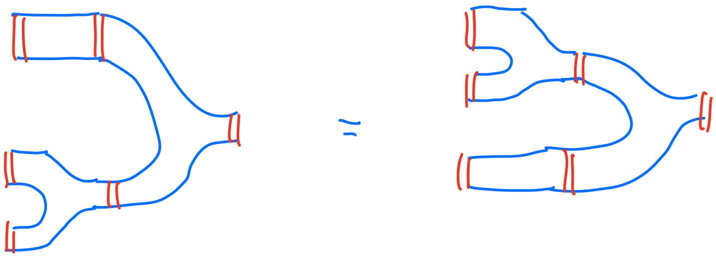


and their mirror images. The last generator is called the zipper and is topologically an annulus with a parametrized interval on one boundary component and a parametrized circle on the other boundary component.

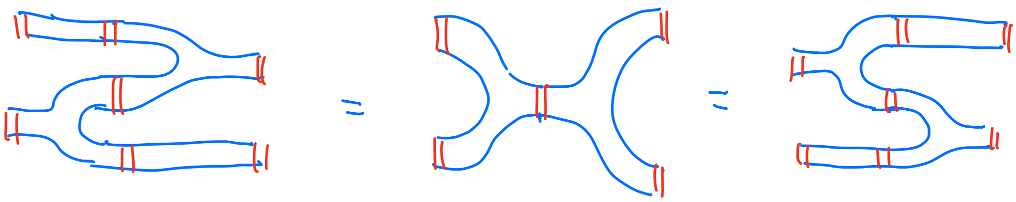
a) Identity relations

b) Unitality relations, e.g.

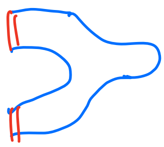
c) Associativity and coassociativity



d) Frobenius relations



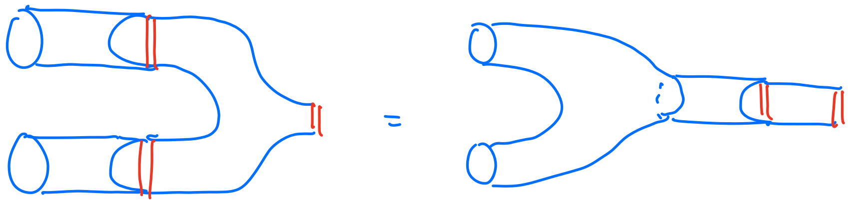
Bilinear form is symmetric



but algebra is not commutative.

Further relations:

①



$$i_*(\phi_1)\, i_*(\phi_2) = i_*(\phi_1 \cdot \phi_2) \qquad \phi_i \in C$$

② is mirror image

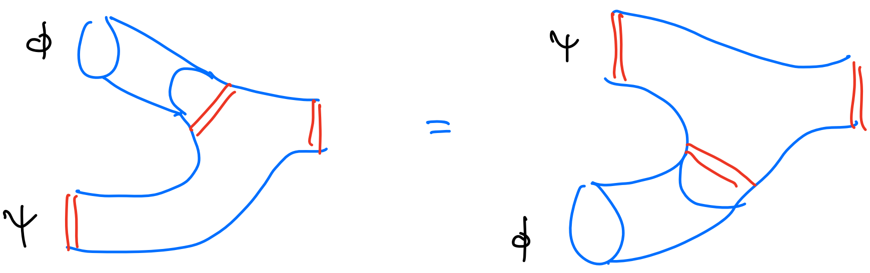
③

$$i_*(1_C) = 1_O \qquad i_* \text{ unital algebra morphism}$$

④ Dually for $i^*: O \to C$ $\qquad \varepsilon_O = \varepsilon_C \circ i^*$

counital coalgebra morphism

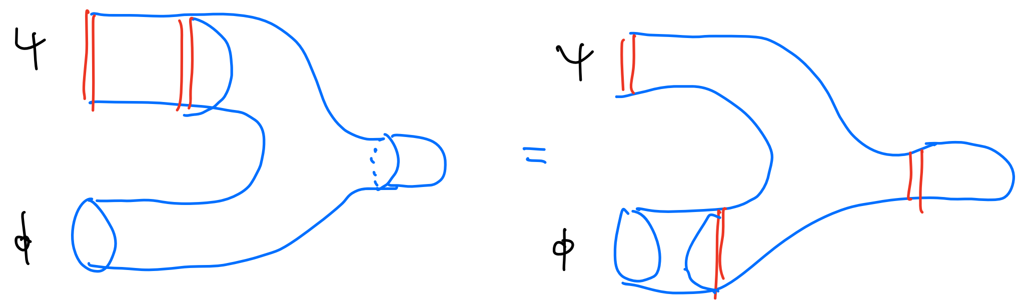
⑤ Relation punctured disc with 2 boundary intervals



$$i_*(\phi)\,\psi = \psi\, i_*(\phi) \qquad \text{for } \phi \in C,\ \psi \in O$$

$$\operatorname{im} i_* \subset Z(O)$$

⑥ Compatibility with Frobenius form



$$\kappa_c(i^*\psi, \phi) = \kappa_o(\psi, i_*(\phi))$$

i^*, i_* are adjoints

⑦ Cardy condition

One finds that the image $O := Z(I)$ of the interval I carries the structure of a Frobenius algebra. C is called the bulk Frobenius algebra, O the boundary Frobenius algebra.

- The Frobenius algebra O is not necessarily commutative: given three disjoint intervals on the boundary of a disk, two of them cannot be exchanged by a diffeomorphism of the disc. This situation is thus rather different from three boundary circles in a sphere, where two of them can be continuously commuted. For this reason, the bulk Frobenius algebra C is commutative.

  Still, the boundary Frobenius algebra is symmetric, i.e. the bilinear form $\kappa_0 : O \otimes O \to \mathbb{K}$ is symmetric: $\kappa_O(a, b) = \kappa_O(b, a)$.

- The zipper gives a linear map $i_* : C \to O$ and the cozipper $i^* : O \to C$. We show graphically that

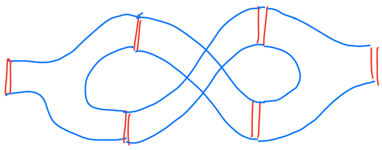
$$\begin{array}{lrcl} (1) & \mu_O \circ (i_* \otimes i_*) & = & i_* \circ \mu_C \\ (2) & (i^* \otimes i^*) \circ \Delta_O & = & \Delta_C \circ i^* \\ (3) & i_*(1_C) & = & 1_O \\ (4) & \epsilon_C \circ i^* & = & \epsilon_O \end{array}$$

  We summarize the relations: $i_* : C \to O$ is a unital algebra morphism. $i^* : O \to C$ is a counital morphism of coalgebras.

- One next shows that the image $i_*(C)$ is in the center $Z(O)$. Moreover, i_* and i^* are adjoints with respect to the Frobenius forms:

$$(6) \qquad \kappa_C(i^*\psi, \phi) = \kappa_O(\psi, i_*\phi) \quad \text{for all} \quad \psi \in O, \phi \in C \ .$$

- Finally, the image of the cobordism



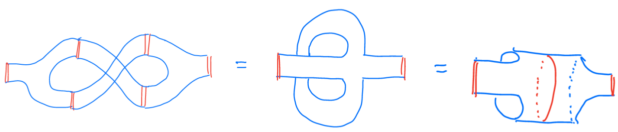
  allows us to use only structure in the Frobenius algebra O to get a map

$$\pi : \quad O \to O \ .$$

  If (b_i) is a basis of O and (b^i) the dual basis, we find

$$\pi : \qquad \psi \mapsto \sum_i \psi b_i \otimes b^i \mapsto \sum_i b^i \otimes \psi b_i \mapsto \sum_i b^i \cdot \psi b_i \ .$$

  The fact that $\sum_i b^i \otimes b_i$ is a Casimir element, cf. observation 3.2.30, implies that $\psi(O) \subset Z(O)$. From the picture



  we obtain the last relation, called the Cardy relation:

$$\pi = i_* \circ i^* \ .$$

- We are thus lead to the definition of a knowledgeable Frobenius algebra: A knowledgeable Frobenius algebra in a symmetric tensor category $\mathcal{D}$ consists of a commutative Frobenius algebra C in $\mathcal{D}$, a not necessarily commutative Frobenius algebra O in $\mathcal{D}$, a unital morphism of algebras
$$i_*: \quad C \to Z(O)$$
such that $\pi = i_* \circ i^*$ with i^* the adjoint of i_* with respect to the Frobenius forms and π defined as before.

  Morphisms of knowledgeable Frobenius algebras are pairs of morphisms of Frobenius algebras, compatible with i_* and i^*. We thus get a category $Frob^{\mathrm{o/cl}}(\mathcal{D})$ of knowledgeable Frobenius algebras and an equivalence of categories
$$[\mathrm{Cob}(2)^{\mathrm{o/cl}}, \mathcal{D}]_{\text{symm. monodial}} = Frob^{\mathrm{o/cl}}(\mathcal{D})$$
which classifies open/closed two-dimensional topological field theories.

- Given the bulk Frobenius algebra C, the boundary Frobenius algebra O is not uniquely determined. Rather, each choice of boundary Frobenius algebra determines a boundary condition for the two-dimensional closed topological field theory based on C. The category of all such boundary conditions carries a natural structure of an *algebroid*, i.e. of a linear category. The Frobenius structure can be encoded in terms of the additional structure of traces on the Hom-spaces.

- As a general reference for this example, we refer to the paper [LP].

This example already illustrates the principle that topological field theories transport geometric structure to algebraic structure.

4.2 Braidings and quasi-triangular bialgebras

It is an obvious question to ask what kind of structure on a Hopf algebra induces the structure of a braiding on its representation category.

Definition 4.2.1

1. *Let H be a bialgebra. The structure of a quasi-cocommutative bialgebra is the choice of an invertible element R in the algebra $H \otimes H$ such that for all $x \in H$*
$$\Delta^{\mathrm{copp}}(x)R = R\Delta(x)\,. \qquad \textit{[QT1]}$$
R is called a universal R-matrix. A quasi-cocommutative Hopf algebra is a Hopf algebra together with the choice of a universal R-matrix.

2. *A quasi-cocommutative bialgebra H is called quasi-triangular, if its universal R-matrix obeys the relations in $H^{\otimes 3}$*
$$\begin{array}{rcll} (\Delta \otimes \mathrm{id}_H)(R) &=& R_{13} \cdot R_{23} & \qquad \textit{[QT2]} \\ (\mathrm{id}_H \otimes \Delta)(R) &=& R_{13} \cdot R_{12} & \qquad \textit{[QT3]} \end{array}$$
with
$$R_{12} := R \otimes 1, \quad R_{23} := 1 \otimes R \quad \text{and} \quad R_{13} := (\tau_{H,H} \otimes \mathrm{id}_H)(1 \otimes R)\ .$$
It is convenient to extend this notation, e.g. by $R_{21} := \tau_{H,H}(R) \in H \otimes H$, and, by some abuse of notation $R_{21} := \tau_{H,H}(R) \otimes 1 \in H \otimes H \otimes H$.

3. *A morphism $f : (H,R) \to (H',R')$ of quasi-triangular Hopf algebras is a morphism $f : H \to H'$ of Hopf algebras such that $R' = (f \otimes f)(R)$.*

Remarks 4.2.2.

1. In Sweedler-like notation $R = R_1 \otimes R_2$, the relations read

$$\begin{array}{rcll} x_{(2)}R_1 \otimes x_{(1)}R_2 & = & R_1 x_{(1)} \otimes R_2 x_{(2)} & \text{[QT1]} \\ (R_1)_{(1)} \otimes (R_1)_{(2)} \otimes R_2 & = & R_1 \otimes R_{1'} \otimes R_2 R_{2'} & \text{[QT2]} \\ R_1 \otimes (R_2)_{(1)} \otimes (R_2)_{(2)} & = & R_1 R_{1'} \otimes R_{2'} \otimes R_2 & \text{[QT3]} \end{array}$$

2. A cocommutative bialgebra has a distinguished structure of a quasi-triangular Hopf algebra with R-matrix $R = 1 \otimes 1$. A quasi-triangular structure on a Hopf algebra can thus be seen as a weakening of cocommutativity. We have already seen in remark 3.3.5 that cocommutative Hopf algebras are not a rich enough structure - over an algebraically closed field of characteristic 0 these are just group algebras.

3. To see a non-trivial quasi-triangular structure, consider the cocommmutative Hopf algebra $\mathbb{K}[\mathbb{Z}_2]$ with $\mathbb{K}$ a field of characteristic different from 2. Write $\mathbb{Z}_2$ multiplicatively as $\{1, g\}$. Then
$$R := \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$$
is a universal R-matrix. A one-parameter family of R-matrices for the four-dimensional Taft Hopf algebra from observation 2.6.1 can be found in [Kassel, p. 174].

4. There is no universally accepted definition for the term quantum group. I would prefer to use the term for quasi-triangular Hopf algebras. Some authors use it as a synonym for Hopf algebras, some for certain subclasses of quasi-triangular Hopf algebras of Lie-theoretic flavour.

Theorem 4.2.3.
Let A be a bialgebra over a field $\mathbb{K}$. Then the tensor category $A-$mod is braided, if and only if A is quasi-triangular. Both structures are in one-to-one correspondence.

Proof.

- Let A be quasi-triangular with R-matrix R. For any pair U, V of left A-modules, we define a linear map

$$\begin{array}{rrcl} c^R_{U,V} : & U \otimes V & \to & V \otimes U \\ & u \otimes v & \mapsto & \tau_{U,V}(R.(u \otimes v)) = R_2.v \otimes R_1.u \,. \end{array}$$

  This is a morphism of A-modules: we have, for all $u \in U, v \in V$ and $h \in A$:

$$\begin{array}{rcll} c^R_{U,V}(h.u \otimes v) & = & R_2 h_{(2)}.v \otimes R_1 h_{(1)}.u & \\ & = & h_{(1)} R_2.v \otimes h_{(2)} R_1.u & \text{[equation QT1]} \\ & = & h.c^R_{U,V}(u \otimes v) \,. & \end{array}$$

  with inverse
$$c^{-1}(v \otimes u) = \overline{R}_1 u \otimes \overline{R}_2 v \,.$$

where $R^{-1} = \overline{R}_1 \otimes \overline{R}_2$ is the multiplicative inverse of R in the algebra $A \otimes A$. This family of morphisms of A-modules is natural for morphisms of A-modules since such morphisms commute with the action of $R \in H \otimes H$.

To check the first hexagon axiom, we compute for $u \in U$, $v \in V$ and $w \in W$:

$$\begin{array}{ll} (\mathrm{id}_V \otimes c^R_{U,W}) \circ (c^R_{U,V} \otimes \mathrm{id}_W)(u \otimes v \otimes w) & \\ = (\mathrm{id}_V \otimes c^R_{UW})(R_2 v \otimes R_1 u \otimes w) & [\text{Defn. of } c^R] \\ = R_2 v \otimes R_{2'} w \otimes R_{1'} R_1 u & [\text{Defn. of } c^R] \\ = (R_2)_{(1)} v \otimes (R_2)_{(2)} w \otimes R_1 u & [\text{equation QT3 for } R\text{-Matrix}] \\ = c^R_{U, V\otimes W}(u \otimes (v \otimes w)) & [\text{Defn. of } c^R] \end{array}$$

The second hexagon follows in complete analogy from equation [QT2].

- Conversely, suppose that the category A-mod is endowed with a braiding. Consider the element
$$R := \tau_{A,A}(c_{A,A}(1_A \otimes 1_A)) \in A \otimes A\,.$$
We have to show that R contains all information on the braiding on the category. To this end, let V be an A-module; for any vector $v \in V$, consider the A-linear map $\overline{v}$ which realizes the isomorphism $V \cong \mathrm{Hom}_A(A, V)$ of vector spaces:
$$\begin{array}{rccc} \overline{v}: & A & \to & V \\ & a & \mapsto & av \end{array}$$
Now consider two A-modules V, W and two vectors $v \in V$ and $w \in W$. The naturality of the braiding c applied to the morphism $\overline{v} \otimes \overline{w}$ implies
$$c_{V,W} \circ (\overline{v} \otimes \overline{w}) = (\overline{w} \otimes \overline{v}) \circ c_{A,A} \qquad (*)$$
and thus
$$\begin{array}{rcll} c_{V,W}(v \otimes w) & = & c_{V,W}(\overline{v} \otimes \overline{w}(1_A \otimes 1_A)) & \\ & = & (\overline{w} \otimes \overline{v}) c_{A,A}(1_A \otimes 1_A) & [\text{naturality, see } (*)] \\ & = & \tau_{V,W}(\overline{v} \otimes \overline{w}(R)) & \\ & = & \tau_{V,W} R.(v \otimes w) & [\text{definition of } \overline{v}, \overline{w}] \end{array}$$
This shows that all the information on a braiding on $A-$mod is contained in the element $R \in A \otimes A$.

We have to derive the three relations on an R-matrix from the properties of a braiding. We have for the action of any $x \in A$ on $c_{A,A}(1 \otimes 1) \in A \otimes A$:
$$x.c_{A,A}(1 \otimes 1) = x.\tau_{A,A}(R) = \Delta(x) \cdot \tau_{A,A}(R)\ ,$$
where the last expression is a product in $A \otimes A$. On the other hand, the braiding $c_{A,A}$ is A-linear. Thus this expression equals
$$c_{A,A}(x.1 \otimes 1) = \tau_{A,A}\left[R \cdot (\Delta(x) \cdot 1 \otimes 1)\right] = \tau_{A,A}\left[R \cdot \Delta(x)\right]\ .$$
Thus $\Delta(x) \cdot \tau_{A,A}(R) = \tau_{A,A}\left[R \cdot \Delta(x)\right]$; applying $\tau_{A,A}$ to this expression yields
$$\begin{array}{rcl} R \cdot \Delta(x) & = & \tau_{A,A}\left[\Delta(x) \cdot \tau_{A,A}(R)\right] = \tau_{A,A}\left[x_{(1)} \cdot R_2 \otimes x_{(2)} \cdot R_2\right] \\ & = & x_{(2)} \cdot R_1 \otimes x_{(1)} \cdot R_2 = \Delta^{\mathrm{opp}}(x) \cdot R\ . \end{array}$$
One can finally derive the two hexagon properties [QT2] and [QT3] of an R-matrix from the hexagon axioms for the braiding.

$\square$

Let A be a quasi-triangular Hopf algebra. We conclude from proposition 4.1.3 that for any A-module V, the automorphism

$$c^R_{V,V} : V \otimes V \to V \otimes V$$

is a solution of the Yang-Baxter equation. This explains the name *universal* R-matrix which is not related to a universal property. We note some properties of this R-matrix.

Proposition 4.2.4.
Let (H, R) be a quasi-triangular bialgebra.

1. Then the universal R-matrix obeys the following equation in $H^{\otimes 3}$:

$$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12}$$

(cf. proposition 4.1.3) and we have

$$(\epsilon \otimes \mathrm{id}_H)(R) = 1 = (\mathrm{id}_H \otimes \epsilon)(R) \ .$$

2. If, moreover, H has an invertible antipode, then

$$(S \otimes \mathrm{id}_H)(R) = R^{-1} = (\mathrm{id}_H \otimes S^{-1})(R)$$

and

$$(S \otimes S)(R) = R \ .$$

Proof.

1. We calculate, using the defining properties of the R-matrix:

$$\begin{aligned} R_{12} \cdot R_{13} \cdot R_{23} &= R_{12}(\Delta \otimes \mathrm{id})(R) \quad \text{[equation QT2]} \\ &= (\Delta^{\mathrm{opp}} \otimes \mathrm{id})(R) \cdot R_{12} \quad \text{[equation QT1]} \\ &= (\tau_{H,H} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) \cdot R_{12} \quad \text{[Defn. of } \Delta^{\mathrm{opp}}] \\ &= (\tau_{H,H} \otimes \mathrm{id})(R_{13}R_{23}) \cdot R_{12} \quad \text{[equation QT2]} \\ &= R_{23} \cdot R_{13} \cdot R_{12} \, . \end{aligned}$$

We now calculate in $H^{\otimes 3}$:

$$\begin{aligned} 1 \otimes R &= (((1\epsilon \otimes \mathrm{id}) \circ \Delta) \otimes \mathrm{id})\,(R) \quad [(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}] \\ &= (1\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13} \cdot R_{23}) \quad \text{[equation QT2]} \\ &= (1\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}) \cdot R_{23} \\ &= (\mathrm{id} \otimes 1\epsilon \otimes \mathrm{id})(R_{23}) \cdot R_{23} \\ &= 1 \otimes ((1\epsilon \otimes \mathrm{id})(R) \cdot R) \end{aligned}$$

Since R is invertible, we get $(\epsilon \otimes \mathrm{id})(R) = 1$. The other equality is derived in complete analogy from equation [QT3].

2. Using the definition of the antipode, we have for all $x \in H$

$$\mu \circ (S \otimes \mathrm{id})\Delta(x) = \epsilon(x)1\,.$$

We tensor this with the identity on H and apply it to $R \in H \otimes H$: we find

$$(\mu \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (1\epsilon \otimes \mathrm{id})R = 1 \otimes 1\ ,$$

where in the last step we used the identity just derived. Now, using equation [QT2], we find

$$1 \otimes 1 = (\mu \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = S(R_1) \cdot R_{1'} \otimes R_2 \cdot R_{2'} = (S(R_1) \otimes R_2) \cdot (R_{1'} \otimes R_{2'})\ .$$

We thus find

$$(S \otimes \mathrm{id})(R) = R^{-1}\ .$$

Recall the notation $R_{21} = \tau_{H,H}(R)$. We observe that equation [QT1] implies $\Delta(x)R_{21} = R_{21} \cdot \Delta^{\mathrm{opp}}(x)$. Thus, there is a quasi-triangular Hopf algebra $(H, \mu, \Delta^{\mathrm{opp}}, S^{-1}, R_{21})$. The corresponding relation for this quasi-triangular Hopf algebra reads

$$(S^{-1} \otimes \mathrm{id})(R_{21}) = R_{21}^{-1}$$

which amounts to

$$(\mathrm{id}_H \otimes S^{-1})(R) = R^{-1}\ . \qquad (*)$$

Finally, we use the two equations just derived to find

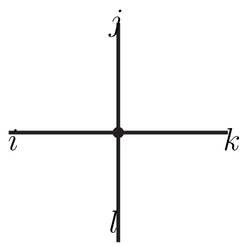
$$\begin{aligned} (S \otimes S)(R) &= (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(R) \\ &= (\mathrm{id} \otimes S)(R^{-1}) \\ &\overset{(*)}{=} (\mathrm{id} \otimes S)(\mathrm{id}_H \otimes S^{-1})(R) = R \end{aligned}$$

□

4.3 Interlude: Yang-Baxter equations and integrable lattice models

Consider the following model: on the lattice points of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, we have "atoms". We are not interested in these atoms, but in their bonds to their nearest neighbour. We describe the state of a bond by a variable taking its values in the finite set $\{1, \dots, n\}$.

Consider a vertex associated to an atom:



To such a vertex, we associate an energy $\epsilon_{ij}^{kl} \in \mathbb{R}$ which is allowed to depend on the type of bond i, j, k, l, but not on the vertex. We include the case that the energy depends on some external parameter $\epsilon_{ij}^{kl}(\lambda)$ which can be thought of as values of external magnetic or electric fields in some applications.

A lattice state is now a map that assigns to each bond a state:

$$\varphi : \quad \text{bonds} \quad \longrightarrow \{1, \ldots n\}$$

The energy of a state φ for given values of the parameters λ is obtained as the sum over atoms:

$$\epsilon_\lambda(\varphi) = \sum_{\text{atoms}} \epsilon^{\varphi(j)\varphi(k)}_{\varphi(i)\varphi(l)}(\lambda) \,.$$

To get a finite sum and thus well-defined expressions, we replace the lattice by a finite part with period boundary conditions, i.e. we consider vertices on $\mathbb{Z}_M \times \mathbb{Z}_N$. The partition function depends on an additional variable $\beta \in \mathbb{R}_+$, with the interpretation of an inverse temperature, $\beta = 1/kT$:

$$Z(\beta, \lambda) := \sum_{\text{states}} \mathrm{e}^{-\beta \epsilon_\lambda(\varphi)} \,.$$

The set of states is now the finite set of functions $S := \mathrm{Fun}(\mathbb{Z}_n \times \mathbb{Z}_m, \{1, \ldots, n\}) = \{1, \ldots, n\}^{N \cdot M}$. We endow it with the structure of a σ-algebra by the power set. We then get a family of probability measures with value

$$p_{\beta,\lambda}(\varphi) = \frac{1}{Z(\beta, \lambda)} e^{-\beta \epsilon_\lambda(\varphi)}$$

on the state $\varphi \in S$. Here, the partition function $Z(\beta, \lambda)$ normalizes the sum of the probabilities to be one.

The random variables, also called observables in this context, are then all measurable functions, i.e. all functions

$$Q : \quad S \to \mathbb{R} \,.$$

The energy ϵ is one example of an observable. The expectation value of a random variable Q is defined, as usual:

$$\mathbb{E}_{\beta,\lambda}[Q] = \frac{\sum_{\varphi \in S} Q(\varphi) e^{-\beta \epsilon_\lambda(\varphi)}}{Z(\beta, \lambda)} \,.$$

For the special case of the energy, we have

$$\mathbb{E}_{\beta,\lambda}[\epsilon] = \frac{\sum_{\text{states}} \epsilon_\lambda(\varphi) e^{-\beta \epsilon_\lambda(\varphi)}}{Z(\beta, \lambda)} = -\frac{\partial}{\partial \beta} \ln Z(\beta, \lambda)$$

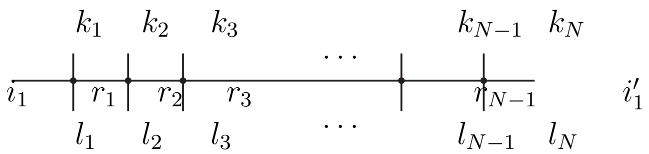
It is an important goal to compute the partition function as a function of β, λ. To this end, we introduce <u>Boltzmann weights</u>

$$R^{kl}_{ij}(\beta, \lambda) := \mathrm{e}^{-\beta \epsilon^{kl}_{ij}(\lambda)} \,.$$

We then get

$$\mathrm{e}^{-\beta \epsilon_\lambda(\varphi)} = \exp(-\beta \sum_{\text{atoms}} \epsilon^{\varphi(j)\varphi(k)}_{\varphi(i)\varphi(l)}(\lambda)) = \prod_{\text{atoms}} R^{kl}_{ij}(\beta, \lambda) \,.$$

Consider the contribution of the atoms in the first row to the partition function where we temporarily allow different values for the leftmost and rightmost bond:



It equals

$$T^{i'_1 l_1 \ldots l_N}_{i_1 k_1 \ldots k_N} = \sum_{r_1 \ldots r_{N-1}} R^{r_1 l_1}_{i_1 k_1} R^{r_2 l_2}_{r_1 k_2} \ldots R^{i'_1 l_N}_{r_{N-1} k_N} \qquad (*)$$

To eliminate indices, we introduce a complex vector space V freely generated on the set $\{1,\dots,n\}$ with basis $\{v_1,\dots v_n\}$ and a family of endomorphisms

$$\begin{array}{rccl} R = R(\beta,\lambda): & V\otimes V & \to & V\otimes V \\ & v_i\otimes v_j & \mapsto & \sum_{k,l} R_{ij}^{kl}(\beta,\lambda) v_k\otimes v_l\,. \end{array}$$

The endomorphism $T\in$ End $(V\otimes V^N)$ with

$$T := R_{01}\cdot R_{02}\cdot R_{03}\cdots R_{0n}$$

is represented by the matrix defined in definition $(*)$. Here we understand that the endomorphism R_{ij} acts on the i-th and j-th copy of V in the tensor product $V\otimes V^N$.

Periodic boundary conditions imply that we have to consider for the first line

$$\mathrm{Tr}_V(T)^{l_1\dots l_N}_{k_1\dots k_N}\,.$$

This endomorphism is called the row-to-row <u>transfer matrix</u>. To sum over all M lines, we take the matrix product and then the trace so that we find:

$$Z = \mathrm{Tr}_{V^{\otimes N}}(\mathrm{Tr}_V(T))^M\,.$$

This raises the problem of understanding the eigenvalues of the endomorphism $\mathrm{Tr}_V(T)\in \mathrm{End}(V^{\otimes N})$: in the thermodynamic limit, we take $M\to\infty$ so that $Z\sim \kappa_N^M$ with κ_N the eigenvalue with the largest modulus.

As usual in eigenvalue problems, we try to find as many endomorphisms of $V^{\otimes N}$ as possible commuting with $\mathrm{Tr}_V(T)$ which allows us to solve the eigenproblem separately on eigenspaces of these operators.

Definition 4.3.1
A vertex model with parameters λ is called <u>integrable</u>, if for any pair μ,ν of values for the parameters there is a value λ such that the equation

$$R_{12}(\lambda)R_{13}(\mu)R_{23}(\nu) = R_{23}(\nu)R_{13}(\mu)R_{12}(\lambda) \qquad \textit{(QYBE)}$$

holds in End $(V\otimes V\otimes V)$. *A specific case is the quantum Yang-Baxter equation with spectral parameters:*

$$R_{12}(\lambda-\mu)R_{13}(\lambda-\nu)R_{23}(\mu-\nu) = R_{23}(\mu-\nu)R_{13}(\lambda-\nu)R_{12}(\lambda-\mu)$$

Finite-dimensional bialgebras are not enough to describe such a structure; Etingof and Varchenko [EV] have instead proposed algebroids.

Lemma 4.3.2.
Consider the tensor product $V\otimes V\otimes V^N$ and denote the index for the first copy of V by 0 and the index for the second copy of V by $\overline{0}$. Then the following equation holds in End $(V\otimes V\otimes V^N)$

$$R_{0\overline{0}}(\lambda)T_0(\mu)T_{\overline{0}}(\nu) = T_{\overline{0}}(\nu)T_0(\mu)R_{0\overline{0}}(\lambda)\,.$$

Proof.
We suppress the spectral parameters and calculate:

$$\begin{array}{rcl} R_{0\overline{0}}T_0T_{\overline{0}} & \overset{\mathrm{def}}{=} & R_{0\overline{0}}R_{01}R_{02}\dots R_{0N}R_{\overline{0}1}\dots R_{\overline{0}N} \\ & = & R_{0\overline{0}}R_{01}R_{\overline{0}1}R_{02}\dots R_{0N}R_{\overline{0}2}\dots R_{\overline{0}N} \\ & \underset{\text{(QYBE)}}{=} & R_{\overline{0}1}R_{01}R_{0\overline{0}}R_{02}\dots R_{0N}R_{\overline{0}2}\dots R_{\overline{0}N} \end{array}$$

Here, we first used that the endomorphisms R_{0j} and $R_{\bar{0}1}$ for $j \geq 2$ act on different factors of the tensor product $V \otimes V \otimes V^N$ and thus commute. Then we applied the integrability equation (QYBE) on the indices $0, \bar{0}, 1$. Repeating this N-times, we get

$$\begin{aligned} &= R_{\bar{0}1} R_{\bar{0}2} \ldots R_{\bar{0}N} R_{01} \ldots R_{0N} R_{0\bar{0}} \\ &= T_{\bar{0}} T_0 R_{0\bar{0}} \end{aligned}$$

□

Proposition 4.3.3.
Suppose that for the integrable lattice model, the endomorphism $R(\lambda)$ is invertible for all values λ of the parameters. Then the endomorphism

$$C(\lambda) := \mathrm{Tr}_V T(\lambda) \in \text{ End } (V^{\otimes N})$$

(which is, of course, just the transfer matrix for the parameter value λ) commutes with $C(\mu)$ for all values λ, μ.

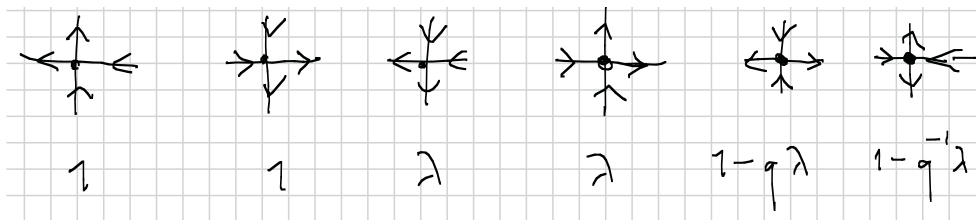
We thus have a set of commuting endomorphisms which make the eigenproblem for any operator $C(\lambda)$ more tractable, hence the name integrable.

Proof.
We take the trace $\mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}$ over the relation in lemma 4.3.2 and use the cyclicity of the trace to get the following equation in End $(V^{\otimes N})$.

$$\begin{aligned} C(\mu) \cdot C(\nu) &= \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}} T_0(\mu) T_{\bar{0}}(\nu) \\ &\overset{4.3.2}{=} \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}} R_{0\bar{0}}(\lambda)^{-1} T_{\bar{0}}(\nu) T_0(\mu) R_{0\bar{0}}(\lambda) \\ &= C(\nu) C(\mu) \end{aligned}$$

□

Example 4.3.4.
We consider the case of two possible states for each bond. One represents the state of a bond by assigning to it a direction, denoted by an arrow. A famous model is then the XXX model or six vertex model. In this case, one assigns Boltzmann weight zero to all vertices, except for those with two ingoing and two outgoing vertices. These are the following six configurations, hence the name of the model:



Since now V is two-dimensional, the R-matrix is a 4×4-matrix

$$R(q, \lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 - q\lambda & 0 \\ 0 & 1 - q^{-1}\lambda & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This model is integrable, if the relation holds

$$\lambda - \mu + \nu + \lambda\mu\nu = (q + q^{-1})\lambda\nu \; .$$

4.4 The Drinfeld center and Yetter-Drinfeld modules

We now present an important class of examples of braided monoidal categories. The following categorical construction works in a rather general situation:

Observation 4.4.1.
Let $\mathcal{C}$ be a strict tensor category.

- We consider a category $\mathcal{Z}(\mathcal{C})$ whose objects are pairs $(V, c_{-,V})$ consisting of an object V of $\mathcal{C}$ and a natural isomorphism $c_{-,V}: \ -\otimes V \xrightarrow{\sim} V \otimes -$, called a half-braiding for V, i.e. isomorphisms for all $X \in \mathcal{C}$

$$c_{X,V}: \quad X \otimes V \to V \otimes X$$

natural in the sense that that for any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} X \otimes V \arrow[r, "c_{X,V}"] \arrow[d, "f\otimes \mathrm{id}_V"'] & V \otimes X \arrow[d, "\mathrm{id}_V \otimes f"] \\ Y \otimes V \arrow[r, "c_{Y,V}"'] & V \otimes Y \end{tikzcd}$$

commutes which obey the additional requirement that for all objects X, Y of $\mathcal{C}$ we have

$$c_{X\otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,V}) \ . \qquad \text{[Hex]}$$

- A morphism $(V, c_{-,V}) \to (W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f: V \to W$ in $\mathcal{C}$ with the property that for all objects X of $\mathcal{C}$ we have

$$(f \otimes \mathrm{id}_X) \circ c_{X,V} = c_{X,W} \circ (\mathrm{id}_X \otimes f) \ . \qquad (**)$$

It is clear that the identity id_V in $\mathcal{C}$ is a morphism in $\mathcal{Z}(\mathcal{C})$ and that if f, g are morphisms in $\mathcal{Z}(\mathcal{C})$ that are composable in $\mathcal{C}$, then $f \circ g$ is a morphism in $\mathcal{Z}(\mathcal{C})$. Thus $\mathcal{Z}(\mathcal{C})$ is a category with composition and identities inherited from $\mathcal{C}$.

We now have examples of braided tensor categories:

Theorem 4.4.2.
Let $\mathcal{C}$ be a tensor category which we assume for simplicity to be strict. Then the category $\mathcal{Z}(\mathcal{C})$ has a natural structure of a strict braided tensor category with

1. Monoidal unit $(\mathbb{I}, \mathrm{id}_{\mathbb{I}})$.
2. The tensor product of two objects $(V, c_{-,V})$ and $(W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is given by

$$(V, c_{-,V}) \otimes (W, c_{-,W}) := (V \otimes W, c_{-,V\otimes W}) \ .$$

Here, given two objects $(V, c_{-,V})$ and $(W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$, we define for any object $X \in \mathcal{C}$ the morphism

$$c_{X,V\otimes W}: \quad X \otimes V \otimes W \to V \otimes W \otimes X$$

by

$$c_{X,V\otimes W} := (\mathrm{id}_V \otimes c_{X,W}) \circ (c_{X,V} \otimes \mathrm{id}_W) \ . \qquad (*)$$

3. The braiding on $\mathcal{Z}(\mathcal{C})$ is given by

$$c_{V,W}: \quad (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V}) \ .$$

This braided monoidal category is called the Drinfeld center of the monoidal category $\mathcal{C}$.

Proof.

1. We have to show that morphism $(*)$ is a half-braiding which implies that the tensor product is indeed an object in $\mathcal{Z}(\mathcal{C})$. It follows immediately that $c_{X,V\otimes W}$ is an isomorphism and natural in X. Hence, we have to check the hexagon [Hex] for any pair $X, Y \in \mathcal{C}$:

$$\begin{aligned}
c_{X\otimes Y,V\otimes W} &\overset{(*)}{=} (\mathrm{id}_V \otimes c_{X\otimes Y,W}) \circ (c_{X\otimes Y,V} \otimes \mathrm{id}_W)\\
&\overset{[\mathrm{Hex}]}{=} (\mathrm{id}_V \otimes c_{X,W} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{V\otimes X} \otimes c_{Y,W}) \circ (c_{X,V} \otimes \mathrm{id}_{Y\otimes W}) \circ (\mathrm{id}_X \otimes c_{Y,V} \otimes \mathrm{id}_W)\\
&= (\mathrm{id}_V \otimes c_{X,W} \otimes \mathrm{id}_Y) \circ (c_{X,V} \otimes \mathrm{id}_{W\otimes Y}) \circ (\mathrm{id}_{X\otimes V} \otimes c_{Y,W}) \circ (\mathrm{id}_X \otimes c_{Y,V} \otimes \mathrm{id}_W)\\
&\overset{(*)}{=} (c_{X,V\otimes W} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,V\otimes W})
\end{aligned}$$

2. Next, we have to show that the tensor product $f \otimes f'$ of morphisms $f : (V, c_{-,V}) \to (W, c_{-,W})$ and $f' : (V', c_{-,V'}) \to (W', c_{-,W'})$ in $\mathcal{Z}(\mathcal{C})$ is again a morphism in $\mathcal{Z}(\mathcal{C})$, i.e. obeys the naturality condition $(**)$

$$\begin{aligned}
(f \otimes f' \otimes \mathrm{id}_X) \circ c_{X,V\otimes V'} &\overset{(*)}{=} (f \otimes \mathrm{id}_{W'} \otimes \mathrm{id}_X) \circ (\mathrm{id}_V \otimes f' \otimes \mathrm{id}_X) \circ (\mathrm{id}_V \otimes c_{X,V'}) \circ (c_{X,V} \otimes \mathrm{id}_{V'})\\
&\overset{(**)}{=} (f \otimes \mathrm{id}_{W'} \otimes \mathrm{id}_X) \circ (\mathrm{id}_V \otimes c_{X,W'}) \circ (\mathrm{id}_V \otimes \mathrm{id}_X \otimes f') \circ (c_{X,V} \otimes \mathrm{id}_{V'})\\
&= (\mathrm{id}_W \otimes c_{X,W'}) \circ (f \otimes \mathrm{id}_X \otimes \mathrm{id}_{W'}) \circ (c_{X,V} \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_X \otimes \mathrm{id}_V \otimes f')\\
&\overset{(**)}{=} (\mathrm{id}_W \otimes c_{X,W'}) \circ (c_{X,W} \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_X \otimes f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_X \otimes \mathrm{id}_V \otimes f')\\
&\overset{(*)}{=} c_{X,W\otimes W'} \circ (\mathrm{id}_X \otimes f \otimes f')
\end{aligned}$$

   We have now defined a tensor product on objects and morphisms of $\mathcal{Z}(\mathcal{C})$. It is a functor, since it is obtained from a tensor product of $\mathcal{C}$. All other axioms are inherited as well. (If $\mathcal{C}$ is not strict, one can show that the associators in $\mathcal{C}$ provide associators for $\mathcal{Z}(\mathcal{C})$.) Thus $\mathcal{Z}(\mathcal{C})$ is a tensor category.

3. We next show that this tensor category is braided. We first show that the component $c_{V,W}$ of the halfbraiding $c_{-,W}$, $c_{V,W} : V \otimes W \to W \otimes V$ is a morphism in $\mathcal{Z}(\mathcal{C})$. We have to show $(**)$:
$$(c_{V,W} \otimes \mathrm{id}_X) \circ c_{X,V\otimes W} = c_{X,W\otimes V} \circ (\mathrm{id}_X \otimes c_{V,W})$$
   for all $X \in \mathcal{C}$. We compute

$$\begin{aligned}
(c_{V,W} \otimes \mathrm{id}_X) \circ c_{X,V\otimes W} &\overset{(*)}{=} (c_{V,W} \otimes \mathrm{id}_X) \circ (\mathrm{id}_V \otimes c_{X,W}) \circ (c_{X,V} \otimes \mathrm{id}_W)\\
&\overset{[\mathrm{Hex}]}{=} c_{V\otimes X,W} \circ (c_{X,V} \otimes \mathrm{id}_W)\\
&= (\mathrm{id}_W \otimes c_{X,V}) \circ c_{X\otimes V,W} \qquad \text{[naturality of braiding]}\\
&\overset{[\mathrm{Hex}]}{=} (\mathrm{id}_W \otimes c_{X,V}) \circ (c_{X,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_X \otimes c_{V,W})\\
&\overset{(*)}{=} c_{X,W\otimes V} \circ (\mathrm{id}_X \otimes c_{V,W})
\end{aligned}$$

4. Note that $c_{V,W}$ is invertible by definition and natural for all morphisms in $\mathcal{C}$, thus in particular for those in $\mathcal{Z}(\mathcal{C})$. To show that it defines a braiding, we have to show the two hexagon axioms from definition 4.1.1. One hexagon axiom has been imposed in [Hex] in the definition of a half-braiding. The second is part $(*)$ of the definition of the tensor product in $\mathcal{Z}(\mathcal{C})$.

□

Remarks 4.4.3.

1. The forgetful functor
$$\begin{array}{rrcl} U: & \mathcal{Z}(\mathcal{C}) & \to & \mathcal{C} \\ & (V, c_{-,V}) & \mapsto & V \end{array}$$
is strict monoidal and exact. It is, in general, neither essentially surjective nor full, but faithful by the definition of morphisms of $\mathcal{Z}(\mathcal{C})$.

2. If $\mathcal{C}$ is a finite tensor category, U as an exact functor has both a left adjoint $L : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ and a right adjoint $R : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. The left adjoint functor is opmonoidal, i.e. comes with morphisms
$$L(X \otimes Y) \to L(X) \otimes L(Y) \quad \text{and} \quad L(\mathbb{I}) \to \mathbb{I}$$
which are in general not isomorphisms. Similarly, the right adjoint is (weakly) monoidal, i.e. comes with morphisms
$$R(X) \otimes R(Y) \to R(X \otimes Y) \quad \text{and} \quad \mathbb{I} \to R(\mathbb{I})$$
which are in general not isomorphisms. In particular, $L(\mathbb{I})$ is naturally a coalgebra in $\mathcal{Z}(\mathcal{C})$ and $R(\mathbb{I})$ a (commutative) algebra in $\mathcal{Z}(\mathcal{C})$.

3. One can show [Sh, Lemma 4.7] that
$$L(D \otimes -) \cong R \cong L(- \otimes D) \quad \text{and} \quad R(D^{-1} \otimes -) \cong L \cong R(- \otimes D^{-1})$$
where D is the distinguished invertible element, cf. observation 3.1.16.4. In particular, if $\mathcal{C}$ is unimodular, left and right adjoint of U coincide. Then $L(1)$ is a commutative Frobenius algebra in $\mathcal{Z}(\mathcal{C})$ [Sh, Theorem 6.1].

4. The pair of adjoint functors $L \dashv U$ defines a monad $U \circ L$ on $\mathcal{C}$, the central monad. The adjunction is monadic. The central monad $U \circ L$ on $\mathcal{C}$ can be used to show that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a finite tensor category is again a finite tensor category. Similarly, $U \dashv R$ defines a comonad on $\mathcal{C}$, the central comonad to which analogous statements apply.

Suppose that the monoidal category $\mathcal{C}$ is given as the category of modules over a Hopf algebra H over a field $\mathbb{K}$. We ask whether the braided monoidal category $\mathcal{Z}(\mathcal{C})$ can be directly realized in terms of the Hopf algebra H.

For the following construction, we do not need to assume that the Hopf algebra H is a Hopf algebra in the braided monoidal category $\mathrm{vect}_{\mathbb{K}}$, and we directly consider a Hopf algebra H in a (strict) braided monoidal category $\mathcal{A}$. A bialgebra R in the braided category $\mathcal{A}$ is called a braided Hopf algebra, if there is a morphism $S : R \to R$ in $\mathcal{A}$ such that
$$S(r^{(1)})r^{(2)} = r^{(1)}S(r^{(2)}) = \eta\varepsilon(r)$$
where $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ for $r \in R$ in slightly modified Sweedler notation – a change of notation is performed in order to avoid confusion in Radford's biproduct below. We will assume that the antipode S is invertible (or that a skew antipode exists).

Definition 4.4.4

1. *Let H be a bialgebra in a braided monoidal category $\mathcal{A}$. A Yetter-Drinfeld module is a triple (V, ρ_V, Δ_V) such that*

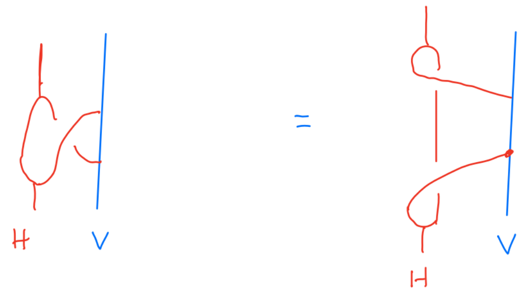
- *(V, ρ_V) is a unital left H-module in $\mathcal{A}$.*
- *(V, Δ_V) is a counital left H-comodule in $\mathcal{A}$:*

$$\begin{array}{rccc} \Delta_V : & V & \to & H \otimes V \\ & v & \mapsto & v_{(-1)} \otimes v_{(0)} \end{array}$$

- *The Yetter-Drinfeld condition*

$$h_{(1)} \cdot v_{(-1)} \otimes h_{(2)}.v_{(0)} = (h_{(1)}.v)_{(-1)} \cdot h_{(2)} \otimes (h_{(1)}.v)_{(0)}$$

holds for all $h \in H$. Graphically, it reads



These pictures are drawn in the strict braided monoidal category $\mathcal{A}$.

2. *Morphisms of Yetter-Drinfeld modules are morphisms of left modules and left comodules. We write ${}^H_H\mathcal{Y}D$ for the category of Yetter-Drinfeld modules over the bialgebra H.*

Example 4.4.5.
Let us consider explicitly Yetter Drinfeld modules over the group algebra $\mathbb{K}[G]$ of a finite group G.

Since Yetter-Drinfeld have the structure of a $\mathbb{K}[G]$-comodule, any Yetter-Drinfeld module has by example 2.2.8 a natural structure of a G-graded vector space,

$$V = \oplus_{g \in G} V_g \ ,$$

which is moreover endowed with a G-action. We evaluate the Yetter-Drinfeld condition for the action of $g \in G$ on a homogeneous element $v_h \in V_h$. We find for the left hand side using $\Delta(g) = g \otimes g$ and $\Delta_V(v_h) = h \otimes v_h$ that $gh \otimes g.v_h$. For the right hand side, we find the sum

$$\sum_{x \in G} xg \otimes (g.v_h)_x \ .$$

The equality $gh \otimes g.v_h = \sum_{x \in G} xg \otimes (g.v_h)_x$ implies that only the term with x such that $xg = gh$ contributes to the sum over x. Thus the Yetter-Drinfeld condition amounts to the condition $g.v_h \in V_{ghg^{-1}}$. Thus the G-action has to cover for the G-grading the action of G on itself by conjugation.

We know that modules over a bialgebra form a tensor category, and so do comodules over a bialgebra. We can thus define as in proposition 2.4.10 and remark 2.4.11 on the tensor product of the objects in $\mathcal{A}$ underlying two Yetter-Drinfeld modules V, W the structure of a module and of a comodule. We also note that the monoidal unit of $\mathcal{A}$ with trivial action

$$H \otimes \mathbb{I} \xrightarrow{\epsilon \otimes \mathrm{id}_{\mathbb{I}}} \mathbb{I} \otimes \mathbb{I} \cong \mathbb{I}$$

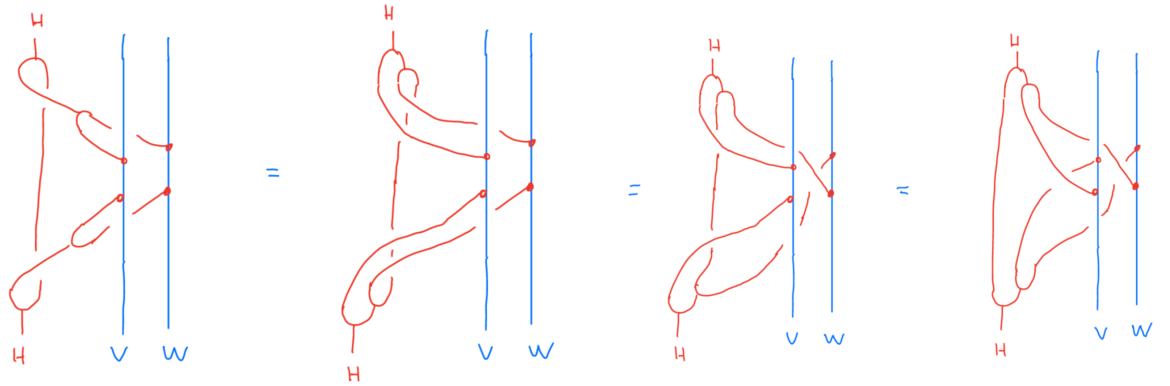
and coaction

$$\mathbb{I} \cong \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta \otimes \mathrm{id}_{\mathbb{I}}} H \otimes \mathbb{I}$$

is trivially a Yetter-Drinfeld module and is a tensor unit for the tensor product.

Proposition 4.4.6.
Let H be a bialgebra. Then the category of Yetter-Drinfeld modules has a natural structure of a tensor category.

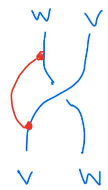
Proof.
Let V, W be Yetter-Drinfeld modules. We only have to show that the vector space $V \otimes W$ with action and coaction defined by the coproduct and product respectively obeys the Yetter-Drinfeld condition.



Here, we first used associativity and coassociativity, then the Yetter-Drinfeld condition on W and then on V. $\square$

Proposition 4.4.7.
Let H be a Hopf algebra. Then the category of Yetter-Drinfeld modules has a natural structure of a braided tensor category with the braiding of two Yetter-Drinfeld modules $V, W \in {}^H_H\mathcal{YD}$

given by $\begin{array}{rcl} c_{V,W} : V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & v_{(-1)}.w \otimes v_{(0)} \end{array}$



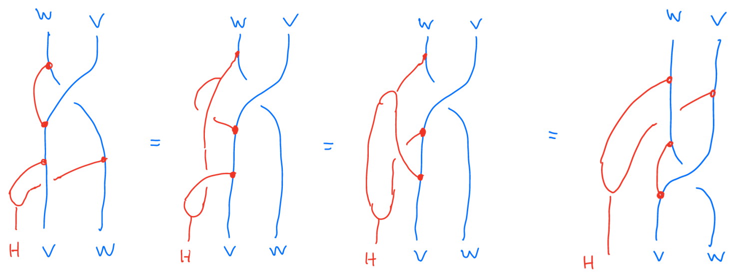
Remark 4.4.8.
We have seen that the category of comodules over a group algebra $H = \mathbb{K}[G]$ is the category of G-graded vector spaces. If G is not abelian, it cannot admit a braiding, since for homogeneous elements $v_g \in V_g$ and $w_h \in W_h$ the tensor product $v_g \otimes w_h$ is homogeneous of degree gh and $w_h \otimes v_g$ of degree hg. It is instructive to see how the combination of action and coaction for Yetter-Drinfeld modules allows for a braiding

$$v_g \otimes w_h \to g.w_h \otimes v_g \quad ;$$

the left hand side is again homogeneous of degree gh, and the right hand side is now homogeneous of degree $ghg^{-1}g = gh$.

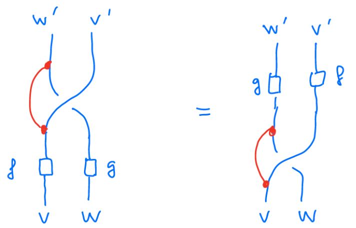
Proof.
The following statements have to be shown:

- The linear map $c_{V,W}$ is a morphism of modules and comodules and thus a morphism of Yetter-Drinfeld modules:



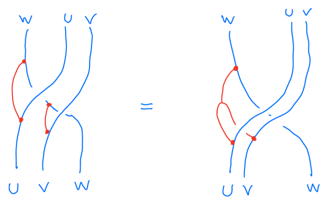
  Here, we first use that we have an H action on W, then the Yetter-Drinfeld condition on V, and finally the action on W.

- The braiding is natural:

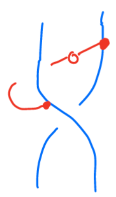


  Here, we use that f is a morphism of comodules and that g is a morphism of modules.

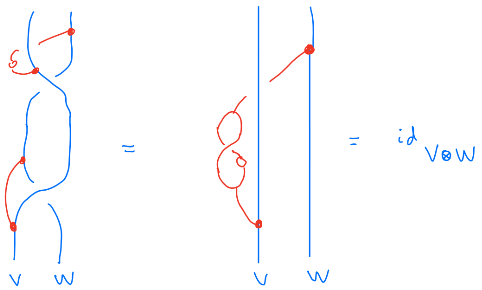
- The morphisms $c_{V,W}$ obey the hexagon axioms, e.g. follows from the fact that we have an H-action:



- Based on lemma 2.5.8, we show that the morphism $c_{V,W}$ has the inverse



  where the inverse S^{-1} of the antipode enters. We show one relation that is needed to show that this is indeed an inverse

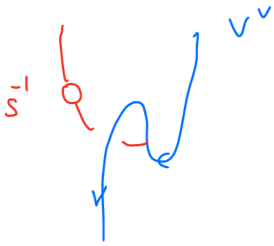


Here, we used the action on W and the coaction on V.

$\square$

To simplify the exposition, let us assume that $\mathcal{A} = \mathrm{vect}_{\mathbb{K}}$. We define as in proposition 2.5.16 the right dual action of H on $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ as the pullback along S of the transpose of the action on V and the left dual action as the pullback of the transpose along S^{-1}. The right dual coaction maps $\beta \in V^*$ to the linear map $\Delta_V^\vee(\beta) \in H \otimes V^* \cong \mathrm{Hom}_{\mathbb{K}}(V, H)$

$$\begin{array}{rrcl} \Delta_V^\vee(\beta): & V & \mapsto & H \\ & v & \mapsto & S^{-1}(v_{(-1)})\beta(v_{(0)}) \end{array}$$



while the left dual coaction maps to

$$\begin{array}{rrcl} {}^\vee\Delta_V(\beta): & V & \mapsto & H \\ & v & \mapsto & S(v_{(-1)})\beta(v_{(0)}) \end{array}$$

Proposition 4.4.9.
Let H be a Hopf algebra. Then the category of *finite-dimensional* Yetter-Drinfeld modules ${}^H_H\mathcal{Y}D$ is rigid.

Proof.
The following statements are easily shown, e.g. by graphical calculations:

- The above definition indeed defines H-coactions on V^*.
- The coaction defined with S^{-1} has the property that the right evaluation and right coevaluation are morphisms of H-comodules. The statement that the coaction defined with S on the left dual is compatible with left evaluation and coevaluation follows in complete analogy.
- The left and right dual actions and coactions obey the Yetter-Drinfeld axiom.

$\square$

Remark 4.4.10.

1. Let H be a Hopf algebra over a field $\mathbb{K}$ with bijective antipode. A bialgebra R in the braided category ${}^H_H\mathcal{Y}D$ is called a braided Hopf algebra.

2. Any $\mathbb{K}$-Hopf algebra is also a braided Hopf algebra over $H = \mathbb{K}$. A super Hopf algebra is nothing but a braided Hopf algebra over the group algebra $H = \mathbb{K}[\mathbb{Z}_2]$.

   The tensor algebra TV of any Yetter–Drinfeld module $V \in {}^H_H\mathcal{Y}D$ is a braided Hopf algebra, where the coproduct Δ is defined in such a way that the elements of V are primitive, i.e. $\Delta(v) = 1 \otimes v + v \otimes 1$ for all $v \in V$, compare example 2.3.2.

3. Consider the largest quotient of TV that is still a braided Hopf algebra and for which the elements of V are the only primitive elements. This is called the Nichols algebra of V. Nichols algebras take the role of quantum Borel algebras in the classification of pointed Hopf algebras, analogously to the classical Lie algebra case. Nichols algebras can be finite-dimensional and infinite-dimensional. Important classification results exist.

   More precisely let $V := x_1\mathbb{C} \oplus x_2\mathbb{C} \oplus \cdots \oplus x_n\mathbb{C}$ be the diagonal Yetter-Drinfel'd module over an abelian group $\Lambda = \mathbb{Z}^n = \langle K_1, \dots, K_n \rangle$ with braiding

$$x_i \otimes x_j \mapsto q_{ij} x_j \otimes x_i \qquad q_{ij} := q^{(\alpha_i, \alpha_j)}$$

   where (α_i, α_j) is the Killing form of a semisimple (finite-dimensional) Lie algebra $\mathfrak{g}$, then the Nichols algebra is the positive part of Lusztig's small quantum group

$$\mathfrak{B}(V) = u_q(\mathfrak{g})^+$$

4. Let H be a Hopf algebra over a field $\mathbb{K}$. For any braided Hopf algebra R in ${}^H_H\mathcal{Y}D$, there exists a natural Hopf algebra $R\#H$ over $\mathbb{K}$ containing R as a subalgebra and H as a Hopf subalgebra. It is called Radford's biproduct or bosonization.

   As a $\mathbb{K}$-vector space, $R\#H$ is just $R \otimes H$. The algebra structure of $R\#H$ is given by

$$(r\#h)(r'\#h') = r(h_{(1)}.r')\#h_{(2)}h',$$

   where $r, r' \in R, \quad h, h' \in H$ and $. : H \otimes R \to R$ is the left action of H on R. Further, the coproduct of $R\#H$ is determined by the formula

$$\Delta(r\#h) = (r^{(1)}\#r^{(2)}{}_{(-1)}h_{(1)}) \otimes (r^{(2)}{}_{(0)}\#h_{(2)}), \quad r \in R, h \in H.$$

   Here $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ denotes the coproduct of r in R, and $\delta(r^{(2)}) = r^{(2)}{}_{(-1)} \otimes r^{(2)}{}_{(0)}$ is the left coaction of H on $r^{(2)} \in R$.

5. The small quantum group $u_q(\mathfrak{g})$ is now obtained by a double bosonization $u_q(\mathfrak{g}) = H\#\mathcal{B}(V)\#\mathcal{B}(V^*)$; its modules are closely related to Yetter-Drinfeld modules of the braided Hopf algebra $\mathcal{B}(V)$ in H-mod.

In the remainder of this section, we want to work with a Hopf algebra H over a field $\mathbb{K}$. We now turn to the the question whether for any given Hopf algebra H over $\mathbb{K}$ the category ${}^H_H\mathcal{Y}D$ can be seen as the category of left modules over a quasi-triangular Hopf algebra $D(H)$.

Observation 4.4.11.

1. To investigate this in more detail, assume that the Hopf algebra H is finite-dimensional and recall from example 2.2.8.1 that a coaction of H then amounts to an action of H^*. Thus the quasi-triangular Hopf algebra $D(H)$ should account for an action of H and H^*. If the two actions would commute, $H^* \otimes H$ with the product structure for algebra and coalgebra would be an obvious candidate. This is not the case, and we have to encode the Yetter-Drinfeld condition in the product on $D(H)$.

2. To match the conventions in [Kassel], it will be convenient to consider a slightly different category ${}_H\mathcal{Y}D^H$ of Yetter-Drinfeld modules: these are triples (V, ρ_V, Δ_V) such that

   - (V, ρ_V) is a unital left A-module.
   - (V, Δ_V) is a counital *right* A-comodule:

$$\begin{array}{rccc} \Delta_V : & V & \to & V \otimes H \\ & v & \mapsto & v_{(V)} \otimes v_{(H)} \end{array}$$

   - The Yetter-Drinfeld condition holds in the form

$$h_{(1)}.v_{(V)} \otimes h_{(2)} \cdot v_{(H)} = (h_{(2)}.v)_{(V)} \otimes (h_{(2)}.v)_{(H)} \cdot h_{(1)} \ .$$

   Morphisms of Yetter-Drinfeld modules in ${}_H\mathcal{Y}D^H$ are morphisms of left modules and right comodules.

This observation leads to the following definition:

Definition 4.4.12
Let H be a finite-dimensional Hopf algebra. Endow the vector space $D(H) := H^* \otimes H$

- *with the structure of a counital coalgebra using the tensor product structure, i.e. for $f \in H^*$ and $a \in H$, we have*

$$\begin{array}{rcl} \epsilon(f \otimes a) & := & \epsilon(a) f(1) \\ \Delta(f \otimes a) & := & (f_{(1)} \otimes a_{(1)}) \otimes (f_{(2)} \otimes a_{(2)}) \ \in \ D(H) \otimes D(H) \ . \end{array}$$

  This encodes the fact that the tensor product of Yetter-Drinfeld modules is the ordinary tensor product of modules and comodules.

- *Define an associative multiplication for $a, b \in H$ and $f, g \in H^*$ by*

$$(f \otimes a) \cdot (g \otimes b) := f \cdot \big(g(S^{-1}(a_{(3)}) ? a_{(1)})\big) \otimes a_{(2)} b \ .$$

  The unit for this multiplication is $\epsilon \otimes 1 \in H^* \otimes H$.

A tedious, but direct calculation (see [Kassel, Chapter IX]) shows:

Proposition 4.4.13.
This defines a finite-dimensional Hopf algebra with antipode given in [Kassel, Theorem IX.2.3]. Moreover, if (e_i) is any basis of H with dual basis (e^i) of H^*, then the element

$$R := \sum_i (1_{H^*} \otimes e_i) \otimes (e^i \otimes 1_H) \in D(H) \otimes D(H) \ ,$$

which, by a standard argument, is independent of the choice of basis, is a universal R-matrix for $D(H)$.

Definition 4.4.14
We call the quasi-triangular Hopf algebra $(D(H), R)$ the <u>Drinfeld double</u> of the Hopf algebra H.

Remarks 4.4.15.

1. The Drinfeld double $D(H)$ contains H and H^* as Hopf subalgebras with embeddings
$$\begin{array}{rrcl} i_H : & H & \to & D(H) \\ & a & \mapsto & 1 \otimes a \end{array}$$
and
$$\begin{array}{rrcl} i_{H^*} : & H^* & \to & D(H) \\ & f & \mapsto & f \otimes 1 \ . \end{array}$$

2. One checks for the product in $D(H)$ that
$$\iota_{H^*}(f) \cdot \iota_H(a) = (f \otimes 1) \cdot (1 \otimes a) = f\epsilon(S^{-1}1_{(3)}?1_{(1)}) \otimes 1_{(2)} \cdot a = f \otimes a$$
and therefore writes $f \cdot a$ instead of $f \otimes a$. The multiplication on $D(H)$ is then determined by the straightening formula
$$a \cdot f = f(S^{-1}a_{(3)}?a_{(1)}) \cdot a_{(2)} \ .$$

3. The Hopf algebra $D(H)$ is quasi-triangular, even if the Hopf algebra H does not admit an R-matrix. If (H, R) is already quasi-triangular, then one can show that the linear map
$$\begin{array}{rrcl} \pi_R : & D(H) & \to & H \\ & fa & \mapsto & f(R_1)R_2 \cdot a \end{array}$$
is a morphism of Hopf algebras. The multiplicative inverse $\overline{R}$ of R gives a second projection $\pi_{\overline{R}} : D(H) \to H$. For more details, we refer to the article [S].

We will see that the three braided monoidal categories
$$\mathcal{T}(H\text{−mod}), \quad {}^H_H\mathcal{Y}D \quad \text{and} \quad D(H)\text{-mod}$$
are equivalent.

The next theorem provides the relation to the Drinfeld center. We can treat a left action ρ_V of H^* on V for a right coaction Δ_V of H by
$$\Delta_V : \quad V \xrightarrow{\mathrm{id}_V \otimes \tilde{b}_H} V \otimes H^* \otimes H \xrightarrow{\tau_{V,H^*} \otimes \mathrm{id}_H} H^* \otimes V \otimes H \xrightarrow{\rho \otimes \mathrm{id}_H} V \otimes H$$
and conversely,
$$\rho_V : \quad H^* \otimes V \xrightarrow{\mathrm{id}_{H^*} \otimes \Delta_V} H^* \otimes V \otimes H \xrightarrow{\mathrm{id}_{H^*} \otimes \tau_{V,H}} H^* \otimes H \otimes V \xrightarrow{d_H \otimes \mathrm{id}_V} V$$
Put differently, we have
$$f.v = \langle f, v_{(H)} \rangle v_{(V)} \quad \text{for all } f \in H^* \ . \qquad (*)$$

Theorem 4.4.16.
Let H be a finite-dimensional Hopf algebra.

1. By replacing the left H^*-action by a right H-coaction as above, any left $D(H)$-module becomes a Yetter-Drinfeld module in ${}_H\mathcal{Y}D^H$.

2. Conversely, any Yetter-Drinfeld module in ${}_H\mathcal{Y}D^H$ has a natural structure of a left module over the Drinfeld double $D(H)$.

This leads to a braided monoidal equivalence ${}_H\mathcal{Y}D^H \cong D(H)$-mod.

Proof.
We note that the structure of a left $D(H)$-module on a vector space V consists of the structure of an H-module and of an H^*-module such that for all $f \in H^*$, $h \in H$ and $v \in V$ the following consequence of the straightening formula holds:

$$a.(f.v) = f(S^{-1}(a_{(3)})?a_{(1)}).\left(a_{(2)}.v\right) \ .$$

To show the second claim, we have to derive this relation from the Yetter-Drinfeld condition:

$$\begin{aligned} f(S^{-1}(a_{(3)}?a_{(1)})).\left(a_{(2)}.v\right) &= \langle f, S^{-1}(a_{(3)})(a_{(2)}v)_{(H)}a_{(1)}\rangle(a_{(2)}v)_{(V)} \quad \text{[equation } (*)] \\ &= \langle f, S^{-1}(a_{(3)})a_{(2)}v_{(H)}\rangle a_{(1)}v_{(V)} \quad \text{[YD condition]} \\ &= \epsilon(a_{(2)})\langle f, v_{(H)}\rangle a_{(1)}v_{(V)} \quad \text{[lemma 2.5.8]} \\ &= \langle f, v_{(H)}\rangle a v_{(V)} = a.(f.v) \quad \text{[equation } (*)] \end{aligned}$$

We leave the proof of the converse to the reader and refer for a more detailed account to [Kassel, Theorem IX.5.2], where Yetter-Drinfeld modules in ${}_H\mathcal{Y}D^H$ are called "crossed H-bimodules". □

Theorem 4.4.17.
For any finite-dimensional Hopf algebra H, the braided tensor categories $Z(H-\text{mod})$ and $D(H)-\text{mod}$ are equivalent as braided monoidal categories.

Proof.

- We construct a functor

  $$Z(H-\text{mod}) \to {}_H\mathcal{Y}D^H \ .$$

  To this end, we define on any object (V, c_{-V}) of the Drinfeld center $Z(H-\text{mod})$ a right H-coaction. Consider

  $$\begin{array}{rcl} \Delta_V : V & \to & V \otimes H \\ v & \mapsto & c_{H,V}(1_H \otimes v) \ . \end{array}$$

  One checks that this defines a coassociative right coaction.

  As in the proof of theorem 4.2.3, the naturality of the braiding allows us to express the braiding in terms of the coaction Δ_V: consider for $x \in X$ the morphism $\overline{x} : H \to X$ with $\overline{x}(1) = x$, i.e. $\overline{x}(h) = h.x$. Then

  $$\begin{aligned} c_{X,V}(x \otimes v) &= c_{X,V} \circ (\overline{x} \otimes \mathrm{id}_V)(1_H \otimes v) = (\mathrm{id}_V \otimes \overline{x}) \circ c_{H,V}(1_H \otimes v) \\ &= v_{(V)} \otimes v_{(H)} \cdot x = \Delta_V(v) \cdot (1_H \otimes x) \qquad (*) \end{aligned}$$

  which is exactly the braiding on the category ${}_H\mathcal{Y}D^H$.

- Next, we use the fact that the braiding is H-linear:

  $$a.c_{X,V}(x \otimes v) = c_{X,V}(a.(x \otimes v))$$

  for all $a \in H$ and $v \in V, x \in X$. Replacing the braiding $c_{X,V}$ by the expression $(*)$ yields the equation

  $$\Delta(a)\Delta_V(v)(1_H \otimes x) = \Delta_V(a_{(2)}v)(1_H \otimes a_{(1)})(1_H \otimes x) \ .$$

  Setting $X = H$ and $x = 1_H$ yields

  $$a_{(1)}.v_{(V)} \otimes a_{(2)} \cdot v_{(H)} = (a_{(2)}v)_V \otimes (a_{(2)}v)_{(H)} \cdot a_{(1)}$$

  which is just the Yetter-Drinfeld condition in ${}_H\mathcal{Y}D^H$.

- For the the proof that this functor is essentially surjective and fully faithful, we refer to [Kassel].

$\square$

We conclude the section with an application of the Drinfeld center to representation theory:

Proposition 4.4.18.
Let H be a finite-dimensional Hopf algebra. Let $t \in H$ be a non-zero right integral and $T \in H^*$ a non-zero left integral. Then $T \otimes t$ is a left *and* right integral for $D(H)$. In particular, the Drinfeld double of any finite-dimensional Hopf algebra is unimodular.

Proof.
We use the following identity for the right integral $t \in H$

$$S^{-1}(t_{(3)})a^{-1}t_{(1)} \otimes t_{(2)} = 1 \otimes t$$

where $a \in H$ is the distinguished group-like element. For the proof, we refer to [Montgomery, p. 192].

We then calculate for $f \in H^*$ and $h \in H$:

$$\begin{aligned}
(T \otimes t) \cdot (f \otimes h) &= Ttfh \\
&= Tf(S^{-1}t_{(3)}?t_{(1)}) \otimes t_{(2)} \cdot h \quad \text{[straightening formula]} \\
&= Tf(S^{-1}t_{(3)}a^{-1}t_{(1)}) \otimes t_{(2)} \cdot h \quad [\text{since } Tf = \langle f, a^{-1}\rangle T] \\
&= T\langle f, 1\rangle \otimes th \quad \text{[preceding identity]} \\
&= \langle f, 1\rangle \epsilon(h) T \otimes t
\end{aligned}$$

Thus $T \otimes t$ is a right integral for $D(H)$. The proof that it is a left integral for $D(H)$ is similar. $\square$

Corollary 4.4.19.
Let H be a finite-dimensional Hopf algebra. Then the following assertions are equivalent:

1. $D(H)$ is semisimple.
2. H is semisimple. and H^* is semisimple.

We have already used in the proof of theorem 3.3.20 that H is semisimple, if and only if H^* is semisimple.

Proof.
If both H and H^* are semisimple, then, by Maschke's theorem 3.2.22 $\epsilon(t) \neq 0$ and $\epsilon^*(T) \neq 0$. By proposition 4.4.18, this implies for $D(H)$ that $\epsilon(T \otimes t) \neq 0$ and thus by Maschke's theorem that $D(H)$ is semisimple. The converse follows by the same type of reasoning. $\square$

4.5 Factorizable Hopf algebras

We now turn to a subclass of quasi-triangular Hopf algebras for which the braiding obeys additional constraints.

Definition 4.5.1
Let (H, R) be a quasi-triangular Hopf algebra.

1. *The invertible element*
$$Q := R_{21} \cdot R_{12} \in H \otimes H$$
is called the monodromy element*. We write $Q = Q_1 \otimes Q_2$ and note that*
$$\Delta(h) \cdot Q = Q \cdot \Delta(h)$$
for all $h \in H$.

2. *The linear map*
$$\begin{array}{rlll} F_R : & H^* & \to & H \\ & \phi & \mapsto & (\mathrm{id}_H \otimes \phi)(R_{21} \cdot R_{12}) = (\mathrm{id}_H \otimes \phi)Q \end{array}$$
is called the Drinfeld map*.*

3. *A quasi-triangular Hopf algebra is called* factorizable*, if the Drinfeld map is an isomorphism of vector spaces.*

Remark 4.5.2.
The word factorizable is justified as follows: let $(b_i)_{i \in I}$ be a basis of H and $(b^i)_{i \in I}$ the dual basis of H^*. If H is factorizable, then the vectors $c_i := F_R(b^i)$ form another basis of H. We write the monodromy element as a linear combination
$$Q = \sum_{i,j} \lambda_{i,j} c_i \otimes b_j$$
with $\lambda_{i,j} \in \mathbb{K}$. We then have
$$c_k = F_R(b^k) = \sum_{i,j} \lambda_{i,j} c_j \otimes b^k(b_i) = \sum_{j \in I} \lambda_{k,j} c_j$$
and thus for the monodromy matrix
$$Q = \sum_{i \in I} c_i \otimes b_i \ ,$$
which explains the word factorizable.

Proposition 4.5.3.
The Drinfeld double $D(H)$ of a finite-dimensional Hopf algebra H is factorizable.

Proof.
Recalling the R-matrix of the Drinfeld double from proposition 4.4.13
$$R = \sum_i (1 \otimes e_i) \otimes (e^i \otimes 1)$$
we find for the monodromy matrix of $D(H)$
$$Q = R_{21} \cdot R_{12} = \sum_{i,j} (e^i e_j) \otimes (e_i e^j) \ .$$

The family $(e^i e_j = e^i \otimes e_j)_{i,j}$ is a basis of $D(H) = H^* \otimes H$. Moreover,

$$S(e_i e^j) = S(e^j) \cdot S(e_i) = S(e^j) \otimes S(e_i)$$

and since S is invertible, the families $(S(e_i))$ and $(S(e^i))$ are bases of H^* and H and hence the family $S(e_i e^j)$ is a basis of $D(H)$. Again by the invertibility of S, the family $(e_i e^j)_{i,j}$ is a basis of $D(H)$ as well. Thus by remark 4.5.2 $D(H)$ is factorizable. $\square$

Remark 4.5.4.
We sketch the categorical meaning of factorizibility:

1. Suppose that A and B are two algebras over the same field $\mathbb{K}$. Then $A \otimes B$ is a $\mathbb{K}$-algebra as well. The Deligne product of two finite abelian categories is defined such that

   $$A \otimes B\text{-mod} \cong A\text{-mod} \boxtimes B\text{-mod} \; .$$

   It can be characterized by a universal property for right exact functors A-mod $\times$ B-mod $\to X$ where X is any finite category. For details, we refer to [D, section 5].

2. Let $\mathcal{C}$ be a braided tensor category. Using the braiding on $\mathcal{C}$ as a half-braiding gives a functor

   $$\begin{array}{rcl} \mathcal{C} & \to & \mathcal{Z}(\mathcal{C}) \\ V & \mapsto & (V, c_{-V}) \end{array}$$

   which is obviously a braided monoidal functor.

3. Taking the inverse braiding

   $$c^{\text{revd}}_{U,V} := c^{-1}_{V,U}$$

   on the same monoidal category, gives another structure of braided tensor category $\mathcal{C}^{\text{revd}}$. We get another functor

   $$\begin{array}{rcl} \mathcal{C}^{\text{revd}} & \to & \mathcal{Z}(\mathcal{C}) \\ V & \mapsto & (V, c^{\text{revd}}_{-V}) \end{array}$$

   which is again a braided monoidal functor.

4. Altogether, we obtain a braided monoidal functor

   $$\mathcal{C}^{\text{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \; .$$

5. Suppose that $\mathcal{C}$ is the category of representations of a quasi-triangular Hopf algebra (H, R). Then (H, R) is factorizable, if and only if the functor $\mathcal{C}^{\text{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is an equivalence of braided monoidal categories.

6. It can be shown that for any tensor category $\mathcal{C}$ the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is factorizable [EGNO, Proposition 8.6.3].

We consider the following subspace of H^*:

$$C(H) := \{ f \in H^* \mid f(xy) = f(yS^2(x)) \quad \text{for all } x, y \in H \}$$

We call this subspace the space of central forms or the space of class functions or the character algebra. We relate it to the center $Z(H)$ of H.

Lemma 4.5.5.
Let H be a finite-dimensional unimodular Hopf algebra with non-zero left cointegral $\lambda \in H^*$. Then by theorem 3.1.14 the map

$$\begin{array}{rcl} H & \to & H^* \\ a & \mapsto & \lambda(a \cdot -) = (\lambda \leftharpoonup a) \end{array}$$

is a bijection. It restricts to a bijection $Z(H) \cong C(H)$. In particular $\dim_{\mathbb{K}} Z(H) = \dim_{\mathbb{K}} C(H)$.

Proof.
If H is unimodular, we have $\alpha = \epsilon$ for the distinguished group-like element α. Then the Nakayama involution for the Frobenius structure given by the right cointegral reads by lemma 3.3.9

$$\rho(h) = \langle \alpha, h_{(1)} \rangle S^{-2}(h_{(2)}) = \langle \epsilon, h_{(1)} \rangle S^{-2}(h_{(2)}) = S^{-2}(h) \ .$$

For the Frobenius structure given by the left integral, one finds $\rho(h) = S^2(h)$ and thus

$$\lambda(a \cdot b) = \lambda(b \cdot S^2(a)) \qquad\qquad (*) \ .$$

Thus for any $a \in H$

$$(\lambda \leftharpoonup a)(yS^2x) = \lambda(ayS^2x) \overset{(*)}{=} \lambda(xay) \ .$$

Thus $(\lambda \leftharpoonup a) \in C(H)$, if and only if for all $x \in H$, we have $\lambda(xa) = \lambda(ax)$. But this amounts to $a \in Z(H)$. □

Theorem 4.5.6 (Drinfeld)**.**
Let (H, R) be a quasi-triangular Hopf algebra with Drinfeld map $F_R : H^* \to H$. Then

1. For all $\beta \in C(H)$, we have $F_R(\beta) \in Z(H)$.
2. For all $\beta \in C(H)$ and $\alpha \in H^*$, we have

$$F_R(\alpha \cdot \beta) = F_R(\alpha) \cdot F_R(\beta) \ .$$

Proof.

- We calculate for $\beta \in C(h)$ and $h \in H$:

$$\begin{array}{rcll} h \cdot F_R(\beta) & = & h \cdot Q_1 \beta(Q_2) & \\ & = & h_{(1)} Q_1 \beta(S^{-1}(h_{(3)}) h_{(2)} Q_2) & [S^{-1} \text{ is a skew antipode}] \\ & = & Q_1 h_{(1)} \beta(Q_2 h_{(2)} S(h_{(3)})) & [Q\Delta = \Delta Q \text{ and } \beta \in C(H)] \\ & = & Q_1 \beta(Q_2) \cdot h & \\ & = & F_R(\beta) \cdot h & \end{array}$$

- For the second statement, consider $\alpha \in H^*$ and $\beta \in C(H)$ and calculate

$$\begin{array}{rcll} F_R(\alpha \cdot \beta) & = & R_2 R'_1 (\alpha \cdot \beta)(R_1 R'_2) & [\text{Defn. Drinfeld map}] \\ & = & R_2 R'_1 (\alpha \otimes \beta) \Delta(R_1 R'_2) & [\text{Defn. product}] \\ & = & R_2 R'_1 (\alpha \otimes \beta) \Delta(R_1) \cdot \Delta(R'_2) & [\text{coproduct is a morphism of algebras}] \\ & = & R_2 r_2 s_1 t_1 \alpha(R_1 t_2) \beta(r_1 s_2) & [[\text{QT2,QT3}] \text{ with } R = r = s = t] \\ & = & R_2 r_2 s_1 \beta(r_1 s_2) t_1 \alpha(R_1 t_2) & \\ & = & R_2 F_R(\beta) t_1 \alpha(R_1 t_2) & [\text{Defn. Drinfeld map}] \\ & = & F_R(\alpha) \cdot F_R(\beta) & [F_R(\beta) \in Z(H)] \end{array}$$

$\square$

We will see below that any factorizable Hopf algebra is unimodular. From this fact we conclude

Corollary 4.5.7.
Let (H, R) be a factorizable Hopf algebra. Then the restriction of the Drinfeld map gives an algebra isomorphism

$$C(H) \xrightarrow{\cong} Z(H) \ .$$

Proof.
By theorem 4.5.6.2, the restriction of the Drinfeld map F to $C(H)$ is a morphism of algebras. It is injective, since the Drinfeld map is injective, due to the assumption that H is factorizable. Using the fact that H is unimodular, lemma 4.5.5 implies equality of dimensions. Hence, the map is surjective also and thus an isomorphism of algebras. $\square$

We finally want to show that factorizable Hopf algebras are unimodular. To this end, we need an alternative point of view on the Drinfeld double of a *quasi-triangular* Hopf algebra.

Let H be a Hopf algebra with invertible antipode. For an invertible element $F \in H \otimes H$ consider the linear map

$$\begin{aligned} \Delta_F : \quad & H \to H \otimes H \\ & \Delta_F(a) = F\Delta(a)F^{-1} \end{aligned}$$

This is obviously a morphism of algebras.

Lemma 4.5.8.

1. A sufficient condition for Δ_F to be coassociative is the identity

   $$F_{12}(\Delta \otimes \mathrm{id}_H)(F) = F_{23}(\mathrm{id}_H \otimes \Delta)(F)$$

   in $H^{\otimes 3}$.

2. A sufficient condition for ϵ to be a counit for Δ_F is the identity

   $$(\mathrm{id} \otimes \epsilon)(F) = (\epsilon \otimes \mathrm{id})(F) = 1 \, .$$

3. Define

   $$v := F_1 \cdot S(F_2) \quad \text{and} \quad v^{-1} := S(G_1)G_2$$

   with $G = F^{-1}$ the multiplicative inverse in the algebra $H \otimes H$. Then S_F with $S_F(h) := v \cdot S(h) \cdot v^{-1}$ is an antipode for the coproduct Δ_F.

Proof.

1. To show coassociativity

   $$(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(a) = (\mathrm{id} \otimes \Delta_F) \circ \Delta_F(a) \ ,$$

   we compute the two sides of this equation separately:

   $$\begin{aligned} (\Delta_F \otimes \mathrm{id})\Delta_F(a) &= (\Delta_F \otimes \mathrm{id})(F\Delta(a)F^{-1}) \\ &= (\Delta_F \otimes \mathrm{id})(F) \cdot F_{12} \cdot (\Delta \otimes \mathrm{id})\Delta(a) \cdot F_{12}^{-1} \cdot (\Delta_F \otimes \mathrm{id})(F^{-1}) \end{aligned}$$

where for the second equality we used that Δ_F is a morphism of algebras. For the right hand side, we find by an analogous computation

$$\begin{aligned}(\mathrm{id}\otimes\Delta_F)\Delta_F(a) &= (\mathrm{id}\otimes\Delta_F)(F\Delta(a)F^{-1})\\ &= (\mathrm{id}\otimes\Delta_F)(F)\cdot F_{23}\cdot(\mathrm{id}\otimes\Delta)\Delta(a)\cdot F_{23}^{-1}\cdot(\mathrm{id}\otimes\Delta_F)(F^{-1})\end{aligned}$$

A sufficient condition for coassociativity to hold is the identity

$$(\Delta_F\otimes\mathrm{id})(F)\cdot F_{12} = (\mathrm{id}\otimes\Delta_F)(F)\cdot F_{23}$$

which, by taking inverses in the algebra $H^{\otimes 3}$, implies

$$F_{12}^{-1}(\Delta_F\otimes\mathrm{id})(F^{-1}) = F_{23}^{-1}(\mathrm{id}\otimes\Delta_F)(F^{-1})$$

and which is equivalent to

$$F_{12}(\Delta\otimes\mathrm{id}_H)(F) = F_{23}(\mathrm{id}_H\otimes\Delta)(F)\ .$$

2. We leave the rest of the proofs to the reader.

□

Definition 4.5.9
Let H be a Hopf algebra with invertible antipode. An invertible element $F\in H\otimes H$ satisfying

$$\begin{aligned}F_{12}(\Delta\otimes\mathrm{id})(F) &= F_{23}(\mathrm{id}\otimes\Delta)(F)\\ (\mathrm{id}\otimes\epsilon)(F) &= (\epsilon\otimes\mathrm{id})(F)=1\end{aligned}$$

is called a 2-cocycle for H or a gauge transformation. We denote the twisted Hopf algebra with coproduct Δ_F, counit ϵ and antipode S_F by H^F.

Examples 4.5.10.

1. Let H be a finite-dimensional Hopf algebra with basis $\{e_i\}$ and dual basis $\{e^i\}$. Consider

$$\widetilde{H} := H^*\otimes H^{\mathrm{opp}}$$

and in $\widetilde{H}\otimes\widetilde{H}$ the basis-independent element

$$\widetilde{F} = \sum_{i=1}^{\dim H}(1_{H^*}\otimes e_i)\otimes(e^i\otimes 1_H)$$

A direct calculation shows that this element is a 2-cocycle for $\widetilde{H}$.

2. Let (H,R) be a finite-dimensional quasi-triangular Hopf algebra. Then

$$F_R := 1\otimes R_2\otimes R_1\otimes 1$$

is a 2-cocycle for $H\otimes H$ with the tensor product Hopf algebra structure. For a proof, we refer to [S, Theorem 4.3].

The proof of the following theorem can be found in [S, Theorem 4.3]:

Theorem 4.5.11.
Let (H, R) be a finite-dimensional quasi-triangular Hopf algebra. The map

$$\begin{array}{rcl} \delta_R : D(H) & \to & (H \otimes H)^{F_R} \\ x & \mapsto & \pi_R(x_{(1)}) \otimes \pi_{\overline{R}}(x_{(2)}) \end{array}$$

with π_R and $\pi_{\overline{R}}$ as in remark 4.4.15.3 is a Hopf algebra morphism. It is bijective, if and only if the quasi-triangular Hopf algebra (H, R) is factorizable. Thus factorizable Hopf algebras are related by a gauge transformation to tensor products.

Corollary 4.5.12.
Let (H, R) be a factorizable Hopf algebra. Then H is unimodular.

Proof.
Let Λ be a left integral in H. Then $\Lambda \otimes \Lambda$ is a left integral in $(H \otimes H)^{F_R}$, since the algebra structure and the counit ϵ are not changed by the twist. Since H is factorizable, the Hopf algebra $(H \otimes H)^{F_R}$ is by theorem 4.5.11 isomorphic to $D(H)$. We have seen in proposition 4.4.18 that the Drinfeld double of any Hopf algebra is unimodular. Thus $\Lambda \otimes \Lambda$ is also a right integral of $(H \otimes H)^{F_R}$. Hence Λ is also a right integral of H. $\square$

One can show the corresponding statement for categories [EGNO, Proposition 8.10.10]: if $\mathcal{C}$ is a factorizable finite tensor category, then $\mathcal{C}$ is unimodular. Since Drinfeld centers of finite tensor categories are factorizable by remark 4.5.4.5, they are in particular unimodular, cf. proposition 4.4.18 for the corresponding statement for Hopf algebras.

5 Topological field theories and quantum codes

5.1 Pivotal categories and pivotal Hopf algebras

We need two last subsections with algebraic preparation. In the rigid monoidal category of finite-dimensional vector spaces, the bidual is canonically isomorphic to the original vector space. For a general monoidal category, this is not necessarily the case, but such an identification is needed for TFT constructions on oriented manifolds. All algebras over fields and their modules in this section will be finite-dimensional; all categories will be finite categories.

Definition 5.1.1

1. *Let $\mathcal{C}$ be a right rigid monoidal category. A* <u>*pivotal structure*</u> *is a monoidal natural isomorphism*
$$\omega : \quad \mathrm{id}_{\mathcal{C}} \to ?^{\vee\vee} \ .$$
A right rigid monoidal category together with a choice of pivotal structure is called a <u>*pivotal category*</u>*.*

2. *A* <u>*pivotal Hopf algebra*</u> *is a pair (H, ω), where H is a Hopf algebra and $\omega \in G(H)$ is a group-like element, called the* <u>*pivot*</u> *such that*
$$S^2(x) = \omega x \omega^{-1} \ .$$

Remarks 5.1.2.

1. Note that we do not require that $\mathcal{C}$ is braided or that H has the structure of a quasi-triangular Hopf algebra.

2. For a given Hopf algebra, the pivot is not necessarily unique. It is determined up to multiplication by an element in the group $G(H) \cap Z(H)$. The choice of a pivot is thus an additional structure on the Hopf algebra H.

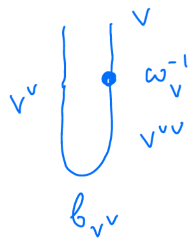
3. Because of the theorem 3.3.20 of Larson-Radford, any finite-dimensional semisimple Hopf algebra H over a field of chacteristic zero admits the unit element 1_H as a pivot. The question whether any fusion category admits a pivot is open.

4. If the category $\mathcal{C}$ is rigid, a pivotal structure implies an identification
$${}^{\vee}X \cong ({}^{\vee}X)^{\vee\vee} \cong X^{\vee}$$
of left and right duals which respects the opposite monoidal structure of the functors of taking duals. Here we first used the isomorphism $\omega_{{}^{\vee}X}$ and then the canonical identification $({}^{\vee}X)^{\vee}$.

5. A pivotal category is called <u>strict</u>, if the unitality and associativity isomorphisms, the pivotal structure ω, and the canonical isomorphism $(V \otimes W)^{\vee} \cong W^{\vee} \otimes V^{\vee}$ are identities. It has been shown in [NgS, Theorem 2.2] that every pivotal category $\mathcal{C}$ is equivalent to a strict pivotal category $\mathcal{C}^{str}$; equivalence as pivotal categories means that the monoidal equivalence $\mathcal{C} \to \mathcal{C}^{str}$ preserves pivotal structures in a suitable sense. In a strict pivotal category, we will denote the dual of V also by V^*.

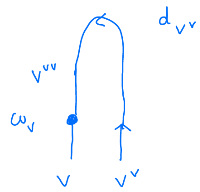
6. If $\mathcal{C}$ is only right rigid, but pivotal, a left evaluation and a left coevaluation can be defined by

$$\tilde{b}_V : \quad \mathbb{I} \xrightarrow{b_{V^\vee}} V^\vee \otimes V^{\vee\vee} \xrightarrow{\mathrm{id}_{V^\vee} \otimes \omega_V^{-1}} V^\vee \otimes V$$



and

$$\tilde{d}_V : \quad V \otimes V^\vee \xrightarrow{\omega_V \otimes \mathrm{id}_{V^\vee}} V^{\vee\vee} \otimes V^\vee \xrightarrow{d_{V^\vee}} \mathbb{I} \ .$$



It is straightforward to show that these morphisms obey the axioms of a left duality. Since ω is natural, one has on morphisms ${}^\vee f = f^\vee$. One obtains a strict pivotal category.

Proposition 5.1.3.
Let H be a finite-dimensional Hopf algebra.

1. If $\omega \in G(H)$ is a pivot for H, then the action with ω endows the category $H{-}\mathrm{mod}_{fd}$ of finite-dimensional H-modules with a pivotal structure.

2. Conversely, if ω is a pivotal structure on the category $H{-}\mathrm{mod}_{fd}$, then $\omega_H(1_H) \in H$ is a pivot for the Hopf algebra H.

Proof.

1. Assume that ω is a pivot for the Hopf algebra H. We know from proposition 2.5.16 that the category $H{-}\mathrm{mod}_{fd}$ is rigid. Let (V, ρ_V) be a finite-dimensional H-module. Identifying canonically the bidual V^{**} of the underlying fiite-dimensional vector space V with V, the right bidual of the H-module (V, ρ_V) is the H-module $(V, \rho_V \circ S^2)$. Use the pivot ω to define the linear isomorphism

$$\begin{array}{rccc} \omega_V : & V & \to & V \\ & v & \mapsto & \omega.v \ . \end{array}$$

This is actually a morphism $V \to V^{\vee\vee}$ of H-modules:

$$a.\omega_V(v) = S^2(a) \cdot \omega.v = \omega \cdot a \cdot \omega^{-1}\omega.v = \omega \cdot a.v = \omega_V(a.v) \ .$$

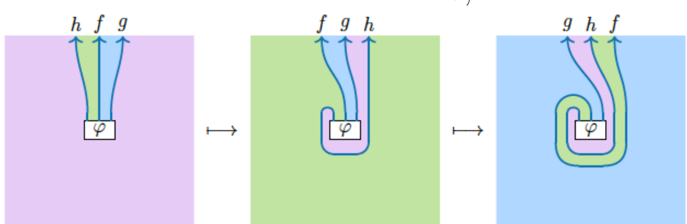
Implicitly assuming that the category of vector spaces has been replaced by an equivalent strict monoidal category, the natural transformation is monoidal by definition 2.4.8.3, if $\omega_V \otimes \omega_W = \omega_{V\otimes W}$. This holds, since $\omega.v \otimes \omega.w = \omega.(v \otimes w)$ for all $v \in V$ and $w \in W$, since ω is a grouplike element. For this reason, the natural transformation (ω_V) is invertible as well.

2. Conversely, suppose that $H-\mathrm{mod}_{fd}$ is a pivotal category. We canonically identify H as a vector space with its bidual H^{**} as a finite-dimensional $\mathbb{K}$-vector space. The action on $H^{\vee\vee}$ then translates to an H-action on H where $h \in H$ acts by $S^2(h)$. We consider the linear endomorphism
$$\omega_H : \quad H \to H^{**} \cong H \ .$$
All right translations by $a \in A$
$$\begin{array}{rccc} R_a : & H & \to & H \\ & h & \mapsto & h \cdot a \end{array}$$
are H-linear. The naturality of the pivotal structure ω thus implies
$$\omega_H(h \cdot a) = \omega_H(h) \cdot a \quad \text{for all} \quad h, a \in H \ . \qquad (*)$$
Since ω_H is a morphism of H-modules, we have
$$\omega_H(a \cdot b) = S^2(a)\omega_H(b) \quad \text{for all} \quad a, b \in H \ . \qquad (**)$$
Altogether, we find
$$S^2(a)\omega_H(1) \stackrel{(**)}{=} \omega_H(a \cdot 1) = \omega_H(1 \cdot a) \stackrel{(*)}{=} \omega_H(1) \cdot a \ .$$
To show that $\omega := \omega_H(1)$ is a pivot for the Hopf algebra H, it remains to show that ω is grouplike. As in the proof of theorem 4.2.3, one shows that for $v \in V$, $\omega_V(v) = \omega_H(1).v = \omega.v$. Monoidality of the natural transformation now implies that ω is grouplike by reversing the arguments in the first part of the proof.

□

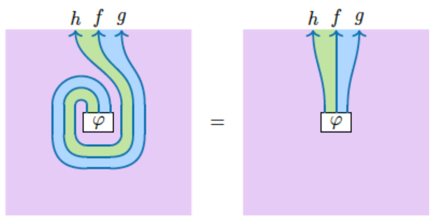
For pivotal categories, the graphical calculus can be cast into a different geometric form.

Lemma 5.1.4.
Let $\mathcal{C}$ be a pivotal category and $X_1, \dots X_n \in \mathcal{C}$. Then there are ismorphisms
$$\mathrm{Hom}(1, X_1 \otimes X_2 \otimes \cdots \otimes X_n) \to \mathrm{Hom}(1, X_2 \otimes X_3 \otimes \cdots \otimes X_n \otimes X_1)$$
of spaces of invariant tensors. For $n = 3$, we have



Applying the corresponding isomorphisms n times yields the identity on $\mathrm{Hom}(1, X_1 \otimes X_2 \otimes \cdots \otimes X_n)$. Thus, up to a canonical isomorphism, the space $\mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_1 \otimes \cdots \otimes V_n)$ only depends on the cyclic order of the objects $V_1, \dots, V_n$.

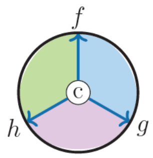
Proof.
We arrive at



where we used that in a strict pivotal category left and right dual morphisms coincide and then applied the zigzag identity for a left duality. $\square$

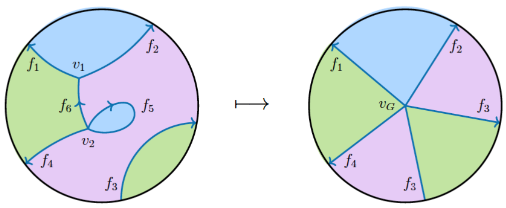
Remarks 5.1.5.

1. For a pivotal tensor category, it is therefore possible to represent morphisms by round coupons,

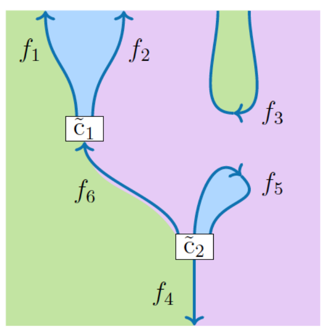
   

   The orientation induces a cyclic ordering of the edges ending at a round coupon. We also include the convention that an oriented edge labellex by $X \in \mathcal{C}$ can be replace by an edge with opposite orientation labelled by X^*.

2. On can then develop a graphical calculus which gives evaluations on discs, rather than on (2-framed) squares:

   To this end, one makes choices to rewrite a disc diagram on a 2-square where ingoing and outgoing edges only end on the upper and lower boundary of the square. One then evaluates it as usual. For example, the right diagram can be evaluated as

   $$(\mathrm{id}_{f_1 \otimes f_2} \otimes b_{f_3}) \circ \left(\tilde{c}_1 \otimes \tilde{d}_{f_3}\right) \circ \tilde{c}_2 : \quad f_4^* \to f_1 \otimes f_2 \otimes f_3 \otimes f_3^*$$

   The axioms of a pivotal tensor category ensure that this evaluation does not depend on the choices.

We need to single out a specific class of pivotal structures. To this end, we introduce the notion of a trace:

Lemma 5.1.6.
In any monoidal category, the associative monoid End $(\mathbb{I})$ is commutative.

Proof.
Identifying $\mathbb{I} \cong \mathbb{I} \otimes \mathbb{I}$, and φ with $\varphi \otimes \mathrm{id}_{\mathbb{I}}$ and φ' with $\mathrm{id}_{\mathbb{I}} \otimes \varphi'$, we see

$$\varphi \circ \varphi' = (\varphi \otimes \mathrm{id}_{\mathbb{I}}) \circ (\mathrm{id}_{\mathbb{I}} \otimes \varphi') = \varphi \otimes \varphi'$$

and

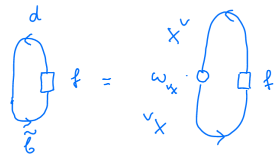
$$\varphi' \circ \varphi = (\mathrm{id}_{\mathbb{I}} \otimes \varphi') \circ (\varphi \otimes \mathrm{id}_{\mathbb{I}}) = \varphi \otimes \varphi' \ .$$

□

Definition 5.1.7
Let $\mathcal{C}$ be a strict pivotal category.

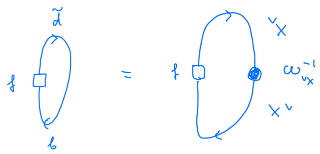
1. *Let X be an object of $\mathcal{C}$ and $f \in \mathrm{End}_{\mathcal{C}}(X)$. We define left and right pivotal traces:*

$$\begin{array}{rrcl} \mathrm{Tr}_l : & \mathrm{End}_{\mathcal{C}}(X) & \to & \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \\ & f & \mapsto & d_X \circ (\mathrm{id}_{X^*} \otimes f) \circ \tilde{b}_X \end{array}$$

$\mathrm{Tr}_l f =$ 

Note that $d_X : X^\vee \otimes X \to \mathbb{I}$ and $\tilde{b}_X : \mathbb{I} \to {}^\vee X \otimes X$, so that we need the pivotal structure to identify the objects $X^\vee$ and ${}^\vee X$.

$$\begin{array}{rl} \mathrm{Tr}_r : & \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \\ & f \mapsto \tilde{d}_X \circ (f \otimes \mathrm{id}_{X^*}) \circ b_X \end{array}$$

$\mathrm{Tr}_r f =$ 

2. *One also defines left and right dimensions:*

$$\dim_l X := \mathrm{Tr}_l \mathrm{id}_X \quad \textit{and} \ \dim_r X := \mathrm{Tr}_r \mathrm{id}_X \ .$$

Note that $\dim_l X \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$ and $\dim_r X \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$.

Some authors call this trace the quantum trace or the categorical trace.

Lemma 5.1.8.
The two traces have the following properties:

1. The traces are cyclic: for any pair of morphisms $g : X \to Y$ and $f : Y \to X$ in $\mathcal{C}$, we have

$$\mathrm{Tr}_l(g \circ f) = \mathrm{Tr}_l(f \circ g) \quad \text{and} \quad \mathrm{Tr}_r(g \circ f) = \mathrm{Tr}_r(f \circ g) \ .$$

2. We have

$$\mathrm{Tr}_l(f) = \mathrm{Tr}_r(f^*) = \mathrm{Tr}_l(f^{**})$$

for any endomorphism f, and similar relations with left and right trace interchanged.

3. Suppose that

$$\alpha \otimes \mathrm{id}_X = \mathrm{id}_X \otimes \alpha \quad \text{for all} \quad \alpha \in \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \text{ and all objects } X \in \mathcal{C} \; . \qquad (*)$$

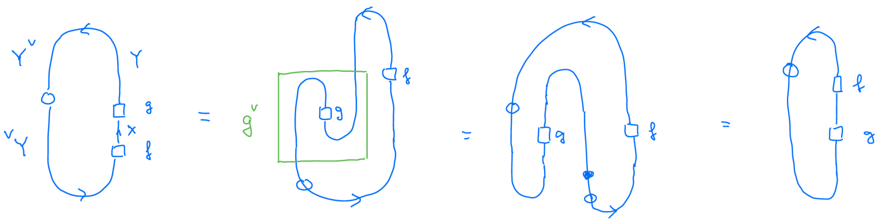
Then the traces are multiplicative for the tensor product:

$$\mathrm{Tr}_l(f \otimes g) = \mathrm{Tr}_l(f) \cdot \mathrm{Tr}_l(g) \quad \text{and} \quad \mathrm{Tr}_r(f \otimes g) = \mathrm{Tr}_r(f) \cdot \mathrm{Tr}_r(g)$$

for all endomorphisms f, g.

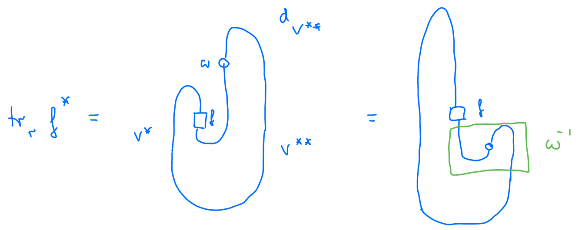
Remark 5.1.9.
The condition $(*)$ always holds for $\mathbb{K}$-linear categories for which $\mathrm{End}_{\mathcal{C}}(\mathbb{I}) \cong \mathbb{K}\mathrm{id}_{\mathbb{I}}$ and thus in particular for categories of modules over Hopf algebras. It also holds for all braided pivotal categories, since $c_{X,\mathbb{I}} \cong c_{\mathbb{I},X} \cong \mathrm{id}_X$, cf. remark 4.1.2.2.

Proof.
We only show the assertions for the left trace. We first show that the left trace is cyclic:



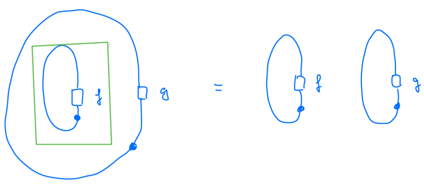
Here, the empty circle stands for the components $\omega_{^\vee X}$ of the pivotal structure and the full circle for its inverse.

For the second assertion, we note that



Using that the pivotal structure ω is monoidal, one shows that the expression in the green box equals ω^{-1} so that we obtain $\mathrm{Tr}_l(f)$.

Finally, the third assertion follows from

$\mathrm{Tr}_l(f \otimes g) =$  $= \mathrm{Tr}_l f \cdot \mathrm{Tr}_l g$

□

Corollary 5.1.10.
From the properties of the traces, we immediately deduce the following properties of the left and right dimensions:

1. Isomorphic objects have the same left and right dimension.

2. $\dim_l X = \dim_r X^* = \dim_l X^{**}$, and similarly with left and right dimension interchanged.

3. $\dim_l \mathbb{I} = \dim_r \mathbb{I} = \mathrm{id}_{\mathbb{I}}$.

4. Suppose, condition $(*)$ holds. Then the dimensions are multiplicative:

$$\dim_l(X \otimes Y) = \dim_l X \cdot \dim_l Y \quad \text{and} \quad \dim_r(X \otimes Y) = \dim_r X \cdot \dim_r Y$$

for all objects X, Y of $\mathcal{C}$.

5. The dimension is additive for exact sequences: from

$$0 \to V' \to V \to V'' \to 0$$

we conclude $\dim V = \dim V' + \dim V''$.

Proof.

1. Choose $f : X \to Y$ and $g : Y \to X$ such that $\mathrm{id}_X = g \circ f$ and $\mathrm{id}_Y = f \circ g$. Then by the symmetry of the trace

$$\dim_l X = \mathrm{Tr}_l \mathrm{id}_X = \mathrm{Tr}_l g \circ f = \mathrm{Tr}_l f \circ g = \mathrm{Tr}_l \mathrm{id}_Y = \dim_l Y \ .$$

2. The axioms of a duality imply that $\mathrm{id}_X^* = \mathrm{id}_{X^*}$. Now the claim follows from the second identity of lemma 5.1.8.

3. Follows from the canonical identification $\mathbb{I} \cong \mathbb{I} \otimes \mathbb{I}^*$.

4. Follows from the identity $\mathrm{id}_{X \otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y$ which is part of the definition of a tensor product.

5. We refer to [AAITC, 2.3.1].

□

Definition 5.1.11

1. *A (trace)spherical category is a pivotal category whose left and right traces are equal,*

$$\mathrm{Tr}_l(f) = \mathrm{Tr}_r(f)$$

for all endomorphisms f.

2. *A pivotal Hopf algebra (H, ω) is called spherical, if for all finite-dimensional representations V of H and all $f \in \mathrm{End}_H(V)$, we have*

$$\mathrm{Tr}_V f \circ \rho_V(\omega) = \mathrm{Tr}_V f \circ \rho_V(\omega^{-1}) \ ,$$

where ω stands for the endomorphism of V given by the left action of the pivot ω. In this case, the pivot ω is called a spherical element.

Remarks 5.1.12.

1. For an example of a pivotal Hopf algebra that is not spherical, we refer to [AAITC, Example 2.2].

2. In a spherical category, we write $\mathrm{Tr}(f)$ and call it the trace of the endomorphism f. In particular, left and right dimensions of all objects are equal, $\dim_l X = \dim_r X$. We call this element of $\mathrm{End}_{\mathcal{C}}(\mathbb{I})$ the dimension $\dim X$ of X.

3. The strictification of a spherical category is again spherical [NgS].

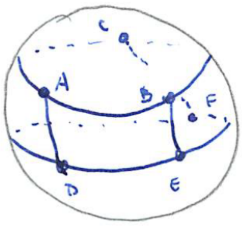
4. An example of a spherical tensor category is the category of super vector spaces which we have seen in remark 4.1.2.5 as a braided category with underyling category $\mathrm{vect}(\mathbb{Z}_2)$. Morphisms are thus grade-preserving linear maps. An endomorphis $f : V_0 \oplus V_1 \to V_0 \oplus V_1$ thus as two components $f_0 : V_0 \to V_0$ and $f_1 : V_1 \to V_1$. The supertrace is then given by $\mathrm{Tr} f_0 - \mathrm{Tr} f_1$. Writing this category as modules over the commutative Hopf algebra $\mathbb{K}(\mathbb{Z}_2)$ of functions on $\mathbb{Z}_2$ with basis (δ_0, δ_1), V_i is the image of δ_i. Hence the pivot is the grouplike elements $\delta_0 - \delta_1$.

5. Suppose that $\mathcal{C} = H-\mathrm{mod}$. Then the traces are given by

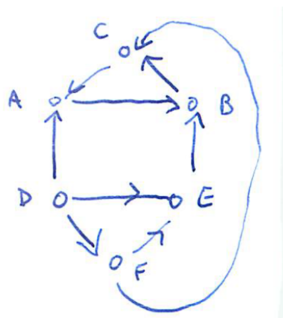
$$\mathrm{Tr}_l(f) = \mathrm{Tr}_V(f\rho_V(\omega^{-1})), \quad \mathrm{Tr}_r(f) = \mathrm{Tr}_V(f\rho_V(\omega)) \quad \text{for} \quad f \in \mathrm{End}_H(V).$$

   Thus, $H-\mathrm{mod}$ is a spherical category, whenever H is a spherical Hopf algebra. One can show [AAITC, Proposition 2.1] that it is sufficient to verify the trace condition on *simple* H-modules to show that a pivotal Hopf algebra is spherical.

6. For spherical categories, the graphical calculus has the following additional property: Consider an oriented graph Γ embedded in the sphere S^2 with standard orientation, where each edge e is colored by an object $V(e) \in \mathbb{C}$, and each vertex v is colored by a morphism $\varphi_v \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V(e_1)^{\pm} \otimes \cdots \otimes V(e_n)^{\pm})$, where $e_1, \ldots, e_n$ are the edges adjacent to vertex v, taken in clockwise order, and $V(e_i)^{\pm} = V(e_i)$ if e_i is outgoing edge, and $V^*(e_i)$ if e_i is the incoming edge.



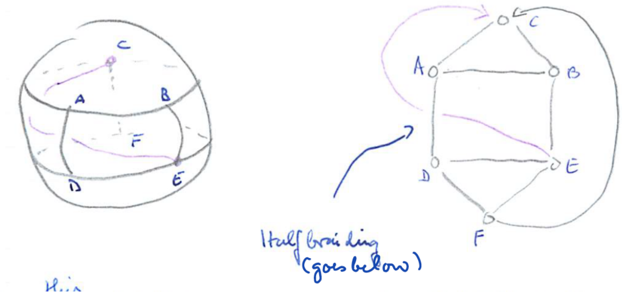
   By removing a point *pt* from S^2 and identifying $S^2 \setminus pt \simeq \mathbb{R}^2$, we can consider Γ as a planar graph



   to which our rules assign an element $Z(\Gamma) \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$.

One shows that this number is an invariant of the coloured graph on the sphere: morphisms represented by diagrams are invariant under isotopies of the diagrams in the two-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. They are thus preserved under pushing arcs through the point ∞. Left and right traces are related by such an isotopy. This explains the name "spherical".

7. Note that up to this point, only graphs on S^2 without crossings were allowed. We generalize this setup by allowing finitely many non-intersecting edges of a different type, labelled by objects of the Drinfeld double $\mathcal{Z}(\mathcal{C})$. These edges are supposed to start and end at the vertices as well. We colour edges of such a graph $\hat{\Gamma}$ with objects in $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ respectively and morphisms as before. We get an invariant $Z(\Gamma) \in \mathbb{K}$ for this type of graph as well:



5.2 Ribbon categories

We also consider analogous additional structure on *braided* tensor categories. Recall from remark 5.1.9 that for a braided category, the trace is always multiplicative.

Definition 5.2.1
Let $\mathcal{C}$ be a braided (strict) pivotal category.

1. *For any object X of $\mathcal{C}$, define the endomorphism*

$$\theta_X = (\mathrm{id}_X \otimes \tilde{d}_X) \circ (c_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes b_X) \ .$$

This endomorphism is called the twist on the object X.

2. *A ribbon category is a braided pivotal category where all twists are selfdual, i.e.*

$$(\theta_X)^* = \theta_{X^*} \quad \text{for all} \quad X \in \mathcal{C} \ .$$

Lemma 5.2.2.
Let $\mathcal{C}$ be a braided pivotal category.

1. The twist is invertible with inverse

$$\theta_X^{-1} = (d_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes c_{X,X}^{-1}) \circ (\tilde{b}_X \otimes \mathrm{id}_X) \ .$$

2. We have $\theta_{\mathbb{I}} = \mathrm{id}_{\mathbb{I}}$ and

$$\theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W) \, .$$

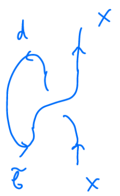
3. The twist is natural: for all morphisms $f : X \to Y$, we have $f \circ \theta_X = \theta_Y \circ f$.
4. A braided pivotal category is a ribbon category, if and only if the identity

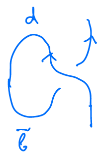
$$\theta_X = (d_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes c_{X,X}) \circ (\tilde{b}_X \otimes \mathrm{id}_X)$$

holds.

Proof.

1. In a pivotal braided category, we have for the twist

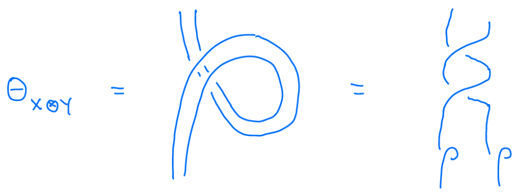


; the inverse of the twist is


We compute

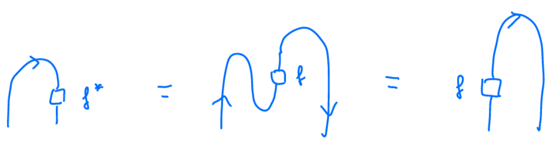
where we use naturality of the braiding and the zig-zag relations for the duality.

2. We compute graphically



3. Using properties of the duality, one shows for $f : U \to V$ that

$$\tilde{d}_U \circ (\mathrm{id}_U \otimes f^*) = \tilde{d}_V \circ (f \otimes \mathrm{id}_{V^*})$$



and similar relations for the other duality morphisms. The naturality of the twist now follows from these relations and the naturality of the braiding and its inverse.

4. Follows from a graphical calculation that is left to the reader.

□

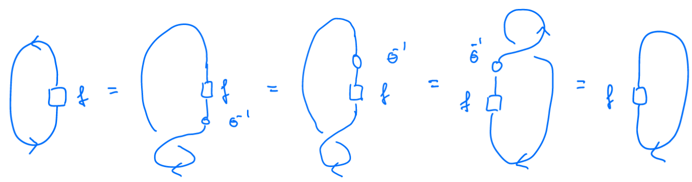
Proposition 5.2.3.
A ribbon category $\mathcal{C}$ is spherical

Proof.
To this end, one notes that

$$\begin{array}{lllll} \overline{d}_V & := & d_V \circ c_{V,V^*} \circ (\theta_V^{-1} \otimes \mathrm{id}_{V^*}) & = & d_V \circ \circ c_{V,V^*} \circ (\mathrm{id}_V \otimes \theta_{V^*}^{-1}) \\ \overline{b}_V & := & (\theta_{V^*}^{-1} \otimes \mathrm{id}_V) \circ c_{V,V^*} \circ b_V & = & (\mathrm{id}_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*} \circ b_V \end{array}$$

form another left duality. Graphically, one has

The proof of this assertion can be found in [Kassel, p. 351-353], with left and right duality interchanged as compared to our statement of the assertion. Since all (left) dualities are equivalent by lemma 2.5.15, the sphericality can be seen as follows:



Here we first insert the definition, then use the naturality of the twist and then of the braiding. □

We finally express these structures on the level of Hopf algebras.

Definition 5.2.4
A ribbon Hopf algebra is a quasi-triangular Hopf algebra (H, R) together with an invertible central element $\nu \in H$ such that

$$\Delta(\nu) = (R_{21} \cdot R)^{-1} \cdot (\nu \otimes \nu) \ , \ \epsilon(\nu) = 1 \quad \text{and} \quad S(\nu) = \nu \, .$$

The element ν is called a ribbon element.

A ribbon element is not unique, but only determined up to multiplication by an element in $\{g \in G(H) \cap Z(H) \mid g^2 = 1\}$, see [AAITC, Definition 2.13]. This reflects the fact that the pivot is *structure*.

Proposition 5.2.5.

1. Let (H, R, ν) be a ribbon Hopf algebra. For $V \in H\text{-mod}_{fd}$, consider the endomorphism

$$\begin{array}{rcl} \theta_V : V & \to & V \\ v & \mapsto & \nu^{-1}.v \end{array}$$

This defines a twist that is compatible with the dualities.

2. Conversely, suppose that (H, R) is a quasi-triangular Hopf algebra and that there is an element $\nu \in H$ such that for any $V \in H-\mathrm{mod}_{fd}$ the endomorphism $\theta_V(v) := \nu^{-1}.v$ is a twist on the braided category $H-\mathrm{mod}_{fd}$. Then ν is a ribbon element.

Proof.

- If ν is central and invertible, then all θ_V are H-linear isomorphisms. Conversely, for any algebra A any natural transformation $\theta : \mathrm{id}_{A-\mathrm{mod}} \to \mathrm{id}_{A-\mathrm{mod}}$ is given by the action of an element of the center, since $Z(A)$ of A, $\mathrm{End}(\mathrm{id}_{A-\mathrm{mod}}) = Z(A)$.
- We compute for $x \in V \otimes W$:

$$c_{W,V} c_{V,W} (\theta_V \otimes \theta_W)(x) = R_{21} R(\nu^{-1} x_{(1)} \otimes \nu^{-1} x_{(2)}) = \Delta(\nu^{-1}) \cdot x = \theta_{V \otimes W}(x) \,.$$

  The compatibility of twist and braiding is thus equivalent to the property $(R_{21}R)^{-1}(\nu \otimes \nu) = \Delta(\nu)$.
- It remains to show that

$$(\theta_{V^*} \otimes \mathrm{id}_V) b_V(1) = (\mathrm{id}_{V^*} \otimes \theta_V) b_V(1) \ .$$

  With $\{e_i\}$ a basis of V, this amounts to

$$\sum_i \nu e_i^* \otimes e_i = \sum_i e_i^* \otimes \nu e_i$$

  Evaluating this identity on any $v \in V$ yields

$$\begin{array}{lccc} & \sum_i \nu e_i^*(v) \otimes e_i & = & \sum e_i^*(v) \otimes \nu e_i \\ \iff & S(\nu) \cdot v & = & \nu \cdot v \end{array}$$

  This shows that $S(\nu) = \nu$ is a sufficient condition. Applying this to $V = H$ and $v = 1$ shows that $S(\nu) = \nu$ is necessary as well.

$\square$

Definition 5.2.6

1. *A link in $\mathbb{R}^3$ is a finite set of disjoint smoothly embedded circles (without parametrization and orientation). A link with a single component, i.e. a smoothly embedded circle, is called a knot.*
2. *An isotopy of a link is a smooth deformation of $\mathbb{R}^3$ which does not induce intersections and self intersections of the link.*
3. *A framed link is a link with a non-zero normal vector field.*
4. *A (k, l)-tangle is a finite set of disjoint circles and intervals that are smoothly embedded in $\mathbb{R}^2 \times [0, 1]$ such that*
   - *The end points of the intervals are precisely the points $(1, 0, 0), \ldots (k, 0, 0)$ and $(1, 0, 1), \ldots, (l, 0, 1)$.*
   - *The circles are contained in the open subset $(\mathbb{R}^2 \times (0, 1))$.*

5. *Isotopies of tangles and framed tangles are defined in complete analogy to 2.*

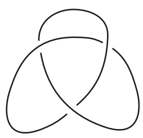
Links in the topological field theories of our interest are framed oriented links.

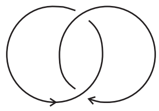
Examples 5.2.7.

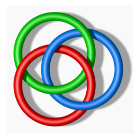
1. A special example is the so-called unknot which is given by the unit circle in the x-y-plane of $\mathbb{R}^3$.

2. Other important examples of well-known knots and links:

trefoil knot — Hopf link — Borromean link





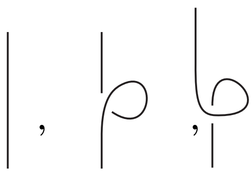


Remark 5.2.8.

1. If one projects a link $L \subset \mathbb{R}^3$ to the plane $\mathbb{R}^2$, we can represent the link by a link diagram. This is a set of circles in $\mathbb{R}^2$ with information about intersections which are, for a generic projection, only double transversal intersections.

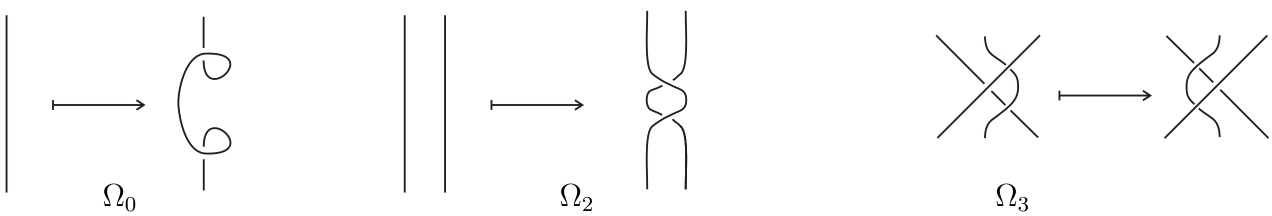
2. By taking the direction orthogonal to the plane containing the link diagram, we obtain a framing for the link represented by a link diagram. Thus any framed link can be represented by a link diagram.

3. Warning: if three knots differ locally by the following configurations,



then they are isotopic as knots, but not as framed knots.

4. Two link diagrams in $\mathbb{R}^2$ represent isotopic framed links in $\mathbb{R}^3$, if they are related by an isotopy of $\mathbb{R}^2$ or one of the Reidemeister moves $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$



Ω_0 Ω_2 Ω_3

These moves are local, i.e. only affect a part of a link contained in a small disc.

5. We define the linking number $\mathrm{lk}(K, K')$ of two knots K, K' in a link as the sum of the signs ± 1 for each over and undercrossing. The matrix of linking numbers is a symmetric link invariant.

   For *framed* knots, one can define the self linking number: one deforms the knot along its normal vector field and defined the self linking number as the linking number of the original knot with its deformation.

Remarks 5.2.9.

1. Since any link in $\mathbb{R}^3$ can be smoothly deformed to a link in $\mathbb{R}^2 \times (0,1)$, we identify links and $(0,0)$ tangles.

2. Tangle diagrams are projections of tangles to $\mathbb{R} \times [0,1]$ with only double transversal intersections. We only consider oriented tangle diagrams.

3. Tangle diagrams represent isotopic tangles, if they are related by an isotopy of $\mathbb{R} \times [0,1]$ or the Reidemeister moves $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$ from remark 5.2.8.4.

Definition 5.2.10

1. *We define a category $\mathcal{T}$ of framed tangles:*

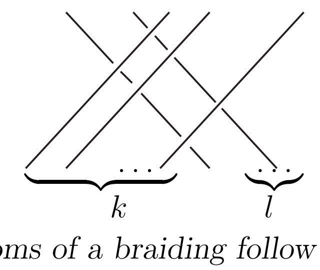
   - *Its objects are the non-negative integers.*
   - *A morphism $k \to l$ is an isotopy class of framed (k,l)-tangles.*

   The composition is concatenation of tangles, followed by a rescaling to the interval $[0,1]$. The identity tangles are given by parallel lines.

2. *We endow $\mathcal{T}$ with a monoidal structure. On objects, we define $k \otimes l := k + l$; on morphisms, we take juxtaposition of tangles. The tensor unit is $0 \in \mathbb{Z}_{\geq 0}$.*

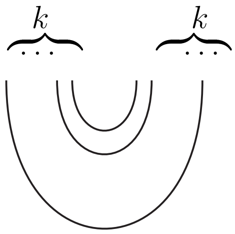
3. *The category $\mathcal{T}$ is endowed with the structure of a braided monoidal category by the following isomorphisms:*

$$c_{k,l} : k \otimes l \to l \otimes k$$

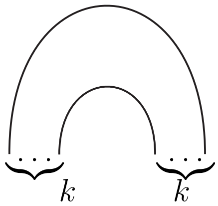


   The axioms of a braiding follow from obvious isotopies.

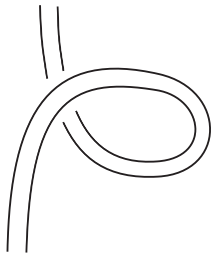
4. *The braided category $\mathcal{T}$ has the dualities*



and



and the twist $\theta_k : k \to k$



which turn it into a ribbon category.

Let now $\mathcal{C}$ be a ribbon category. We describe the category $\mathcal{T}_\mathcal{C}$ of $\mathcal{C}$-coloured framed oriented tangles.

Observation 5.2.11.

1. Tangles are now assumed to be framed and oriented. Each component of a tangle is labelled with an object of $\mathcal{C}$. Isotopies preserve the orientation, framing and $\mathcal{C}$-coloring.

2. The objects of $\mathcal{T}_\mathcal{C}$ are finite sequences of pairs
$$(V_1, \epsilon_1) \dots (V_n, \epsilon_n) \qquad V_i \in \ \mathcal{C} \quad \epsilon_i \in \{\pm 1\}\,,$$
including the empty sequence.

3. Morphisms are isotopy classes of framed oriented tangles. If the source object has label $\epsilon = +1$, the tangle is upward directed and labelled with V. It has to end on either an object $(V, +1)$ at $t = 1$ or at $(V, -1)$ at $t = 0$, where $t \in [0, 1]$ parametrizes the tangle.

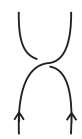
4. The category $\mathcal{T}_\mathcal{C}$ is endowed with a ribbon structure in complete analogy to the ribbon structure on the category $\mathcal{T}$ of framed oriented tangles.

The following theorem describes the graphical calculus for ribbon categories:

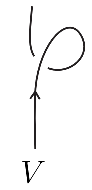
Proposition 5.2.12.
Let $\mathcal{C}$ be a ribbon category. Then there is a *unique* braided tensor functor
$$F = F_\mathcal{C} : \quad \mathcal{T}_\mathcal{C} \to \mathcal{C},$$
such that

1. F acts on objects as $F(V, +) = V$ and $F(V, -) = V^*$.

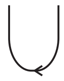
2. For all objects V, W of $\mathcal{C}$, we have



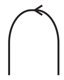
$$\longmapsto \qquad c_{V,W}$$



$\longmapsto \quad \theta_V$



$\longmapsto \quad b_V$



$\longmapsto \quad d_V$

Proof.
One can show that any tangle can be decomposed into the building blocks listed above. One then has to show the compatibility of F with the Reidemeister moves. This follows from the axioms of a ribbon category. $\square$

Definition 5.2.13
A modular tensor category is a finite ribbon category in which the braiding is non-degenerate in the sense that the braided monoidal functor

$$\mathcal{C}^{\mathrm{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$$

from remark 4.5.4.4 is an equivalence.

Remarks 5.2.14.

1. For a modular *fusion* category, we choose representatives $(V_i)_{i\in I}$ for the isomorphism classes of simple objects, assuming without loss of generality that $V_0 = \mathbb{I}$. The non-degeneracy condition is then equivalent to the statement that the $|I| \times |I|$-matrix with entries
$$S_{ij} = \mathrm{Tr}\, c_{V_j,V_i} \circ c_{V_i,V_j} \in \ \mathrm{End}\ (\mathbb{I}) \cong \mathbb{K}$$
is invertible over $\mathbb{K}$. The symmetry of the trace implies that the matrix S is symmetric, $S_{ij} = S_{ji}$.

2. The matrix element S_{ij} of a modular fusion category equals the invariant of the Hopf link with the two components coloured by the objects V_i and V_j.

3. Let H be a complex semi-simple ribbon factorizable Hopf algebra. Then the category $H{-}\mathrm{mod}_{fd}$ is a modular tensor category.

4. Let H be a semi-simple complex Hopf algebra. Then the category of finite-dimensional modules over its Drinfeld double $D(H)$-mod_{fd} is modular.

   One can show that the Drinfeld center of any *spherical* finite tensor category is a ribbon category [Sh, Theorem 5.11]. Here, if $\mathcal{C}$ is not semisimple, spherical is not trace-spherical in the sense of definition 5.1.11, but is defined as in [DSPS1, Definition 4.5.2] (the pivotal structure squares to the Radford isomorphism).

5.3 Stringnets and extended TFTs

We now work towards the construction of a three-dimensional topological field theory. Our input datum is a (strictly) spherical tensor category $\mathcal{C}$, and our first goal is to construct a vector space for a compact oriented surface Σ, possibly with boundaries.

The idea is to globalize the graphical calculus on discs for pivotal tensor categories that we developped in remark 5.1.5.

Definition 5.3.1

1. *A boundary datum* b *consists of finitely many points on the boundary $\partial\Sigma$, where we require that each boundary component contains at least one point. For every point, an object of $\mathcal{C}$ is chosen as a label.*

2. *Denote by $\mathrm{G}(\Sigma, \mathrm{b})$ the set of all finite $\mathcal{C}$-colored graphs on Σ with prescribed boundary datum* b*, and by $\mathbb{K}\mathrm{G}(\Sigma, \mathrm{b})$ the $\mathbb{K}$-vector space freely generated by it.*

3. *A <u>null graph</u> on Σ is an element $\sum_i \lambda_i \Gamma_i$ of $\mathbb{K}\mathrm{G}(\Sigma, \mathrm{b})$ such that there exists an embedding $\varphi\colon D \hookrightarrow \mathrm{int}(\Sigma)$ of the standard disk D to the interior of Σ that satisfies the following requirements:*

   - *the circle $\varphi(\partial D)$ does not contain any vertex of any of the graphs Γ_i;*
   - *any intersection of $\varphi(\partial D)$ and an edge of any of the graphs Γ_i is transversal;*
   - *on the complement $\Sigma \setminus \varphi(D)$ all graphs Γ_i coincide;*
   - *and the values of the graphs pulled back by φ sum up to zero, $\sum_i \lambda_i \langle \Gamma_i \cap \varphi(D) \rangle_D = 0$. Here, we use the evaluation explained in remark 5.1.5 on the disc D.*

4. *The (bare)* string-net space $\mathrm{SN}(\Sigma, \mathrm{b})$ *is the quotient*

$$\mathrm{SN}(\Sigma, \mathrm{b}) := \mathbb{K}\mathrm{G}(\Sigma, \mathrm{b}) / \mathrm{N}(\Sigma, \mathrm{b}) \, ,$$

   where $\mathrm{N}(\Sigma, \mathrm{b})$ is the subspace of $\mathbb{K}\mathrm{G}(\Sigma, \mathrm{b})$ spanned by all null graphs on Σ. We call elements of $\mathrm{SN}(\Sigma, \mathrm{b})$ string-nets.

Examples 5.3.2.

1. For any spherical category, the string-net space associated to the sphere S^2 is one-dimensional and spanned by the empty graph.

2. Assume for simplicity that $\mathcal{C}$ is a spherical fusion category. For the torus T^2, we find

$$\mathrm{SN}(T^2) = \oplus_{X,Y} \mathrm{Hom}(X, Y \otimes X \otimes {}^\vee Y) \ ,$$

   where the sum is over isomorphism classes of simple objects of $\mathcal{C}$.

   Taking $\mathcal{C} = \mathrm{vect}_{\mathbb{C}}\mathbb{Z}_2$, we obtain a four-dimensional vector space. This shows that the dimension of the string-net space is sensitive to the topology of Σ.

Remark 5.3.3.

For a three-dimensional topological field theory, we indeed need to associate vector spaces to surfaces. We do not discuss arbitrary three-manifolds with boundary, representing a general cobordism. We rather observe that any diffeomorphism φ of Σ gives a cobordism $\Sigma \to \Sigma$, namely the cylinder $\Sigma \times [0,1]$ with identification

$$\overline{\Sigma} \sqcup \Sigma \xrightarrow{\mathrm{id} \sqcup \varphi} \Sigma \times [0,1] \ .$$

We call this a φ-twisted cylinder. Mapping class group elements are isotopy classes of diffeomorphisms. Diffeomorphisms act on string-nets, isotopic diffeomorphisms map a string-net to two string-nets that differ by a null graph. Hence, the mapping class group acts *geometrically* on string-nets.

We therefore introduce a symmetric monoidal subcategory $\mathrm{Cob}_{2+\epsilon,2}$ of $\mathrm{Cob}_{3,2}$ that contains as morphisms only (classes of) twisted cylinders. The string-net construction then provides for any pivotal finite tensor category $\mathcal{C}$ a symmetric monoidal functor

$$\mathrm{Cob}_{2+\epsilon,2} \to \mathrm{vect} \ \ .$$

It can be shown [B22] that if $\mathcal{C}$ is $\mathbb{C}$-linear and semisimple, i.e. a pivotal complex fusion category, this functor extends to a three-dimensional topological field theory

$$\mathrm{tft}_{\mathcal{C}} : \quad \mathrm{Cob}_{3,2} \to \mathrm{vect}$$

(The construction uses ideas from Morse theory to build a a three-manifold.)

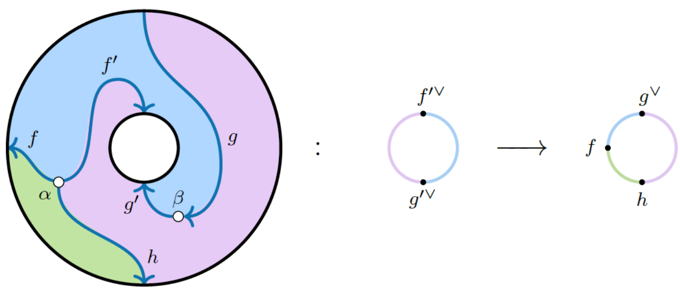
Here, we extend the construction to one-dimensional manifolds.

Definition 5.3.4
Let $\mathcal{C}$ be a strictly pivotal category and ℓ a closed oriented 1-manifold. If ℓ is non-empty, the cylinder category $\mathrm{Cyl}(\mathcal{C}, \ell)$ *for $\mathcal{C}$ over ℓ is the following category:*

- *An object of $\mathrm{Cyl}(\mathcal{C}, \ell)$ is a $\mathcal{C}$-boundary datum on ℓ.*
- *A morphism of $\mathrm{Cyl}(\mathcal{C}, \ell)$ between two boundary data is given by a string-net on the cylinder $\ell \times I$ that matches the boundary data at $\ell \times \{0\}$ and $\ell \times \{1\}$.*
- *The composition of morphisms is given by the concatenation of string-nets.*

For the empty 1-manifold $\emptyset$, we set $\mathrm{Cyl}(\mathcal{C}, \emptyset) :=$ vect.

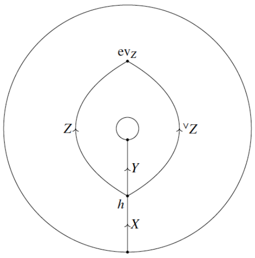
Example 5.3.5.
For instance, for any choice of α and β,



is a morphism in $\mathrm{Cyl}(\mathcal{C}, S^1)$.

Remarks 5.3.6.

1. For the circle $\ell = S^1$, we describe the category more explicity. Using the equivalence relation in the definition of string-nets, any object is isomorphic to an object given by a boundary datum with one point. Moreover, any morphism can be brought to the following standard form:

   Seeing $X, Y \in \mathcal{C}$ as elements in the cylinder category, we find, if $\mathcal{C}$ is a spherical fusion category

   $$\mathrm{Hom}_{\mathrm{Cyl}(\mathcal{C},S^1)}(X,Y) = \oplus_Z \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y \otimes {}^\vee Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \oplus_Z Z \otimes Y \otimes {}^\vee Z)$$

   where the sum is over isomorphism classes of simple objects of $\mathcal{C}$.

2. Now recall the forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. It has a left adjoint $I \dashv U$ which for the case when $\mathcal{C}$ is a fusion category gives the object

   $$\oplus_Z Z \otimes c \otimes {}^\vee Z$$

   of $\mathcal{C}$ with a certain half-braiding. Thus

   $$UI(c) = \oplus_Z Z \otimes c \otimes {}^\vee Z \ .$$

   We thus find for the cylinder category

   $$\mathrm{Hom}_{\mathrm{Cyl}(\mathcal{C},S^1)}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, UI(Y)) \cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(IX, IY) \ .$$

   We have thus recovered (part of the) Drinfeld center from the string-net construction; for more information, we refer to [Ki11, Section 6].

Example 5.3.7.
Let $\mathcal{C}$ be a monoidal category. We then obtain a bicategory $B\mathcal{C}$ with a single object and endomorphism category $\mathcal{C}$. This is a higher-categorical generalization of the construction that obtains from an associative monoid M a category with a single object $*$ and $\mathrm{Hom}(*,*) = M$.

These results suggest to extend the idea of topological field theories and to consider (at least) three-layered structures. For a three-dimensional extended topological field theory, we want to associate categories to one-manifolds. To this end, we have to go beyond categories, and our topological field theory should take values in Cat, consisting of (Categories, functors, natural transformations). This is a bicategory:

Definition 5.3.8
A bicategory $\mathcal{B}$ consists of the following data subject to the following axioms. The data are

- *A class* ob $\mathcal{B}$ *with elements $A, B, \ldots$ which we depict as 0-cells.*
- *Categories* $\mathrm{Hom}(A,B)$ *for each pair $A, B \in$ ob $\mathcal{B}$, whose objects f, g we call a 1-cells or 1-morphisms and whose arrows $\alpha, \beta, \ldots$ we call 2-cells or 2-morphisms.*

- *Composition functors for any triple $A, B, C \in \mathrm{ob}\,\mathcal{B}$*

$$\begin{array}{rrcl} c_{ABC}: & \mathrm{Hom}(B,C)\times\mathrm{Hom}(A,B) & \to & \mathrm{Hom}(A,C) \\ & (g,f) & \mapsto & g\circ f \\ & (\beta,\alpha) & \mapsto & \beta\circ\alpha \end{array}$$

and an identity functor $\mathrm{Id}_A : 1 := *//\mathrm{id}_* \to \mathrm{Hom}(A,A)$ for any object $A \in \mathrm{ob}\,\mathcal{B}$. Note that this gives for each object A of a bicategory an identity 1-morphism Id_A and an identity 2-morphism $\mathrm{Id}_A \to \mathrm{Id}_A$.

- *Natural isomorphisms a, r, l of functors expressing associativity:*

$$\begin{tikzcd} \mathrm{Hom}(C,D)\times\mathrm{Hom}(B,C)\times\mathrm{Hom}(A,B) \arrow[r, "\mathrm{id}\times c_{ABC}"] \arrow[d, "c_{BCD}\times\mathrm{id}"'] & \mathrm{Hom}(C,D)\times\mathrm{Hom}(A,C) \arrow[d, "c_{ACD}"] \\ \mathrm{Hom}(B,D)\times\mathrm{Hom}(A,B); \arrow[r, "c_{ABD}"'] \arrow[ur, Rightarrow, "a_{ABCD}"] & \mathrm{Hom}(A,D) \end{tikzcd}$$

and unitality

$$\begin{tikzcd} \mathrm{Hom}(A,B)\times 1 \arrow[d, "1\times\mathrm{Id}_A"] \arrow[dr, "\sim"] & \\ \mathrm{Hom}(A,B)\times\mathrm{Hom}(A,A); \arrow[r, "c_{AAB}"'] \arrow[r, phantom, shift left=3, "\Rightarrow r_{AB}"] & \mathrm{Hom}(A,B) \end{tikzcd}$$

and

$$\begin{tikzcd} 1\times\mathrm{Hom}(A,B) \arrow[d, "\mathrm{Id}_B\times 1"] \arrow[dr, "\sim"] & \\ \mathrm{Hom}(B,B)\times\mathrm{Hom}(A,B); \arrow[r, "c_{ABB}"'] \arrow[r, phantom, shift left=3, "\Rightarrow l_{AB}"] & \mathrm{Hom}(A,B) \end{tikzcd}$$

thus 2-cells

$$\begin{array}{rccc} a_{hgf}: & (hg)f & \tilde{\to} & h(gf) \\ r_f: & f\circ I_A & \tilde{\to} & f \\ l_f: & I_B\circ f & \tilde{\to} & f. \end{array}$$

Axioms: the following diagrams commute:

- *Pentagon diagrams*

$$\begin{tikzcd} & (kh)(gf) \arrow[dr, "\alpha"] & \\ ((kh)g)f \arrow[ur, "\alpha"] \arrow[d, "\alpha\mathrm{id}"'] & & k(h(gf)) \\ (k(hg))f \arrow[rr, "\alpha"'] & & k((hg)f) \arrow[u, "\mathrm{id}\alpha"'] \end{tikzcd}$$

- *Triangle diagrams*

$$\begin{tikzcd} (g\mathrm{id})f \arrow[rr, "a"] \arrow[dr, "r*1"'] & & g(\mathrm{id}f) \arrow[dl, "1*l"] \\ & gf & \end{tikzcd}$$

We should now proceed and introduce the notion of a (symmetric) monoidal bicategory. These notions encode a huge amount of structure. The following examples are relevant for us:

Definition 5.3.9
$\mathrm{Cob}_{3,2,1}$ *and* $\mathrm{Cob}_{2+\epsilon,2,1}$ *are the following two symmetric monoidal bicategories:*

- *Objects are compact, closed, oriented 1-manifolds S.*
- *1-Morphisms are 2-dimensional, compact, oriented collared cobordisms $S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$.*
- *2-Morphisms*
  - *of* $\mathrm{Cob}_{3,2,1}$ *are generated by diffeomorphisms of cobordisms fixing the collar and 3-dimensional collared, oriented cobordisms with corners M, up to diffeomorphisms preserving the orientation and boundary.*
  - *of* $\mathrm{Cob}_{2+\epsilon,2,1}$ *are twisted cylinders over surfaces.*
- *Composition of 1-morphisms is by gluing along collars.*
- *The monoidal structure is given by disjoint union with the empty set $\emptyset$ as the monoidal unit.*

We are now ready to present the definition of an (once) extended topological field theory and of a modular functor:

Definition 5.3.10
Let $\mathcal{S}$ be any symmetric monoidal bicategory.

1. *A <u>once extended topological field theory</u> with values in the target category $\mathcal{S}$ is a symmetric monoidal 2-functor*
$$\mathrm{tft}: \quad \mathrm{Cob}_{3,2,1} \to \mathcal{S} \ .$$
2. *A <u>modular functor</u> with values in $\mathcal{S}$ is a symmetric monoidal 2-functor*
$$\mathrm{MF}: \quad \mathrm{Cob}_{2+\epsilon,2,1} \to \mathcal{S} \ .$$

An important target category $\mathcal{S}$ is the following symmetric monoidal bicategory:

Definition 5.3.11
Let $\mathbb{K}$ be a field. The bicategory Lex *is defined as follows:*

1. *Objects are $\mathbb{K}$-linear finite tensor categories.*
2. *1-morphisms are left exact $\mathbb{K}$-linear functors.*
3. *2-morphisms are $\mathbb{K}$-linear transformations.*
4. *The <u>Deligne tensor product</u> $\boxtimes$ endows this bicategory with the structure of a symmetric monoidal bicategory. The category of finite-dimensional $\mathbb{K}$-vector spaces is the monoidal unit.*

Remarks 5.3.12.

1. The full subbicategory whose objects are semisimple finite categories is called the bicategory $2\mathrm{vect}(\mathbb{K})$ of 2-vector spaces.

2. One can also consider right exact functors as 1-morphisms and obtains another symmetric monoidal bicategory.

For an account of extended topological field theories, see [L, Section 1.2] and for an informal account see [NS]. We justify the terminology *extended* topological field theory.

Remark 5.3.13.

1. Consider an extended topological field theory with values in the bicategory of 2-vector space. Thus we have as a zeroth layer finitely semisimple categories associated to closed oriented 1-manifolds. At the first layer, we will have not only closed surfaces, but also 2-manifolds with boundary. After having chosen an object for each boundary circle, we get a vector space which depends functorially on the choice of objects. On the third level, we have three-manifolds with corners relating the 2-manifolds with boundaries. In particular, we obtain invariants of knots and links in three-manifolds generalizing the constructions of the previous subsection and thus to representations of braid groups.

2. We note that the monoidal 2-functor tft has to send the monoidal unit $\emptyset$ in $\mathrm{Cob}_{3,2,1}$ to the monoidal unit which is the category $\mathrm{vect}(\mathbb{K})$. The 2-functor tft restricts to a monoidal functor $\mathrm{tft}|_\emptyset$ from the endomorphisms of $\emptyset$ in $\mathrm{Cob}_{1,2,3}$ to the endomorphisms of $\mathrm{vect}(\mathbb{K})$.

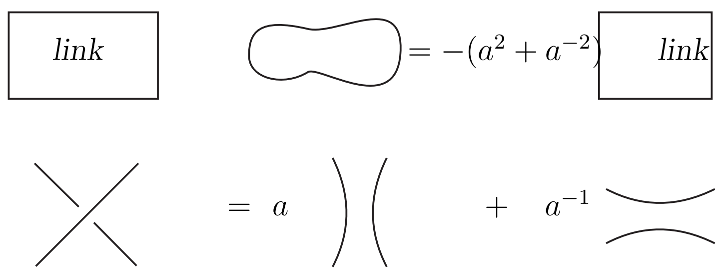
3. It follows directly from the definition that
$$\mathrm{End}_{\mathrm{Cob}_{1,2,3}}(\emptyset) \cong \mathrm{Cob}_{3,2} \ .$$
Using the fact that the morphisms are additive (which follows from $\mathbb{K}$-linearity of functors in the definition), it is also easy to see that the equivalence of categories $\mathrm{End}_{2\mathrm{vect}(\mathbb{K})}\big(\mathrm{vect}(\mathbb{K})\big) \cong \mathrm{vect}(\mathbb{K})$ holds. This equivalence maps a $\mathbb{K}$-linear functor $\phi \in \mathrm{End}_{2\mathrm{vect}(\mathbb{K})}(\mathrm{vect}_\mathbb{K})$ to $\phi(\mathbb{K}) \in \mathrm{vect}(\mathbb{K})$.

4. We have seen that for the spherical fusion category $\mathcal{C} := \mathrm{vect}_\mathbb{C}(\mathbb{Z}_2)$, the string-net space is four-dimensional. The Drinfeld center $\mathcal{Z}(\mathcal{C})$ has four simple objects. Indeed, $\mathrm{SN}(T^2)$ has a basis labelled by simple objects of $\mathcal{Z}(\mathcal{C})$.

We add some comments which relate a famous link invariant to four-dimensional topological field theories:

Definition 5.3.14

Fix $a \in \mathbb{C}^\times$. Let $E(a)$ be the complex vector space, freely generated by all link diagrams up to isotopy of $\mathbb{R}^2$ modulo the two Kauffman relations:



The vector space $E(a)$ is called the <u>*skein module*</u>*. The class of a link diagram D determines a vector $\langle D\rangle(a) \in E(a)$.*

Theorem 5.3.15.

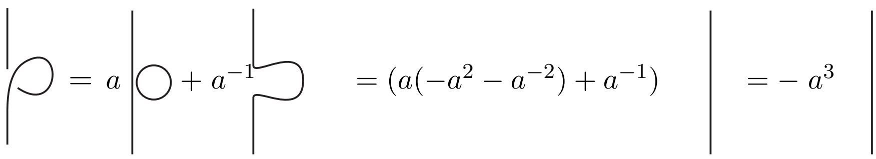
1. The skein module is one-dimensional, $\dim_{\mathbb{C}} E(a) = 1$. A generator is given by the skein class $\langle\emptyset\rangle$ of the empty knot which we use to identify it with $\mathbb{C}$.

2. The skein class of a link is invariant under the Reidemeister moves $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$ and thus an isotopy invariant of links, cf. remark 5.2.8.4.

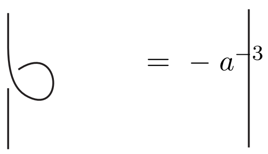
Proof.

1. The Kauffman relations are sufficient to unknot any knot. The unknot is the identified with the complex number $-a^2 - a^{-2}$.

2. To show invariance under the Reidemeister move Ω_0 from remark 5.2.8.4, we compute:

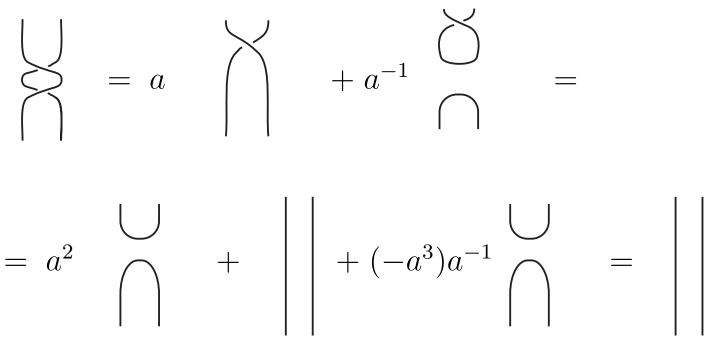
   

   In a similar way, we show for the opposite curl:

   We conclude invariance under the Reidemeister move $\Omega_0^{\pm 1}$.

3. Invariance under the Reidemeister move $\Omega_2^{\pm 1}$ is shown by a similar computation:

   In the third identity, we used the result of 2. for the positive curl. We leave it to the reader to show invariance under the Reidemeister move $\Omega_3^{\pm 1}$.

□

Remark 5.3.16.
The string-net construction can be generalized to higher dimensions. In particular, given a ribbon fusion category, one can develop a graphical calculus on full three-balls. One uses projections to the plane which yields link diagrams which can be evaluated by proposition 5.2.12. For suitable values of the parameter a, the skein module is then the vector space associated to S^3 for a ribbon fusion category associated to $sl(2)$. (For this category, labels can be eliminated since the decomposition of tensor products the defining two-dimensional representation contain all simple objects.)

Definition 5.3.17
Let $a \in \mathbb{C}^\times$ be such that $a^2 + a^{-2} \neq 0$. Let L be a framed link. Choose any link diagram D representing L. Then the bracket polynomial of L is defined by

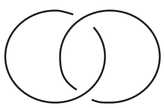
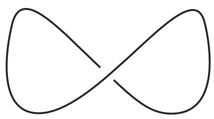
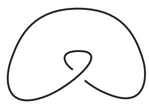
$$\langle L \rangle(a) = \frac{\langle D \rangle(a)}{-a^2 - a^{-2}} \, .$$

This is a Laurent polynomial in a. This function of a is an isotopy invariant of the link L.

Examples 5.3.18.

1. It is obvious that the unknot with trivial framing has bracket polynomial $\langle L \rangle(a) = 1$.

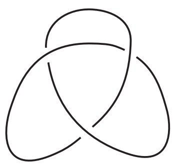
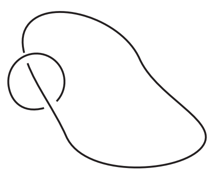
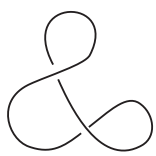
2. We obtain for the Hopf link by applying the Kauffman relation at the upper braiding the following element of $E(a)$:


$= a$

$+a^{-1}$


$$= a(-a^{+3}) + a^{-1}(-a^{-3}) = -a^4 - a^{-4}$$

Here we used the results for the positive and negative curl obtained in the proof of theorem 5.3.15.

3. We obtain for the trefoil knot by applying the Kauffman relation to the upper right braiding the following element of $E(a)$:


$= a$

$+a^{-1}$


$$\begin{aligned} &= a(-a^{+4} - a^{-4}) + a^{-1}(a^{-3})^2 \\ &= -a^5 - a^{-3} + a^{-7} \end{aligned}$$

One should check that this Laurent polynomial is again divisible by $-a^2 - a^{-2}$. Here we used in the second equality the results for the Hopf link and the curls. We remark that the invariant of the trefoil knot and the unknot are different. Hence the trefoil knot is not isotopic to the trivial knot. One can show that for the mirror image $\overline{L}$ of a link L, we have $\langle \overline{L} \rangle(a) = \langle L \rangle(a^{-1})$. We conclude that the trefoil knot is not isotopic to its mirror.

In passing, we mention:

Definition 5.3.19
Let L be an oriented link in $\mathbb{R}^3$ without framing. Choose a framing for each component L_i such that the self-linking number of L_i is

$$-\sum_{j\neq i} lk(L_i, L_j)$$

to obtain a framed link L^f. The Jones polynomial [3] *for L is the Laurent polynomial*

$$V_L(q) = \langle L^f \rangle (q^{-1/2}) \, .$$

5.4 State sum TFT

We now discuss the construction of an extended three-dimensional topological field theory in the sense of definition 5.3.10. Our input is a spherical fusion category over the field $\mathbb{C}$ of complex numbers. Our exposition closely follows [BK1]. In particular, we have to achieve the following goals:

- To a closed oriented three-manifolds we want to assign an invariant with values in $\mathrm{End}_{\mathcal{C}}(\mathbb{I})$. This invariant should be a topological invariant.
- To closed oriented two-manifolds, we want to assign a finite-dimensional complex vector space. To a three-manifold M with boundary representing a cobordism $\partial_- M \xrightarrow{M} \partial_+ M$, we want to assign a linear map. This expresses a locality property of our invariants.
- We want to obtain an extended topological field theory and thus assign categories to closed oriented 1-manifolds.

The following proposition will be used:

Proposition 5.4.1. [ENO, Theorem 2.3]
If $\mathcal{C}$ is a spherical fusion category over the field $\mathbb{C}$, then the so-called global dimension of $\mathcal{C}$ is non-zero:

$$\mathcal{D}^2 := \sum_{i\in I} (\dim V_i)^2 \neq 0 \; .$$

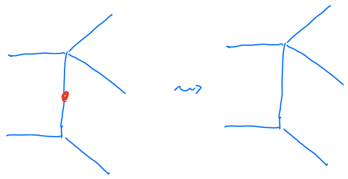
(We do not suppose that a square root $\mathcal{D}$ of the right hand side has been chosen; the notation will just be convenient later.)

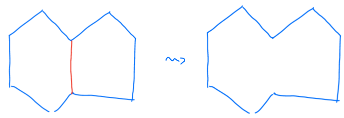
Observation 5.4.2.

1. All manifolds are compact, oriented and piecewise linear. We fix as a combinatorial datum a polytope decomposition Δ, in which we allow individual cells to be arbitrary polytopes (rather than just simplices). Moreover, we allow the attaching maps to identify some of the boundary points, for example gluing polytopes so that some of the vertices coincide. On the other hand, we do not want to consider arbitrary polytope decompositions, since it would make describing the elementary moves between two such decompositions more complicated. We call a piecewise linear manifold M with a polytope decomposition Δ a *combinatorial manifold.*

[3] Vaughan F. R. Jones, 1952-2020, Field medal 1990 with Drinfeld, Mori and Witten.

2. The moves are then:



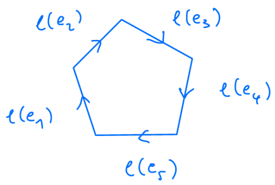


(M1): Removing a (regular) vertex (M2) Removing a (regular) edge
(M3) Removing a (regular) 2-cell

Observation 5.4.3.
Let $\mathcal{C}$ be a spherical fusion category over the field of complex numbers.

1. A (simple) labeling l of a combinatorial manifold (M, Δ) is a map that assigns to each edge e of Δ a (simple) object of $\mathcal{C}$ such that $l(\overline{e}) = l(e)^*$ for the edge $\overline{e}$ with opposite orientation.

2. We assign to any 2-cell C with labeling l

   the vector space of invariant tensors

   $$H(C, l) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_1) \otimes \ldots \otimes l(e_n)) ,$$

   cf. lemma 5.1.4. Here the edges $e_1, \ldots, e_n$ of the 2-cell C are taken counterclockwise with respect to the orientation of C. We have shown that the properties of a spherical category imply:

   - Up to canonical isomorphism, the vector space $H(C, l)$ does not depend on the choice of starting point in the counterclockwise enumeration of the edges $e_1, \ldots, e_n$, cf. lemma 5.1.4.
   - To the 2-cell with the reversed orientation, we assign the vector space

     $$H(\overline{C}, l) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_n)^* \otimes \ldots l(e_1)^*)$$

     which is canonically in duality with $H(C, l)$.

3. Let now (Σ, Δ) be a combinatorial 2-manifold with labelling l. We assign to a labelled combinatorial surface (Σ, Δ, l) the vector space

   $$H(\Sigma, \Delta, l) = \bigotimes_{C \in \Delta} H(C, l) ,$$

   i.e. the tensor product over the vector spaces of invariant tensors assigned to all faces C of the polytope decomposition Δ of Σ, and then sum over all labelings by simple objects,

   $$H(\Sigma, \Delta) := \bigoplus_{l} H(\Sigma, \Delta, l) .$$

- This vector space depends on the choice of polytope decomposition Δ and is therefore *not* the vector space assigned to Σ by the topological field theory we want to construct.
- The assignment is tensorial: for a disjoint union $\Sigma_1 \sqcup \Sigma_2$ of 2-manifolds with polytope decomposition $\Delta_1 \sqcup \Delta_2$, we obtain the vector space

$$H(\Sigma_1 \sqcup \Sigma_2, \Delta_1 \sqcup \Delta_2) = H(\Sigma_1, \Delta_1) \otimes H(\Sigma_2, \Delta_2) \ .$$

- Upon change of orientation, we obtain the dual vector space

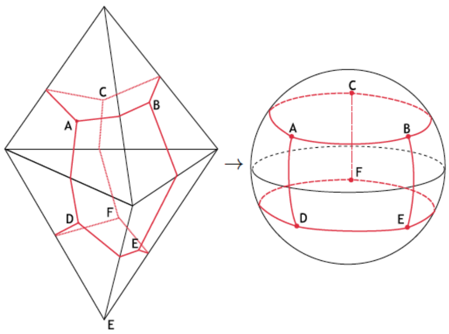
$$H(\overline{\Sigma}, \Delta) \cong H(\Sigma, \Delta)^* \ .$$

4. Our next goal is to assign to a 3-cell F with labeling l a vector

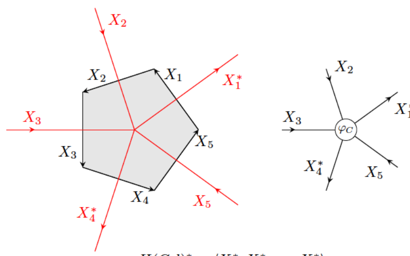
$$H(F, l) \in H(\partial F, l)$$

in the vector space associated to the boundary ∂F with the induced labelling ∂l and induced polytope decomposition $\partial\Delta$.

The boundary ∂F has the form of a sphere with an embedded graph whose surfaces are faces and thus carry vector spaces $H(C, l)$. Take the dual graph Γ on S^2



The vertices of the dual graph are the faces of the original graph and thus labelled by vector spaces $H(C, l)$ which are Hom-spaces of $\mathcal{C}$. Its edges are labeled by (simple) objects,



$\varphi_C \in H(C, l)^* = \langle X_5^*, X_4^*, \ldots, X_1^* \rangle$

Choose for every face $C \in \partial F$ an element in the dual vector space

$$\varphi_C \in H(C, l)^* \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_n)^* \otimes \ldots \otimes l(e_1)^*)$$

It defines a valid labelling of a graph Γ on the sphere S^2 so that remark 5.1.12.6 gives a number $Z(\Gamma)$. We define the vector $H(F, l) \in H(\partial F, l)$ by its values on the vectors in the dual vector space:

$$\langle H(F, l), \otimes_{C \in \partial F} \varphi_C \rangle = Z(\Gamma) \ .$$

5. We now assign to a combinatorial labeled 3-manifold (M, Δ, l) with boundary the combinatorial surface $(\partial F, \partial\Delta)$ a vector
$$H(M, \Delta, l) \in H(\partial M, \partial\Delta, \partial l) \ .$$
We note that
$$\otimes_{F\in\Delta} H(F, l) \in \otimes_F H(\partial F, \partial\Delta, \partial l) = H(\partial M, \partial l) \otimes \bigotimes_c H(c', \partial l) \otimes H(c'', \partial l)$$
where F runs over all 3-cells of M and c runs over all 2-cells in the interior of M, which appear for two faces, with opposite orientation. The associated vector spaces are thus in duality and we can contract the corresponding components in $\otimes_{F\in M} H(F, l)$ by applying the evaluation to them. We thus define
$$H(M, \Delta, l) := \mathrm{ev}\Big(\bigotimes_{F\in\Delta} H(F, l)\Big) \in H(\partial M, \partial\Delta, l)$$

6. We have finally to get rid of the labelling. This is done by a summation with weighting factors which involve dimensions, hence depend on the pivotal structure:
$$Z_{TV}(M, \Delta) := \mathcal{D}^{-2v(M)} \sum_l \Big(H(M, \Delta, l) \prod_e d_{l(e)}^{n_e}\Big)$$
where
   - the sum is taken over all equivalence classes of simple labelings l of Δ,
   - the product over e runs over the set of all (unoriented) edges of Δ
   - $\mathcal{D}$ is the dimension of the category $\mathcal{C}$ from proposition 5.4.1 and
   $$v(M) := \text{number of internal vertices of } M + \frac{1}{2}(\text{number of vertices on } \partial M)$$
   - $d_{l(e)}$ is the categorical dimension of $l(e)$ and $n_e = 1$ for an internal edge, and 1/2 for an edge in the boundary ∂M. Here, we assume that some square root has been chosen for each dimension of a simple object.

7. Consider a combinatorial 3-cobordism (M, Δ) between two combinatorial surfaces (N_1, Δ_1) and (N_2, Δ_2), i.e. a combinatorial 3-manifold (M, Δ) with boundary $\partial M = \overline{N_1} \sqcup N_2$ and the induced combinatorial structure $\partial\Delta = \Delta_1 \sqcup \Delta_2$ on the boundary. Then
$$H(\partial M, \partial\Delta) \cong H(N_1, \Delta_1)^* \otimes H(N_2, \Delta_2) \cong \mathrm{Hom}_{\mathbb{K}}(H(N_1, \Delta_1), H(N_2, \Delta_2))$$
so that we have a linear map
$$H(M, \Delta)\colon H(N_1, \Delta_1) \to H(N_2, \Delta_2) \ .$$

One now proves, using the moves in obervation 5.4.2

Theorem 5.4.4.

1. For a closed PL manifold M, the scalar $Z_{TV}(M, \Delta) \in \mathbb{K}$ does not depend on the choice of polytope decomposition Δ. We write $Z_{TV}(M)$.

2. More generally, if M is a 3-manifold with boundary and Δ, Δ' are two polytope decompositions of M that agree on the boundary, $\partial\Delta = \partial\Delta'$, then we have the equality of vectors
$$H(M,\Delta) = H(M,\Delta') \in H(\partial M, \partial\Delta) = H(\partial M, \partial\Delta') \ .$$

3. For a combinatorial 2-manifold (N,Δ), consider the linear maps associated to the cylinders
$$A_{N,\Delta} := H(N \times [0,1]) : \quad H(N,\Delta) \to H(N,\Delta) \ .$$
The composition of two cylinders is again a cylinder. Thus, as a consequence of 2, the maps are idempotents: $A^2_{N,\Delta} = A_{N,\Delta}$. If we already had a topological field theory, this should be the identity, though.

4. To a combinatorial 2-manifold (N,Δ), we therefore assign the vector space
$$Z_{TV}(N,\Delta) := \mathrm{Im}\,(A_{N,\Delta}) \subset H(N,\Delta) \ .$$
It is an invariant of PL manifolds: for different polytope decompositions, one has *canonical* isomorphisms $Z_{TV}(N,\Delta) \cong Z_{TV}(N,\Delta')$. We write $Z_{TV}(N)$.

5. We denote this vector space by $Z_{TV}(N)$. For a cobordism $N_1 \xrightarrow{M} N_2$, we denote by $Z_{TV}(M)$ the restriction of the linear map $H(M,\Delta)$ to $Z_{TV}(N)$. This defines a three-dimensional topological field theory $Z_{TV} : \mathrm{Cob}(3,2) \to \mathrm{vect}(\mathbb{K})$.

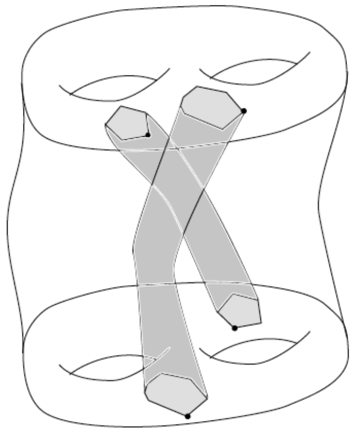
For the proof of all these statements, we refer to [BK1]; an excellent introduction that uses a Poincaré dual picture (and a different combinatorial description of manifolds) is the book [TV]. The essential step is to show that the properties of a finitely semisimple spherical category imply the independence under the three moves changing the polytope decomposition.

In our construction, we have assigned objects of a spherical fusion category to edges; no braiding on this category is required.

Observation 5.4.5.

1. We now allow surfaces with boundaries. To reduce them to closed surfaces, we glue a disc to the boundary circle and work with surfaces with marked discs instead. These discs are supposed to be faces of the triangulation and actually are faces of a new type. For later use, we mark a vertex on the boundary of the disc. We assign to a marked disc an object in the Drinfeld double $\mathcal{Z}(\mathcal{C})$.

2. The three-manifolds are now manifolds with corners. Three-manifolds with corners and surfaces with marked discs form an extended cobordism bicategory, cf. definition 5.3.9. They contain two types of tubes: open tubes, ending at the boundaries or closed tubes. They will lead to 3-cells of a new type. We suppose that all components of tubes are labelled with objects in $\mathcal{Z}(\mathcal{C})$.



3. We extend the TV invariants to such extended surfaces and cobordisms:

   (a) Define, for every labelled extended surface N, a vector space $Z_{TV}(N, \{Y_\alpha\})$ which

   - functorially depends on the colors $Y_\alpha \in \mathcal{Z}(\mathcal{C})$,
   - is functorial under homeomorphisms of extended surfaces,
   - has natural isomorphisms $Z_{TV}(\overline{N}, \{Y_\alpha^*\}) = Z_{TV}(N, \{Y_\alpha\})^*$,
   - satisfies the gluing axiom for surfaces.

   (b) Define, for any colored extended 3-cobordism M between colored extended surfaces N_1, N_2, a linear map $Z_{TV}(M)\colon Z_{TV}(N_1) \to Z_{TV}(N_2)$ so that this satisfies the gluing axiom for extended 3-manifolds.

4. We repeat the steps in the previous construction, with the following modifications:

   (a) There is now an additional type of 2-cell corresponding to an embedded disc with label $Y \in \mathcal{Z}(\mathcal{C})$. To such a 2-cell, we assign the vector space

   $$\mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, U(Y) \otimes l(e_1) \otimes \ldots \otimes l(e_n)) \ .$$

   Here we applied the forgetful functor $U: \ \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ from proposition 4.4.2. We needed to specify a point to get a linear order on the objects, because now the position of $U(Y) \in \mathcal{C}$ matters. We then continue to define vector spaces $H(N, \Delta, \{Y_\alpha\})$ as above by summing over labellings of inner edges.

   (b) There are now two different types of 3-cells: tube cells and usual cells. To assign vectors to tube cells, we use observation 5.1.12.7. Then the construction continues as above.

5. A new feature is now the fact that we have a gluing axiom for extended surfaces. We refer to [BK1, Theorem 8.5].

One can now show [Ki11, Theorem 5.1]:

Theorem 5.4.6.
Let $\mathcal{C}$ a spherical fusion category. The vector spaces assigned to a closed oriented surface Σ by the string-net construction and the Turaev-Viro construction are canonically isomorphic.

5.5 Quantum codes and pivotal tensor categories

There are two basic tasks in computing, both for classical and quantum computing:

- Storing information in a medium and transmitting information.
- Doing computations by processing information.

The first question leads to the mathematical notion of codes, the second to the notion of gates. We start our discussion with classical computing.

5.5.1 Classical codes

Implicitly, assumptions made on storage devices and manipulation of information in classical information theory is based on classical physics, as opposed to quantum mechanics.

Information is stored in the form of binary numbers, hence in terms of elements of the standard vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2 = \{0,1\}$ of two elements. (Note that the standard basis of $\mathbb{F}_2^n$ plays a distinguished role.) We identify the elements of $\mathbb{F}_2 = \{0,1\}$ with either on/off or with the truth values $0 =$ false and $1 =$ true. If we are dealing with an element of $\mathbb{F}_2^n$, we say that we have n bits of information.

For storing and transmitting information, it is important that errors occurring in the transmission or by the dynamics of the storage device can be corrected. For this reason, only a subset $C \subset \mathbb{F}_2^n$ should correspond to valid information.

Definition 5.5.1

1. *A subset $C \subset (\mathbb{F}_2)^n$ is called a code. The natural number n is called the length of the code. One says that a code word $c \in C$ is composed of n bits.*
2. *A code $C \subset (\mathbb{F}_2)^n$ is called linear, if C is a vector subspace. Then $\dim_{\mathbb{F}_2} C =: k$ is called the dimension of the code.*

One can also allow instead of the field $\mathbb{F}_2$ an arbitrary finite field. We will not discuss this in more detail.

To deal with error correction, one defines:

Definition 5.5.2
Let $\mathbb{K} = \mathbb{F}_2$ and $V = \mathbb{K}^n$. The map

$$d_H : V \times V \to \mathbb{N}$$
$$d_H(v,w) := |\{j \in \{1,\dots,n\} |\, v_j \neq w_j\}|$$

is called Hemming distance. It equals the number of components (bits) in which the two code words v and w differ.

Lemma 5.5.3.
The Hemming distance has the following properties:

1. $d_H(v,w) \geq 0$ for all $v,w \in V$ and $d_H(v,w) = 0$, if and only $v = w$
2. $d_H(v,w) = d_H(w,v)$ for all $v,w \in V$ (symmetry)
3. $d_H(u,w) \leq d_H(u,v) + d_H(v,w)$ for all $u,v,w \in V$ (triangle inequality)

4. $d_H(v, w) = d_H(v + u, w + u)$ for all $u, v, w \in V$ (translation invariance)

Definition 5.5.4
For $\lambda \in \mathbb{N}$, a subset $C \subset (\mathbb{F}_2)^n$ is called a λ–error correcting code, if

$$d_H(u, v) \geq 2\lambda + 1 \quad \text{for all } u, v \in C \quad \text{with } u \neq v \; .$$

The reason for this name is the following

Lemma 5.5.5.
Let $C \subset V$ be a λ-error correcting code. Then for any $v \in V$, there is at most one $w \in C$ with $d_H(v, w) \leq \lambda$.

Proof.
Suppose we have $w_1, w_2 \in C$ with $d_H(v, w_i) \leq \lambda$ for $i = 1, 2$. Then the triangle inequality yields

$$d_H(w_1, w_2) \leq d_H(w_1, v) + d_H(v, w_2) \leq 2\lambda \; .$$

Since the code C is supposed to be λ-error correcting, we have $w_1 = w_2$. □

Remarks 5.5.6.

1. It is important to keep in mind the relative situation: a code C is a subspace of $\mathbb{F}_2^n$. The Hemming distance gives an indication to what extent the subspace C of code words is spread out in V.
2. We say that information is stored in the code, if an element $c \in C$ is selected.
3. If $C \subset \mathbb{F}_2^n$, we say that a codeword of C is composed of n bits. If C is a linear code with $\dim_{\mathbb{F}_2} C = k$, we refer to a $[n, k]$ code. Denote by

   $$d := \min_{c \in C \setminus \{0\}} d_H(c, 0)$$

   the minimal distance of a code. We refer to an $[n, k, d]$ code. In practice, the length n of the code has to be kept small, because this causes costs for storing and transmitting. The minimal distance d has to be big, since by lemma 5.5.5 this allows to many correct errors. The dimension k of the code has to be big enough to allow enough code words. From elementary linear algebra, one derives the singleton bound

   $$k + d \leq n + 1$$

   which shows that these goals are in competition.

Lemma 5.5.7.
Let q be the size of the alphabet of a code C, e.g. $q = 2$ for a code over $\mathbb{F}_2$. Let n be the length of the code and d the minimum distance. Then we have

$$|C| \leq q^{n-d+1} \; .$$

In particular, for a linear code over the field $\mathbb{F}_q$ of dimension k, we have $|C| = q^k$ so that we obtain from $|C| = q^k \leq q^{n-d+1}$ the singleton bound above.

Proof.
Suppose that we have $|C| > q^{n-d+1}$. Consider the first $n-d+1$ letters of the code words. The offer q^{n-d+1} possibilities for the entries. Since we have more code words in C, by the pigeon hole principle, there are two different code words $c_1, c_2 \in C$, $c_1 \neq c_2$ whose first $n-d+1$ letters coincide. Hence, they can differ in at most $n-(n-d+1) = d-1$ letters, which contradicts the assumption that the distance is d. □

Remarks 5.5.8.

1. Classical storage devices are typically localized, either in space (e.g. an electron or a nuclear spin) or in momentum space (e.g. a photon polarization).
2. Many storage devices are magnetic, i.e. a collection of coupled spins. The Hamiltonian is such that it favours the alignment of spins. So if one spin is kicked out by thermal fluctuation, the Hamiltonian tends to push it back in the right position. Thus errors in the storage device are corrected by the dynamics of the system. This idea will also enter in the construction of quantum codes.

5.5.2 Classical gates

To process information, we need logical gates: A logical gate takes as an input n bits of information an yields m bits as an output.

Definition 5.5.9
Let $\mathbb{K} = \mathbb{F}_2$.

1. *A gate is map $f : \mathbb{K}^n \to \mathbb{K}^m$. Typically, one requires a gate to act non-trivially only on few, two or three) bits, i.e. to act as the identity on all except for a few summands of $\mathbb{K}^n$.*
2. *A gate is called linear, if the map f is $\mathbb{K}$-linear.*
3. *If the map f is invertible, the gate is called reversible.*
4. *A finite set of gates is called a library of gates. One then applies to $\mathbb{F}_2^n$ a sequence of gates in the library acting on any subset of summands in $\mathbb{F}^n$ and as the identity elsewhere. The composition of such maps is called a circuit.*
5. *A library of gates is called universal, for any Boolean function $f(x_1, x_2, ..., x_m)$, there is a circuit consisting of gates in the library which takes $x_1, x_2, \ldots, x_m$ and some extra bits set to 0 or 1 and outputs $x_1, x_2, \ldots, x_m, f(x_1, x_2, \ldots, x_m)$, and some extra bits (called garbage). Essentially, this means that one can use the gates in the library to build systems that perform any desired Boolean function computation.*

We wish to use gates to implement the basic Boolean operations:

Examples 5.5.10.

1. Basic gates include negation NOT, AND and OR:

A	$\neg A$
t	f
f	t

A	B	$A \wedge B$
t	t	t
t	f	f
f	t	f
f	f	f

A	B	$A \vee B$
t	t	t
t	f	t
f	t	t
f	f	f

These are gates acting on one bit resp. mapping two bits to one bit.

2. Also in use are the following gates acting on two bits:

A	B	NAND
t	t	f
t	f	t
f	t	t
f	f	t

A	B	NOR
t	t	f
t	f	f
f	t	f
f	f	t

A	B	XOR
t	t	f
t	f	t
f	t	t
f	f	f

3. It is an important theoretical question whether a library of gates is universal. For example, the NAND gate is universal:

   - To get the NOT gate, double the input and feed it into a NAND gate.
   - To get the AND gate, take a NAND gate, followed by a NOT gate, which can be constructed from a NAND gate.
   - To get an OR gate, use de Morgan's law: apply NOT gates to both inputs and feed it into a NAND gate.

4. The Toffoli gate is the linear map

$$T: \mathbb{F}^3 \to \mathbb{F}^3$$

given by the truth table

INPUT			OUTPUT		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

It is the identity on the first two bits. If the first two bits are both one, then the last bit is flipped. It thus acts

$$\begin{aligned} \mathbb{F}^3 &\to \mathbb{F}^3 \\ (a, b, c) &\mapsto (a, b, c + ab) \end{aligned}$$

It is not linear, but universal: one can use Toffoli gates to build systems that will perform any desired boolean function computation in a reversible manner.

5. To add two bits A and B, double the bits and feed them into a XOR gate to get the last digit S of the sum and into an AND gate to get a carry-on bit C:

A	B	S=XOR	C=AND
t=1	t=1	f=0	t=1
t=1	f=0	t=1	f=0
f=0	t=1	t=1	f=0
f=0	f=0	f=0	f=0

In such a way, one realizes the arithmetic operations on natural numbers.

5.5.3 Codes and quantum computing

Quantum computation is based on quantum mechanical systems. Now states can be superposed, which leads to a richer structure. On the other hand, the uncertainty principle introduces new limitations, e.g. quantum information cannot be copied: there is no complete set of observables characterizing a state completely that can be measured simultaneously. Moreover, there is no canonical linear map $V \to V \otimes V$ that can be defined without choosing a basis first.

We use the simplest possible quantum mechanical system: The state of the system is now not a vector in $\mathbb{F}_2^n$, but rather a vector in the following space: denote by $H = \mathbb{C}[\mathbb{Z}_2]$ the *complex* group algebra of the cyclic group $\mathbb{Z}_2$. It will be essential that this is a finite-dimensional semisimple Hopf algebra H with a two-sided integral. As a vector space, $H \cong \mathbb{C}^2$, with a selected basis. We can interpret this system as a non-interacting spin 1/2-particle with basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$.

The analogue of the ambient vector space $\mathbb{F}_2^n$ is the tensor power $V := H^{\otimes n}$ which we can think of as n coupled spins. A quantum code is a linear subspace $C \subset H^{\otimes n}$ on which the dynamics should afford an error correction. We will say that a code vector $v \in C$ is composed of n qubits. We should mention that H has a natural unitary scalar product by declaring the vectors of the canonical basis to be orthonormal.

To get a framework for quantum computing, we need to set up:

- Codes, i.e. interesting subspaces of $H^{\otimes n}$. To make quantum computing fault tolerant, these subspaces should have special properties. In particular, in a physical realization, the dynamics of the system should suppress errors.
- Gates, i.e. unitary operators acting on $H^{\otimes n}$ that preserve these subspaces.

5.5.4 Quantum gates

First let us discuss quantum gates: for quantum computation, we need unitary operators $H^{\otimes n} \to H^{\otimes n}$ to be realized by some time evolution. Unitarity implies reversibility.

Definition 5.5.11

1. *A quantum gate on $H^{\otimes n}$ is a unitary map $H^{\otimes n} \to H^{\otimes n}$ that acts as the identity on at least $n-2$ tensorands.*
2. *Consider a fixed finite set $\{U_i\}_{i\in I}$ of quantum gates, i.e. $U_i \in \mathrm{U}(H)$ or $U_i \in \mathrm{U}(H \otimes H)$, called a library of quantum gates. Denote by $U_i^{\alpha\beta}$ the gate U_i acting on the α and β tensorand resp. U_i^α acting on the α tensorand of H^n. A quantum circuit based on this library is a finite product of U_i^α and $U_i^{\alpha\beta}$. It is a unitary endomorphism of $H^{\otimes n}$.*
3. *A library of quantum gates is called universal, if for any n, the subgroup of $\mathrm{U}(H^{\otimes n})$ generated by all circuits is dense.*

Examples 5.5.12.

1. An important example of a gate is the CNOT gate (controlled not gate) which acts on two qubits: $H^{\otimes 2} \to H^{\otimes 2}$. The CNOT gate flips the second qubit (called the target qubit), if and only if the first qubit (the control qubit) is 1. Here we write $1 = |\uparrow\rangle$ and $0 = |\downarrow\rangle$.

Before		After	
Control	Target	Control	Target
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

The resulting value of the second qubit corresponds to the result of a classical XOR gate while the control qubit is unchanged.

An experimental realization of the CNOT gate was afforded by a single Beryllium ion in a trap already in 1995 with a reliability of 90%. The two qubits were encoded into an optical state and into the vibrational state of the ion.

2. The relative phase gate $H \to H$ acting on one qubit, a popular choice of which is in the selected basis $|\uparrow\rangle, |\downarrow\rangle$:
$$\begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi \mathrm{i}/5) \end{pmatrix}$$
Similarly, the three Pauli matrices give rise to so-called Pauli gates acting on a single qubit.

3. The library consisting of the CNOT gate and the relative phase gate can be shown to be universal.

4. A universal gate is the Deutsch gate which depends on an angular parameter θ
$$\begin{array}{rcl} H^3 & \to & H^3 \\ (a,b,c) & \mapsto & \begin{cases} \mathrm{i}\cos\theta(a,b,c) + \sin\theta(a,b,1-c) & \text{if} \quad a=b=1 \\ (a,b,c) \text{ else} \end{cases} \end{array}$$
For $\theta = \frac{\pi}{2}$, we recover the classical Toffoli gate. This is taken as an argument that all operations possible in classical computing are possible in quantum computing.

5.5.5 Quantum codes

We can now define quantum codes. For a general reference, see [FKLW].

Definition 5.5.13
Denote by H again the complex group algebra of $\mathbb{Z}_2$ which is a Hopf algebra. We call a tensor product $V := H^{\otimes n}$ a discrete quantum medium. (Think of a system composed of n spin $1/2$ particles.)

1. *A quantum code is a linear subspace $W \subset V$ of a quantum medium V. Sometimes, a quantum code is also called a protected space.*

2. *Let $0 \le k \le n$. A k-local operator is a linear map $O : V \to V$ which is the identity on $n-k$ tensorands of V. (By definition 5.5.11, quantum gates are thus at least 2-local.)*

3. *Denote by $\pi_W : V \to W$ the orthogonal projection. A quantum code $W \subset V$ is called a k-code, if the linear operator*
$$\pi_W \circ O : W \to W$$
is multiplication by a scalar for any k-local operator O.

One can show the following analogue of a lemma 5.5.5:

Lemma 5.5.14.
If W is a k-code, then information cannot be degraded from errors operating on less than $\frac{k}{2}$ of the n particles.

Remarks 5.5.15.

1. A first attempt to realize qubits might be to take *isolated* trapped particles, individual atoms, trapped ions or quantum dots. Such a configuration is fragile and one has to minimize any external interaction. On the other hand, external interaction is need to write and read off information.

   The idea of topological quantum computing is to use non-local degrees of freedom to produce fault tolerant subspaces. Concretely, one needs non-abelian anyons in quasi two-dimensional systems.

2. Storage devices are typically effectively two-dimensional. Thus the protected subspace should be the space of states of a three-dimensional topological field theory. Maps describing gates and circuits are obtained from colored cobordisms, i.e. three-manifolds containing links. For example, the quantum analogue of the XOR gate, the CNOT gate can be realized to arbitrary precision by braids.

3. A theorem of Freedman, Kitaev and Wang asserts that quantum computers and classical computers can perform exactly the same computations. But their efficiency is different, e.g. for problems like factoring integers into primes.

5.5.6 Topological quantum computing and Turaev-Viro models

We now present a toy model for a system providing a quantum code where the Hamiltonian describes a dynamics which tends to correct errors. It generalizes Kitaev's toric code (which is literally not suitable for quantum computing since it does not allow for universal gates for which one needs more complicated, nonabelian representations of the braid group).

Since storage devices are (quasi)two-dimensional, we take a compact oriented surface Σ on which the physical degrees of freedom are located. To get a discrete structure, take a polytope decomposition Δ of Σ.

Our input is a complex semisimple finite-dimensional $*$-Hopf algebra H. We give the definition and refer for more information to [KS97, Section 1.2.7].

Definition 5.5.16

1. *Let H be a complex algebra. A conjugate-linear map $* : H \to H$ which satisfies*

$$(x^*)^* = x \qquad \text{and} \qquad (xy)^* = y^* x^* \quad \text{for all } x, y \in H$$

   is called an involution; an algebra with an involution is called a $*$-algebra. For a $*$-algebra, we have $1_H^* = 1_H$.

2. *Let H be a coalgebra. A conjugate-linear map $* : H \to H$ which satisfies*

$$(x^*)^* = x \qquad \text{and} \qquad \Delta(x^*) = \Delta(x)^* \quad \text{for all } x \in H$$

   is called a $*$-coalgebra. Here the involution of $H \otimes H$ is defined by $(x \otimes y)^* = x^* \otimes y^*$. In a $*$-coalgebra one always has $\epsilon(x^*) = \overline{\epsilon(x)}$, where the bar denotes complex conjugation.

3. *A bialgebra H with an involution for which it is both a $*$-algebra and a $*$-coalgebra is called a $*$-bialgebra.*

If a Hopf algebra H also has the structure of a $*$-bialgebra then the interplay between antipode and involution is already determined:

$$S\big(S(x^*)^*\big) = x. \quad \text{for all } x \in H$$

Consequently, a Hopf algebra with the structure of a $*$-bialgebra is called a Hopf $*$-algebra. Its antipode is always invertible, even if H is not finite-dimensional.

Finally, one can show that the dual of a Hopf $*$-algebra H is again a Hopf $*$-algebra with the involution given by

$$f^*(x) = \overline{f\big(S(x)^*\big)} \quad \text{for all } f \in H^* \ .$$

A finite-dimensional Hopf $*$-algebra H has a normalized two-sided integral, $\epsilon(\Lambda) = 1$, called the defindHaar integral. The Haar-integral $\lambda \in H^*$ allows to endow H with the structure of a finite-dimensional Hilbert space by

$$\langle h_1, h_2 \rangle := \lambda(h_1 \cdot h_2^*) \ .$$

We allocate degrees of freedom to edges e of Δ and consider as the discrete quantum medium the $\mathbb{K}$-vector space

$$V(\Sigma, \Delta) := \otimes_{e \in \Delta} H \ .$$

Here we should first choose an orientation of the edges, and identify $x \mapsto S(x)$ if the orientation is reversed. Since $S^2 = \mathrm{id}$, this isomorphism is well defined. It is clear that the discrete quantum medium depends on the choice of a polytope decomposition.

To construct subspaces for quantum codes $W \subset V$, we need linear endomorphisms on V.

Definition 5.5.17

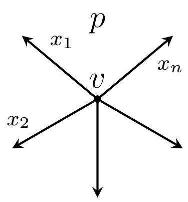
1. *Let Σ be a two-dimensional manifold with a polytope decomposition Δ. A <u>site</u> of Δ is a pair (v, p), consisting of a face p and a vertex v adjacent to p.*

2. *For every site (v, p) of (Σ, Δ) and every element $a \in H$, we define an endomorphism*

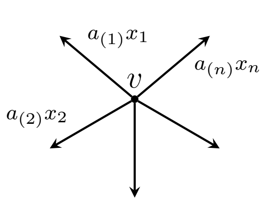
$$A_{(v,p)}(a) : \quad V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

by a multiple coproduct and the left action of H on itself:

$A_{(v,p)}(a)$:



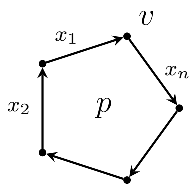
$\mapsto$



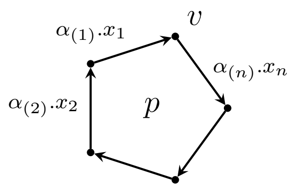
where the edges incident to the vertex v are indexed counterclockwise starting from p. Here all edges incident to the vertex v are assumed to point away from v. Using the antipode to change orientation, we see that for edges oriented towards the vertex v, the left regular action has to be replaced by the following left action: instead of $a_{(i)}x_i$, we have $S\left(a_{(i)}S(x_i)\right) = x_iS(a_{(i)})$.

3. *Given a site $s = (v,p)$ of the polytope decomposition Δ and every element $\alpha \in H^*$, the plaquette operator*
$$B_{(v,p)}(\alpha) : V(\Sigma,\Delta) \to V(\Sigma,\Delta)$$
is defined by a multiple coproduct in H^* and a left action where $\alpha.x = \alpha \rightharpoondown h$ of H^* on H.

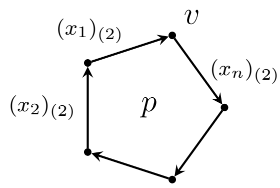
$B_{(v,p)}(\alpha)$:



$\mapsto$



$= \langle \alpha, S(x_n)_{(1)} \dots (x_1)_{(1)} \rangle$



We need the following

Lemma 5.5.18.
Let X be a representation of H, and Y a representation of H^*. For $h \in H$, $\alpha \in H^*$, define the endomorphisms $p_h, q_\alpha \in \mathrm{End}(H \otimes X \otimes Y \otimes H)$ by

$$p_h(u \otimes x \otimes y \otimes v) = h_{(1)}u \otimes h_{(2)}x \otimes y \otimes vS(h_{(3)})$$
$$q_\alpha(u \otimes x \otimes y \otimes v) = \alpha_{(3)} \rightharpoondown u \otimes x \otimes \alpha_{(2)}.y \otimes \alpha_{(1)} \rightharpoondown v$$

Then these endomorphisms satisfy the straightening formula of $D(H)$. Then the map

$$D(H) \to \mathrm{End}(H \otimes X \otimes Y \otimes H)$$
$$h \otimes \alpha \mapsto p_h q_\alpha$$

is a morphism of algebras.

Proof.
It is obvious that we have actions $a \mapsto p_a$ and $\alpha \mapsto p_\alpha$ of H and H^*. It remains to show that these endomorphisms satisfy the straightening formula of $D(H)$, cf. remark 4.4.15.2. This is done in a direct, but tedious calculation in [BMCA, Lemma 1, Theorem 1]. $\square$

This allows us to show:

Theorem 5.5.19.

1. If v, w are distinct vertices of Δ, then the operators $A_{(v,p)}(a)$, $A_{(w,p')}(b)$ commute for any pair $a, b \in H$.

2. Similarly, if p, q are distinct plaquettes, then the operators $B_{(v,p)}(\alpha)$, $B_{(v',q)}(\beta)$ commute for any pair $\alpha, \beta \in H^*$.

3. If the sites are different, then the operators $A_{(v,p)}(h)$ and $B_{(v',p')}(\alpha)$ commute.

4. For a given site $s = (v, p)$, the operators $A_{(v,p)}(h)$ and and $B_{(v,p)}(\alpha)$ satisfy the commutation relations of the Drinfeld double $Z(H)$: the map

$$\rho_s\colon D(H) \to \mathrm{End}(V(\Sigma, \Delta)) \tag{4}$$
$$a \otimes \alpha \mapsto A_{(v,p)}(a)B_{(v,p)}(\alpha) \tag{5}$$

is an algebra morphism.

Proof.

1. The operators $A^{-}_{(v,-)}$, $A^{-}_{(w,-)}$ obviously commute if the edges incident to the vertex v and those incident to the vertex w are disjoint. We therefore assume that the vertices v and w are adjacent, i.e. at least one edge connects them. Clearly, we need only to check that the actions of $A^{-}_{(v,-)}$ and $A^{-}_{(w,-)}$ commute on their common support. Suppose such an edge e is oriented so that it points from the vertex v to the vertex w. Then $A^{-}_{(v,-)}$ acts on the corresponding copy of H via the left regular representation, and $A^{-}_{(w,-)}$ acts on the copy of H associated the edge e via the right regular representation. These are commuting actions by associativity.

2. This statement is dual to 1, using coassociativity.

3. Follows by the same type of argument.

4. Follows from lemma 5.5.18.

□

Observation 5.5.20.

1. Let $h \in H$ be a cocommutative element, i.e. $\Delta(h) = \Delta^{opp}(h)$. Then the multiple coproduct $\Delta^{(n)}(h) \in H^{\otimes n}$ is cyclically invariant. As a consequence, the endomorphism $A_{(s,p)}(h)$ is independent of the plaquette p which was previously used to construct a linear order on the edges incident to the vertex v. We denote the endomorphism by $A_s(h)$. Similarly, $B_p(f)$ for a cocommutative element $f \in H^*$ is independent on the vertex.

2. Recall that both H and H^* have, as *-Hopf algebras, Haar integrals, i.e. normalized two-sided integrals $\Lambda \in H$ and $\lambda \in H^*$. Two-sided integrals are cocommutative. We thus get an endomorphism $A_v := A_v(\Lambda)$ for each vertex and $B_p := B_p(\lambda)$ for each plaquette.

Lemma 5.5.21.
All endomorphisms A_v and B_p commute with each other and are idempotents,

$$A_v^2 = A_v \quad \text{and} \quad B_p^2 = B_p \ .$$

Proof.
For a normalized integral, we have $\Lambda \cdot \Lambda = \epsilon(\Lambda)\Lambda = \Lambda$. Theorem 5.5.19 now implies that the endomorphisms are idempotents. A two-sided integral is central, $\Lambda \cdot h = \epsilon(h)\Lambda = h \cdot \Lambda$ for all $h \in H$, which implies again with theorem 5.5.16 that the endomorphisms commute. □

One shows that with respect to the scalar product on the quantum medium $H^{\otimes n}$, these endomorphisms are hermitian. We now define as a Hamiltonian the sum of these commuting endomorphisms:

$$H := \sum_v (\mathrm{id} - A_v) + \sum_p (\mathrm{id} - B_p) \ .$$

As a sum over commuting hermitian endomorphisms, the Hamiltonian is Hermitian and diagonalizable. Note that this Hamiltonian has to property to favour allignment of spins.

Definition 5.5.22
The ground state or protected subspace is the zero eigenspace of H:

$$K(\Sigma, \Delta) := \{v \in H^{\otimes n} : Hv = 0\}$$

It is a quantum code.

Note that the information is not stored here in a localized way, which gives hope that the code will be fault tolerant.

Remarks 5.5.23.

1. One shows that $x \in K(\Sigma, \Delta)$, if and only if $A_v x = x$ and $B_p x = x$ for all vertices v and all plaquettes p.

2. Up to canonical isomorphism, the ground space does not depend on the choice of polytope decomposition Δ of Σ.

3. In the case of a group algebra of a finite group, $H = \mathbb{C}[G]$, we use the distinguished basis of H consisting of group elements of G. (Kitaev's toric code uses the cyclic group $\mathbb{Z}_2$.) A basis of $V(\Sigma, \Delta)$ is given by assigning to any edge of Δ a group element g. We interpret the group elements g as the holonomy of a connection along the edge.

   - The projection by the operator A_v implements gauge invariance at the vertex v by averaging with respect to the Haar measure.
   - The projection by the operator B_p implements that locally on the face p the field strength vanishes, i.e. that the connection is locally flat. Indeed, integrals project to invariants and thus for the holonomy around a plaquette, we have $\epsilon(g_1 \cdot g_2 \cdot \ldots \cdot g_n) = 1$ which amounts to the flatness condition $g_1 \cdot g_2 \cdot \ldots \cdot g_n = e$.

4. One can then easily modify at single sites the projection condition: instead of requiring invariance under the action of the double associated to the site, one only keeps states transforming in a specific representation of the double. In this way, again the category of $D(H-\mathrm{mod})$ appears in the description of degrees of freedoms at insertions.

One can now show [BK2]:

Theorem 5.5.24.
Let H be a finite-dimensional semisimple Hopf algebra. The vector spaces of ground states constructed from the Hopf algebra H are canonically isomorphic to the vector spaces of the Turaev-Viro topological field theory based on H and thus, by theorem 5.4.6 also to the vector spaces appearing in the string-net construction.

A Facts from linear algebra

A.1 Free vector spaces

Let $\mathbb{K}$ be a fixed field. To any $\mathbb{K}$-vector space, we can associate the underlying set. Any $\mathbb{K}$-linear map is in particular a map of sets. We thus have a so-called forgetful functor $U : \mathrm{vect}(\mathbb{K}) \to \mathrm{Set}$. The functor U is faithful, but neither full nor essentially surjective. We also need a functor from sets to $\mathbb{K}$-vector spaces.

Definition A.1.1
Let S be a set and $\mathbb{K}$ be a field. A $\mathbb{K}$-vector space $V(S)$, together with a map of sets $\iota_S : S \to V(S)$, is called the free vector space on the set S, if for any $\mathbb{K}$-vector space W and any map $f : S \to W$ of sets, there is a unique *$\mathbb{K}$-linear map $\tilde{f} : V(S) \to W$ such that $\tilde{f} \circ \iota_S = f$. As a commuting diagram:*

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[dr, "f"'] & V(S) \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & W \end{tikzcd}$$

Remarks A.1.2.

1. A free vector space, if it exists, is unique up to unique isomorphism. Suppose that (V', ι') is another free vector space on the same set S. Consider the commutative diagram

$$\begin{tikzcd} & V(S) \arrow[d, dashed, "\exists ! \phi"] \\ S \arrow[ur, "\iota_S"] \arrow[r, "\iota'"] \arrow[dr, "\iota_S"'] & V' \arrow[d, dashed, "\exists ! \phi'"] \\ & V(S) \end{tikzcd}$$

   By the defining property of $V(S)$, applied to the map ι' of sets, we find a (unique) $\mathbb{K}$-linear map $\phi : V(S) \to V'$ such that the upper triangle commutes. By the defining property of V', applied to the map ι_S of sets, we find a (unique) $\mathbb{K}$-linear map $\phi' : V' \to V(S)$ such that the lower triangle commutes. Thus the outer triangle commutes, $\iota_S = \phi' \circ \phi \circ \iota_S$. On the other hand, $\iota_S = \mathrm{id}_{V(S)} \circ \iota_S$, and by the defining property of $V(S)$, such a map is unique. Thus $\phi' \circ \phi = \mathrm{id}_{V(S)}$. Exchanging the roles of $V(S)$ and V', we find $\phi \circ \phi' = \mathrm{id}_V$. Thus $V(S)$ and V are isomorphic with distinguished isomorphisms.

2. The free vector space exists: take $V(S)$ the set of maps $S \to \mathbb{K}$ which take value zero $0 \in \mathbb{K}$ almost everywhere. Adding the values of two maps $(f + g)(s) := f(s) + g(s)$ and taking scalar multiplication on the values $(\lambda f)(s) := \lambda \cdot f(s)$ endows $V(S)$ with the structure of a $\mathbb{K}$-vector space. To define the map ι_S, let $\iota_S(s)$ for $s \in S$ be the map which takes value 1 on the element s and zero on all other elements,

$$\iota_S(s)(s) = 1 \qquad \iota_S(s)(t) = 0 \quad \text{for } t \neq s \ .$$

   Then the set $\iota_S(S) \subset V(S)$ is a $\mathbb{K}$-basis. The condition $\tilde{f} \circ \iota_S(s) = f(s)$ then fixes $\tilde{f}$ uniquely by its values on a basis. This shows that we have constructed the free vector space on the set S.

3. Let $f : S \to S'$ be any map of sets. Applying in the diagram

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[d, "f"'] & V(S) \arrow[d, dashed, "V(f)"] \\ S' \arrow[r, "\iota_{S'}"'] & V(S') \end{tikzcd}$$

the defining property of the free vector space $V(S)$ to the map $\iota_{S'} \circ f$ of sets, we find a unique $\mathbb{K}$-linear map $V(f) : V(S) \to V(S')$.

One checks that this defines a functor $V : \mathrm{Set} \to \mathrm{vect}(\mathbb{K})$. For any $\mathbb{K}$-vector space W and any set S, we have a bijection of morphism spaces:

$$\mathrm{Hom}_{\mathbb{K}}(V(S), W) \cong \mathrm{Hom}_{\mathrm{Set}}(S, U(W))$$

that is compatible with morphism of sets and $\mathbb{K}$-linear maps. The functor V assigning to a set the vector space generated by the set is thus a left adjoint to the forgetful functor U, cf. definition 2.5.22.

A.2 Tensor products of vector spaces

We summarize some facts about tensor products of vector spaces over a field $\mathbb{K}$.

Definition A.2.1
Let $\mathbb{K}$ be a field and let V, W and X be $\mathbb{K}$-vector spaces. A $\mathbb{K}$-bilinear map is a map

$$\alpha : V \times W \to X$$

that is $\mathbb{K}$-linear in both arguments, i.e. $\alpha(\lambda v + \lambda' v', w) = \lambda\alpha(v, w) + \lambda'\alpha(v', w)$ and $\alpha(v, \lambda w + \lambda' w') = \lambda\alpha(v, w) + \lambda'\alpha(v, w')$ for all $\lambda, \lambda' \in \mathbb{K}$ and $v, v' \in V$, $w, w' \in W$.

Given any $\mathbb{K}$-linear map $\varphi : X \to X'$, the map $\varphi \circ \alpha : V \times W \to X'$ is $\mathbb{K}$-bilinear as well. This raises the question of whether for two given $\mathbb{K}$-vector spaces V, W, there is a "universal" $\mathbb{K}$-vector space with a universal bilinear map such that *all* bilinear maps out of $V \times W$ can be described in terms of linear maps out of this vector space.

Definition A.2.2
The tensor product of two $\mathbb{K}$-vector spaces V, W is a pair, consisting of a $\mathbb{K}$-vector space $V \otimes W$ and a bilinear map

$$\begin{array}{cccc} \kappa : & V \times W & \to & V \otimes W \\ & (v, w) & \mapsto & v \otimes w \end{array}$$

with the following universal property: for any $\mathbb{K}$-bilinear *map*

$$\alpha : V \times W \to U$$

there exists a unique linear *map $\tilde{\alpha} : V \otimes W \to U$ such that*

$$\alpha = \tilde{\alpha} \circ \kappa\,.$$

As a diagram:

$$\begin{tikzcd} V \times W \arrow[rr, "\alpha"] \arrow[dr, "\kappa"'] & & U \\ & V \otimes W \arrow[ur, dashed, "\exists!\tilde{\alpha}"'] & \end{tikzcd}$$

Remarks A.2.3.

1. This reduces bilinear maps to linear maps.

2. We first show that the tensor product, if it exists, is unique up to unique isomorphism. Suppose we have two bilinear maps

$$\kappa : V \times W \to V \otimes W \qquad \tilde{\kappa} : V \times W \to V \tilde{\otimes} W .$$

   having each the universal property.

   Using the universal property of κ for the specific bilinear map $\tilde{\kappa}$, we find a unique linear map $\Phi_{\tilde{\kappa}} : V \otimes W \to V \tilde{\otimes} W$ with $\Phi_{\tilde{\kappa}} \circ \kappa = \tilde{\kappa}$.

   Exchanging the roles of κ and $\tilde{\kappa}$, we obtain a linear map $\Phi_\kappa : V \tilde{\otimes} W \to V \otimes W$ with $\Phi_\kappa \circ \tilde{\kappa} = \kappa$. The maps $\kappa = \mathrm{id}_{V\otimes W} \circ \kappa$ and $\Phi_\kappa \circ \Phi_{\tilde{\kappa}} \circ \kappa$ describe the same bilinear map $V \times W \to V \otimes W$. The uniqueness statement in the universal property implies $\Phi_\kappa \circ \Phi_{\tilde{\kappa}} = \mathrm{id}_{V\otimes W}$. Similarly, we conclude $\Phi_{\tilde{\kappa}} \circ \Phi_\kappa = \mathrm{id}_{V\tilde{\otimes}W}$. Note that this is the same argument as in remark A.1.2.

3. To show the existence of the tensor product, chose a basis $\mathcal{B} := (b_i)_{i\in I}$ of V and $\mathcal{B}' := (b'_i)_{i\in I'}$ of W. Since a bilinear map is uniquely determined by its values on all pairs $(b_i, b'_j)_{i\in I, j\in I'}$, we need a vector space with a basis indexed by these pairs. Thus define $V \otimes W$ as the vector space freely generated by the set of these pairs. We denote by $b_i \otimes b'_j$ the corresponding element of the basis of $V \otimes W$.

   The bilinear map κ is then defined by $\kappa(b_i, b'_j) := b_i \otimes b'_j$. It has the universal property: to any bilinear map $\alpha : V \times W \to X$, we associate the linear map $\tilde{\alpha} : V \otimes W \to X$ with $\tilde{\alpha}(b_i \otimes b'_j) = \alpha(b_i, b'_j)$. By the uniqueness argument in 1. any two realizations, e.g. based on different bases, are isomorphic up to unique isomorphism.

4. As a corollary, we conclude that for finite-dimensional vector spaces V, W, the dimension of the tensor product is $\dim V \otimes W = \dim V \cdot \dim W$.

5. The elements of $V \otimes W$ are called <u>tensors</u>; elements of the form $v \otimes w$ with $v \in V$ and $w \in W$ are called <u>simple tensors</u>. The simple tensors span $V \otimes W$, but there are elements of $V \otimes W$ that are not tensor products of a vector $v \in V$ and $w \in W$.

Observation A.2.4.
Given $\mathbb{K}$-linear maps

$$\varphi : V \to V' \qquad \psi : W \to W'$$

we obtain a $\mathbb{K}$-linear map

$$\varphi \otimes \psi : V \otimes W \to V' \otimes W'$$

on the tensor products. To this end, consider the commuting diagram:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"'] \arrow[d, "\varphi\times\psi"'] & V \otimes W \arrow[d, dashed, "\exists!\varphi\otimes\psi"] \\ V' \times W' \arrow[r, "\otimes"'] & V' \otimes W' \end{tikzcd}$$

Since the map $\otimes \circ (\varphi \times \psi)$ is bilinear, the universal property of the tensor product implies the existence of a map $\varphi \otimes \psi$ for which the identity

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w)$$

holds.

Remarks A.2.5.

1. The bilinearity of κ implies that the tensor product of linear maps is bilinear:

$$\begin{array}{rcl}(\lambda_1\varphi_1+\lambda_2\varphi_2)\otimes\psi & = & \lambda_1\varphi_1\otimes\psi+\lambda_2\varphi_2\otimes\psi\\ \varphi\otimes(\lambda_1\psi_1+\lambda_2\psi_2) & = & \varphi\otimes\lambda_1\psi_1+\varphi\otimes\lambda_2\psi_2\end{array}$$

2. Similarly, one deduces the following compatibility with direct sums:

$$(V_1\oplus V_2)\otimes W\cong(V_1\otimes W)\oplus(V_2\otimes W)\,,$$

   and analogously in the second argument.

3. There are canonical isomorphisms

$$\begin{array}{rcrcl}a_{U,V,W} & : & (U\otimes V)\otimes W & \xrightarrow{\sim} & U\otimes(V\otimes W)\\ & & u\otimes(v\otimes w) & \mapsto & (u\otimes v)\otimes w\end{array}$$

   which allow to identify the $\mathbb{K}$-vector spaces $U\otimes(V\otimes W)$ and $(U\otimes V)\otimes W$. With this identification, the tensor product is strictly associative.

   One verifies that the following diagram commutes:

$$\begin{tikzcd}(U\otimes V)\otimes(W\otimes X) \arrow[r, equal] & (U\otimes V)\otimes(W\otimes X) \arrow[d, "\alpha_{U,V,W\otimes X}"]\\ ((U\otimes V)\otimes W)\otimes X \arrow[u, "\alpha_{U\otimes V,W,X}"] \arrow[d, "\alpha_{U,V,W}\otimes\mathrm{id}_X"'] & U\otimes(V\otimes(W\otimes X))\\ (U\otimes(V\otimes W))\otimes X \arrow[r, "\alpha_{U,V\otimes W,X}"'] & U\otimes((V\otimes W)\otimes X) \arrow[u, "\mathrm{id}_U\otimes\alpha_{V,W,X}"']\end{tikzcd}$$

   One can show that, as a consequence, any bracketing of multiple tensor products gives canonically isomorphic vector spaces.

4. There are canonical isomorphisms

$$\begin{array}{rcl}\mathbb{K}\otimes V & \xrightarrow{\sim} & V\\ \lambda\otimes v & \mapsto & \lambda\cdot v\end{array}$$

   with inverse map $v\mapsto 1\otimes v$ which allow to consider the ground field $\mathbb{K}$ as a unit for the tensor product. There is also a similar canonical isomorphism $V\otimes\mathbb{K}\cong V$. We will tacitly apply the identifications described in 3. and 4. This shows that $\mathrm{vect}(\mathbb{K})$ is a monoidal category.

5. For any pair U,V of $\mathbb{K}$-vector spaces, there are canonical isomorphisms

$$\begin{array}{rcrcl}c_{U,V} & : & U\otimes V & \to & V\otimes U\\ & & u\otimes v & \mapsto & v\otimes u\ ,\end{array}$$

   which allow to permute the factors. One has $c_{V,U}\circ c_{U,V}=\mathrm{id}_{U\otimes V}$ as well as the identity

$$(c_{V,W}\otimes\mathrm{id}_U)\circ(\mathrm{id}_V\otimes c_{U,W})\circ(c_{U,V}\otimes\mathrm{id}_W)=(\mathrm{id}_W\otimes c_{U,V})\circ(c_{U,W}\otimes\mathrm{id}_V)\circ(\mathrm{id}_U\otimes c_{V,W})$$

   Note that in this identify, we have tacitly used the identification $(U\otimes V)\otimes W\cong U\otimes(V\otimes W)$ from 3. This shows that $\mathrm{vect}(\mathbb{K})$ is a braided (even symmetric) monoidal category.

6. For any pair of $\mathbb{K}$-vector spaces vector spaces V, W, the canonical map

$$\begin{array}{lll} V^* \otimes W^* & \to & (V \otimes W)^* \\ \alpha \otimes \beta & \mapsto & (v \otimes w \mapsto \alpha(v) \cdot \beta(w)) \end{array}$$

is an injection. If both V and W are finite-dimensional, this is an isomorphism. (Give an example of a pair of infinite-dimensional vector spaces and an element that is not in the image!)

7. For any pair of $\mathbb{K}$-vector spaces V, W, the canonical map

$$\begin{array}{lll} V^* \otimes W & \to & \mathrm{Hom}_{\mathbb{K}}(V, W) \\ \alpha \otimes w & \mapsto & (v \mapsto \alpha(v) w) \end{array}$$

is an injection. If both V and W are finite-dimensional, this is an isomorphism. (Give again an example of a pair of infinite-dimensional vector spaces and an element that is not in the image!)

B Abelian categories

We start by requiring additional algebraic structure on the morphisms sets of categories. For example, in the category $R-$Mod the morphisms sets were abelian groups. It is then natural to study the class of functors that respect these structures.

Definition B.0.1

1. *A category $\mathcal{C}$ is called* <u>*additive*</u>*, if*

   (a) All Hom*-sets are abelian groups and the composition $\circ$ is bilinear.*

   (b) All finite products and coproducts exist in $\mathcal{C}$.

2. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called* <u>*additive*</u>*, if for every pair X, Y of objects of $\mathcal{C}$ the map $F\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a group homomorphism.*

Remark B.0.2.

1. These axioms have many implications. In particular, additive categories have an initial object, i.e. an object $0 \in \mathcal{C}$, from which there exists a unique morphism to every object of $\mathcal{C}$, and a terminal object. Initial and terminal object coincide.

2. For a given category it is thus a *property* to be an additive category, and this does not require choosing any additional structure.

Definition B.0.3

1. *Let $f\colon a \to b$ be a morphism in an additive category $\mathcal{C}$. A morphism $\iota\colon k \to a$ is called* <u>*kernel*</u> *of f, and we write $\iota = \ker(f)$, if $f \circ \iota = 0$ and for every morphism $d \xrightarrow{g} a$ with $f \circ g = 0$ there exists a unique morphism $d \to k$, such that the diagram*

$$\begin{tikzcd} k \arrow[r, "\iota"] & a \arrow[r, "f"] & b \\ & d \arrow[ul, dashed, "\exists!"] \arrow[u, "g"] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes.

2. *Dually, a morphism $p\colon b \to c$ is called* <u>cokernel</u> *of f, and we write $p = \mathrm{coker}(f)$, if for every morphism $g : b \to d$ with $g \circ f$ there exists a unique morphism $c \to d$ such that the following diagram commutes.*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "0"'] & b \arrow[r, "p"] \arrow[d] & c \arrow[dl, dashed, "\exists!"] \\ & d & \end{tikzcd}$$

In a general category, not every morphism has to have a kernel resp. cokernel. As usual for object defined via universal properties, one can show that kernels, cokernels and images are unique up to unique isomorphism, if they exist.

Definition B.0.4 *Let $\mathcal{C}$ be a category.*

1. *A morphism $\iota\colon a \to b$ is called a* <u>monomorphism</u>*, if $\iota \circ f = \iota \circ f'$ implies $f = f'$ for all morphism f and f' with source b and equal target. (In an additive category it is sufficient to require this for $f' = 0$.)*

2. *A morphism $p\colon a \to b$ is called an* <u>epimorphism</u>*, if $f \circ p = f' \circ p$ implies $f = f'$ for all morphisms f, f' with target a and equal source.*

Lemma B.0.5.
Let $\mathcal{C}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism that has a kernel $K \xrightarrow{\iota} A$. Then the kernel ι is a monomorphism. Dually, if f has a cokernel, then the cokernel is an epimorphism.

Proof.
For an arbitrary object X we consider the zero morphism $X \xrightarrow{0} A$. By the universal property of the kernel there exists a unique morphism ϕ, such that the diagram

$$\begin{tikzcd} \ker f \arrow[r] & A \arrow[r, "f"] & B \\ & X \arrow[ul, dashed, "\exists\phi"] \arrow[u, "0"'] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes. For the choice 0 for ϕ the diagram commutes as well, so we must have $\phi = 0$. Given $X \xrightarrow{g} \ker f$ with $\ker f \circ g = 0$, then we must have $g = 0$, so $\ker f$ is a monomorphism. $\square$

Definition B.0.6
Let $\mathcal{C}$ be a category. An additive category is called an <u>abelian category</u>*, if every morphism has a kernel and a cokernel, and the following compatibility condition holds:*

– *For every monomorphism $\iota\colon a \to b$ one has $\iota = \ker(\mathrm{coker}(\iota))$. In the diagram for the monomorphism*

$$\begin{tikzcd} a \arrow[r, "\iota"] & b \arrow[r, "\mathrm{coker}\iota"] & \mathrm{coker}\iota \\ & \ker\mathrm{coker}\iota \arrow[u, "\ker"] & \end{tikzcd}$$

the left horizontal arrow and the vertical arrow have the same universal property.

– *For every epimorphism $p\colon a \to b$ one has $p = \text{coker}(\ker(p))$.*

Remarks B.0.7.

1. For a given category it is again a property to be an abelian category, without choice of an additional structure.

2. If a morphism in an arbitrary category $f\colon a \to b$ can be decomposed as $f = \iota \circ p$, where $p\colon a \to x$ is an epimorphism and $\iota\colon x \to b$ a monomorphism, then the object x is called an <u>image</u> of f and we write $x = \text{Im}\,(f)$.

3. In abelian categories every morphism f can be written as $f = \iota \circ p$ with ι a monomorphism and p an epimorphism, and so all images exist.

Examples B.0.8.

1. For every ring R the category of R-modules is abelian, because kernels and cokernels are defined on the level of the underlying abelian groups.

   The converse also holds and is known as the full embedding theorem: every small abelian category can be fully faithfully embedded in the category of modules over a suitable ring, such that exactness properties are preserved, See e.g. B. Mitchell, Theory of categories, Academic Press 1965, London-New York, p. 151.

2. If $\mathcal{C}$ is an abelian category, then so is the opposite category $\mathcal{C}^{opp}$. The kernels in $\mathcal{C}^{opp}$ are the cokernels in $\mathcal{C}$ and vice versa. Categorical language generally follows the design principle that an X in $\mathcal{C}$ corresponds to a co-X in $\mathcal{C}^{opp}$.

We now have all required notions to make sense of exact sequences in abelian categories. Abelian categories are thus a natural framework for homological algebra.

Definition B.0.9
Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories. We consider all short exact sequences $0 \to a' \to a \to a'' \to 0$ in $\mathcal{C}$. Now F is called

<u>left exact</u>,	*if $0 = F(0) \to F(a') \to F(a) \to F(a'')$ is exact for all short exact sequences in $\mathcal{C}$;*
<u>right exact</u>,	*if $F(a') \to F(a) \to F(a'') \to 0$ is exact for all short exact sequences in $\mathcal{C}$;*
<u>exact</u>,	*if $0 \to F(a') \to F(a) \to F(a'') \to 0$ is exact for all short exact sequences in $\mathcal{C}$.*

Examples B.0.10.

1. Let M be an R-right module. The functor $M \otimes_R -\colon R{-}\text{Mod} \to \text{Ab}$ is right exact. It is exact if and only if the module M is flat.

2. Let M be an R-module. The functor $\text{Hom}_R(M, -)\colon R{-}\text{Mod} \to \text{Ab}$ is left exact. It is exact if and only if the module M is projective.

3. Let M be an R-module. The functor $\text{Hom}_R(-, M)\colon (R{-}\text{Mod})^{opp} \to Ab$ is left exact.

C Tanaka-Krein reconstruction

Let $\mathbb{K}$ be a field. In this subsection, we explain under what conditions a finite tensor category over a field $\mathbb{K}$ can be described as the category of modules or comodules over a Hopf algebra. We assume that the field $\mathbb{K}$ is algebraically closed of characteristic zero and that all categories are essentially small, i.e. equivalent to a small category, a category in which the class of objects is a set.

Definition C.0.1
Let $\mathcal{C}, \mathcal{D}$ be abelian tensor categories. A fibre functor is an exact faithful tensor functor $\Phi : \mathcal{C} \to \mathcal{D}$.

Examples C.0.2.

1. Let H be a bialgebra over a field $\mathbb{K}$. The forgetful functor

$$\mathcal{F} : H\text{−mod} \rightarrow \text{vect}(\mathbb{K}),$$

   is a strict tensor functor. It is faithful, since by definition $\text{Hom}_{H-\text{mod}}(V,W) \subset \text{Hom}_{\text{vect}(\mathbb{K})}(V,W)$. It is exact, since the kernels and images in the categories H−mod and vect($\mathbb{K}$) are the same. Thus the forgetful functor $\mathcal{F}$ is a fibre functor.

2. There are tensor categories that do not admit a fibre functor to vector spaces. Deligne has characterized [D, Theorem 7.1] those $\mathbb{K}$-linear semisimple ribbon categories over a field $\mathbb{K}$ of characteristic zero that admit a fibre functor: these are those categories for which all objects have categorical dimensions that are non-negative integers.

We only sketch the proof of the following result:

Theorem C.0.3.
Let $\mathbb{K}$ be a field and $\mathcal{C}$ a $\mathbb{K}$-linear abelian tensor category and

$$\Phi : \mathcal{C} \to \text{vect}_{fd}(\mathbb{K})$$

a fibre functor in the category of finite-dimensional $\mathbb{K}$-vector spaces. Then there is a $\mathbb{K}$-Hopf algebra H and an equivalence of tensor categories

$$\omega : \mathcal{C} \xrightarrow{\sim} \text{comod- } H\,,$$

such that the following diagram of monoidal functors commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\omega"'] \arrow[dr, "\Phi"] & \\
\text{comod}-H \arrow[r, "\mathcal{F}"'] & \text{vect}_{fd}(\mathbb{K})
\end{tikzcd}$$

where $\mathcal{F}$ is the forgetful functor.

Proof.
We only sketch the idea and refer for details to the book by Chari and Pressley.

- For any $\mathbb{K}$-vector space M, consider the functor

$$\begin{array}{rrcl} \Phi \otimes M : & \mathcal{C} & \to & \mathrm{vect}_{fd}(\mathbb{K}) \\ & U & \mapsto & \Phi(U) \otimes_{\mathbb{K}} M \end{array} ,$$

which is not monoidal, in general. We use these functors to construct a functor

$$\begin{array}{rrcl} \mathrm{Nat}(\Phi, \Phi \otimes -) : & \mathrm{vect}_{fd}(\mathbb{K}) & \to & \mathrm{vect}_{fd}(\mathbb{K}) \\ & M & \mapsto & \mathrm{Nat}\ (\Phi, \Phi \otimes M) \end{array}$$

which is representable: there is a vector space $H \in \mathrm{vect}_{fd}(\mathbb{K})$ and a natural isomorphism of functors $\tau : \mathrm{vect}_{fd}(\mathbb{K}) \to \mathrm{vect}_{fd}(\mathbb{K})$

$$\mathrm{Hom}(H, -) \to \mathrm{Nat}\ (\Phi, \Phi \otimes -)$$

i.e. natural isomorphisms of vector spaces for any $M \in \mathrm{vect}_{fd}(\mathbb{K})$

$$\tau_M : \mathrm{Hom}_{\mathbb{K}}(H, M) \xrightarrow{\sim} \mathrm{Nat}\ (\Phi, \Phi \otimes M)\,.$$

- The natural identification

$$e : \Phi \to \Phi \otimes \mathbb{K} \in \mathrm{Nat}(\Phi, \Phi \otimes \mathbb{K})$$

gives a linear form $\epsilon := \tau_{\mathbb{K}}^{-1}(e) \in \mathrm{Hom}\ (H, \mathbb{K})$.

- Consider

$$\delta := \tau_H(\mathrm{id}_H) \in \mathrm{Nat}(\Phi, \Phi \otimes H)$$

which gives for any object U of $\mathcal{C}$ a natural $\mathbb{K}$-linear map

$$\delta_U : \Phi(U) \to \Phi(U) \otimes_{\mathbb{K}} H\ .$$

Consider the natural transformation

$$(\delta \otimes \mathrm{id}_H) \circ \delta : \quad \Phi \to \Phi \otimes H \to (\Phi \otimes H) \otimes H \cong \Phi \otimes (H \otimes H)\ .$$

Then define

$$\Delta := \tau_{H \otimes H}^{-1}\Big((\delta \otimes \mathrm{id}_H) \circ \delta\Big) \in \mathrm{Hom}(H, H \otimes H)\,.$$

One can show that ϵ and Δ endow the vector space H with the structure of a counital coassociative coalgebra over the field $\mathbb{K}$.

- For the algebra structure on H, we use the monoidal structure on the functor Φ: consider

$$\begin{array}{rcl} m_{U,V} & : & \Phi(U) \otimes \Phi(V) \cong \Phi(U \otimes V) \xrightarrow{\delta_{U \otimes V}} \Phi(U \otimes V) \otimes H \\ & & \cong \Phi(U) \otimes \Phi(V) \otimes H \end{array}$$

which is an element in

$$\mathrm{Nat}\ (\Phi^2, \Phi^2 \otimes H) \cong \mathrm{Hom}(H \otimes H, H)\ .$$

This gives an associative product with unit element

$$\eta : \mathbb{K} \cong \Phi(\mathbb{I}) \xrightarrow{\delta_{\mathbb{I}}} \Phi(\mathbb{I}) \otimes H \cong H\,.$$

- In a similar way, one uses the duality on $\mathcal{C}$ to obtain an antipode on H and shows that H becomes a Hopf algebra.
- One finally shows that $H-\mathrm{mod} \cong \mathcal{C}$ as monoidal categories.

$\square$

Remark C.0.4.
There exist generalizations: by [BLV, Theorem 7.6], any finite tensor category $\mathcal{C}$ over a field $\mathbb{K}$ is equivalent, as a tensor category, to the category of modules over a finite-dimensional left Hopf algebroid over $\mathbb{K}$.

D Glossary German-English

For the benefit of German speaking students, we include a table with German versions of important notions.

English	German
abelian Lie algebra	abelsche Lie-Algebra
absolutely simple object	absolut einfaches Objekt
additive tensor category	additive Tensorkategorie
adjoint functor	adjungierter Funktor
alternating algebra	alternierende Algebra
antipode	Antipode
associator	Assoziator
augmentation ideal	Augmentationsideal
autonomous category	autonome Kategorie
Boltzmann weights	Boltzmann-Gewichte
braid	Zopf
braid group	Zopfgruppe
braided tensor category	verzopfte Tensorkategorie
braided tensor functor	verzopfter Tensorfunktor
braided vector space	verzopfter Vektorraum
braiding	Verzopfung
character	Charakter
class function	Klassenfunktion
coaction	Kowirkung
code	Code
coevaluation	Koevaluation
coinvariant	Koinvariante
commutativity constraint	Kommutatitivitätsisomorphismus
convolution product	Konvolutionsprodukt, Faltungsprodukt
coopposed coalgebra	koopponierte Algebra
counitality	Kounitarität
derivation	Derivation
distinguished group-like element	ausgezeichnetes gruppenartiges Element
dominating family	dominierende Familie
Drinfeld center	Drinfeld Zentrum
Drinfeld double	Drinfeld-Doppel
enriched category	angereicherte Kategorie
enveloping algebra	einhüllende Algebra
error correcting code	fehlerkorrigierender Code
essentially small category	wesentlich kleine Kategorie
evaluation	Evaluation
exterior algebra	äußere Algebra

English	German
factorizable Hopf algebra	faktorisierbare Hopf-Algebra
fibre functor	Faserfunktor
forgetful functor	Vergissfunktor
free vector space	freier Vektorraum
fundamental groupoid	Fundamentalgruppoid
fusion category	Fusionskategorie
gate	Gatter
gauge transformation	Eichtransformation
group-like element	gruppenartiges Element
groupoid	Gruppoid
Hamming distance	Hamming-Abstand
Haar integral	Haarsches Maß
hexagon axioms	Hexagon-Axiome
Hopf algebra	Hopf-Algebra
isotopy	Isotopie
knot	Knoten
left adjoint functor	linksadjungierter Funktor
left integral	Linksintegral
left module	Linksmodul
link	Verschlingung
linking number	Verschlingungszahl
modular category	modulare Kategorie
modular element	modulares Element
monodromy element	Monodromie-Element
opposite algebra	opponierte Algebra
pentagon axiom	Pentagon-Axiom
pivotal category	pivotale Kategorie
primitive element	primitives Element
projective module	projektiver Modul
pullback of a representation	Pullback einer Darstellung
quantum circuit	Quantenschaltkreis
quantum code	Quantenkode
quantum gate	Quantengatter
quantum group	Quantengruppe
quasi-triangular bialgebra	quasitrianguläre Bialgebra
Reidemeister move	Reidemeister-Bewegung
representation	Darstellung
right adjoint functor	rechtsadjungierter Funktor
right integral	Rechtsintegral

English	German
semisimple algebra	halbeinfache Algebra
semisimple module	halbeinfacher Modul (der)
separable algebra	separable Algebra
simple module	einfacher Modul (der)
skein	Gebinde
skew antipode	Schiefantipode
spherical category	sphärische Kategorie
surgery	Chirurgie
tensor unit	Tensoreins
trace	Spur
trefoil knot	Kleeblattknoten
triangle axiom	Dreiecksaxiom
trivial module	trivialer Modul (der)
unimodular Hopf algebra	unimodulare Hopf-Algebra
universal R-matrix	universelle R-Matrix
universal enveloping algebra	universelle einhüllende Algebra
universal quantum gate	universelles Quantengatter
unknot	trivialer Knoten, Unknoten
Yang-Baxter equation	Yang-Baxter-Gleichung
Yetter-Drinfeld module	Yetter-Drinfeld-Modul (der)

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