Introduction to conformal field theory

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Contents

Literature:

References I used to prepare the course and references to research papers are at the end of these lecture notes.

The current version of these notes can be found under
`http://www.math.uni-hamburg.de/home/schweigert/skripten/cftskript.pdf`
as a pdf file.

Please send comments and corrections to `christoph.schweigert@uni-hamburg.de`!

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1 Introduction

1.1 Overview

Two-dimensional conformal field theory has lead to an interplay of various mathematical disciplines

- Representation theory, in particular representation theory of Hopf algebras, loop groups, Kac-Moody algebras, vertex algebras.
- Algebraic geometry, e.g. the moduli spaces of complex curves via the theory of non-abelian theta functions.
- Topology via three-dimensional topological field theory and knot invariants.

It has various applications in physics

- The world sheet theories of closed and open strings (and superstrings) are full local conformal field theories.
- Universality classes of critical two-dimensional systems.
- Quasi-one-dimensional systems with impurities, so-called Kondo systems, are described full local conformal field theories with boundaries.
- Universality classes of quanten Hall fluids are described by chiral conformal field theories.

It is important to note that the word "CFT" is used for two rather different theories:

- Chiral conformal field theory, which is a theory defined on Riemann surfaces without boundaries. Insertions are allowed, hence the importance of the moduli space of complex curves with marked points. Chiral conformal field theories are defined on *oriented* surfaces. The orientation is explicitly given by the external magnetic field in quantum Hall systems.
- Full local conformal field theory which is defined on conformal surfaces. These surfaces can have (physical) boundaries. There is also a version defined on unoriented surfaces that is important for string theories of type I.

In this class, we do not assume any familiarity with the notion of a quantum field theory, but assume a general mathematical background. We will also explain some basic notions of representation theory. The plan of these lectures is as follows:

- We use the free massless boson to explain what kind of structure we expect to be present in a two-dimensional full conformal field theory.
- From a first analysis of correlation functions of a quantum field theory, we extract an infinite set of meromorphic sections of line bundles on the Riemann sphere. We show how to extract from this set the algebraic structure of a vertex algebra.
- We show how affine Lie algebras arise in this context. We review some aspects of their basic theory, then we show how to construct conformal blocks from them and investigate some of their properties. In particular we explain the Knizhnik-Zamolodchikov connection.
- We then show how much structure that exists on the system of conformal blocks can be encoded in the structure of a modular functor. We explain how this is related to a three-dimensional topological field theory.
- We finally use three-dimensional topological field theories to construct full local two-dimensional rational conformal field theories.

1.2 Classical field theories

Typically, a field theory is defined for a certain category of manifolds; moreover, a certain dimension is distinguished in a field theories. Keeping boundary terms in mind, it is not surprising that manifolds of lower dimensions ultimately play a role in an n-dimensional field theory as well. The manifolds involved can have additional structure (bundles, metrics, ...). We thus start our considerations with a category $\mathcal{M}$ of manifolds.

Examples 1.2.1.

1. Mechanical systems depend on one variable, with the interpretation of time. The relevant category is the category of one-dimensional oriented manifolds with a metric. (For the moment, we take manifolds without boundary, i.e. circles.)

2. Topological field theory in d-dimensions is defined on the category of d-dimensional smooth manifolds, with morphisms being diffeomorpisms.

3. For gauge theories in n-dimensions with structure group typically a finite-dimensional Lie group G, the objects in the category are smooth n-dimensional manifolds, together with a G-bundle (and possibly more structure).

4. Theories with fermions are defined on categories of manifolds with spin bundles.

5. Full local two-dimensional conformal field theories are defined on two-dimensional manifolds with a conformal structure, i.e. with a class of Riemannian metrics up to local rescaling: two metrics g, g' on a manifold M are thus identified, if there is a nowhere vanishing function $\lambda : M \to \mathbb{R}_+$ such that $g'(x) = \lambda(x)g(x)$ for all $x \in M$. (Lorentzian signature is also considered in the literature.)

   There are two classes of theories: for oriented conformal field theories, one considers the category of oriented conformal manifolds $\mathcal{M}_{or}$; there is also a variant without orientation $\mathcal{M}_{unor}$.

In a classical field theory, we assign for each $M \in \mathcal{M}$ a set $\mathcal{F}(M)$ of field configurations. Later on, the structure of a set will not be enough to define dynamics and to implement covariance.

Examples 1.2.2.

1. For a mechanical system describing a particle moving on a smooth manifold T, we associate to an oriented one-dimensional manifold I the space of smooth maps to T,

   $$\mathcal{F}_T(I) = C^\infty(I, T) \; .$$

   This is the space of trajectories in T parametrized by the one-dimensional time manifold I.

2. For a (complex or real) free boson, we associate to $M \in \mathcal{M}$ the space of smooth (real- or complex-valued) functions on M, for the real boson

   $$\mathcal{F}_\mathbb{R}(M) = C^\infty(M, \mathbb{R}) \; .$$

3. These two examples suggest to consider the following construction: take any smooth manifold T and consider for $M \in \mathcal{M}$

   $$\mathcal{F}_T(M) = C^\infty(M, T) \; .$$

Such a model is called a sigma model with target space T.

An important example is obtained by taking an n-dimensional lattice $L \subset \mathbb{R}^n$.

Consider $\mathbb{R}^n$ with the standard structure of a Euclidean vector space. A lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subset that spans $\mathbb{R}^n$. Alternatively, it is a free abelian group Λ of rank an, together with a positive definite bilinear form.

Then $\mathcal{F}_{\mathbb{R}^n/L}(-)$ describes the classical field configurations for a free boson compactified on the torus $T = \mathbb{R}^n/L$. Another important class of sigma models have a Lie group G as a target space.

4. For a gauge theory with structure group G, we assign to a manifold M the category of all G-bundles on M, with morphisms being gauge transformations. Systems in which field configurations are (higher) categorical objects are thus crucial for physics. While this is also of relevance for two-dimensional conformal field theory, e.g. for gauged sigma-models, we do not pursue this any further in these lectures.

Functions can be pulled back. Given a function $N \to T$ and a map $f : M \to M'$ of manifolds, we have a map of functions

$$\begin{array}{rccl} f^*: & C^\infty(M', T) & \to & C^\infty(M, T) \\ & \varphi & \mapsto & \varphi \circ f \end{array}$$

This has the property that for given maps $M \xrightarrow{f} M' \xrightarrow{g} M''$, we have

$$(g \circ f)^* \varphi = \varphi \circ (g \circ f) = (g^* \varphi) \circ f = f^*(g^* \varphi)\,.$$

We keep this feature since it allows to build in locality and covariance with respect to the category $\mathcal{M}$ on which the theory is defined. We are thus ready to describe the kinematical setup for classical field theories:

Definition 1.2.3
A kinematic classical field theory in dimension d consists of a category $\mathcal{M}$ of d-dimensional manifolds (with possibly additional structure), together with a presheaf on $\mathcal{M}$ with values in Set *(or possible a richer category or even a bicategory), i.e. a contravariant functor*

$$\mathcal{F}: \quad \mathcal{M}^{\mathrm{opp}} \to \mathrm{Set}\ .$$

We now need dynamics for classical theories. Take a mechanical system of a particle moving on a target space T. For each one-dimensional manifold S, we have an equation of motion, typically a differential equation. Only a subset of the trajectories in $\mathcal{F}_T(S)$ are a solution to this equation of motion. We thus single out a subspace $Sol_M \subset \mathcal{F}(M)$ for each $M \in \mathcal{M}$.

For example, given a Riemannian metric on T, we might take Sol_M to be the subspace of trajectories that are geodesics. The dynamics should respect covariance, and we should impose more conditions on the subspaces Sol_M, e.g. it should be a subpresheaf of $\mathcal{F}$. (If Sol_M is specified by differential equations, they should be natural.) If the differential equation is given as the Euler-Lagrange equations of an action,

$$S_M : \mathcal{F}(M) \to \mathbb{C}$$

many aspects of covariance are easier to build in. One should notice that an action is a function on the space of field configurations.

We thus define:

Definition 1.2.4
A classical field theory consists of a category $\mathcal{M}$ of manifolds, together with a presheaf $\mathcal{F}$ on $\mathcal{M}$ with values in Set*, called the sheaf of field configurations (or trajectories) and a subpresheaf Sol of $\mathcal{F}$ called the presheaf of classical solutions.*

Examples 1.2.5.

1. Consider a mechanical system given by a target space (T, g) which is a Riemannian manifold. A field configuration in $s \in \mathcal{F}_{(T,g)}(I)$ is a trajectory $s : I \to T$. We can consider the action
$$S_I(s) := \int_I \mathrm{d}t \, g(\dot{s}(t), \dot{s}(t)) \quad \text{with } \dot{s} := \frac{\mathrm{d}s}{\mathrm{d}t} \ .$$
The solutions for the corresponding Euler-Lagrange equation are geodesics on T.

2. This directly generalizes to higher dimensional field theories, defined on a category $\mathcal{M}$ of Riemannian manifolds, i.e. objects of $\mathcal{M}$ are pairs (M, h). Then field configurations are maps $s : M \to T$ and a natural action, the Nambu-Goto action, is
$$S_{M,h}(s) := \int_M \mathrm{d}\mu_h \, h^{\alpha\beta} g(\partial_\alpha s(t), \partial_\beta s(t))$$
One can also consider a theory defined on manifolds without metric and use the metric on M induced from the metric on T via s to define the action:
$$S_{M,h}(s) = \int_M \sqrt{s^* \det h} \ .$$
Then the classical solutions are minimal surfaces.

What are interesting questions to ask? In a mechanical system, we look for solutions φ of the equation of motion on an interval $I = [a, b]$ such that given values ξ_a, ξ_b are taken at the boundary,
$$\varphi(a) = \xi_a \quad \varphi(b) = \xi_b$$
and for example count their number (or understand the structure of this space). In this way, we also finally get time *intervals*. An example would be the space of geodesics on (T, g) connecting two given points $\xi_a, \xi_b \in T$.

This leads us to extend the type of theories we consider:

- we now consider a manifold with boundary. There might have been manifolds with boundary already in $\mathcal{M}$, but now we deal with the boundary in a special way: we prescribe values there. Hence, we rather consider a cobordism. For a mechanical system, we consider a diagram of manifolds of the form

$$\{a\} \nearrow [a,b] \nwarrow \{b\}$$

In general, we need the following definition:

Definition 1.2.6
Let n be any positive integer. We define a category $\mathrm{Cob}_{n,n-1}$ *of n-dimensional <u>cobordisms</u> as follows:*

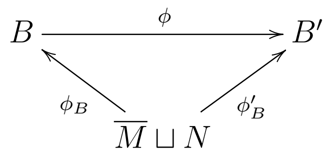
1. *An object of* $\mathrm{Cob}_{n,n-1}$ *is a closed oriented* $(n-1)$*-dimensional smooth manifold.*

2. *Given a pair of objects* $M, N \in \mathrm{Cob}_{n,n-1}$*, a morphism* $M \to N$ *is a class of cobordisms from* M *to* N*. A cobordism is an oriented,* n*-dimensional smooth manifold* B *with boundary, together with an orientation preserving diffeomorphism*

$$\phi_B : \quad \overline{M} \sqcup N \xrightarrow{\sim} \partial B \ .$$

   Here $\overline{M}$ *denotes the same manifold with opposite orientation.*

   Two cobordisms B, B' *give the same morphism in* $\mathrm{Cob}_{n,n-1}$*, if there is an orientation-preserving diffeomorphism* $\phi : B \to B'$ *such that the following diagram commutes:*



3. *For any object* $M \in \mathrm{Cob}_{n,n-1}$*, the identity map is represented by the product cobordism* $B = M \times [0, 1]$*, i.e. the so-called cylinder over* M*.*

4. *Composition of morphisms in* $\mathrm{Cob}_{n,n-1}$ *is given by gluing cobordisms: given objects* $M, M', M'' \in \mathrm{Cob}_{n,n-1}$*, and cobordisms* $B : M \to M'$ *and* $B' : M' \to M''$*, the composition is defined to be the morphism represented by the manifold* $B \sqcup_{M'} B'$*. (To get a smooth structure on this manifold, choices like collars are necessary. They lead to diffeomorphic glued cobordisms, however.)*

Example 1.2.7.

1. The objects of $\mathrm{Cob}_{1,0}$ are collections of finitely many oriented points. Thus objects are finite disjoint unions of $(\bullet, +)$ and $(\bullet, -)$. The morphisms are oriented one-dimensional manifolds, possibly with boundary, i.e. unions of intervals and circles.

2. The objects of $\mathrm{Cob}_{2,1}$ are finite disjoint unions of oriented circles. There are six elementary morphism: the cylinder, the cap, the trinion or pair of pants and their inverses and two exchanging cylinders.

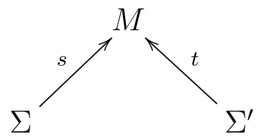
Remarks 1.2.8.

1. The definition of a cobordism uses smooth manifolds. We will need categories in which the morphisms will be manifolds with more structure. This will be possibly at the expense of loosing literally the structure of a category. (If cylinders have to have a length, we will not have identities any longer.)

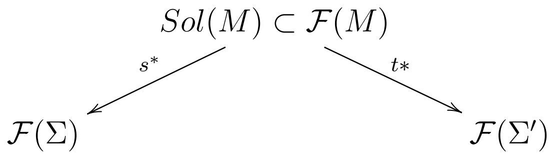
2. There is an obvious short-coming in our definition: one would like to incorporate a notion when two manifolds that are morphisms are "close to each other". We drop this aspect in these lectures.

We also have to extend the field configurations we consider:
▪ we include field configurations on the boundary. Since the boundary has a different dimension, we have to assume for this type of questions that the presheaf of field configurations $\mathcal{F}$ is also defined on a class of $n-1$-dimensional manifolds.

Given a cobordism in any dimension,



we get a so-called span of sets



Here the maps s^* and t^* are restrictions of field configurations. We can now ask questions like: given field configurations $\varphi \in \mathcal{F}(\Sigma)$ and $\varphi' \in \mathcal{F}(\Sigma')$, how many solutions to the equations of motions are there with these boundary values, i.e. study the space

$$Sol(M)_{\varphi,\varphi'} := \{\Phi \in Sol(M) \,|\, s^*\Phi = \varphi, t^*\Phi = \varphi'\} \;;$$

this is a subspace of the following space of field configurations:

$$\mathcal{F}(M)_{\varphi,\varphi'} := \{\Phi \in \mathcal{F}(M) \,|\, s^*\Phi = \varphi, t^*\Phi = \varphi'\}$$

We thus have

Definition 1.2.9
A classical $(n, n-1)$-dimensional field theory consists of a "category" Cob *of n-dimensional cobordisms of manifolds, together with a presheaf $\mathcal{F}$ with values in* Set *defined at least on all manifolds appearing either as objects or morphisms in* Cob*. This presheaf is called the presheaf of field configurations (or trajectories, in the case of one-dimensional systems).*

Moreover, we have a subpresheaf Sol of $\mathcal{F}$ defined on manifolds appearing as a morphism of Cob*, called the presheaf of classical solutions. (We definitely neglect here some important structure on the boundary, think about the role of boundary terms in Noether's theorem!)*

For later use, we present the following

Example 1.2.10.
For a two-dimensional σ-model given by a target space T, we have boundaries which are closed oriented 1-manifolds S and thus disjoint unions of circles. Thus

$$\mathcal{F}_T(S) = \mathcal{F}_T(S^1 \sqcup \ldots \sqcup S^1) = C^\infty(S^1 \sqcup \ldots \sqcup S^1, T) = C^\infty(S^1, T) \times \ldots \times C^\infty(S^1, T)\ .$$

We thus get a Cartesian product of loop spaces $LM := C^\infty(S^1, T)$. In the case of loop groups, LG, in particular when the target space is a compact Lie group G, there are tools from representation theory and infinite-dimensional analysis that are used in certain approaches to conformal field theory.

1.3 Quantum field theories

We have now to motivate the structures appearing in a quantum field theory. We use the "path" integral as a heuristic tool. The word "path integral" is appropriate for one-dimensional systems; in the general case, we have integrals over spaces of field configurations.

Observation 1.3.1.

- Suppose we are given a classical n-dimensional field theory with action S, defined on a category $\mathrm{Cob}_{n,n-1}$.
- Closed n-dimensional manifolds are endomorphisms $M \in \mathrm{End}_{\mathrm{Cob}}(\emptyset)$. The heuristic idea is to introduce an invariant $Z(M)$ for an n-manifold M – a "partition function"– by integration over all field configurations on M:

$$Z(M) := \text{"} \int_{\mathcal{F}(M)} \mathrm{d}\phi \; e^{iS[\phi]} \text{"}.$$

  In general, this path integral has only a heuristic meaning.

- We consider a n-manifold M with a $(n-1)$-dimensional boundary $\Sigma := \partial M$. We fix a boundary field configuration $\phi_1 \in \mathcal{F}(\Sigma)$ and consider the space $\mathcal{F}_{\varphi_1}(M)$ of all fields ϕ on M that restrict to the given boundary values φ_1.

  Once we have fixed boundary values ϕ_1 of the field, we can think about performing a path integral over the space of field configurations $\mathcal{F}_{\varphi_1}(M)$. We therefore introduce, again at a heuristic level, the complex number

$$Z(M)_{\varphi_1} := \text{"} \int_{\mathcal{F}_{\varphi_1}(M)} d\phi \; e^{iS[\phi]} \text{"}. \tag{1}$$

  Any n-manifold M with boundary Σ thus provides an assignment

$$\begin{array}{rccl} \psi_M : & \mathcal{F}(\Sigma) & \to & \mathbb{C} \\ & \phi_1 & \mapsto & Z(M)_{\varphi_1} \; ; \end{array}$$

  We are thus lead to assign to a codimension 1 manifold Σ a vector space $\mathcal{H}_\Sigma$, the *state space*

$$\mathcal{H}_\Sigma := \text{"} L^2\big(\mathcal{F}(\Sigma), \mathbb{C}\big) \text{"}$$

  of "wave functions". Any n-manifold M with boundary Σ specifies a wave function in the state sapce $\mathcal{H}_\Sigma$. Our idea will be to find an independent starting point for the construction of these state spaces.

- The transition from field configurations to wave functions amounts to a *linearization*. The notation L^2 should be taken with a grain of salt; it should not suggest the existence of any distinguished measure on a general category.
- The situation naturally generalizes to cobordisms $\Sigma \to M \leftarrow \Sigma'$ to which we wish to associate a linear map

$$Z(M) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

  by giving its matrix elements ("transition amplitudes") in terms of the path integral

$$Z(M)_{\phi,\phi'} := \text{"} \int_{\mathcal{F}_{\varphi,\varphi'}(M)} \mathrm{d}\Phi \; e^{iS[\Phi]} \text{"}$$

  with fixed boundary field configurations $\varphi \in \mathcal{F}(\Sigma)$ and $\varphi' \in \mathcal{F}(\Sigma')$. Here $\mathcal{F}_{\phi,\phi'}(M)$ is the space of field configurations on M that restrict to the field configuration ϕ on the ingoing boundary Σ and to the field configuration ϕ' on the outgoing boundary Σ'.

- The linear maps $Z(M)$ should be compatible with gluing of cobordisms along boundaries.

In a two-dimensional theory, one has finite disjoint unions of circles as boundaries. to see further properties of the assignment, we consider a two-dimensional sigma model with target space T. We recall that the classical configuration space is the space functions on $LT := C^\infty(S^1, T)$, the loop space for T. For the vector space, one might take functions $L^2(LT, \mathbb{C})$ on the loop space (or sections in line bundles over it). We thus expect that the vector space assigned to the one-dimensional manifold $S \cong S^1 \sqcup S^1$ is

$$L^2(C^\infty(S^1, T) \times C^\infty(S^1, T)) \cong L^2(C^\infty(S^1, T)) \otimes L^2(C^\infty(S^1, T)) \ .$$

We therefore postulate that the assignment should be such that $Z(S^1 \sqcup S^1) \cong Z(S^1) \otimes Z(S^1)$.

No measure exists on $\mathcal{F}(\Sigma)$ which is typically an infinite-dimensional space, e.g. a loop space. Therefore, such a space is not an easy starting point. We therefore do the following formal argument and consider cobordisms. Let us take the simplest example of a target space, $T = \mathbb{R}$. With the action

$$S_{(M,h)}[X] = \int_M \sqrt{\det h}\, h^{\mu\nu} \partial_\mu X \partial_\nu X \tag{2}$$

one obtains a conformal field theory that is called massless free real boson. Suppose for simplicity that we have $\Sigma' = \emptyset$, Σ is the disjoint union of n circles and M is the complement of n discs in $P^1 \cong \mathbb{C} \cup \{\infty\}$. Denote the boundary circles by C_i, with $i = 1, \ldots, n$. We prescribe boundary conditions $f_i \in \mathcal{F}_\mathbb{R}(S^1)$ and thus fix n functions

$$f_i : S^1 \to \mathbb{R} \ .$$

We then should "integrate" over the space all functions $X : M \to \mathbb{R}$ that restrict to f_i on C_i. Thus we want to give sense to

$$\begin{aligned} & \int \mathcal{D}X \mathrm{e}^{\mathrm{i}S(X)} \prod \delta(X(c) - f(c)) \\ = & \int \mathcal{D}X \mathrm{e}^{\mathrm{i}S(X)} \prod \int \mathrm{e}^{\mathrm{i}k_i(X(c)-f(c))} dk_i \\ = & \int dk_i \mathrm{e}^{-\mathrm{i}k_i f(c)} \int \mathcal{D}X \mathrm{e}^{\mathrm{i}S(X)} \prod \mathrm{e}^{\mathrm{i}k_i(X(c))} \end{aligned}$$

We thus wish to give some sense for each n-tuple $(k_1, \ldots, k_n) \in \mathbb{R}^n$ and each n-tuple $(z_1, \ldots, z_n)$ of distinct points in $\mathbb{C}$ to the expression

$$\int \mathcal{D}X \mathrm{e}^{\mathrm{i}S(X)} \prod \mathrm{e}^{\mathrm{i}k_i(X(z_i))} =: \langle \mathrm{e}^{\mathrm{i}k_1 X(z_1)} \ldots \mathrm{e}^{\mathrm{i}k_n X(z_n)} \rangle \tag{3}$$

We call the right hand side a correlator. We consider correlators (and not actions and path integrals) as the basic building blocks of quantum field theories. We consider a correlator as a function on the configuration space of n distinct points on P^1. The expression $\mathrm{e}^{\mathrm{i}k_1 X(z_1)}$ does not have, for the moment, any independent meaning. We will relate it to vertex operators only later.

1.4 A discrete model

To find a reasonable definition for the expression (3), we replace the d-dimensional manifold M by a finite d-dimensional cubic lattice with periodic boundary conditions

$$M_{\text{disc}} = (\mathbb{Z}_m)^d$$

On this space, we construct a probabilistic model that has essential features of the free boson. We need:

- A measure space Ξ with probability measure μ
- Random variables are measurable functions $f : \Xi \to \mathbb{R}$ for which we would like to study expectations

$$E(f) = \langle f \rangle = \int f \mathrm{d}\mu$$

We realise such a model as follows:

- for Ξ, take the finite-dimensional real vector space

$$\Xi := C^\infty(M_{\text{disc}}, \mathbb{R}) \cong \mathbb{R}^{M_{\text{disc}}}$$

space of maps X from M_{disc} to the target space $\mathbb{R}$. A field configuration $X \in \Xi$ is a vector $(X(p))_{p \in M_{\text{disc}}}$ and $X(p)$ is the value of the field X in the point $p \in M_{\text{disc}}$.

- The probability measure that specifies our model is continuous with respect to the Lebesgue measure on the finite-dimensional vector space Ξ and is given by the exponential of the action S, which is now a function

$$\mathrm{e}^{\mathrm{i}S_M} : \quad C^\infty(M_{\text{disc}}, \mathbb{R}) \to \mathbb{C}$$

Thus the measure is

$$\mathrm{d}\mu(X) = \mathrm{e}^{-S_M(X)}/Z \mathrm{d}\mu_M$$

We have included a normalization factor Z, called the partition function, to ensure that the measure μ has total weight 1.

For the action, we replace differentials by differences and as a replacement for a metric on M_{disc}, we choose a symmetric matrix $(K_{p_1 p_2})_{p_1, p_2 \in M_{\text{disc}}}$:

$$S(X) = \frac{\beta_0}{4\pi} \sum_{|t-s|=1} |X(s) - X(t)|^2 =: \frac{1}{2}(X, KX)$$

The sum is thus over nearest neighbors. We simplify our model by the assumption that the matrix K is positive definite and thus in particular non-degenerate.

We are thus lead to compute Gaussian integrals. Gaussian integrals are essentially combinatorial. Denoting by A the inverse of the matrix K, we have the following integral over a finite-dimensional vector space:

$$\int_\Xi \mathrm{e}^{-\frac{1}{2}(X,KX)} DX = (\det A)^{1/2} (2\pi)^{|M_{\text{disc}}|/2}$$

This fixes the normalization constant:

$$Z = \int_\Xi \mathrm{e}^{-\frac{1}{2}(X,KX)} DX = (\det A)^{1/2} (2\pi)^{|M_{\text{disc}}|/2} \tag{4}$$

Determinants of operators acting on spaces of functions are thus important. There is a good mathematical theory for this, which is not the subject of these lectures.

As a trick to get access to correlators, we regard Z as in (4) as a function of the "metric" K on M_{disc}. (This is a discrete classical background field.) Differentiating $Z(K)$ with respect to K_{p_1,p_2} gives the identity

$$\frac{\int_\Xi X(p_1) X(p_2) \mathrm{e}^{-\frac{1}{2}(X,KX)} DX}{\int_\Xi \mathrm{e}^{-\frac{1}{2}(X,KX)} DX} = A_{p_1,p_2}$$

One can see this as a 2-point correlator for the "field" X. By repeating this, we obtain for the n-point correlator zero, if n is odd, for symmetry reasons, and for $n = 2m$

$$\langle X(p_1)\dots X(p_{2m})\rangle := \frac{\int_\Xi X(p_1)X(p_2)\dots X(p_{2m})\mathrm{e}^{-\frac{1}{2}(X,KX)}DX}{\int_\Xi \mathrm{e}^{-\frac{1}{2}(X,KX)}DX} = \sum \prod_{\text{pairs } \sigma} A_{\sigma_+,\sigma_-} \tag{5}$$

where the sum is over all possibilities to group the variables $p_1 \dots p_{2m}$ in pairs and then one multiplies the corresponding elements of the matrix A. The matrix A is also called the propagator.

A convenient trick to summarize the information in (5) is the following: introduce a vector J in $C^\infty(M_{\text{disc}}, \mathbb{R}) \cong \mathbb{R}^{M_{\text{disc}}}$, called a *source*; this is to be seen as a function on M_{disc}. Consider the function

$$\begin{array}{rcl} \mathbb{R}^{M_{\text{disc}}} & \to \mathbb{R} & \\ Z[J] & := & \int_\Xi DX\mathrm{e}^{-\frac{1}{2}(X,KX)+(J,X)} \end{array} \tag{6}$$

Then, the n-point correlators in (5) can be recovered by repeated partial differentiation:

$$\frac{1}{Z}\frac{\partial^n Z[J]}{\partial J_{p_1}\dots J_{p_n}}|_{J=0} = \langle X(p_1)\dots X(p_n)\rangle$$

In the Gaussian model, there is a simple quadratic completion of the exponent to reduce the integral (6) to a Gaussian integral and one computes

$$Z[J] = (\det A)^{1/2}(2\pi)^{|M_{\text{disc}}|}\mathrm{e}^{\frac{1}{2}\langle J,AJ\rangle}$$

The same trick allows to obtain the correlators

$$\langle \mathrm{e}^{k_1X(p_1)}\dots \mathrm{e}^{k_nX(p_n)}\rangle = \frac{1}{Z}\int_\Xi DX \exp(\sum_i k_iX(p_i))\exp(X,KX)$$

from a source function of the form

$$J(z) = \sum_{i=1}^n k_i\delta(z-p_i)$$

as

$$\langle \mathrm{e}^{k_1X(p_1)}\dots \mathrm{e}^{k_nX(p_n)}\rangle = (\det A)^{1/2}\mathrm{e}^{1/2J^tAJ} = \exp(\sum_{ij} k_ik_j\frac{1}{2}A_{p_ip_j}) = \prod_{i,j}\exp(\frac{1}{2}A_{p_ip_j})^{k_ik_j} \ .$$

This should be seen as a discrete version of the correlator (3) to which we have to give sense (and which will be the starting point of our more rigorous considerations).

1.5 The free boson: aspects of the continuum theory

We now come back to the free boson, a theory defined on two-dimensional conformal manifolds with target space $T = \mathbb{R}$ with the standard metric. Correlators will be the basis of our theory, but not probability measures.

We fix a metric h on M in the conformal class and consider the action

$$S_{(M,h)}(X) = \int_M \mathrm{d}z\sqrt{h}h^{\mu\nu}\partial_\mu X\partial_\nu X = \int_M \mathrm{d}X \wedge \star_h \mathrm{d}X$$

One verifies that the value of the action does not change under local conformal transformations

$$h(z) \mapsto \mathrm{e}^{\omega(z)}h(z) \ .$$

To arrive at a set of correlation functions, we use a formal analogy to the previous subsection. More rigorous methods are available, but not the topic of these lectures. For more details, we refer to [G]. For the metric h, we have a Laplacian Δ_h which depends on the metric and not only on the conformal structure. The integrand in the action becomes, after a partial integration in the action

$$K(p_1,p_2)X(p_1)X(p_2) = X(p_1)(-\Delta_h)X(p_2)\delta(p_1,p_2)$$

It has a kernel – harmonic functions – which has to be dealt with separately. The role of the inverse A of K is played by the Green function that obeys

$$-\Delta G'(\sigma_1,\sigma_2) = h^{-1/2}\delta(\sigma_1,\sigma_2) - vol(M) \ .$$

For a sphere with quasi global coordinate z and metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\omega(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z} \tag{7}$$

the Green function reads

$$G' = -\frac{1}{4\pi}\log|z_1-z_2|^2 + f(z_1,\bar{z}_1) + f(z_2,\bar{z}_2) \tag{8}$$

for a certain function f depending on the conformal function ω (this function will be neglected soon).

In this way, one is lead to consider the expression

$$\begin{aligned} &\langle \exp(\mathrm{i}k_1X(p_1))\ldots\exp(\mathrm{i}k_nX(p_n))\rangle \\ = \ &\underbrace{\mathcal{N}(\det\frac{-\Delta_h}{2\pi})^{-1}\exp\left(-\frac{1}{4\pi}\sum_i k_i^2\omega(p_i)\right)}\delta(\textstyle\sum_i k_i)\prod_{i<j}|p_i-p_j|^{k_ik_j/2} \end{aligned}$$

Notice that the first two terms depend on the conformal factor in the metric (7). We will neglect them and consider the functions on the complex plane with flat metric

$$\langle \exp(\mathrm{i}k_1X(p_1))\ldots\exp(\mathrm{i}k_nX(p_n))\rangle = \epsilon(k_1,k_2,\ldots,k_n)\delta(\sum_i k_i)\prod_{i<j}|p_i-p_j|^{k_ik_j} \tag{9}$$

Such expressions are the starting point for us to derive the algebraic notion of a vertex algebra.

1.6 Holomorphic factorization

Correlation functions are functions on the configuration space $\mathcal{M}_{0,n}$ of n marked points on S^2 *as a conformal manifold* (or sections in a line bundle over it). Two dimensional conformal manifolds and complex one-dimensional manifolds are closely related: a complex one-dimensional manifold is an orientable conformal manifold, together with a choice of orientation.

For any manifold M, the orientation cover $\hat{M}$ is a two-fold cover $\hat{M}\xrightarrow{\pi}M$ whose total space consists of points $p\in M$, together with a choice of local orientation in p.

Examples 1.6.1.

1. The orientation cover of the sphere S^2 consists of two copies of S^2 with opposite orientation. If the oriented sphere $S^2\cong\mathbb{C}\cup\{\infty\}$ is described by a quasi-global coordinate $z\in\mathbb{C}$, denote the coordinates on the two connected components of $\hat{S}^2$ by z, z^*; the two points in the same fibre are related by the $\mathbb{Z}_2$-action $z\mapsto z^*=\bar{z}$, i.e. by complex conjugation (which indeed reverses the orientation).

2. More generally, for any closed orientable manifold M, the orientation cover $\hat{M}$ consists of two disjoint copies of M with opposite orientation.

3. The orientation cover of a Klein bottle is the torus. More precisely, consider a torus given by identifying the opposite edges of the rectangle $(0, 1, 2\mathrm{i}t, 2\mathrm{i}t + 1)$ with $t \in \mathbb{R}_+$ in the complex plane, modulo the anticonformal involution

$$z \mapsto 1 - \overline{z} + \mathrm{i}t \ ,$$

which does not have any fixed points.

4. For a two-dimensional manifold with boundary, we consider a version of the cover in which boundary circles are identified. An example is given by the Möbius strip. The cover is a parallelogram in the complex plane with vertices $(0, 1, \frac{1}{2} + \frac{\mathrm{i}t}{2}, \frac{3}{2} + \frac{\mathrm{i}t}{2})$ and anticonformal involution $z \mapsto 1 - \overline{z}$ which has fixed points.

There is a natural map

$$\mathrm{doub}: \quad \mathcal{M}_n(M) \to \mathcal{M}_{2n}(\hat{M})$$

from the moduli space of n points in M to the moduli space of $2n$ points in $\hat{M}$ which associates to an n-tuple of points in M the $2n$-tuple of their preimages under $\pi : \hat{M} \to M$. Later, we will also include moduli of the conformal structure of M and of the complex structure of $\hat{M}$. (This has already been done in the example of the Möbius strip and the Klein bottle.)

In the case of the Riemann sphere with a quasi-global coordinate z, and an orientation double with coordinates z and z^*, we have

$$\begin{array}{rccc} \mathrm{doub}: & \mathcal{M}_n(P^1) & \to & \mathcal{M}_{2n}(P^1 \sqcup \overline{P}^1) \\ & (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_n; \overline{z}_1, \ldots, \overline{z}_n) \end{array}$$

The correlator (9) which is a meromorphic section on $\mathcal{M}_n(M)$ for M a conformal surface, is the product of the meromorphic functions

$$\prod_{i<j} (z_i - z_j)^{k_i k_j} \quad \text{and} \quad \prod_{i<j} (z_i^* - z_j^*)^{k_i k_j} \ , \tag{10}$$

with z_i^* set to $\overline{z}_i$. In general, products can be related to pullback. Consider e.g. a manifold M and the diagonal map $\Delta : M \to M \times M$. A smooth function on $M \times M$ given by $f(x, y) = g_1(x) g_2(y)$ is then pulled back to $\Delta^* f(x) = g_1(x) \cdot g_2(x)$.

The idea is thus that correlators on M are pullbacks along the map doub of *holomorphic* functions on $\mathcal{M}_{2n}(\hat{M})$ defined on the moduli space of points for the double $\hat{M}$. Actually, we should consider not only functions, but allow for sections in non-trivial bundles as well. These sections will be called conformal blocks and are the *building blocks* of local correlators.)

A technically more convenient starting point for us is a different set of correlators than (9). We differentiate and study

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \langle X(z_1) X(z_2) \rangle = \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \log |z_1 - z_2| = \frac{1}{(z_1 - z_2)^2}$$

We see this as a 2-point correlator of a new field $J(z) = \partial X(z)$.

This gives a purely holomorphic expression. Repeating the differentiation, we get for the n-point correlators purely holomorphic expressions as well:

$$\langle J(z_1) \ldots J(z_{2n}) \rangle = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{j=1}^{n} \frac{1}{\left(z_{\pi(j)} - z_{\pi(j+n)}\right)^2} \tag{11}$$

and that these expressions transform as a density over the moduli space of $2n$ points on the sphere, i.e.

$$\langle J(z_1)\dots J(z_{2n})\rangle \mathrm{d}z_1 \dots \mathrm{d}z_{2n}$$

is independent of the choice of coordinates.

2 Amplitudes, vertex algebras and representations

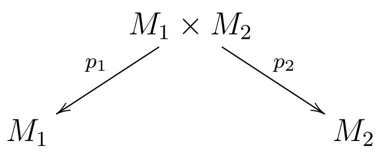
We now show how to obtain from a set of amplitudes of finitely many fields for an arbitrary number of insertions on a Riemann sphere the algebraic structure of a vertex algebra and its representations. We follow [GG].

2.1 From sets of amplitudes to vertex operators

To describe the input data of our construction, we first need the following

Observation 2.1.1.

1. The Cartesian product $M_1 \times M_2$ of two manifolds comes with canonical projections



   Suppose we are given line bundles $L_1 \to M_1$ and $L_2 \to M_2$. Then

$$L_1 \boxtimes L_2 := (p_1)^* L_1 \otimes_{\mathbb{C}} (p_2)^* L_2$$

   is a line bundle on $M_1 \times M_2$. (This construction obviously generalizes to vector bundles.)

2. Given a Riemann surface Σ, we can consider the holomorphic tangent bundle $T\Sigma$. It is a holomorphic line bundle. Sections in its dual are meromorphic differential forms. In local coordinates, they read $\omega = \omega(z)\mathrm{d}z$ and transform as differential form under the change of local holomorphic coordinates.

Definition 2.1.2
A consistent set of amplitudes consists of the following data:

- *A a $\mathbb{Z}$-graded vector space $V = \oplus_{h\in\mathbb{Z}} V_h$.*

  We call a homogeneous vector $v \in V_h$ a quasi-primary field of scaling dimension or conformal weight $h \in \mathbb{Z}$.

- *For any n-tuple $(v_1, v_2, \dots, v_n) \in V^n$ of homogeneous vectors of degree* $\deg(v_i) = h_i$, *we have a meromorphic section, called the amplitude, of the bundle*

$$T^{-h_1}P^1 \boxtimes T^{-h_2}P^1 \boxtimes \dots \boxtimes T^{-h_n}P^1$$

  over $\mathcal{M}_n(P^1) = (P^1)^{\times n}$ which we write

$$f(v_1,\dots,v_n; z_1,\dots,z_n) \equiv \langle V(v_1,z_1)V(v_2,z_2)\cdots V(v_n,z_n)\rangle \prod_{j=1}^{n} (\mathrm{d}z_j)^{h_j} \ .$$

The amplitudes are required to obey the following axioms:

- *They are multilinear in the vectors v_i.*
- *(Meromorphicity:) They are analytic, except possibly for finite order poles, if points coincide, $z_i = z_j$.*
- *(Locality:) They are invariant under exchange $(v_i, z_i) \leftrightarrow (v_j, z_j)$,*

$$\begin{aligned} & f(v_1, \ldots, v_i, \ldots, v_j \ldots, v_n; z_1, \ldots, z_i, \ldots, z_j, \ldots z_n) \\ = \; & f(v_1, \ldots, v_j, \ldots, v_i \ldots, v_n; z_1, \ldots, z_j, \ldots, z_i, \ldots z_n) \end{aligned}$$

- *(Non-degeneracy:) If all amplitudes involving a given $\psi \in V$ vanish, then $\psi = 0$.*

We should warn the reader that there are interesting chiral conformal field theories, so-called logarithmic conformal field theories, which are not covered by these axioms and where one has to admit for logarithms in the amplitudes.

Since the behaviour as a differential form is fixed by the conformal weight, we follow the usual convention: we fix a quasi-global coordinate z on P^1 and drop all basis elements $\mathrm{d}z$.

Examples 2.1.3.

1. Free boson, Heisenberg algebra:
   In this case, V is one-dimensional in degree 1. We fix a generator J of V, i.e. $V = \mathbb{C}J$ and call the corresponding field an abelian current. The amplitude of an odd number of J-fields is defined to vanish, and in the case of an even number it is given, with the shorthand $J(z) := V(J, z)$ by the expression in (11),

$$\langle J(z_1) \cdots J(z_{2n}) \rangle = \frac{k^n}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\pi(j)} - z_{\pi(j+n)})^2}$$

   where k is an arbitrary (real) constant and S_{2n} is the permutation group on $2n$ object. This defines the amplitudes on a basis of V and we extend the definition by multilinearity. It is clear that the amplitudes are meromorphic in z_j, and that they satisfy the locality condition.

2. Affine Lie algebras:
   We generalise this example to the case of an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$.

Definition 2.1.4

(a) *A Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space, together with a bilinear map, called the Lie bracket,*

$$[\cdot, \cdot] \quad \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that is antisymmetric, $[x, x] = 0$ for all $x \in \mathfrak{g}$, and for which the Jacobi identity holds,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \textit{for all } x, y, z \in \mathfrak{g} \; .$$

(b) *A morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a $\mathbb{K}$-linear map such that $[\phi(x), \phi(y)] = \phi([x, y])$ for all $x, y \in \mathfrak{g}$.*

Important examples of Lie algebras include the Lie algebra $\mathrm{gl}(V) = \mathrm{End}_{\mathbb{K}}(V)$ for an any K-vector space, with the Lie bracket given by the commutator. For $\dim_{\mathbb{K}} V < \infty$, traceless matrices form a Lie subalgebra $\mathrm{sl}(V)$.

For any element $t \in \mathfrak{g}$, we consider the endomorphism $\mathrm{ad}_t := [t, -]$ of $\mathfrak{g}$. The Jacobi identity implies

$$\mathrm{ad}_{t_1} \circ \mathrm{ad}_{t_2} - \mathrm{ad}_{t_2} \circ \mathrm{ad}_{t_1} = \mathrm{ad}_{[t_1,t_2]} \ ;$$

thus $t \mapsto \mathrm{ad}_t$ defines the adjoint action of $\mathfrak{g}$ on itself. Let K be any endomorphism of $\mathfrak{g}$ commuting with the adjoint action of $\mathfrak{g}$, e.g. the identity. For any $n \in \mathbb{N}$, define an n-linear map

$$\begin{array}{rcl} \kappa : \mathfrak{g} \times \ldots \times \mathfrak{g} & \to & \mathbb{C} \\ (t_1, t_2, \ldots, t_n) & \mapsto & \mathrm{tr}_{\mathfrak{g}}(K \circ \mathrm{ad}_{t_1} \circ \mathrm{ad}_{t_2} \circ \ldots \circ \mathrm{ad}_{t_n}) \ . \end{array}$$

These forms have cyclic symmetry and, due to the Jacobi identity, obey

$$\kappa(t_1, t_2, t_3, \ldots, t_{m-1}, t_m) - \kappa(t_2, t_1, t_3 \ldots, t_{m-1}, t_m) = \kappa([t_1, t_2], t_3 \ldots, t_{m-1}, t_m) \, .$$

Our amplitudes are now defined by taking V equal to $\mathfrak{g}$ in degree 1. We call the elements of V non-abelian currents and use for $t \in \mathfrak{g}$ the shorthand $t(z) := V(t, z)$.

To a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_m) \equiv (i_2, \ldots, i_m, i_1)$ and $(t_1, \ldots, t_m) \in \mathfrak{g}^m$, we associate the function on $\mathcal{M}_m(P^1)$

$$f_\sigma^{t_1,t_2\ldots t_m}(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) = \frac{\kappa(t_1, t_2, \ldots, t_m)}{(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \cdots (z_{i_{m-1}} - z_{i_m})(z_{i_m} - z_{i_1})} \, .$$

A permutation $\rho \in S_n$ without fixed points can be written as the product of disjoint cycles of length at least two, $\rho = \sigma_1 \sigma_2 \ldots \sigma_M$. We associate to ρ the product f_ρ of functions $f_{\sigma_1} f_{\sigma_2} \ldots f_{\sigma_M}$ and define $\langle t_1(z_1) t_2(z_2) \ldots t_n(z_n) \rangle$ to be the sum of such functions f_ρ over permutations $\rho \in S_n$ with no fixed point.

The amplitudes are evidently local and meromorphic.

3. Lattice Theories:
These theories appear in the description of free bosons compactified on lattices.

Let $\Lambda \subset V$ be an even n-dimensional Euclidean lattice; an even lattice is a lattice for which $\langle k, k \rangle \in 2\mathbb{Z}$ for all $k \in \Lambda$. For any pair $v_1, v_2 \in \Lambda$ of lattice vectors of an even lattice, we have by polarization

$$\langle v_1, v_2 \rangle = \frac{1}{2} \left(\langle v_1 + v_2, v_1 + v_2 \rangle - \langle v_1, v_1 \rangle - \langle v_2, v_2 \rangle \right) \in \mathbb{Z} \ .$$

As a consequence, the dual lattice

$$\Lambda^* := \{ v \in V \mid \langle v, w \rangle \in \mathbb{Z} \quad \text{for all } w \in \Lambda \}$$

of an even lattice obeys $\Lambda^* \supset \Lambda$. Then $A_\Lambda := \Lambda^*/\Lambda$ is a finite abelian group on which the quadratic form

$$v \mapsto \langle v, v \rangle \bmod \mathbb{Z}$$

is well-defined.

We need to construct auxiliary quantities:
Fix a basis $e_1, e_2, \ldots, e_n$ of Λ as a free $\mathbb{Z}$-module. Consider an algebra with generators γ_j,

$1 \leq j \leq n$ and relations $\gamma_j^2 = 1$ and $\gamma_i\gamma_j = (-1)^{\langle e_i, e_j\rangle}\gamma_j\gamma_i$. For $k = m_1e_1 + m_2e_2 + \ldots + m_ne_n \in \Lambda$, we define $\gamma_k := \gamma_1^{m_1}\gamma_2^{m_2}\ldots\gamma_n^{m_n}$. Then one can show that

$$\gamma_{k_1}\gamma_{k_2}\ldots\gamma_{k_N} = \epsilon(k_1, k_2, \ldots, k_N)\gamma_{k_1+k_2+\ldots+k_N}$$

with $\epsilon(k_1, k_2, \ldots, k_N) \in \{\pm 1\}$.

We take V to be the (infinite-dimensional) vector space $\mathbb{C}[\Lambda]$ freely generated by the set underlying the lattice Λ. It has a natural basis $\{\psi_k : k \in \Lambda\}$; we define a $\mathbb{Z}$-grading on V by assigning grade $\frac{1}{2}k^2 \in \mathbb{Z}$ to ψ_k. All homogeneous components of V are finite-dimensional.

Writing $V(\psi_k, z) = V(k, z)$, we define the n-point amplitudes to be

$$\langle V(k_1, z_1)V(k_2, z_2)\cdots V(k_N, z_n)\rangle = \epsilon(k_1, k_2, \ldots, k_N) \prod_{1\leq i<j\leq N} (z_i - z_j)^{k_i\cdot k_j}$$

if $k_1 + k_2 + \ldots + k_N = 0$ and zero otherwise. This expression should be compared to the one in equation (9). Correspondingly, we write in this case

$$\langle V(k_1, z_1)V(k_2, z_2)\cdots V(k_N, z_n)\rangle = \langle \mathrm{e}^{k_1X(z_1)}\mathrm{e}^{k_2X(z_2)}\ldots\mathrm{e}^{k_nX(z_n)}\rangle \ .$$

Our idea is now to generate a "space of states" from a consistent set of amplitudes. We will see that we rather get a family of topological vector spaces. A word of warning is in order: the construction we present will *not* give us all states that appear in a conformal field theory, but only the "descendants of the vacuum".

Observation 2.1.5 (1.step)**.**
We fix an open set $\mathcal{C} \subset P^1 \setminus \{\infty\}$. Denote by $\mathcal{M}_n(\mathcal{C})$ the space of n distinct ordered points in $\mathcal{C}$. Consider the vector space

$$\mathcal{V}_\mathcal{C} := \oplus_{n=0}^\infty V^{\otimes n} \otimes \mathrm{span}_\mathbb{C}(\mathcal{M}_n(\mathcal{C})) \ ;$$

This vector space contains in particular elements which we write quite formally in the form

$$\vec{\psi} := \prod_{i=1}^n V(\psi_i, z_i)\Omega \quad \text{for} \quad \psi_1 \otimes \ldots \psi_n \in V^{\otimes n} \text{ and } (z_1, \ldots, z_n) \in \mathcal{M}_n(\mathcal{C})$$

with all ψ_i homogeneous elements. We call the set of such elements $\mathcal{B}_\mathcal{C}$; they form an (uncountable) set of generators for $\mathcal{V}_\mathcal{C}$. The elements of $\mathcal{B}_\mathcal{C}$ are thus just finitely many points, with a homogeneous vector in V attached to each point. We interpret the vector $\vec{\psi}$ as being generated by the quasi-primary field $V(\psi_i)$ at the point z_i by acting on the vacuum Ω.

The vector space $\mathcal{C}$ has not enough structure: we have not implemented the idea that the same field $V(\psi)$ at insertion points z_i and z_i' that are close to each other should give "neighbouring" states. To this end, we need a topology on $\mathcal{V}_\mathcal{C}$. The only tool we have available to this end is the set of amplitudes from definition 2.1.2.

Observation 2.1.6 (2.step)**.**

1. Let $\mathcal{O}$ be an open subset of P^1 in the complement of $\mathcal{C}$, i.e. $\mathcal{O}\cap\mathcal{C} = \emptyset$. Each vector $\vec{\phi} \in \mathcal{B}_\mathcal{O}$ defines a map on $\mathcal{V}_\mathcal{C}$ by the amplitude

$$\eta_{\vec{\phi}}(\vec{\psi}) = \left\langle \prod_{i=1}^m V(\vec{\phi}_i, \zeta_i) \prod_{j=1}^n V(\vec{\psi}_j, z_j) \right\rangle \ .$$

2. One could say that we are using states attached to points in $\mathcal{O}$ to "test" the states attached to points in $\mathcal{C}$.

3. We now define a topology on $\mathcal{V}_\mathcal{C}$. For any compact subset $K \subset \mathcal{O}$ and any $\epsilon > 0$, consider the compact set of m distinct ordered points in K of distance at least ϵ

$$K_\epsilon^m := \{(\zeta_1, \dots, \zeta_m) \quad \text{with } \zeta_i \in K, |\zeta_i - \zeta_j| \geq \epsilon\} \subset \mathcal{M}_n(\mathcal{O}) \ .$$

   We say that a sequence (ψ_i) of elements in $\mathcal{V}_\mathcal{C}$ is an $\mathcal{O}$-Cauchy sequence, the sequence of complex numbers

$$\eta_{\vec{\phi}}(\vec{\psi}_i)$$

   converges uniformly for all $\vec{\phi} \in \mathcal{B}_\mathcal{O}$ with insertion points $\vec{\zeta} \in K_\epsilon^m$.

4. One can show that the limit $\lim_{i\to\infty} \eta_{\vec{\phi}}(\vec{\psi}_i)$ is a meromorphic function of the ζ_i, with singularities only for coinciding points. This yields on $\mathcal{V}_\mathcal{C}$ the structure of a topological vector space, which depends on the choice of $\mathcal{O}$.

5. The corresponding completion $\tilde{\mathcal{V}}_\mathcal{C}^\mathcal{O}$ is still too big: we have to divide out those elements $\vec{\psi} \in \tilde{\mathcal{V}}_\mathcal{C}^\mathcal{O}$ for which

$$\eta_{\vec{\phi}}(\psi) = 0 \quad \text{ for all } \quad \vec{\phi} \in \mathcal{B}_\mathcal{O} \ ,$$

   because they cannot be distinguished by the amplitudes. We thus divide out the radical of all forms $\eta_{\vec{\phi}}$ used to induce the topology and denote the corresponding quotient by $\mathcal{V}_\mathcal{C}^\mathcal{O}$. (This should be compared to the construction of the Banach space $L^1(\mathbb{R}^n)$ by dividing out all integrable functions such that $\int_{\mathbb{R}^n} |f| = 0$.)

   It can be shown that the topology of $\mathcal{V}_\mathcal{C}^\mathcal{O}$ is given by a countable family of seminorms. Under the assumption of cluster decomposition (see below), the span of $\mathcal{B}_\mathcal{C}$ can be identified with a dense subspace of any $\mathcal{V}_\mathcal{C}^\mathcal{O}$.

One then shows the following

Theorem 2.1.7.

1. If the complement of $\mathcal{O}$ is path connected, then $\mathcal{V}_\mathcal{C}^\mathcal{O}$ is independent of $\mathcal{C}$.

2. If $\mathcal{O} \subset \mathcal{O}'$, then $\mathcal{V}^{\mathcal{O}'} \subset \mathcal{V}^\mathcal{O}$ as a dense subset.

Remarks 2.1.8.

1. The idea of the proof of 1. is the fact that a meromorphic function is already uniquely specified on any open subset.

2. The idea in the proof of 2. is the fact, that for the larger set $\mathcal{O}'$, we impose more convergence conditions on sequences vectors in $\mathcal{V}_\mathcal{C}$ to call them a Cauchy sequence.

3. One can show independence of the choice of global coordinate.

4. We summarize our findings on the space of states: For any open set $\mathcal{O}$ such that the complement is path connected, we have a topological vector space $\mathcal{V}^\mathcal{O}$. For $\mathcal{O} \subset \mathcal{O}'$, then $\mathcal{V}^{\mathcal{O}'} \subset \mathcal{V}^\mathcal{O}$. Moreover, for any open set $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{O} = \emptyset$, the set of vectors $\mathcal{B}_\mathcal{C}$ is a dense subset of $\mathcal{V}^\mathcal{O}$.

We can now define vertex operators. It is crucial that they are *not* endomorphisms. Rather, they are a whole collection of linear maps between different topological vector spaces. To define a vertex operator at a point $z \in P^1 \setminus \{\infty\}$ for a vector $\varphi \in V$, we need to fix open sets $\mathcal{O}, \mathcal{O}'$ such that $z \in \mathcal{O}' \setminus \mathcal{O}$. Moreover, we fix an open set $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{O}' = \emptyset$ and define a linear map

$$V(\varphi, z) : \mathcal{V}^{\mathcal{O}'} \to \mathcal{V}^{\mathcal{O}}$$

(i.e. as an operator going to a *larger* space) by its action on the dense subset $\mathcal{B}_{\mathcal{C}}$

$$V(\varphi, z)\vec{\psi} := V(\varphi, z) \underbrace{V(\psi_1, z_1)V(\psi_2, z_2) \cdots V(\psi_n, z_n)\Omega}_{\vec{\psi}} \; .$$

One then shows:

Proposition 2.1.9.

1. This is well-defined, i.e. independent of the choice of $\mathcal{C}$, and $V(\psi, z)$ is continuous.

2. If $z, \zeta \in \mathcal{O}$, $z \neq \zeta$, and $\phi, \psi \in V$, then it follows directly from the locality property of the amplitudes that
$$V(\phi, z)V(\psi, \zeta) = V(\psi, \zeta)V(\phi, z)$$
as an identity on $\mathcal{V}^{\mathcal{O}}$.

2.2 Möbius invariance

We have now to impose one more requirements on our amplitudes to get a reasonable theory. Recall that the group $\mathrm{PSL}(2, \mathbb{C})$ acts on the Riemann sphere by

$$\gamma(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{C},\ ad - bc = 1 \; .$$

This group has the following one-parameter-subgroups:

- Translations
$$\mathrm{e}^{\lambda L_{-1}}(z) = z + \lambda \text{ with } \lambda \in \mathbb{C} \; .$$

- Scaling transformations
$$\mathrm{e}^{\lambda L_0}(z) = \mathrm{e}^{\lambda} z \text{ with } \lambda \in \mathbb{C} \; .$$

- Special conformal transformations
$$\mathrm{e}^{\lambda L_1}(z) = \frac{z}{1 - \lambda z} \text{ with } \lambda \in \mathbb{C} \; .$$

One verifies that in the Lie algebra of $PSL(2, \mathbb{C})$

$$[L_n, L_m] = (m - n)L_{m+n} \quad \text{for } m, n \in \{0, \pm 1\} \; .$$

We impose covariance of the amplitudes under these transformations:

Definition 2.2.1

A Möbius-covariant consistent set of amplitudes obeys in addition the axiom that for all $\gamma \in$ $\mathrm{PSL}(2,\mathbb{C})$ *and any n-tuple $(\psi_1,\ldots,\psi_n)$ of homogeneous vectors in V of degree* $\deg(\psi_j) = h_j$

$$\left\langle \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle \prod_{j=1}^{n} (dz_j)^{h_j} = \left\langle \prod_{j=1}^{n} V(\psi_j, \zeta_j) \right\rangle \prod_{j=1}^{n} (d\zeta_j)^{h_j}, \quad \text{where } \zeta_j = \gamma(z_j)\,,$$

or, equivalently,

$$\left\langle \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle = \left\langle \prod_{j=1}^{n} V(\psi_j, \gamma(z_j)) \right\rangle \prod_{j=1}^{n} (\gamma'(z_j))^{h_j} \,.$$

Remarks 2.2.2.

1. The power of the derivative $\gamma'(z)^h$ takes into account that the amplitude is really a section of a line bundle.
2. The amplitudes for the free boson, the non-abelian currents and the lattice theory presented in example 2.1.3 are all Möbius covariant.

Observation 2.2.3.

1. The Möbius group also acts on open subsets of P^1 by

   $$\mathcal{O} \mapsto \mathcal{O}_\gamma := \{\gamma(z) : z \in \mathcal{O}\}\,.$$

   To implement the symmetries on the space of states, we we define an operator $U(\gamma)$: $\mathcal{V}^{\mathcal{O}} \to \mathcal{V}^{\mathcal{O}_\gamma}$. Again we choose $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{O} = \emptyset$ and define $U(\gamma)$ on the dense subset $\mathcal{B}_{\mathcal{C}}$ by

   $$U(\gamma)\vec{\psi} = \prod_{j=1}^{n} V(\psi_j, \gamma(z_j)) \prod_{j=1}^{n} (\gamma'(z_j))^{h_j}\, \Omega \in \mathcal{V}^{\mathcal{O}_\gamma} \,.$$

2. It follows immediately from the definition of $U(\gamma)$ that $U(\gamma)\Omega = \Omega$. This is the (Möbius-)invariance of the vacuum. Furthermore,

   $$U(\gamma)V(\psi, z)U(\gamma^{-1}) = V(\psi, \gamma(z))\gamma'(z)^h, \quad \text{for } \psi \in V_h. \tag{12}$$

3. Then the definition is extended from the dense subspace $\mathcal{B}_{\mathcal{C}}$ to $\mathcal{V}^{\mathcal{O}}$.
4. Fix any point $z_0 \notin \mathcal{O}$. One can conclude from Möbius invariance of the amplitudes that the map

   $$\begin{array}{ccc} V & \to & \mathcal{V}^{\mathcal{O}} \\ \psi & \mapsto & V(\psi, z_0)\Omega \end{array}$$

   is an injection. For, if

   $$\langle \prod_{i=1}^{n} V(\psi_i, z_i) V(\psi, z_0) \rangle$$

   vanishes for all ψ_i and z_i, then by the invariance property, the same holds for an infinite number of points that are Möbius-images of z_0. Regarded as a function of z_0, the preceding expression defines a meromorphic function with infinitely many zeros; it therefore vanishes identically, thus implying by the non-degeneracy axiom of the amplitudes that $\psi = 0$.

5. Without loss of generality, we can chose in a Möbius covariant theory $z_0 = 0 \in \mathbb{C} \subset P^1$ as a reference point and then identify

$$\psi = V(\psi, 0)\Omega \in \mathcal{V}^{\mathcal{O}} .$$

This is the first instance of a field-state-correspondence.

6. We next notice that for an element $\gamma \in \mathrm{SL}(2, \mathbb{C})$, we have

$$U(\gamma) = \exp\left(\frac{b}{d} L_{-1}\right) \left(\frac{\sqrt{ad - bc}}{d}\right)^{L_0} \exp\left(-\frac{c}{d} L_1\right) .$$

Now the Möbbius covariance relation (12) implies

$$U(\gamma)\psi = U(\gamma)V(\psi, 0)\Omega = U(\gamma)V(\psi, 0)U(\gamma)^{-1}\Omega = \lim_{z\to 0} V(\psi, \gamma(z))\Omega\gamma'(z)^h$$

From this relation, we conclude that

$$L_0\psi = h\psi , \qquad L_1\psi = 0 , \qquad L_{-1}\psi = V'(\psi, 0)\Omega . \tag{13}$$

Vectors obeying the relation (13) are called quasi-primary states. For example, for the subgroup generated by L_0, we have

$$\gamma'(\lambda) = (\mathrm{e}^\lambda)' = \gamma(\lambda)$$

and thus

$$U(\mathrm{e}^{\lambda L_0})\psi = (\mathrm{e}^\lambda)^h \lim_{z\to 0} V(\psi, \mathrm{e}^\lambda z)\Omega = (\mathrm{e}^\lambda)^h \psi$$

and thus by differentiating with respect to λ, we get $L_0\psi = h\psi$.

Note that L_0 acts as the grading operator on V. (This is a consequence of our axioms, but not necessarily true for all conformal field theories considered in the literature. In so-called logarithmic conformal field theories, L_0 can have a nilpotent part.)

7. One then defines vertex operators for the state $\vec{\psi} = \prod_{j=1}^n V(\psi_j, z_j)\Omega \in \mathcal{B}_\mathcal{C}$ by

$$V(\vec{\psi}, z) = \prod_{j=1}^n V(\psi_j, z_j + z) .$$

Then $V(\vec{\psi}, z)$ is a continuous operator $\mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2}$ for suitably chosen open subsets $\mathcal{O}_1, \mathcal{O}_2$.

We can further extend the definition of $V(\vec{\psi}, z)$ by linearity from $\vec{\psi} \in \mathcal{B}_\mathcal{C}$ to vectors $\Psi \in \mathrm{V}_\mathcal{C}^\mathcal{O}$, the image of $\mathcal{V}_\mathcal{C}$ in the completion $\mathcal{V}_\mathcal{C}^\mathcal{O}$, to obtain a continuous linear operator

$$V(\Psi, z) : \mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2} ,$$

where $\mathcal{C}_z \cap \mathcal{O}_2 = \emptyset$, $\mathcal{O}_2 \subset \mathcal{O}_1$ and $\mathcal{C}_z \subset \mathcal{O}_1$ for $\mathcal{C}_z = \{\zeta + z : \zeta \in \mathcal{C}\}$. It can, however, not be extended to all states in the closure $\mathcal{V}_\mathcal{C}^\mathcal{O} \cong \mathcal{V}^\mathcal{O}$ of $\mathrm{V}_\mathcal{C}^\mathcal{O}$.

Proposition 2.2.4.

1. For the vertex operator associated to $\Psi \in \mathrm{V}_\mathcal{C}^\mathcal{O}$, we again have

$$\mathrm{e}^{\lambda L_{-1}} V(\Psi, z)\mathrm{e}^{-\lambda L_{-1}} = V(\Psi, z + \lambda) \quad \text{and} \quad V(\Psi, 0)\Omega = \Psi .$$

2. Furthermore, we have for $\zeta \neq z$ that

$$\begin{aligned} V(\Psi, z)V(\phi,\zeta) &= V(\phi,\zeta)V(\Psi,z)\,, \\ V(\Psi,z)\Omega &= \mathrm{e}^{zL_{-1}}\Psi \end{aligned}$$

for any $\phi \in V$. Thus L_{-1} plays the role of a translation operator.

3. The two conditions in 2. characterise the vertex operator for the state $\Psi \in \mathrm{V}_{\mathcal{C}}^{\mathcal{O}}$ uniquely. This can be called a field-state correspondence.

4. (Duality:) If $\Psi \in \mathrm{V}_{\mathcal{C}}^{\mathcal{O}}$ and $\Phi \in \mathrm{V}_{\mathcal{C}}^{\mathcal{O}'}$, then

$$V(\Psi,z)V(\Phi,\zeta) = V(V(\Psi,z-\zeta)\Phi,\zeta)\,. \tag{14}$$

This can be seen as a generalized version of associativity.

Proof.
We only show uniqueness and duality:

1. To show uniqueness in 3., we assume that we have an operator $W(z)$ such that for all $\phi \in V$

$$W(z)V(\phi,\zeta) = V(\phi,\zeta)W(z) \quad \text{and} \quad W(z)\Omega = \mathrm{e}^{zL_{-1}}\Psi\ .$$

It follows for all $\Phi \in \mathrm{V}_{\mathcal{C}'}^{\mathcal{O}'}$ that

$$\begin{aligned} W(z)\mathrm{e}^{\zeta L_{-1}}\Phi &= W(z)V(\Phi,\zeta)\Omega && \text{[creation property of } V] \\ &= V(\Phi,\zeta)W(z)\Omega && \text{[locality assumption on } W] \\ &= V(\Phi,\zeta)e^{zL_{-1}}\Psi && \text{[creation assumption on } W] \\ &= V(\Phi,\zeta)V(\Psi,z)\Omega && \text{[creation property of } V] \\ &= V(\Psi,z)V(\Phi,\zeta)\Omega && \text{[locality property of } V] \\ &= V(\Psi,z)\mathrm{e}^{\zeta L_{-1}}\Phi && \text{[Möbius covariance of } V] \end{aligned}$$

Since this holds on the dense subspace $\mathrm{V}_{\mathcal{C}'}^{\mathcal{O}'}$ of $\mathcal{V}_{\mathcal{C}'}^{\mathcal{O}'}$ we have $W(z) = V(\Psi,z)$.

2. To see duality, we use Möbius covariance to find:

$$\begin{aligned} V(\Phi,z)V(\Psi,\zeta)\Omega &= V(\Phi,z)e^{\zeta L_{-1}}V(\Psi,0)e^{-\zeta L_{-1}}\Omega = V(\Phi,z)e^{\zeta L_{-1}}\Psi \\ &= \mathrm{e}^{\zeta L_{-1}}\mathrm{e}^{-\zeta L_{-1}}V(\Phi,z)e^{\zeta L_{-1}}\Psi = \mathrm{e}^{\zeta L_{-1}}V(\Phi,z-\zeta)\Psi \\ &= V(V(\Phi,z-\zeta)\Psi),\zeta)\Omega. \end{aligned}$$

The uniqueness result in 3. now implies (14).

□

2.3 Modes, graded vector spaces and OPE's

We now want to get rid of families of topological vector spaces and reduce the structure to a $\mathbb{Z}$-graded vector space with finite-dimensional homogeneous components.

Observation 2.3.1.

1. It is straightforward to see that we can construct contour integrals of vectors in the topological vector space $\mathcal{V}^{\mathcal{O}}$ of the form

$$\oint_{C_1} dz_1 \oint_{C_2} dz_2 \ldots \oint_{C_r} dz_r \mu(z_1, z_2, \ldots, z_r) \prod_{i=1}^{n} V(\psi_i, z_i)\Omega \,,$$

where $r \leq n$ and the weight function μ is analytic in some neighbourhood of the contours $C_1 \times C_2 \times \cdots \times C_r$ and the distances $|z_i - z_j|$, $i \neq j$, are bounded away from 0 on this set. In this way we can define the modes

$$V_n(\psi) = \frac{1}{2\pi \mathrm{i}} \oint_C z^{h+n-1} V(\psi, z) dz \,, \qquad \text{for } \psi \in V_h \,,$$

as linear operators on $\mathcal{V}^{\mathcal{O}}_{\mathcal{C}}$, where the contour C encircles $\mathcal{C}$ and $C \subset \mathcal{O}$ with $\infty \in \mathcal{O}$ and $0 \in \mathcal{C}$.

2. The meromorphicity of the amplitudes allows us to prove the identity

$$V(\psi, z) = \sum_{n=-\infty}^{\infty} V_n(\psi) z^{-n-h} \tag{15}$$

with convergence with respect to the topology of $\mathcal{V}^{\mathcal{O}'}$ for an appropriate open set $\mathcal{O}'$.

The definition of the mode operator $V_n(\psi)$ is independent of the contour C if it is taken to be a simple contour encircling the origin once positively. Further, if $\mathcal{O}_2 \subset \mathcal{O}_1$, $\mathcal{V}^{\mathcal{O}_1} \subset \mathcal{V}^{\mathcal{O}_2}$ and if $\infty \in \mathcal{O}_2, 0 \notin \mathcal{O}_1$, the definition of $V_n(\psi)$ on $\mathcal{V}^{\mathcal{O}_1}$, $\mathcal{V}^{\mathcal{O}_2}$, agrees on $\mathcal{V}^{\mathcal{O}_1}$, which is dense in $\mathcal{V}^{\mathcal{O}_2}$, so that we may regard the definition as independent of $\mathcal{O}$. $V_n(\psi)$ depends on our choice of 0 and ∞ but different choices can be related by Möbius transformations.

3. We define the graded vector space $\mathcal{H}^{\mathcal{O}} \subset \mathcal{V}^{\mathcal{O}}$ to be the space spanned by finite linear combinations of vectors of the form

$$\Psi = V_{n_1}(\psi_1) V_{n_2}(\psi_2) \cdots V_{n_N}(\psi_N)\Omega \,,$$

where $\psi_j \in V$ and $n_j \in \mathbb{Z}$, $1 \leq j \leq N$. Then, by construction, $\mathcal{H}^{\mathcal{O}}$ has a countable basis. It is easy to see that $\mathcal{H}^{\mathcal{O}}$ is dense in $\mathcal{V}^{\mathcal{O}}$. Further it follows from the independence of the definition of modes that $\mathcal{H}^{\mathcal{O}}$ is independent of $\mathcal{O}$, and, where there is no ambiguity, we shall denote it simply by $\mathcal{H}$. It does however depend on the choice of 0 and ∞, but different choices will be related by the action of the Möbius group again.

4. It follows that

$$V(\psi, 0)\Omega = \psi \quad \text{ for } \psi \in V \,,$$

so that we can identify $V \subset \mathcal{H}$. Using the mode expansion (15), we find

$$V_n(\psi)\Omega = 0 \qquad \text{if } n > -h \quad \text{ and } \quad V_{-h}(\psi)\Omega = \psi \,.$$

Observation 2.3.2.

1. The duality property (14) of the vertex operators can be rewritten in terms of modes as the operator product expansion (OPE) for $\Psi, \Phi \in \mathcal{H}$ with $L_0\Psi = h_\Psi \Psi$ and $L_0 \Phi = h_\Phi \Phi$

$$V(\Phi, z) V(\Psi, \zeta) = V(V(\Phi, z-\zeta)\Psi, \zeta) = \sum_n V(V_n(\Phi)\Psi, \zeta)(z-\zeta)^{-n-h_\Phi} \,. \tag{16}$$

2. We can then use contour deformation techniques to derive from this formula commutation relations for the respective modes. Indeed, the commutator of two modes $V_m(\Phi)$ and $V_n(\Psi)$ acting on $\mathcal{B}_{\mathcal{C}}$ is defined by

$$\begin{aligned}[V_m(\Phi),\, V_n(\Psi)] &= \frac{1}{(2\pi \mathrm{i})^2}\oint \mathrm{d}z \oint \mathrm{d}\zeta_{|z|>|\zeta|} z^{m+h_\Phi-1}\zeta^{n+h_\Psi-1}V(\Phi,z)V(\Psi,\zeta) \\ &- \frac{1}{(2\pi \mathrm{i})^2}\oint \mathrm{d}z \oint \mathrm{d}\zeta_{|\zeta|>|z|} z^{m+h_\Phi-1}\zeta^{n+h_\Psi-1}V(\Phi,z)V(\Psi,\zeta)\,,\end{aligned}$$

where the contours on the right-hand side encircle C anti-clockwise. We can then insert (16) and deform the two contours to find:

$$[V_m(\Phi),\, V_n(\Psi)] = \frac{1}{(2\pi \mathrm{i})^2}\oint_0 \zeta^{n+h_\Psi-1}\mathrm{d}\zeta \oint_\zeta z^{m+h_\Phi-1}\mathrm{d}z \sum_l V(V_l(\Phi)\Psi,\zeta)(z-\zeta)^{-l-h_\Phi}\,,$$

where the ζ contour is a positive circle about 0 and for fixed ζ the z contour is a small positive circle about ζ. Now recall the Cauchy-Riemann formula

$$\frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}z \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w)\ .$$

Only terms with $l \geq 1-h_\Phi$ contribute to the z-integral, which becomes

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}z\frac{z^{m+h_\Phi-1}}{(z-\zeta)^{l+h_\Phi}} &= \frac{1}{(l+h_\phi-1)!}\frac{\mathrm{d}^{l+h_\Phi-1}}{\mathrm{d}\zeta^{l+h_\Phi-1}}\zeta^{m+h_\Phi-1} \\ &= \frac{1}{(l+h_\phi-1)!}\frac{(m+h_\Phi-1)!}{(m+h_\Phi-1-(l+h_\Phi-1))!}\zeta^{m+h_\Phi-1-l-h_\Phi+1} \\ &= \begin{pmatrix} m+h_\Phi-1 \\ m-l \end{pmatrix}\zeta^{m-l}\end{aligned}$$

We thus find

$$\begin{aligned}[V_m(\Phi), V_n(\Psi)] &= \sum_{k,l}\begin{pmatrix} m+h_\Phi-1 \\ m-l \end{pmatrix}\frac{1}{2\pi \mathrm{i}}\oint_0 \mathrm{d}\zeta\ \zeta^{n+h_\Psi-1}\zeta^{m-l}\, V_k(V_l(\Phi)\Psi)\zeta^{-k-(h_\Psi-l)} \\ &= \textstyle\sum_l \begin{pmatrix} m+h_\Phi-1 \\ m-l \end{pmatrix} V_{n+m}(V_l(\Phi)\Psi)\end{aligned}$$

We thus find the following commutation relations between the modes:

$$[V_m(\Phi), V_n(\Psi)] = \sum_{N=-h_\Phi+1}^{\infty}\begin{pmatrix} m+h_\Phi-1 \\ m-N \end{pmatrix} V_{m+n}(V_N(\Phi)\Psi)\,. \tag{17}$$

Consider the case when

$$m \geq -h_\Phi+1 \quad \text{and} \quad n \geq -h_\Psi+1\ .$$

The first identity implies that $m-N \geq 0$ in the sum (17); as a consequence $m+n \geq N+n \geq N-h_\Psi+1$. This implies that the modes $\{V_m(\Psi) : m \geq -h_\Psi+1\}$ close as a Lie algebra.

The cluster decomposition property to be discussed later guarantees that the spectrum of L_0 is bounded below by 0. If this is the case then the sum in (17) is also bounded above by h_Ψ.

We present a few more tools and structures that can be derived from the operator product expansion (16):

$$V(\Phi, z)V(\Psi, \zeta) = \sum_{n=-n_0}^{\infty} C_{(n)}(z-\zeta)^n \,.$$

Definition 2.3.3
We call

$$\underbrace{V(\Phi, z)V(\Psi, \zeta)} = \sum_{n=-n_0}^{-1} C_{(n)}(z-\zeta)^n \,.$$

the contraction of the two vertex operators $V(\Phi, z)$ and $V(\Psi, \zeta)$ and

$$: V(\Phi, z)V(\Psi, z) : \quad := C_{(0)}(z)$$

the normal ordered product of the two vertex operators. We then have

$$V(\Phi, z)V(\Psi, \zeta) = \underbrace{V(\Phi, z)V(\Psi, \zeta)} + : V(\Phi, z)V(\Psi, z) : + O(z-\zeta) \ .$$

We take now up our previous examples and extract algebraic structure.

Examples 2.3.4.

1. For the free boson from example 2.1.3, we find from the amplitudes the contraction

$$\underbrace{J(z)J(w)} = \frac{k}{(z-w)^2} \,.$$

we then obtain the commutation relations

$$[J_n, J_m] = \frac{1}{2\pi \mathrm{i}} \oint_0 \mathrm{d}z z^n \frac{1}{2\pi \mathrm{i}} \oint_z \mathrm{d}w \, w^m \frac{k}{(z-w)^2} = k \frac{1}{2\pi \mathrm{i}} \oint_0 \mathrm{d}z z^n \frac{\mathrm{d}z^m}{\mathrm{d}z} = nk\delta_{n,-m} \,.$$

This defines an infinite-dimensional Lie algebra, the Heisenberg algebra $\hat{u}(1)$. The value of the central extension is fixed in a chiral conformal field theory.

2. The amplitudes in example 2.1.3 that are given by a Lie algebra and a central element determine the contraction of the operator product expansion to be of the form

$$\underbrace{J^a(z)J^b(w)} = \frac{\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)} \,,$$

and the algebra therefore becomes

$$[J^a_m, J^b_n] = f^{ab}{}_c J^c_{m+n} + m\kappa^{ab}\delta_{m,-n} \,.$$

This is centrally extended loop algebra; the central extension is fixed in a conformal field theory. In the particular case where $\mathfrak{g}$ is simple, κ is a certain multiple of the Killing form and $\hat{\mathfrak{g}}$ is an untwisted affine Lie algebra $\hat{\mathfrak{g}}$.

For later use, we give a definition of affine Lie algebras.

Definition 2.3.5

The affine Lie algebra $\widehat{\mathfrak{g}}$ associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((z))$ by $\mathbb{C}$:

$$\widehat{\mathfrak{g}} = \big(\mathfrak{g} \otimes \mathbb{C}((z))\big) \oplus \mathbb{C}c \ ,$$

the bracket of two elements of $\mathfrak{g} \otimes \mathbb{C}((z))$ being given by

$$[X \otimes f, Y \otimes g] = [X,Y] \otimes fg \ + \ c \cdot \kappa(X,Y)\mathrm{Res}(g\,\mathrm{d}f)$$

with κ the Killing form. We denote by $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ the subspaces $\mathfrak{g} \otimes z\mathbb{C}[[z]]$ positive modes and $\mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}]$ of $\widehat{\mathfrak{g}}$, so that we have a triangular decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_+ \ .$$

By the formula for the Lie bracket, each summand is actually a Lie subalgebra of $\widehat{\mathfrak{g}}$.

We investigate the normal ordering in a bit more detail.

Observation 2.3.6.

1. From the definition 2.3.3 of the normal ordering, one derives the formula

$$: A(z)B(z) := \frac{1}{2\pi\mathrm{i}} \oint_z \mathrm{d}w \frac{A(w)B(z)}{z-w}$$

one finds for commutators, cf. [F, (3.1.23)]

$$: [A(z), B(z)] :\overset{\mathrm{def}}{=}: A(z)B(z) : - : B(z)A(z) := \sum_{n=1} \frac{(-1)^{n+1}}{n!} \partial^n C_{-n}(z) \ ,$$

where $C_{(n)}$ are the fields appearing in the OPE of the vertex operators $A(z)$ and $B(w)$. so that normal ordering is not commutative, but commutative up to derivative fields. It can be shown to be associative [Z96, Section 4.4].

2. An explicit contour integration [F, (3.1.29)] shows that the modes of the normal ordering are

$$c_m = \sum_{n\in\mathbb{Z}} : a_n b_{m-n} :$$

with

$$: a_m b_n : \quad := \begin{cases} a_m b_n & \text{for } m \le -h_A \\ b_n a_m & \text{for } m > -h_A \end{cases}$$

3. Another normal ordering prescription is obtained which differs from the normal ordering by a finite number of terms:

$$N_0(A,B)_n = \sum_{m\ge 0} a_m b_{n-m} + \sum_{m<0} b_{n-m} a_m \ ;$$

in terms of contour integrals, one has

$$N_0(A,B)(0) = \frac{1}{2\pi\mathrm{i}} \oint_0 \frac{(1+z)^{h_A}}{z} A(z)B(0) \ .$$

This product is not associative; however, the violation of associativity can be expressed in terms expressions of the type $\partial\Phi + h_\Phi\Phi$: modulo normally ordered products of such fields, the product is associative.

4. The quotient $C_2(\mathcal{H})$ of $\mathcal{H}$ by such fields is an associative algebra, called Zhu's algebra. If the quotient is finite-dimensional, then $\mathcal{H}$ is called C_2-cofinite.

2.4 Cluster decomposition

So far the axioms we have formulated do not impose any restrictions on the relative normalisation of amplitudes involving a different number of vectors in V.

Definition 2.4.1
A Möbius-covariant consistent set of amplitudes is said to have the <u>*cluster decomposition property*</u>*, if the following property holds. If we partition the variables of an amplitude into two sets and scale one set towards a fixed point (e.g. 0 or ∞) the behaviour of the amplitude is dominated by the product of two amplitudes, corresponding to the two sets of variables, multiplied by an appropriate power of the separation:*

$$\left\langle \prod_i V(\phi_i,\zeta_i) \prod_j V(\psi_j,\lambda z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i,\zeta_i) \right\rangle \left\langle \prod_j V(\psi_j,z_j) \right\rangle \lambda^{-\Sigma h_j} \quad \text{as } \lambda \to 0\,,$$

where $\phi_i \in V_{h'_i}, \psi_j \in V_{h_j}$. Here, we assume $\eta_i \neq 0$ for all i. Here we write for functions $f(\lambda) \sim g(\lambda)$ for $\lambda \to 0$, if $\lim_{\lambda\to 0} \frac{f(\lambda)}{g(\lambda)} = 1$.

Remarks 2.4.2.

1. It follows from the invariance under the Möbius transformation $z \mapsto \lambda z$, that the cluster decomposition property is equivalent to

$$\left\langle \prod_i V(\phi_i,\lambda\zeta_i) \prod_j V(\psi_j,z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i,\zeta_i) \right\rangle \left\langle \prod_j V(\psi_j,z_j) \right\rangle \lambda^{-\Sigma h'_i} \quad \text{as } \lambda \to \infty\,.$$

2. The cluster decomposition property extends also to vectors $\Phi_i, \Psi_j \in \mathcal{H}$.

3. It is not difficult to check that the examples 2.1.3 satisfy this condition.

Proposition 2.4.3.
Given a Möbius-covariant consistent set of correlators obeying the cluster decomposition property, the spectrum of L_0 is non-negative and the vacuum is unique up to scalars.

Proof.
Consider the endomorphism of $\mathcal{H}$ defined by

$$P_N = \frac{1}{2\pi \mathrm{i}} \oint_0 u^{L_0-N-1} \mathrm{d}u, \qquad \text{for } N \in \mathbb{Z}\,.$$

In particular, we have

$$P_N \prod_j V(\psi_j,z_j)\Omega = \frac{1}{2\pi \mathrm{i}} \oint u^{h-N-1} V(\psi_j,uz_j)\Omega \mathrm{d}u\,,$$

where $h = \sum_j h_j$. This implies that the P_N are projection operators onto the eigenspaces of L_0,

$$L_0 P_N = N P_N\,.$$

For $N \leq 0$, the cluster decomposition property implies

$$\begin{aligned} \left\langle \prod_i V(\phi_i,\zeta_i) P_N \prod_j V(\psi_j,z_j) \right\rangle &= \oint_0 u^{\Sigma h_j - N-1} \left\langle \prod_i V(\phi_i,\zeta_i) \prod_j V(\psi_j,uz_j) \right\rangle \mathrm{d}u \\ &\sim \left\langle \prod_i V(\phi_i,\zeta_i) \right\rangle \left\langle \prod_j V(\psi_j,z_j) \right\rangle \oint_{|u|=\rho} u^{-N-1} \mathrm{d}u, \end{aligned}$$

which, by taking $\rho \to 0$, is seen to vanish for $N < 0$ and, for $N = 0$, to give

$$P_0 \prod_j V(\psi_j, z_j)\Omega = \Omega \left\langle \prod_j V(\psi_j, z_j) \right\rangle ,$$

and so $P_0\Psi = \Omega \langle \Psi \rangle$. □

Remark 2.4.4.
The absence of negative eigenvalues of L_0 gives an upper bound on the order of the pole in the operator product expansion of two vertex operators, and thus to an upper bound in the sum in (16): if $\Phi, \Psi \in \mathcal{H}$ are of degree $L_0\Phi = h_\Phi \Phi, L_0\Psi = h_\Psi \Psi$, we have that $V_n(\Phi)\Psi = 0$ for $n > h_\Psi$ because otherwise $V_n(\Phi)\Psi$ would have a negative eigenvalue, $h_\Psi - n$, with respect to L_0. One says that the action is locally finite. In particular, this shows that the leading singularity in the OPE $V(\Phi, z)V(\Psi, \zeta)$ is at most $(z - \zeta)^{-h_\Psi - h_\Phi}$.

2.5 Vertex algebras

Our field theoretic discussion thus leads to the following algebraic definition:

Definition 2.5.1
A vertex algebra consists of the following data:

- *a graded vector space*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{(n)}$$

  whose homogeneous subspaces $\mathcal{H}_{(n)}$ are finite-dimensional, $\dim \mathcal{H}_{(n)} < \infty$;

- *a non-zero vacuum vector $\Omega \in \mathcal{H}$;*

- *a shift operator*

$$T = L_{-1} : \quad \mathcal{H} \to \mathcal{H} \,;$$

- *and a field-state correspondence Y involving a formal variable z:*

$$Y : \quad \mathcal{H} \to \mathrm{End}(\mathcal{H})[[z, z^{-1}]] \,; \tag{18}$$

  $Y(v, \cdot)$ is also called the vertex operator for the vector $v \in \mathcal{H}$.

These data are subject to the conditions

- *that the field for the vacuum Ω is the identity, $Y(\Omega, z) = \mathrm{id}_{\mathcal{H}}$;*

- *that the field-state correspondence respects the grading, i.e. if $v \in \mathcal{H}_{(h)}$, then all endomorphisms v_m appearing in $Y(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-h-m}$ have grade m: $v_m(\mathcal{H}_{(p)}) \subseteq \mathcal{H}_{(p+m)}$;*

- *that one recovers states by acting with the corresponding fields on the vacuum and 'sending z to zero', or more precisely,*

$$Y(v, z)\Omega \in v + z\,\mathcal{H}[[z]]$$

  for every $v \in \mathcal{H}$;

- *that T implements infinitesimal translations,*

$$[T, Y(v,z)] = \partial_z Y(v,z)\,; \tag{19}$$

and that the vacuum is translation invariant, $T\Omega = 0$.

- *Finally, the most non-trivial constraint – called* locality *– is that commutators of fields have poles of at most finite order. More precisely, for any two $v_1, v_2 \in \mathcal{H}$ there must exist a number $N = N(v_1, v_2)$ such that*

$$(z_1 - z_2)^N \, [Y(v_1, z_1), Y(v_2, z_2)] = 0\,. \tag{20}$$

Remarks 2.5.2.

1. The requirement in equation (20) is a kind of weak commutativity. This constraint only makes sense because we consider formal series in the z_i, which can extend to both arbitrarily large positive and negative powers. Had we restricted ourselves to ordinary Laurent series, i.e. series without arbitrarily large negative powers, (20) would already imply that the commutator vanishes.

2. The vectors in $\mathcal{H} \setminus \{\mathbb{C}\Omega\}$ are also called descendants of the vacuum.

We take up our examples:

Examples 2.5.3.

1. For a free boson, we have the Heisenberg Lie-algebra $\mathfrak{g}$ with generators $(J_n)_{n\in\mathbb{Z}}$ and relations

$$[J_n, J_m] = n\,\delta_{n+m,0}\,.$$

Consider the Lie subalgebra $\mathfrak{b}_+ := \mathrm{span}(J_n)_{n\geq 0}$. Fix any finite-dimensional vector space W together with an endomorphism $\varphi \in \mathrm{End}(W)$. Then W can be endowed with the structure of a $\mathfrak{b}_+$-module by $J_n.w = 0$ for all $n \geq 1$ and all $w \in W$ and $J_0 w = \varphi(w)$. The induced $\mathfrak{g}$-module

$$\mathcal{F}(W, \varphi) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b}_+)} W$$

is called the Fock space for W. The grading on the Heisenberg Lie algebra induces a natural structure of a graded vector space.

The underlying vertex algebra is defined on the graded vector space $\mathcal{H} = \mathcal{F}(\mathbb{C})$, where $\mathbb{C}$ is endowed with the zero endomorphism. The vacuum $\Omega = 1 \in \mathbb{C} \subset \mathcal{F}(W)$ is the ground state in this Fock space. The field-state correspondence is expressed in terms of the abelian currents

$$J(z) = \sum_{n\in\mathbb{Z}} J_n\, z^{-n-1}$$

by setting $Y(J_{-1}\Omega, z) = J(z)$ and, more generally,

$$Y(J_{n_1} \cdots J_{n_k}\Omega, z) = \frac{1}{(n_1{-}1)! \cdots (n_k{-}1)!} :\partial_z^{n_1-1} J(z) \cdots \partial_z^{n_k-1} J(z):\,,$$

where the colons indicate a normal ordering, the lowest order regular part of an OPE. This prescription indeed yields the structure of a vertex algebra. (It is not a trivial exercise, though, to check that this works out.)

2. We also describe the generalization to lattice vertex algebras. Let $L \subset \mathbb{R}^d$ be an even Euclidean lattice. We now take the d-fold tensor product $\mathrm{U} := (\mathrm{U}(\mathfrak{g}))^{\otimes d}$ of the Heisenberg algebra with generators $(J_n^a)_{n\in\mathbb{Z},a=1,\dots d}$. Denote by $\mathfrak{b}_+$ again the subalgebra spanned by non-negative modes. A Fock space is now constructed as follows: take d commuting endomorphisms $(\varphi_i)_{i=1,\dots,d}$ of a finite-dimensional vector space W to endow W with the structure of a $\mathfrak{b}_+$-module
$$J_n^a.w = \delta_{n,0}\varphi_a(w) \ .$$
Then the Fock space is again
$$\mathcal{F}(W) := \mathrm{U} \otimes_{\mathrm{U}(\mathfrak{b}_+)} W$$
As a particular example, take $W \cong \mathbb{C}$ to be one-dimensional and for $q \in \mathbb{R}^d$ the action $\varphi_a(w) = q_a \cdot w$. Denote the corresponding Fock space by $\mathcal{F}_q$

For an even Euclidian lattice $\Lambda \subset \mathbb{R}^n$, the vector space
$$\mathcal{H} = \oplus_{q\in L}\mathcal{F}_q$$
has the structure of a vertex algebra. Vertex operators are normally ordered expressions of the type
$$Y(J_{n_1}\cdots J_{n_k}q, z) = \frac{1}{(n_1-1)!\cdots(n_k-1)!} : \partial_z^{n_1-1}J(z)\cdots\,\partial_z^{n_k-1}J(z)\,\exp(iqX(z)) : ,$$
Here the grade of a vector is $\frac{q^2}{2}$ plus the natural grade in the Fock space.

3. In the case of untwisted affine Lie algebras based on a finite-dimensional simple Lie algebra $\mathfrak{g}$ with a certain multiple $\ell \in \mathbb{N}$ of the Killing form, a vector space $\mathcal{H}^{(\ell)}$ will be constructed for each $\ell \in \mathbb{N}$ from representations of certain infinite-dimensional Lie algebras, untwisted affine Lie algebras in section 2.7. The multiple $\ell \in \mathbb{N}$ is called the level of the theory.

4. The classes of vertex algebras we described are not disjoint. For example, for any simply laced finite-dimensional simple Lie algebra $\mathfrak{g}$, the vertex algebra at level $\ell = 1$ and the lattice algebra for the root lattice of $\mathfrak{g}$ are the same. This is known as the Frenkel-Kac-Segal construction.

For many applications, e.g. in string theory, one needs to make sense of a chiral conformal field theory not only on a the Riemann sphere, but on a general compact Riemann surface. To this end, one needs more structure on a vertex algebra than Möbius invariance.

Definition 2.5.4

1. *A conformal structure of Virasoro-central charge $c \in \mathbb{C}$ on a vertex algebra $\mathcal{H}$ is a vector $v_{Vir} \in \mathcal{H}_{(2)}$ such that the operators appearing in the mode expansion of the so-called (chiral) stress-energy tensor*
$$T(z) := Y(v_{Vir}, z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$$
possess the following properties: $L_{-1} = T$ gives the translations; L_0 is semisimple and reproduces the grading of $\mathcal{H}$; L_0 acts as $n\,\mathrm{id}$ on $\mathcal{H}_{(n)}$; and finally,
$$L_2 v_{Vir} = \frac{1}{2}\,c\,\Omega\,.$$

2. *A vertex algebra, together with a choice of conformal structure, is called a conformal vertex algebra.*

Remarks 2.5.5.

1. The axioms of a conformal vertex algebra imply that the modes L_n span an infinite-dimensional Lie algebra, the Virasoro algebra of central charge c. It has has basis $(\{L_n, n \in \mathbb{Z}, C\}$ where C is central and the other Lie brackets read

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}m(m^2-1)\delta_{m+n,0} \tag{21}$$

In a chiral conformal field theory, the central element C acts as the same multiple of the identity in all sectors.

2. The choice of a conformal structure for a vertex algebra should not be confused with the choice of a conformal structure of the world sheet; it is, rather, related to (chiral) properties of the target space.

3. For applications to string theory, one needs vertex algebras with even more structure, e.g. a superconformal structure.

Examples 2.5.6.

The examples we have discussed all carry the structure of a conformal vertex algebra:

1. For the vertex algebra based on the Heisenberg Lie algebra, there is a one-parameter family of conformal vectors in the Fock space:

$$v_{Vir}(\lambda) = \left(\frac{1}{2}(J_{-1})^2 + \lambda J_{-2}\right)\Omega \in \mathcal{F}_0$$

with Virasoro central charge $c(\lambda) = 1 - 12\lambda^2$. The corresponding stress-energy tensor reads

$$T_\lambda(z) = \frac{1}{2} : J(z)J(z) : + \lambda\partial J(z) \ .$$

2. For the chiral free boson, but also for the lattice vertex algebras, the case $\lambda = 0$ is crucial. In particular, the lattice vertex algebra for a lattice $\Lambda \subset \mathbb{R}^n$ has Virasoro central charge $c = n$. The Virasoro algebra is realized by the operators

$$L_m := \frac{1}{2}\sum_{n\in\mathbb{Z}} : J_n J_{m-n} :$$

where we have the normal ordering from observation 2.3.6.2

$$: J_n J_m : := \begin{cases} J_n J_m & \text{for } n \leq -h_J = -1 \\ J_m J_n & \text{for } n > -1 \end{cases}$$

A careful discussion of normal ordering is e.g. in [F, Section 3.1].

3. For an affine Lie algebra at level ℓ, one fixes a basis $(J^a)_{a=1,\dots \dim \mathfrak{g}}$ of the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$ that is orthonormal with respect to the Killing form and takes

$$\frac{1}{2(\ell + h^\vee)} \sum_a (J^a_{-1})^2 \Omega^{(\ell)} \in \mathcal{H}^{(\ell)} \ ,$$

where $h^\vee$ is the dual Coxeter number, a natural number associated to each finite-dimensional simple Lie algebra $\mathfrak{g}$. The Virasoro central charge can be computed [F, Section 3.2] to be

$$c(\ell) = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee} \ . \tag{22}$$

We again write the modes of the stress energy tensor explicitly:

$$L_m := \frac{1}{2(\ell + h^\vee)} \sum_{n\in\mathbb{Z}} \sum_{a,b} \kappa_{a,b} : J^a_n J^b_{m-n} : \tag{23}$$

with an analogous normal ordering convention based on the mode numbers. This construction of a Virasoro algebra is called the (affine) Sugawara construction. One should note the similarity to the second order Casimir operator.

2.6 Representations

Vertex algebras are formalizing aspects of a chiral symmetry algebra, and symmetries in a quantum theory should be represented on the space of states. We thus expect that more vector spaces contribute to the space of states of the theory. These are additional (superselection/solitonic) sectors of the theory. In the applications of chiral conformal field theories to quantum Hall systems, these sectors describe non-trivial quasi-particle excitations. We are thus lead to study the representation theory of vertex algebras.

Definition 2.6.1
A representation of a vertex algebra is a graded vector space

$$M = \bigoplus_{n\in\mathbb{Q}_{\geq 0}} M_{(n)}$$

with

- *a translation operator $T_M\colon M \to M$ of degree 1 and*
- *a representation map*
$$Y_M : \quad \mathcal{H} \to \mathrm{End}(M)[[z, z^{-1}]] \,,$$

such that

- *for $v \in \mathcal{H}_{(n)}$ all components of $Y_M(v, z)$ are endomorphisms of M of degree $-n$;*
- $Y_M(\Omega) = \mathrm{id}_M$;
- $[T_M, Y_M(v, z)] = \partial_z Y_M(v, z)$;
- *and the "duality" relation*
$$Y_M(v_1, z_1)\, Y_M(v_2, z_2) = Y_M(Y(v_1, z_1{-}z_2)v_2, z_2) \tag{24}$$
holds.

Morphisms of representations can be defined in the obvious way: it is a linear map $F : M \to M'$ such that

$$F(Y_M(a,z),v) = Y_{M'}(a,z)F(v) \quad \textit{for all } v \in M \quad \textit{and } a \in V \ .$$

Remarks 2.6.2.

1. One can show that the vertex algebra furnishes a representation of itself; this is called the vacuum representation. This implies that the identity (24) is in particular valid for $\mathcal{H}$, i.e. (24) remains true when Y_M is replaced by Y. This expresses a kind of associativity of the vertex algebra. Thus for vertex algebras 'associativity' in the sense of (24) is a consequence of 'commutativity' in the sense of (20). This is, of course, the duality of (14) of proposition 2.2.4.

2. The definition of representations is also possible in terms of a set of generalized amplitudes on a finite cover of the Riemann sphere. See [GG, Section 8] for details. Note that then one simultaneously defines a representation and its dual.

3. Once we know what representations and what morphisms of representations are, we can consider equivalence classes of isomorphic irreducible representations. They form a set

$$I := \pi_0(\mathcal{H}\text{-mod}) \ .$$

   We will call the elements of I *labels* or also, by an abuse of terminology, primary fields or sectors. We use lower case greek letters for elements in I, and denote by $\mathcal{H}_\mu$ the underlying vector space for a representation isomorphic to $\mu \in I$.

4. When the vertex algebra is conformal, every module M is in particular, by restriction, a module over the Virasoro algebra (21). It follows directly from the definition of a conformal vertex algebra that in each CFT model the central element C of the Virasoro algebra acts as $C = c\,\mathrm{id}$ with one and the same value of the number c in every representation that occurs in the model.

5. If every finitely generated module over a conformal vertex algebra decomposes into a finite direct sum of irreducible Virasoro modules, the vertex algebra is called a Virasoro-minimal model. (In the case of logarithmic minimal models, this definition has to be modified.)

6. When v is an eigenstate of the Virasoro zero mode L_0, its eigenvalue Δ_v is called the *conformal weight* of v. The conformal weights of different vectors in the same irreducible module differ by integers, or in other words, $\mathrm{e}^{2\pi \mathrm{i} L_0}$ acts as a multiple of the identity on every irreducible module. The endomorphism $\theta_M = \mathrm{e}^{-2\pi \mathrm{i} L_0}$, called the twist, commutes with the action of all vertex operators. In particular, it acts as a scalar multiple on each irreducible module.

7. Vectors $v \in M$ which obey the conditions in (13)

$$L_1 v = 0 \quad \text{and} \quad L_0 v = hv$$

   are called quasi-primary states of conformal weight h. Vectors $v \in \mathcal{H}$ which obey

$$L_n v = 0 \ \text{ for all } n > 0 \quad \text{and } L_0 v = hv$$

   are called Virasoro-primary states of conformal weight h.

8. It is a remarkable result [Z96] that under the condition of C_2-cofiniteness from observation 2.3.6.4, the character of any simple module M

$$\chi_M(\tau) := \mathrm{tr}_M \exp(2\pi \mathrm{i}\tau(L_0 - c/24))\,. \tag{25}$$

is a holomorphic functions in τ on the complex upper half-plane H.

Definition 2.6.3
A rational vertex algebra $\mathcal{H}$ is a conformal vertex algebra that obeys the following finiteness conditions:

1. *The homogeneous subspace of $\mathcal{H}$ of weight 0 is spanned by the vacuum Ω.*
2. *The graded dual $\mathcal{H}' = \oplus_n \mathcal{H}^*_{(n)}$ is isomorphic to $\mathcal{H}$ as a $\mathcal{H}$-module.*
3. *Every $\mathcal{H}$-module is completely reducible.*
4. *$\mathcal{H}$ is C_2-cofinite.*

Remarks 2.6.4.

1. The homogeneous subspaces $M_{(n)}$ of the graded vector space underlying an irreducible representation M of a rational vertex algebra are finite-dimensional.
2. A rational vertex algebra $\mathcal{H}$ has only finitely many inequivalent irreducible representations. They are in bijection to isomorphism classes of irreducible representations of Zhu's algebra $C_2(\mathcal{H})$.
3. Moreover, the action of the group $\mathrm{SL}(2,\mathbb{Z})$ on the complex upper half plane

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \text{with } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1$$

induces an action on the irreducible characters.

4. The vertex algebra for an even lattice is rational. The same applies to the vertex algebra associated to an irreducible highest weight module over an affine Lie algebra.

   An example of a non-rational vertex algebra is provided by the vertex algebra for a single free boson. It has infinitely many inequivalent irreducible representations, one for each complex number q. The Fock space for an endomorphism φ that is not semisimple is an example of a non fully reducible representation.

5. Notice, however, that not every vertex algebra that has only finitely many inequivalent irreducible representations is rational in the sense of definition 2.6.3.
6. It has been proven [H05] that the category of representations of a rational vertex algebra carries even more structure, namely the one of a modular tensor category. We will explain this structure later.

Examples 2.6.5.
We make the structure of the representation categories explicit in the first two of our examples.

1. A single free boson is not rational. The indecomposable representations obeying finiteness conditions on the homogeneous components are given by generalized Fock spaces for which the endomorphism consists of a single Jordan block. In this case, J_0 and hence L_0 has a nilpotent part so that we are strictly speaking leaving our framework by including these representations. For the irreducible representations, we have an endomorphism φ of a one-dimensional vector space, i.e. a complex number q. The conformal weight is then $q^2/2$.

   On the finite abelian group Λ^*/Λ this gives the quadratic form

$$\begin{array}{rcl} \Lambda^*/\Lambda & \to & \mathbb{C}^* \\ [\mu] & \mapsto & \mathrm{e}^{2\pi\mathrm{i}\frac{\langle\mu,\mu\rangle}{2}} \end{array} .$$

   This is well-defined: for $\mu \in \Lambda^*$ and $\lambda \in \Lambda$, we have

$$\frac{\langle \mu+\lambda, \mu+\lambda\rangle}{2} = \frac{\langle\mu,\mu\rangle}{2} + \langle\mu,\lambda\rangle + \frac{\langle\lambda,\lambda\rangle}{2} = \frac{\langle\mu,\mu\rangle}{2} \bmod \mathbb{Z} ,$$

   since $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ for λ in the even lattice Λ and $\langle\lambda,\lambda\rangle$ since the lattices Λ and Λ^* are dual.

2. The irreducible representations of a lattice vertex algebra based on an even lattice $\Lambda \subset \mathbb{R}^n$ are in bijection to the finite abelian group Λ^*/Λ. For $\mu \in \Lambda^*$, the direct sum of Fock spaces

$$\oplus_{\lambda\in\Lambda}\mathcal{F}_{\mu+\lambda}$$

   carries an irreducible representation of the lattice vertex algebra for Λ. The dual lattice inherits from the Euclidian scalar product on $\mathbb{R}^n$ a quadratic form $q : \Lambda \to \mathbb{R}$. The conformal weight is then given by $q(\lambda)^2/2$.

   To compute the character of a Fock space for the Heisenberg algebra, introduce the Lie subalgebra $\mathfrak{n}_- := \mathrm{span}(J_n)_{n<0}$ of the Heisenberg algebra and note that $\mathcal{F}_q$ is free as an $\mathrm{U}(\mathrm{n}_-)$ module:

$$\mathcal{F}_q = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b}_+)} \mathbb{C} \cong \mathrm{U}(\mathrm{n}_-) \otimes_{\mathbb{C}} \mathbb{C}$$

   so that the Poincaré-Birkhoff-Witt theorem gives us a basis,

$$\chi_q(\tau) = \mathrm{Tr}_{\mathcal{F}_q} \exp(2\pi\mathrm{i}\tau(L_0 - 1/24)) = \frac{\mathrm{e}^{2\pi\mathrm{i}\tau\frac{q^2}{2}}}{\eta(\tau)}$$

   where

$$\eta(q) := q^{1/24} \cdot \prod_{n=1}^{\infty}(1-q^n) \quad \text{with } q = \mathrm{e}^{2\pi\mathrm{i}\tau}$$

   is the Dedekind eta-function. The character of the lattice model thus is a quotient of a theta function for the lattice $\Lambda \subset \mathbb{R}^n$,

$$\theta_\mu(\tau) = \sum_{\lambda\in\Lambda} \mathrm{e}^{2\pi\mathrm{i}\tau\frac{\langle\mu+\lambda,\mu+\lambda\rangle}{2}} \quad \text{for } \mu \in \Lambda^*$$

   and the eta-function:

$$\chi_\mu(\tau) = \frac{\theta_\mu(\tau)}{\eta(\tau)^n} .$$

One can use Poisson resummation to compute the representation of $\mathrm{SL}(2,\mathbb{Z})$ on the characters explicitly. We find for all $\lambda \in \Lambda^*/\Lambda$

$$\chi_\lambda(\tau+1) = T_\lambda \chi_\lambda(\tau) \quad \text{and } \chi_\lambda(-\frac{1}{\tau}) = \sum_{\mu\in\Lambda^*/\Lambda} S_{\lambda,\mu}\chi_\mu(\tau)$$

with a diagonal unitary matrix with entries

$$T_\lambda = \exp\left[2\pi \mathrm{i}\left(\frac{(\lambda,\lambda)}{2} - \frac{n}{24}\right)\right]$$

and a symmetric unitary matrix

$$S_{\lambda,\mu} = (\sqrt{|\Lambda^*/\Lambda|})^{-1/2} \exp\left[-2\pi \mathrm{i}(\lambda,\mu)\ .\right] \tag{26}$$

3. Zhu's algebra has been computed [L96] for the vertex algebra based on the Heisenberg algebra where it is the polynomial algebra, $C_2(\mathcal{H}(\mathrm{U}(1))) \cong \mathbb{C}[X]$ whose irreducible representations are in bijection to the complex numbers and for the even lattice $\sqrt{2}\mathbb{Z} \subset \mathbb{R}$ where one finds a direct sum of two matrix algebras of 2×2 matrices.

4. For more background about vertex algebras, we refer to [K-Ver].

2.7 Affine Lie algebras

For the case of affine Lie algebras, we need some notions of Lie theory. From now on, we closely follow [B96].

We start recalling some basic facts from the theory of finite-dimensional simple Lie algebras. For the reader not sufficiently familiar with this theory, we exemplify our statements in the case of $\mathfrak{g} = \mathrm{sl}(N)$, the Lie algebra of traceless $N \times N$ matrices. Denote in this context by $(E_{ij})_{1\le i,j\le n}$ the matrix units which form a basis of the space of all matrices and by E^*_{ij} the corresponding dual matrix.

Observation 2.7.1.

1. We fix a simple complex Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For $\mathfrak{g} = \mathrm{sl}(N)$, $\mathfrak{h}$ are the traceless diagonal matrices.

   One then has a root system $R(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{h}^*$. For $\mathfrak{g} = \mathrm{sl}(N)$, these are the $n^2 - n$ vectors $\pm(E^*_{ii} - E^*_{jj})$ for $i \neq j$.

   Denote by $H_\alpha \in \mathfrak{h}$ the coroot associated to a root α. For the root $\pm(E^*_{ii} - E^*_{jj})$ of $\mathrm{sl}(N)$, this is the diagonal matrix $\pm(E_{ii} - E_{jj})$ We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in R(\mathfrak{g},\mathfrak{h})} \mathfrak{g}^\alpha$. We also fix a basis $(\alpha_1,\ldots,\alpha_r)$ of simple roots, the root system, which provides us with a partition of the roots into positive and negative ones. In the case of $\mathrm{sl}(N)$, we take $\alpha_i := E^*_{ii} - E^*_{i+1,i+1}$ for $i = 1,\ldots n-1$. The positive roots are then of the form $E^*_{ii} - E^*_{jj}$ for $i < j$.

2. The weight lattice $P \subset \mathfrak{h}^*$ is the additive group of linear forms $\lambda \in \mathfrak{h}^*$ such that $\lambda(H_\alpha) \in \mathbb{Z}$ for all roots α. A weight λ is *dominant* if $\lambda(H_\alpha) \geq 0$ for all positive roots α; we denote by P_+ the set of dominant weights.

3. To each dominant weight λ is associated a simple $\mathfrak{g}$-module V_λ, unique up to isomorphism, containing a *highest weight vector* v_λ with weight λ (this means that v_λ is annihilated by $\mathfrak{g}^\alpha$ for $\alpha > 0$ and that $H\, v_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$). The map $\lambda \mapsto [V_\lambda]$ is a bijection of P_+ onto the set of isomorphism classes of finite-dimensional simple $\mathfrak{g}$- modules.

4. The *normalized Killing form* $(\ |\ )$ on $\mathfrak{g}$ is the unique $\mathfrak{g}$-invariant nondegenerate symmetric form on $\mathfrak{g}$ satisfying $(H_\beta \,|\, H_\beta) = 2$ for every long root β. We denote by the same symbol the non-degenerate form induced on $\mathfrak{h}$ and the inverse form on $\mathfrak{h}^*$. We will use these normalized forms.

5. Let θ be the highest root of $R(\mathfrak{g}, \mathfrak{h})$, and H_θ the corresponding coroot. In the case of $\mathrm{sl}(N)$, we have $\theta = E^*_{11} - E^*_{n,n}$. We choose elements X_θ in $\mathfrak{g}^\theta$ and $X_{-\theta}$ in $\mathfrak{g}^{-\theta}$ satisfying

$$[H_\theta, X_\theta] = 2X_\theta \quad , \quad [H_\theta, X_{-\theta}] = -2X_{-\theta} \quad , \quad [X_\theta, X_{-\theta}] = -H_\theta \ .$$

These elements span a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, isomorphic to sl_2, which will play an important role. In the case of $\mathrm{sl}(N)$,

$$H_\theta = E_{11} - E_{nn} \quad X_\theta = E_{1,n} \quad X_{-\theta} = E_{n1} \ .$$

Observation 2.7.2.

1. We fix an integer $\ell > 0$, called the level. We need the irreducible representations of the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ which are of level ℓ, i.e. such that the central element c of $\widehat{\mathfrak{g}}$ acts as multiplication by ℓ. Let P_ℓ be the set of dominant weights λ of $\mathfrak{g}$ such that $\lambda(H_\theta) \leq \ell$. This set is finite. For example, for $\mathfrak{g} = \mathrm{sl}(2)$, the set is $P_\ell = \{0, 1, \ldots, \ell\}$. A fundamental result of the representation theory of $\widehat{\mathfrak{g}}$ asserts that reasonable irreducible representations of level ℓ are classified by P_ℓ. For more, see observation 2.7.4.

2. More precisely, for each $\lambda \in P_\ell$, there exists a simple $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ of level ℓ, characterized up to isomorphism by the following property: *The subspace of $\mathcal{H}_\lambda$ annihilated by the subalgebra $\widehat{\mathfrak{g}}_+$ of positive modes is isomorphic, as a $\mathfrak{g}$-module, to the irreducible highest weight module V_λ of $\mathfrak{g}$.*

   In the sequel we will identify V_λ with the subspace of $\mathcal{H}_\lambda$ annihilated by $\widehat{\mathfrak{g}}_+$. The subcategory of direct sums of such representations is semi-simple.

   For later use, we note that the modes L_n of the Virasoro algebra (23) given by the affine Sugawara construction act on $\mathcal{H}_\lambda$ such that the relation

$$[L_m, X \otimes t^n] = -nX \otimes t^{m+n} \tag{27}$$

   holds for all $X \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

Remarks 2.7.3.

1. The vertex algebra $\mathcal{H}^{(\ell)}$ introduced in Example 2.5.3.3 is defined on the $\mathbb{Z}_+$-graded vector space underlying the module $\mathcal{H}_0$ at level ℓ.

2. It is known [FZ] that Zhu's algebra is in this case isomorphic to a quotient of the universal enveloping algebra,

$$C_2(\mathcal{H}) \cong \mathrm{U}(\mathfrak{g})/\langle X_\theta^{\ell+1}\rangle \ .$$

   It has been computed [B98] using Gröbner basis techniques in the case of $\mathfrak{g} = \mathrm{sl}(2)$:

$$C_2(\mathcal{H}(\mathrm{sl}(2), \ell)) = \oplus_{\lambda \in P_\ell} \mathrm{End}_{\mathbb{C}}(V_\lambda) \ .$$

   This shows that the vertex algebra is C_2-cofinite.

3. The simple objects of $\mathcal{H}^{(\ell)}$-mod are thus in bijection to elements in P_ℓ. Their characters can be computed explicitly, see the Weyl-Kac character formula, cf. e.g. [F, Section 2.6] or [FS, Section 14]. The characters can be expressed as a quotient of an alternating sum over theta-functions for the coroot-lattice of $\mathfrak{g}$.

4. One can therefore compute the representation of $\mathrm{SL}(2,\mathbb{Z})$ on the characters explicitly. We find, see e.g. [F, Section 2.7], for all $\lambda \in P_\ell$:

$$\chi_\lambda(\tau+1) = T_\lambda^{(\ell)} \chi_\lambda(\tau) \quad \text{and } \chi_\lambda(-\frac{1}{\tau}) = \sum_{\mu\in P_\ell} S_{\lambda,\mu}^{(\ell)} \chi_\mu(\tau)$$

with a diagonal unitary matrix with entries

$$T_\lambda^{(\ell)} = \exp\left[2\pi\mathrm{i}\left(\frac{(\Lambda,\Lambda+2\rho)}{2(\ell+h^\vee)} - \frac{c(\ell)}{24}\right)\right]$$

with the Virasoro central charge $c(\ell)$ as in (22) and $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$. One should note that $(\Lambda,\Lambda+2\rho)$ is the eigenvalue of the second order Casimir operator. We moreover, have the symmetric unitary matrix

$$S_{\lambda,\mu}^{(\ell)} = \mathcal{N}\sum_{w\in W} \mathrm{sign}(w) \exp\left[-\frac{2\pi\mathrm{i}}{\ell+h^\vee}(w(\lambda+\rho),\mu+\rho)\right] \tag{28}$$

where the sum is over the Weyl group W of $\mathfrak{g}$ and the normalization constant $\mathcal{N}$ is chosen such that S is unitary and that $S_{0\lambda}>0$ for all $\lambda \in P_\ell$.

Observation 2.7.4.

1. We will need a few more technical details about the $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$. Let us first recall its construction. Let $\mathfrak{p}$ be the Lie subalgebra $\mathfrak{p} := \mathfrak{g}\oplus\mathbb{C}c\oplus\widehat{\mathfrak{g}}_+$ of non-negative modes in $\widehat{\mathfrak{g}}$, a so-called parabolic subalgebra. We extend the representation of $\mathfrak{g}$ on V_λ to $\mathfrak{p}$ by letting the subalgebra $\widehat{\mathfrak{g}}_+$ of positive modes act trivially and c as $\ell\,\mathrm{Id}_{V_\lambda}$; we denote by $\mathcal{V}_\lambda$ the induced $\widehat{\mathfrak{g}}$- module $\mathrm{U}(\widehat{\mathfrak{g}})\otimes_{U(\mathfrak{p})} V_\lambda$. It is called a parabolic Verma module. It contains a unique maximal $\widehat{\mathfrak{g}}$-submodule $\mathcal{Z}_\lambda$. The quotient $\mathcal{V}_\lambda/\mathcal{Z}_\lambda$ is the irreducible module $\mathcal{H}_\lambda$.

   Denote by $\widehat{\mathfrak{g}}_-$ the Lie subalgebra of negative modes. Since $\mathrm{U}(\widehat{\mathfrak{g}})$ is isomorphic as a $\mathrm{U}(\widehat{\mathfrak{g}}_-)$-module to $\mathrm{U}(\widehat{\mathfrak{g}_-})\otimes_{\mathbb{C}}\mathrm{U}(\mathfrak{p})$, we see that *the natural map* $\mathrm{U}(\widehat{\mathfrak{g}}_-)\otimes_{\mathbb{C}} V_\lambda \longrightarrow \mathcal{V}_\lambda$ *is an isomorphism of* $\widehat{\mathfrak{g}}_-$*- modules.*

   We identify the $\mathfrak{g}$-module V_λ to the $\mathfrak{g}$-submodule $1\otimes V_\lambda$ of $\mathcal{V}_\lambda$. The unique maximal submodule $\mathcal{Z}_\lambda$ is generated by the element $(X_\theta\otimes z^{-1})^{\ell-\lambda(H_\theta)+1}\,v_\lambda$ [K-Lie, Exercise 12.12]; this element is annihilated by the Lie subalgebra $\widehat{\mathfrak{g}}_+$ of positive modes. We still have an embedding $V_\lambda\subset\mathcal{H}_\lambda$.

2. The representation theory of $\widehat{\mathfrak{g}}$ is essentially independent of the choice of the local coordinate z. Let $u=u(z)$ be an element of $\mathbb{C}[[z]]$ with $u(0)=0$, $u'(0)\neq 0$ describing a change of coordinates $z\mapsto u(z)$. Then, we have an automorphism

$$\begin{array}{rccc}\gamma_u: & \widehat{\mathfrak{g}} & \to & \widehat{\mathfrak{g}} \\ & f\otimes X & \mapsto & (f\circ u)\otimes X\end{array}$$

   Let $\lambda\in P_\ell$; since γ_u preserves $\widehat{\mathfrak{g}}_+$ and is a multiple of the identity on $\mathfrak{g}$, the representation $\pi_\lambda\circ\gamma_u$ is irreducible, and the subspace annihilated by $\widehat{\mathfrak{g}}_+$ is exactly V_λ. Therefore the representation $\pi_\lambda\circ\gamma_u$ is isomorphic to π_λ. In other words, there is a canonical linear automorphism Γ_u of $\mathcal{H}_\lambda$ such that

$$\Gamma_u\big((X\otimes f)v\big) = (X\otimes f\circ u)\;\Gamma_u(v) \text{ for } v\in\mathcal{H}_\lambda \text{ and } X\otimes f\in\widehat{\mathfrak{g}}$$

   and $\Gamma_u(v)=v$ for $v\in V_\lambda$.

3 From affine Lie algebras to WZW conformal blocks

3.1 Conformal blocks

Observation 3.1.1.

1. We started out with a Möbius covariant consistent set of amplitudes
$$\langle V(v_1, z_1)V(v_2, z_2)\cdots V(v_n, z_n)\rangle \tag{29}$$
for $\psi \in V$ and $\vec{z} = (z_1, \ldots, z_p) \in \mathcal{M}_{0,p}$. Then we enlarged the graded vector space V to first the $\mathbb{Z}_+$-graded vector space $\mathcal{H}$ on which the vertex algebra is defined and next to representations $\mathcal{H}_\mu$ of the vertex algebra.

2. We would like to have a consistent set of expressions available that generalize the amplitudes (29) and where v_i can be a vector in any $\mathcal{H}$-module. At the same time, we would like to take the points $\vec{z}$ to be distinct points in any given complex curve C of genus g. Denote by $\mathcal{M}_{m,g}$ the moduli space of complex curves of genus g with m distinct marked points. In this general situation, we might not be able to pinpoint an amplitude (29) exactly. Rather, we will introduce a vector space of possible amplitudes obeying necessary conditions.

3. We now change the perspective and consider instead of (29) for every $(\vec{z}, C) \in \mathcal{M}_{m,g}$ linear functionals
$$\begin{array}{rcl}\beta_{\vec{z},C} : \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \mathcal{H}_{\lambda_m} & \to & \mathbb{C} \\ (v_1, \ldots, v_m) & \mapsto & \langle V(v_1, z_1)V(v_2, z_2)\cdots V(v_n, z_n)\rangle\end{array}$$

4. The correlators of a full local conformal field theory should be built from these linear functionals. Not any linear functional should qualify, though. Put differently, we want to identify a subbundle $\mathcal{V}_{n,g}(\vec{\lambda})$ of the trivial bundle $(\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \mathcal{H}_{\lambda_m})^* \times \mathcal{M}_{m,g} \to \mathcal{M}_{m,g}$ by imposing consistency conditions that come from the vertex algebra and depend on the point in $\mathcal{M}_{m,g}$.

5. This can be done for a general vertex algebra, for more details see the book [FB-Z] or the review [F]. It is a non-trivial result that for a rational vertex algebra, for any m, $\vec{\lambda} := (\lambda_1, \ldots, \lambda_m)$ and g, a finite-dimensional subbundle can be identified. This bundle is called the bundle of <u>conformal blocks</u> or <u>chiral blocks</u>. In fact, it is not only a vector bundle, but even comes with a projectively flat connection.

6. To simplify the presentation, we decided not to cover the general case of a rational vertex algebra, but rather restrict to the case of vertex algebras given by an affine Lie algebra. In this case, Lie theoretic tools are available. Moreover, a simple set of necessary conditions turns out to be sufficient to determine a collection of conformal blocks obeying non-trivial properties.

We now present some basic Lie-theoretic notions to understand conformal blocks for affine Lie algebras.

Definition 3.1.2
Let $\mathfrak{a}$ be a Lie algebra and V an $\mathfrak{a}$-module.

1. *The space of <u>coinvariants</u> of V, denoted by $[V]_{\mathfrak{a}}$, is the largest quotient of V on which $\mathfrak{a}$ acts trivially, that is the quotient of V by the subspace spanned by the vectors $X.v$ for all $X \in \mathfrak{a}$ and all $v \in V$.*

2. *The space of <u>invariants</u> of V, denoted by $[V]^{\mathfrak{a}}$, is the largest subspace of V on which $\mathfrak{a}$ acts trivially, that is the subspace*

$$[V]^{\mathfrak{a}} := \{v \in V \mid X.v = 0 \text{ for all } X \in \mathfrak{a}\} \ .$$

Remarks 3.1.3.

1. The space of coinvariants $[V]_{\mathfrak{a}}$ equals $V/\mathrm{U}^+(\mathfrak{a})V$, where $\mathrm{U}^+(\mathfrak{a})$ is the augmentation ideal of the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$.

2. Let V and W two $\mathfrak{a}$-modules. Using the canonical anti-involution σ of $U(\mathfrak{a})$ (characterized by $\sigma(X) = -X$ for any X in $\mathfrak{a}$) we can consider V as a right $U(\mathfrak{a})$-module as well.

   Then the space of coinvariants $[V \otimes W]_{\mathfrak{a}}$ is the tensor product $V \otimes_{\mathrm{U}(\mathfrak{a})} W$.

   To see this, observe that both vector spaces are equal to the quotient of $V \otimes W$ by the subspace spanned by the elements $Xv \otimes w + v \otimes Xw$ for all $X \in \mathfrak{a}$, $v \in V$ and $w \in W$.

3. The algebraic dual V^* of an $\mathfrak{a}$-module V carries the structure of an $\mathfrak{a}$-module by $X.\beta(-) = \beta(\sigma(X).-)$ as well. Then

$$[V^*]^{\mathfrak{a}} \xrightarrow{\sim} ([V]_{\mathfrak{a}})^*$$

   An $\mathfrak{a}$-invariant linear form $\beta : V \to \mathbb{C}$ is precisely a linear form vanishing on the subspace $\mathrm{U}^+(\mathfrak{a})V \subset V$. This is precisely the space of linear forms descending under the projection

$$\pi : V \to [V]^{\mathfrak{a}} = V/\mathrm{U}^+(\mathfrak{a})V$$

   to linear forms on the space $[V]^{\mathfrak{a}}$ of coinvariants.

The last remark allows us to work with spaces of coinvariants to understand conformal blocks. This avoids the use of algebraic duals.

Observation 3.1.4.

1. Let C be a smooth, connected complex curve. For each open set $U \subset X$, we denote by $\mathcal{O}(U)$ the ring of holomorphic functions on U, and by $\mathfrak{g}(U)$ the Lie algebra $\mathfrak{g} \otimes \mathcal{O}(U)$.

   We want to associate a vector space to the data of C, a finite subset $\vec{P} = \{P_1, \ldots, P_p\}$ of C, and an highest weight λ_i of P_ℓ attached to each insertion point P_i.

2. In order to do this, we consider the $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1} \otimes \ldots \otimes \mathcal{H}_{\lambda_p}$. We choose a local coordinate z_i at each P_i, and denote by f_{P_i} the Laurent series at P_i of an element $f \in \mathcal{O}(C \setminus \vec{P})$ with at most finite order poles at the points $P_1, \ldots, P_p$. This defines for each i a ring homomorphism $\mathcal{O}(C \setminus \vec{P}) \longrightarrow \mathbb{C}((z))$, hence a Lie algebra homomorphism $\mathfrak{g}(C \setminus \vec{P}) \longrightarrow \mathfrak{g} \otimes \mathbb{C}((z))$. We define an action of $\mathfrak{g}(C \setminus \vec{P})$ on $\mathcal{H}_{\vec{\lambda}}$ by the formula

$$(X \otimes f).(v_1 \otimes \ldots \otimes v_p) = \sum_{i=1}^{p} v_1 \otimes \ldots \otimes (X \otimes f_{P_i})v_i \otimes \ldots \otimes v_p$$

   That this is indeed a Lie algebra action:

$$\begin{aligned}
[X \otimes f, Y \otimes g].(v_1 \otimes \ldots \otimes v_p) &= \textstyle\sum_{i=1}^p v_1 \otimes \ldots \otimes [X \otimes f_{P_i}, Y \otimes g_{P_i}]v_i \otimes \ldots \otimes v_p \\
&= \textstyle\sum_{i=1}^p v_1 \otimes \ldots \otimes [X,Y] \otimes (fg)_{P_i} v_i \otimes \ldots \otimes v_p \\
&\quad + \ell\kappa(X,Y)\mathrm{Res}_{P_i}\left(f_{P_i} \mathrm{d}g_{P_i}\right) \\
&= [X,Y] \otimes fg.(v_1 \otimes \ldots \otimes v_p)
\end{aligned}$$

   since the sum over the residues vanishes by the residue formula $\sum_i \mathrm{Res}_{P_i}\left(f_{P_i}\mathrm{d}g_{P_i}\right) = 0$.

3. Using the notation of definition 3.1.2, we introduce the vector spaces

$$V_C(\vec{P},\vec{\lambda}) := [\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C\setminus\vec{P})} \qquad \text{and} \qquad V_C^\dagger(\vec{P},\vec{\lambda}) := \mathrm{Hom}_{\mathfrak{g}(C\setminus\vec{P})}(\mathcal{H}_{\vec{\lambda}},\mathbb{C}) = [\mathcal{H}^*_{\vec{\lambda}}]^{\mathfrak{g}(C\setminus\vec{P})},$$

where $\mathbb{C}$ is considered as a trivial $\mathfrak{g}(C\setminus\vec{P})$- module. We call $V_C(\vec{P},\vec{\lambda})$ the space of WZW-conformal blocks for the curve C with insertions at the points $\vec{P}$ labelled by the weights $\vec{\lambda}$.

4. By remark 3.1.3.3, $V_C^\dagger(\vec{P},\vec{\lambda})$ is in natural duality with $V_C(\vec{P},\vec{\lambda})$. By observation 2.7.2, these spaces do not depend – up to a *canonical* isomorphism – on the choice of the local coordinates $z_1,\dots,z_p$. On the other hand it is important to keep in mind that they depend on the Lie algebra $\mathfrak{g}$ and the integer ℓ, though neither of these appear in the notation.

5. We give an interpretation of the space $V_C^\dagger(\vec{P},\vec{\lambda})$ which shows that they have the properties expected for an amplitude (29).

   - There is one consistency condition for any meromorphic function f on C that has at most finite order poles at the points $z_1,\dots,z_m$ and any element $x\in\mathfrak{g}$. Heuristically, we might think of this space of Lie-algebra valued functions $\mathcal{F}_\mathfrak{g}$ as a "globalization" to the curve $(C,z_1,\dots,z_m)$ of the vertex algebra. Indeed, given a conformal vertex algebra $\mathcal{H}$, a bundle of vertex algebras can be defined for any complex curve with marked points; for details, we refer to the book [FB-Z] or the review [F, Section 4.2].
   - We present a heuristic understanding of the conditions:
   consider a contour C_P encircling a point $P\in C$. Let z be a local coordinate centered at P. We note that for any meromorphic function f on C with Laurent series $f(z) = \sum a_n z^n$ we have for the vertex operator $V(J,z,) =: J(z)$:

   $$\frac{1}{2\pi\mathrm{i}}\oint_{C_P} f(z)J(z) = \frac{1}{2\pi\mathrm{i}}\oint_{C_P}\sum_{n,m} a_n z^n J_m z^{-m-1} = \sum_n a_n J_m$$

   We use the fact that on the vacuum $\Omega\in\mathcal{H}_0$, we have $J_n\Omega = 0$ for all $n\geq 0$ for all $J\in\mathfrak{g}$ with $\mathfrak{g}$ the underlying Lie algebra. Consider a conformal block on a curve of genus g and insert a "small" contour that does not encircle any insertion. We imagine at a point z_0 inside the contour a vacuum inserted which should not change the block (this will be shown in corollary 3.1.6). Thus for any meromorphic function f on C that has at most finite order poles at the points $z_1,\dots,z_m$, but not at z_0, we find

   $$0 = \frac{1}{2\pi\mathrm{i}}\langle V(v_1,z_1)V(v_2,z_2)\cdots V(v_n,z_n)\oint_{C_{z_0}} f(z)J(z)V(\Omega,z_0)\rangle\ ,$$

   where the action is again defined in terms of local coordinates.
   Using a contour deformation such that we get contours around all other insertion points then yields the identity

   $$0 = \sum_{i=1}^n \langle V(v_1,z_1)V(v_2,z_2)\dots\frac{1}{2\pi\mathrm{i}}\oint_{C_{z_i}} f(z)J(z)V(v_i,z_i)\dots V(v_n,z_n)\rangle$$

   This motivates the idea that the building blocks of correlators on conformal surfaces of arbitrary genus should be invariant under the action of Lie-algebra valued functions.

The following proposition makes the situation quite explicitly accessible:

Proposition 3.1.5.
Let $\vec{P} = \{P_1, \ldots, P_p\}$, $\vec{Q} = \{Q_1, \ldots, Q_q\}$ be two finite nonempty subsets of C, without common point; let $\lambda_1, \ldots, \lambda_p$; $\mu_1, \ldots, \mu_q$ be integrable weights in P_ℓ. We let the Lie algebra $\mathfrak{g}(C \setminus \vec{P})$ act on the finite-dimensional $\mathfrak{g}$-modules V_{μ_j} through the evaluation map $X \otimes f \mapsto f(Q_j)X$. The inclusions $V_{\mu_j} \hookrightarrow \mathcal{H}_{\mu_j}$ induce an isomorphism

$$[\mathcal{H}_{\vec{\lambda}} \otimes V_{\vec{\mu}}]_{\mathfrak{g}(C\setminus\vec{P})} \xrightarrow{\sim} [\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{\mu}}]_{\mathfrak{g}(C\setminus\vec{P}\setminus\vec{Q})} = V_C(\vec{P} \cup \vec{Q}, (\vec{\lambda}, \vec{\mu})) \ .$$

We refer to [B96, Section 3] for a proof of this proposition. This means that we can trade for all but one insertion infinite $\hat{\mathfrak{g}}$-modules for finite-dimensional $\mathfrak{g}$-modules. A similar description of conformal blocks on the sphere can be obtained for any vertex algebra, see [FZ, Theorem 1.5.2,1.5.3].

We immediately have the following consequences:

Corollary 3.1.6.

1. Let $Q \in C \setminus \vec{P}$. There is a canonical isomorphism

$$V_C(\vec{P}, \vec{\lambda}) \xrightarrow{\sim} V_C(\vec{P} \cup Q, (\vec{\lambda}, 0))$$

   This isomorphism is, rather misleadingly, called "propagation of vacua" in the mathematics literature.

2. Let $Q \in C \setminus \vec{P}$. There is a canonical isomorphism

$$V_C(\vec{P}, \vec{\lambda}) \xrightarrow{\sim} [\mathcal{H}_0 \otimes V_{\vec{\lambda}}]_{\mathfrak{g}(C\setminus Q)} \ .$$

Proof.

1. This is the special case $\vec{Q} = \{Q\}$ with label $\mu = 0$.

2. Apply 1., then the proposition 3.1.5 inverting the role of $\vec{P}$ and $\vec{Q}$.

□

The expression for $V_C(\vec{P}, \vec{\lambda})$ given by corollary 3.1.6.2 is quite flexible. It allows to get rather explicit expressions for the conformal blocks on the sphere P^1 and to relate conformal blocks for surfaces at different genus.

Proposition 3.1.7.
Let C be a complex curve with n marked points $\vec{P}$. Assume that the curve is singular and has an ordinary double point in $c \in C$. Let $\nu : \tilde{C} \to C$ be a partial desingularization in c and let $\nu^{-1}(c) = \{a, b\} \subset \tilde{C}$.

Then there is an isomorphism of vector spaces

$$\oplus_{\mu \in P_l} V_{\tilde{C}}(\vec{P}, a, b, ; \vec{\lambda}, \mu, \mu^*) \tilde{\to} V_C(\vec{P}; \vec{\lambda}) \ ,$$

where V_{μ^*} the the finite-dimensional irreducible $\mathfrak{g}$-representation dual to V_μ. The isomorphism is unique up to a scalar.

Proof.
We only sketch the proof. Let $U \subset C$ be the complement of the marked points and $\tilde{U} := \nu^{-1}(U) \subset \tilde{C}$. Proposition 3.1.5 implies that it suffices to find a canonical isomorphism

$$\delta: \quad \oplus_{P_\ell} \mathrm{Hom}_{\mathfrak{g}(\tilde{U})}(\mathcal{H}_{\vec{\lambda}} \otimes V_\mu \otimes V_{\mu^*}, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}(U)}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}) \ .$$

The ideal $\mathcal{I}$ of meromorphic functions on U vanishing in the double point c equals the the ideal of functions vanishing on $\tilde{U}$ in the two points a and b. Moreover, we have an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}(U)}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{g}\otimes\mathcal{I}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})^{\mathfrak{g}}$$

and, with the following analogue of the regular representation

$$\mathcal{K}_\ell := \oplus_{P_\ell} V_\mu \otimes V_{\mu^*}$$

we have

$$\oplus_{P_\ell} \mathrm{Hom}_{\mathfrak{g}(U)}(\mathcal{H}_{\vec{\lambda}} \otimes V_\mu \otimes V_{\mu^*}, \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{g}\otimes\mathcal{I}}(\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{K}_\ell, \mathbb{C})^{\mathfrak{g}\times\mathfrak{g}} \ .$$

Since V_{μ^*} is the dual representation of the irreducible representation V_μ, the vector space of invariants $(V_\mu \otimes V_{\mu^*})^{\mathfrak{g}}$ is one-dimensional. We fix a non-zero element γ_μ in it.

Given $\psi_\mu \in \mathrm{Hom}_{\mathfrak{g}(U)}(\mathcal{H}_{\vec{\lambda}} \otimes V_\mu \otimes V_{\mu^*}, \mathbb{C})$, we set

$$\delta(\psi_\mu): u \mapsto \psi_\mu(u \otimes \gamma_\mu) \quad \text{for } u \in \mathcal{H}_{\vec{\lambda}} \ .$$

One shows that this is a conformal block and that the map is an isomorphism of vector spaces. □

To show that the conformal blocks are finite-dimensional, we need the following

Lemma 3.1.8.
Let $\mathfrak{a}$ be a Lie algebra and H an $\mathfrak{a}$-module of finite type, i.e. there is a finite-dimensional subspace $L \subset H$ such that $\mathrm{U}(\mathfrak{a})L = H$. Suppose that there is a basis (e_i) of $\mathfrak{a}$ such that all e_i act on H in a locally finite way, i.e. for any $u \in H$ the subspace generated by $((e_i)^l u)_{i=0,\ldots,}$ is finite-dimensional.

Let $\mathfrak{a}_+ := \{x \in \mathfrak{a} \,|\, x.L = 0\}$ and suppose that $\mathfrak{k} \subset \mathfrak{a}$ is a Lie subalgebra such that $\mathfrak{k} + \mathfrak{a}_+$ has finite codimension in $\mathfrak{a}$. Then the quotient $H/\mathfrak{k}H$ is finite-dimensional.

Proof.
The hypotheses are such that we can find finitely many locally finite basis elements $(e_i)_{i=1,\ldots N}$ in $\mathfrak{a}$ such that $\mathfrak{a} = [\mathfrak{k} + \mathfrak{a}_+] \oplus \left(\oplus_{i=1}^N \mathbb{C} e_i\right)$. By the Poincaré-Birkhoff-Witt theorem, we then have

$$\mathrm{U}(\mathfrak{a}) = \sum_{m_1,\ldots,m_N} \mathrm{U}(\mathfrak{k}) \otimes e_1^{m_1} \ldots \otimes e_N^{m_N} \otimes \mathrm{U}(\mathfrak{a}_+) \ .$$

By definition of $\mathfrak{a}_+$, we have $\mathrm{U}(\mathfrak{a}_+)L = L$; thus

$$H = \sum_{m_1,\ldots,m_N} \mathrm{U}(\mathfrak{k}) e_1^{m_1} \ldots e_N^{m_N} L$$

Since all e_i act in a locally finite way, one can show by induction that the subspace $e_1^{m_1} \ldots e_N^{m_N} L$ is finite-dimensional. Thus $H = \mathrm{U}(\mathfrak{k})\tilde{L}$ with $\tilde{L}$ a finite-dimensional subspace. Thus the map $\tilde{L} \to H/\mathfrak{k}H$ induced from the embedding $\tilde{L} \to H$ is surjective which shows the claim. □

Proposition 3.1.9.
The WZW conformal blocks are all finite-dimensional, $\dim_{\mathbb{C}} V_C(\vec{P},\vec{\lambda}) < \infty$ for all choices $\vec{\lambda}$ of sectors and all curves, of any genus with any number of marked points.

Proof.
We can assume by proposition 3.1.5 that there is just a single marked point $p \in C$ with label $\lambda \in P_\ell$. We choose a local coordinate z around p.

Take in the lemma $\mathfrak{a} = \hat{\mathfrak{g}}$ and $\mathfrak{a}_+ = \hat{\mathfrak{g}}_+$. Take $\mathfrak{k} := \mathfrak{g}(C \setminus \{p\})$. Then $\mathfrak{k} + \mathfrak{a}_+$ are $\mathfrak{g}$-valued functions on C which are holomorphic everywhere, except for at most a finite order pole in p. By Riemann-Roch, this space is of finite codimension in $\mathfrak{a}$.

We still have to find a basis of $\mathfrak{a}$ obeying the conditions in the lemma: take any basis of the Lie subalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$ of positive modes and add the center and elements $X(m)$ with X ad-nilpotent and m negative. □

We will give in the next section an explicit formula for their dimension.

3.2 Conformal blocks on the Riemann sphere and the Verlinde formula

We can be even more explicit for conformal blocks on the Riemann sphere. We fix a quasi-global coordinate t on P^1. We remind the reader of the Lie subalgebra $\mathfrak{s} = \{X_\theta, X_{-\theta}, H_\theta\} \subset \mathfrak{g}$ isomoprhic to sl(2) introduced in Observation 2.7.1 using the highest root θ of $\mathfrak{g}$.

Proposition 3.2.1.
Let $P_1, \ldots, P_p$ be distinct points of P^1 with coordinates $t_1, \ldots, t_p$, and let $\lambda_1, \ldots, \lambda_p$ be elements of P_ℓ. Let T be the endomorphism of the tensor product $V_{\vec{\lambda}} = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ of finite-dimensional $\mathfrak{g}$-modules defined by

$$T(v_1 \otimes \ldots \otimes v_p) = \sum_{i=1}^{p} t_i \, v_1 \otimes \ldots \otimes X_\theta \, v_i \otimes \ldots \otimes v_p \; .$$

1. The space of conformal blocks $V_{\mathbf{P}^1}(\vec{P},\vec{\lambda})$ is canonically isomorphic to the largest quotient of $V_{\vec{\lambda}}$ on which $\mathfrak{g}$ and $T^{\ell+1}$ act trivially.
2. The space $V^\dagger_{\mathbf{P}^1}(\vec{P},\vec{\lambda})$ is isomorphic to the space of $\mathfrak{g}$- invariant p-linear forms $\varphi : V_{\lambda_1} \times \ldots \times V_{\lambda_p} \longrightarrow \mathbb{C}$ such that $\varphi \circ T^{\ell+1} = 0$.

Proof.
We apply corollary 3.1.5 with $Q = \infty$, with the understanding that the local coordinate z at $Q = \infty$ is t^{-1}. This gives an isomorphism of $V_{\mathbf{P}^1}(\vec{P},\vec{\lambda})$ onto $[\mathcal{H}_0 \otimes V_{\vec{\lambda}}]_{\mathfrak{g}(\mathbb{C})}$.

Now the Lie algebra $\mathfrak{g}(\mathbb{C})$ is the sum of the zero-mode algebra $\mathfrak{g}$ and the subalgebra of negative modes $\widehat{\mathfrak{g}}_-$; it follows from observation 2.7.2 that the $U(\mathfrak{g}(\mathbb{C})$-module $\mathcal{H}_0$ is generated by the highest weight vector v_0, with the relations $\mathfrak{g}\, v_0 = 0$ and $(X_\theta \otimes z^{-1})^{\ell+1} \, v_0 = 0$. Therefore the space and by remark 3.1.3.2

$$[\mathcal{H}_0 \otimes V_{\vec{\lambda}}]_{\mathfrak{g}(\mathbb{C})} \cong \mathcal{H}_0 \otimes_{U(\mathfrak{g}(\mathbb{C}))} V_{\vec{\lambda}}$$

which is canonically isomorphic to $V_{\vec{\lambda}}/(\mathfrak{g}V_{\vec{\lambda}} + \mathrm{Im}\, T^{\ell+1})$, where T is the action of the Lie algebra valued function $X_\theta \otimes t$ which is given by the endomorphism of $V_{\vec{\lambda}}$ given by the above formula.

The description of $V^\dagger_{\mathbf{P}^1}(\vec{P},\vec{\lambda})$ follows by duality. □

When $p = 3$, one can describe the space $V_{\mathbf{P}^1}(a,b,c;\lambda,\mu,\nu)$ of three-point blocks in an even more concrete way. Let us first consider the case when $\mathfrak{g} = \mathrm{sl}_2$. We denote by E the standard 2-dimensional representation of $\mathfrak{g}$. We will identify P_ℓ with the set of integers p with $0 \le p \le \ell$ by associating to such an integer the irreducible representation S^pE. By proposition 3.2.1, $V^\dagger_{\mathbf{P}^1}(a,b,c;p,q,r)$ is the space of linear forms $F \in \mathrm{Hom}_{\mathfrak{g}}(S^pE \otimes S^qE \otimes S^rE, \mathbb{C})$ such that $F \circ T^{\ell+1} = 0$.

Lemma 3.2.2.

1. The space $\mathrm{Hom}_{\mathfrak{g}}(S^pE \otimes S^qE \otimes S^rE, \mathbb{C})$ is either zero- or one-dimensional. It is nonzero, if and only if $p+q+r$ is even, say $= 2m$, and p,q,r are $\le m$.
2. The subspace $V^\dagger_{\mathbf{P}^1}(a,b,c;p,q,r)$ is nonzero if and only if $p+q+r$ is even and $\le 2\ell$.

Proof.
The first assertion is an immediate consequence of the Clebsch-Gordan formula. The second is an explicit calculation for which we refer to [B96, Section 4]. □

For the general case, we consider the Lie subalgebra $\mathfrak{s} \cong \mathrm{sl}_2$ of $\mathfrak{g}$ for the highest root with basis $(X_\theta, X_{-\theta}, H_\theta)$. For $\lambda \in P_\ell$, the $\mathfrak{g}$-module V_λ is completely reducible as an $\mathfrak{s}$-module: we denote the isotypical components by $V^{(i)}_\lambda$ with $0 \le i \le \ell/2$.

Proposition 3.2.3.
Let $\mathfrak{g}$ be any finite-dimensional simple Lie algebra.

1. The space $V_{P^1}(a,b,c;\lambda,\mu,\nu)$ is canonically isomorphic to the quotient of the space $\mathfrak{g}$-invariants
$$[V_\lambda \otimes V_\mu \otimes V_\nu]_{\mathfrak{g}}$$
by the image of the subspaces $V^{(p)}_\lambda \otimes V^{(q)}_\mu \otimes V^{(r)}_\nu$ for $p+q+r > \ell$.
2. The space $V^\dagger_{\mathbf{P}^1}(a,b,c;\lambda,\mu,\nu)$ is canonically isomorphic to the space of $\mathfrak{g}$- invariant linear forms $\varphi : V_\lambda \otimes V_\mu \otimes V_\nu \longrightarrow \mathbb{C}$ which vanish on the subspaces $V^{(p)}_\lambda \otimes V^{(q)}_\mu \otimes V^{(r)}_\nu$ whenever $p+q+r > \ell$.

Proof.
The two assertions are equivalent, we prove 2. By proposition 3.2.1, we have to reformulate the condition $\varphi \circ T^{\ell+1} = 0$ for a $\mathfrak{g}$-invariant linear form $\varphi : V_\lambda \otimes V_\mu \otimes V_\nu \longrightarrow \mathbb{C}$.

To this end, we decompose into isotypical $\mathfrak{s}$-components

$$V_\lambda = \oplus_{p=0}^{\ell/2} V^{(p)}_\lambda, \quad V_\mu = \oplus_{p=0}^{\ell/2} V^{(p)}_\mu \quad \text{and} \quad V_\nu = \oplus_{p=0}^{\ell/2} V^{(p)}_\nu$$

The subspaces $V^{(p)}_\lambda \otimes V^{(q)}_\mu \otimes V^{(r)}_\nu$ are stable under $\mathfrak{s}$ and hence under the endomorphism T, so we have to find out when the restriction φ_{pqr} of φ to any of these subspaces vanishes on $\mathrm{Im}\, T^{\ell+1}$. By lemma 3.2.2.2, the restriction automatically vanishes, if $p+q+r \le \ell$, while we have to impose $\varphi_{pqr} = 0$ when $p+q+r > \ell$, hence the proposition. □

We note a direct consequence:

Corollary 3.2.4.
One has the following statements for one-point and two-point blocks on P^1:

$$V_{\mathbf{P}^1}(P,\lambda) = 0 \quad \text{for} \quad \lambda \neq 0 \quad , \quad V_{\mathbf{P}^1}(P,0) \cong V_{\mathbf{P}^1}(\emptyset) \cong \mathbb{C}$$
$$V_{\mathbf{P}^1}(P,Q,\lambda,\mu) = 0 \quad \text{for} \quad \mu \neq \lambda^* \quad , \quad V_{\mathbf{P}^1}(P,Q,\lambda,\lambda^*) \cong \mathbb{C} \ .$$

Combining now the factorization result 3.1.7 with the rather explicit description of the three-point blocks on the Riemann sphere in 3.2.3 allows to compute the dimension of the space of conformal blocks. For a detailed derivation, we refer to [B96]; the result is

Proposition 3.2.5 (Verlinde formula)**.**
1. Assume that $\mathfrak{g}$ is of type A, B, C, D or G. One has

$$\dim V_C(\vec{P},\vec{\lambda}) = |T_\ell|^{g-1} \sum_{\mu\in P_\ell} \mathrm{Tr}_{V_{\vec{\lambda}}}(\exp 2\pi i \frac{\mu+\rho}{\ell+h^\vee}) \prod_{\alpha>0} \left| 2 \sin \pi \frac{(\alpha \,|\, \mu+\rho)}{\ell+h} \right|^{2-2g} \ .$$

   with $|T_\ell| = (\ell+h^\vee)^r f q$, where r is the rank of $\mathfrak{g}$, f its connection index, $f = |P/Q|$, and q the index of the sublattice Q_{lg} of long roots in the root lattice Q. A glance at the tables gives $q = 2$ for B_r, 2^{r-1} for C_r, 4 for F_4, 6 for G_2, and of course 1 otherwise.

2. It is more instructive to express the result in terms of the symmetric unitary matrix S appearing in (28) in the description of the modular transformation properties of the characters:

$$\dim V_C(\vec{P},\vec{\lambda}) = \sum_{\mu\in P_\ell} \prod_{i=1}^{m} \frac{S_{\lambda_i,\mu}}{S_{0\mu}} \ (S_{0,\mu})^{2-2g} \ .$$

3.3 The Knizhnik-Zamolodchikov connection

In this subsection, we follow [K, Section 1.5]. We now show that the WZW-conformal blocks form a vector bundle with projectively flat connection over the configuration space $\mathcal{M}_m(P^1)$ of m distinct points on the complex plane. To this end, we need to vary the insertion points.

Observation 3.3.1.
1. We fix $m \in \mathbb{N}$ and an m-tuple $\vec{\lambda}$ of integrable highest weights at level ℓ. Recall that conformal blocks are a subbundle of the trivial bundle

$$E(\vec{\lambda}) := \mathcal{M}_m(P^1) \times (\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m})^* \to \mathcal{M}_m(P^1)$$

   over the configuration space $\mathcal{M}_m(P^1)$.

2. Consider the projection $\pi : (P^1)^{m+1} \to (P^1)^m$ on the first m factors.

   For any open subset $U \subset \mathcal{M}_m(P^1)$, consider the infinite-dimensional Lie algebra $\mathfrak{g}(U)$ of meromorphic $\mathfrak{g}$-valued functions on $\pi^{-1}(U)$ with singularities at most poles of finite order along the divisors D_i of $(P^1)^{m+1}$ that are defined by $z_i = z_{m+1}$.

   The coordinates $z_1, \ldots, z_m$ parametrize the insertion points and can be seen as coordinates on the configuration space. The last coordinate z_{m+1} will be used to describe meromorphic functions on P^1 with singularities at most finite order poles at the insertion points.

   One should keep in mind that $f \in \mathfrak{g}(U)$ is a Lie algebra-valued function of $m+1$-variables. Along the divisor D_j, any $f \in \mathfrak{g}(U)$ has a Laurent expansion in $t_j := z_{m+1} - z_j$ of the form

$$f_{D_j}(t_j) = \sum_{n=-N}^{\infty} a_n(z_1, \ldots, z_m) t_j^n$$

with holomorphic $\mathfrak{g}$-valued functions $a_n(z_1, \ldots, z_m)$.

Denote by $\mathcal{O}(U)$ the set of holomorphic functions on $U \subset \mathcal{M}_m(P^1)$. The Laurent expansion gives a map

$$\begin{array}{rccl} \tau_j : & \mathfrak{g}(U) & \rightarrow & \mathfrak{g} \otimes \mathcal{O}(U)((t_j)) \\ & f & \mapsto & f_{D_j}(t_j) \end{array}$$

We regard this as an element in the loop algebra $\mathbb{C}((t))$ depending holomorphically on the m insertion points $(z_1, \ldots, z_m)$.

3. The expansion map τ_i allows us again to define an action of the Lie algebra $\mathfrak{g}(U)$ depending on $(z_1, \ldots, z_m)$ on the tensor product $(\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m})^*$.

   We denote by $\mathcal{V}_m(\vec{\lambda})(U)$ the set of smooth sections

   $$\Psi : U \to U \times (\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m})^* =: E(\vec{\lambda})|_U$$

   of the trivial vector bundle $E(\vec{\lambda})$ that is invariant under this action of $\mathfrak{g}(U)$ at any $\vec{P} := (p_1, \ldots, p_m) \in U$. Specifically, the section $\Psi \in \mathcal{V}_m(\vec{\lambda})(U)$ obeys

   $$\sum_{j=1}^{m} \Psi(p_1, \ldots, p_m)(v_1, \ldots, \tau_j(f)v_j, \ldots, v_m) = 0$$

   for all $f \in \mathfrak{g}(U)$. Then $\Psi(\vec{P}) \in V^{\dagger}_{\mathbf{P}^1}(\vec{P}, \vec{\lambda})$ for any $\vec{P} \in U$.

4. For $X \in \hat{\mathfrak{g}}$ and any $\Psi \in (\mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m})^*$ we let

   $$[X^{(j)}\Psi](v_1, \ldots, v_m) := \Psi(v_1, \ldots, Xv_j, \ldots v_m) \ .$$

   We extend this notation to the modes of the Virasoro algebra obtained by the Sugawara construction (23).

Lemma 3.3.2.
If Ψ is a smooth local section with values in $\mathcal{V}_m(\vec{\lambda})(U)$, then

$$\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi$$

is a smooth local section with values in $\mathcal{V}_m(\vec{\lambda})(U)$ as well.

Proof.

- We first note that for $f \in \mathfrak{g}(U)$, also the partial derivative $f_j := \frac{\partial f}{\partial z_j}$ with $j = 1, \ldots, m$ is in $\mathfrak{g}(U)$. Its Laurent expansion along D_j is

  $$\tau_j(f_j) = \sum_{n=-N}^{\infty} \left(\frac{\partial a_n}{\partial z_j} t_j^n - a_n n t_j^{n-1} \right) \ ,$$

  since $t_j = z_{m+1} - z_j$ depends on z_j as well.

- The partial derivative

  $$\partial_j : \mathfrak{g} \otimes \mathcal{O}(U) \to \mathfrak{g} \otimes \mathcal{O}(U)$$

  is extended to

  $$\partial_j : \mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j)) \to \mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$$

  with the trivial action on the last tensorand. With this notation,

  $$\tau_j(f_j) = \sum_{n=-N}^{\infty} \left(\frac{\partial a_n}{\partial z_j} t_j^n - a_n n t_j^{n-1} \right) = \partial_j \tau_j(f) - \frac{\partial}{\partial t_j} \tau_j(f) \ .$$

- Recall from 2.7.2 the relation (27) which implies for the action on $\hat{\mathfrak{g}}$-modules:

$$\frac{\partial}{\partial t_j}\tau_j(f) = -[L_{-1},\tau_j(f)] \ .$$

  Thus

$$(*) \qquad \tau_j(f_j) = \partial_j\tau_j(f) + [L_{-1},\tau_j(f)] \ ;$$

  moreover, one has

$$\tau_i(f_j) = \partial_j\tau_i(f) \quad \text{for } i \neq j \ .$$

- We have to show that

$$\sum_{i=1}^{m}\left(\frac{\partial\Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m) = 0$$

  for all $v_i \in \mathcal{H}_{\lambda_i}$ and all $f \in \mathfrak{g}(U)$. We first note that

$$(**) \qquad \frac{\partial}{\partial z_j}[\Psi(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m)] =$$
$$\frac{\partial\Psi}{\partial z_j}(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m) + \Psi(v_1,\ldots,\partial_j\tau_i(f)v_i,\ldots,v_m)]$$

  Summing over i and using $(*)$ and $(**)$, we find

$$\begin{aligned}
&\sum_{i=1}^{m}\left(\frac{\partial\Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m)\\
\overset{(**)}{=}\ &\sum_{i=1}^{m}\frac{\partial}{\partial z_j}[\Psi(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m)]\\
&-\sum_{i=1}^{m}\Psi(v_1,\ldots,\partial_j\tau_i(f)v_i,\ldots,v_m)\\
&-\sum_{i=1,i\neq j}^{m}\Psi(v_1,\ldots,\tau_i(f)v_i,\ldots,L_{-1}v_j,\ldots,v_m)\\
&-\Psi(v_1,\ldots,v_i,\ldots,L_{-1}\tau_j(f)v_j,\ldots,v_m)\\
\overset{(*)}{=}\ &\sum_{i=1}^{m}\frac{\partial}{\partial z_j}[\Psi(v_1,\ldots,\tau_i(f)v_i,\ldots,v_m)]\\
&-\sum_{i=1}^{m}\Psi(v_1,\ldots,\tau_i(f_j)v_i,\ldots,v_m)\\
&-\sum_{i=1}^{m}\Psi(v_1,\ldots,\tau_i(f)v_i,\ldots,L_{-1}v_j,\ldots,v_m)
\end{aligned}$$

  .

  Since both f and f_j are in $\mathfrak{g}(U)$ and since Ψ is required to be invariant under the action of $\mathfrak{g}(U)$, the all three terms on right hand side vanish.

□

We therefore introduce the linear operators

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial z_j}}: \quad \mathcal{V}_m(\vec{\lambda})(U) &\rightarrow \mathcal{V}_m(\vec{\lambda})(U)\\
\Psi &\mapsto \tfrac{\partial\Psi}{\partial z_j} - L_{-1}^{(j)}\Psi \ .
\end{aligned}$$

Theorem 3.3.3.
The family of conformal blocks $\mathcal{V}_m(\vec{\lambda})$ over $\mathcal{M}_m(P^1)$ is endowed by these operators with the structure of a vector bundle E with flat connection:

$$\begin{array}{rcl} \nabla : \Gamma(E) & \to & \Gamma(T^*\mathcal{M}_m \otimes E) \\ \Psi & \mapsto & \nabla\Psi = \mathrm{d}\Psi - \sum_{i=1}^m L_{-1}^{(i)} \Psi \mathrm{d}z_i \end{array}$$

This connection is called the Knizhnik-Zamolodchikov connection.

Proof.
Consider

$$\omega := \sum_{i=1}^m L_{-1}^{(i)} \mathrm{d}z_i \ ;$$

this is a 1-form on the configuration space $\mathcal{M}_m(P^1)$ with values in $\mathrm{End}(E(\vec{\lambda}))$. Since $L_{-1}^{(i)}$ does not depend on $z \in \mathcal{M}$, the form is closed, $\mathrm{d}\omega = 0$. Moreover, $[L_{-1}^{(i)}, L_{-1}^{(j)}] = 0$ for all i, j implies $\omega \wedge \omega = 0$. Thus we have a flat connection on the trivial bundle $E(\vec{\lambda})$.

Lemma 3.3.2 implies that the connection ∇ restricts to the subsheaf $\mathcal{V}_m(\vec{\lambda}) \subset E$. Horizontal local sections for the connection provide a local frame which locally trivializes $\mathcal{V}_m(\vec{\lambda})$. Thus this is a vector bundle, and the restriction of the connection ∇ to it is still flat. □

By an explicit calculation using the affine Sugawara (23) construction from Observation 2.5.6, one gets a concrete equation. To state it, denote by $(t^a)_{a=1,\dots \dim \mathfrak{g}}$ a basis of $\mathfrak{g}$ that is orthonormal with respect to the Killing form of $\mathfrak{g}$. Put

$$\Omega := \sum_{a=1}^{\dim \mathfrak{g}} t^a \otimes t^a \in \mathfrak{g} \otimes \mathfrak{g}$$

and denote by $\Omega^{(ij)}$ for $i \neq j$ its action on the i-th and j-th component in the tensor product $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_m}$.

Corollary 3.3.4.
Let Ψ be a local horizontal section of the bundle $\mathcal{V}_m(\vec{\lambda})$. Then the restriction Ψ_0 of Ψ to the finite-dimensional subspace $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_m} \subset \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m}$ satisfies the following system of partial differential equations:

$$\frac{\partial \Psi_0}{\partial z_i} = \frac{1}{\ell + h^\vee} \sum_{j; j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j} \qquad \text{for } 1 \leq i \leq n \ .$$

This system is called the Knizhnik-Zamolodchikov equation.

.

Remarks 3.3.5.

1. As with any vector bundle with flat connection, we get a representation of the fundamental group $\pi_1(\mathcal{M}_m(P^1))$ of the base space. This is a braid group, and as a consequence, we have an action of the braid group on the WZW-conformal blocks on the Riemann sphere.

2. The construction can be generalized to conformal blocks of higher genus, but then the Knizhnik-Zamolodchikov connection is only projectively flat. The fundamental group of the moduli space $\mathcal{M}_{m,g}$ is called mapping class group. We thus get a projective representation of the relevant mapping class group on WZW conformal blocks at any genus.

3. The mapping class group for a torus is the modular group $\mathrm{SL}(2,\mathbb{Z})$. Hence the zero-point WZW-blocks on the torus carry an action of the modular group. It turns out to be the action of $\mathrm{SL}(2,\mathbb{Z})$ on the characters from remark 2.7.3.4.

4. The structure we have discussed in this subsection is expected to generalize to any rational conformal vertex algebra: one can define finite-dimensional vector bundles of conformal blocks that obey factorization rules and that are endowed with a projectively flat connection. See the book [FB-Z] for some steps in this direction. In particular, we have a projective action of mapping class groups on the spaces of conformal blocks

This is the basis for our next axiomatization.

4 Modular functors and modular tensor categories

The system of conformal blocks endows the abelian category $\mathcal{H}$-mod of modules over a vertex algebra with much additional structure induced from the flat connection. The additional structure is captured by the notion of a modular functor (this is really a family of functors).

4.1 Modular functors

We note that the category $\mathcal{H}$-mod is a $\mathbb{C}$-linear abelian category.

Definition 4.1.1
A $\mathbb{C}$-linear abelian category is a category such that the set of morphisms between any two objects U, V is a $\mathbb{C}$-vector space $\mathrm{Hom}(U,V)$*, and such that compositions are bilinear.*

Let A be a complex algebra. Then the category A-mod of A-modules is a $\mathbb{C}$-linear category.

It is helpful to understand the structure on the collection of such categories: we have a three-layered structure, a bicategory:

- Its objects a $\mathbb{C}$-linear abelian categories.
- Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $\mathbb{C}$-linear abelian categories. A 1-morphism $F:\mathcal{C}_1\to\mathcal{C}_2$ is a $\mathbb{C}$-linear functor.
- Given two $\mathbb{C}$-linear functors $F_1, F_2 : \mathcal{C}_1 \to \mathcal{C}_2$, a two-morphism $F_1 \to F_2$ is a natural transformation.

It is crucial to note that the bicategory $\mathcal{C}at(\mathbb{C})$ comes with a categorified tensor product, the Deligne tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Rather than giving an explicit definition, it suffices to say:

- If $\mathcal{C}_i$ is equivalent to the category of modules A_i-mod of a $\mathbb{C}$-algebra A_i, then the category $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is equivalent to the category $A_1 \otimes A_2$-mod.
- If both categories $\mathcal{C}_i$ are finitely semisimple, then $\mathcal{C}_1\boxtimes\mathcal{C}_2$ is finitely semisimple with simple objects of the form $S_1 \boxtimes S_2$, where S_i runs over all simple objects of $\mathcal{C}_i$.

Conformal blocks are associated to Riemann surfaces; to keep track of the representations of the mapping class group they furnish, we have to introduce a suitable class of surfaces.

Definition 4.1.2
An extended surface is a compact oriented smooth *two-dimensional manifold Σ, possibly with*

boundary, together with a choice of a marked point on each connected component of the boundary $\partial\Sigma$. The set of boundary components of Σ is denoted by $A(\Sigma)$ and we write extended surfaces as $(\Sigma, \{e_a\}_{a\in A(\Sigma)})$. A morphism of extended surfaces is a smooth map that preserves marked points.

Definition 4.1.3
Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category. A gluing object $\mathcal{R} \in \mathcal{C} \boxtimes \mathcal{C}$ is a symmetric object, i.e. $\mathcal{R} \cong \mathcal{R}^{op}$. Here $\mathcal{R}^{op}$ is obtained by the permutation action on the two factors.

A $\mathcal{C}$-extended modular functor consists of the following data:

1. *Functors for extended surface:*
   For every extended surface Σ, we have a functor
   $$\tau(\Sigma, \{e_a\}_{a\in A(\Sigma)}) : \boxtimes_{a\in A(\Sigma)}\mathcal{C} \to \mathrm{vect} \ , \tag{30}$$
   We write $\tau(\Sigma; \{V_a\})$ for the value of the functor on a family $\{V_a\}$ of objects in $\mathcal{C}$.

2. *Functorial isomorphisms for morphisms of extended surfaces:*
   For every isomorphism $f : (\Sigma, \{e_a\}_{a\in A(\Sigma)}) \to (\Sigma', \{e'_a\}_{a\in A(\Sigma')})$ of extended surfaces a functorial isomorphism
   $$f_* : \tau(\Sigma, \{e_a\}_{a\in A(\Sigma)}) \to \tau(\Sigma', \{e'_a\}_{a\in A(\Sigma')})$$
   that depends only on the isotopy class of f.

3. *Functorial isomorphisms $\tau(\emptyset) \cong \mathbb{C}$ and $\tau(\Sigma \sqcup \Sigma') \cong \tau(\Sigma) \otimes_{\mathbb{C}} \tau(\Sigma')$.*

4. *Functorial gluing isomorphisms:*
   Let $(\Sigma, \{e_a\}_{a\in A(\Sigma)})$ be an extended surface and let $\alpha, \beta \in A(\Sigma), \alpha \neq \beta$. We can glue Σ along the boundary components α and β. We require the existence of functorial gluing isomorphisms
   $$G_{\alpha,\beta} : \quad \tau(\Sigma; \{V_a\}, \mathcal{R}) \xrightarrow{\sim} \tau(\sqcup_{\alpha,\beta}\Sigma; \{V_a\}) \tag{31}$$
   This is well defined by symmetry of $\mathcal{R}$. Here $\sqcup_{\alpha,\beta}\Sigma$ is the surface with the boundary components α and β glued.

These data are subject to the following conditions:

- *$(f \circ g)_* = f_* \circ g_*$ and $id_* = id$.*
- *All morphisms in 3. and 4. are functorial in $(\Sigma, \{e_a\}_{a\in A(\Sigma)})$ and compatible with each other. Examples include the compatibility of gluing with disjoint union and the associativity of gluing [BK, Section 5.1].*
- *Symmetry of the gluing, $G_{\alpha,\beta} = G_{\beta,\alpha}$.*
- *Normalization: $\tau(S^2) \cong \mathbb{C}$*

Definition 4.1.4
A $\mathcal{C}$-extended modular functor is called non-degenerate, if for every non-zero object V in $\mathbb{C}$, there is a an extended surface Σ and a collection of objects $\{V_a\}$, such that $\tau(\Sigma, V, \{V_a\})$ is non-zero.

Observation 4.1.5.

1. We take $\mathcal{C}$ to be the representation category of a vertex algebra $\mathcal{H}$. We then can think of the vector space $\tau(M;\{V_a\})$ as the space of conformal blocks associated to a complex curve with underlying smooth manifold M and insertions labelled by $\mathcal{H}$-modules V_a. The functors introduced in 1. then express that taking invariants as in the definition 3.1.4.3 of conformal blocks is functorial in the representation. The projectively flat connection then leads to representations of the mapping class group which are captured by the data 2. Finally the gluing data in 4. capture the factorization of conformal blocks described in proposition 3.1.7.

2. We can view this definition as follows: a two-dimensional topological field theory is a symmetric monoidal function
$$\mathrm{tft} : \mathrm{Cob}_{2,1} \to \mathrm{vect}\ .$$
A modular functor is now, morally, a category-valued topological field theory:
$$\mathrm{tft} : \mathrm{Cob}_{2,1} \to \mathcal{C}at(\mathbb{C})$$

3. We can also describe the information we have suppressed. For each one-dimensional manifold S, we need a functor
$$H_S : \mathrm{tft}(S) \to \mathrm{vect}$$
with values in graded vector spaces, together in particular with coherent equivalences $H_S \otimes H_{S'} \to H_{S\sqcup S'}$ of functors.

A non-degenerate modular functor endows the category $\mathcal{C} := \tau(S^1)$ with much additional structure. To this end, we need the following notions:

Definition 4.1.6

1. *A monoidal category or tensor category $\mathbb{C}$ is a category, together with a functor*
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
and certain associativity constraints. We also require the existence of a monoidal unit $\mathbb{I}$ and coherent isomorphisms $\mathbb{I}\otimes V \cong V \cong V \otimes \mathbb{I}$. Such a category is called strict, if for any objects U, V, W, we have identitites $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, and $V \otimes \mathbb{I} = \mathbb{I} \otimes V = V$. We restrict to strict tensor categories from now on.

2. *For a $\mathbb{C}$-linear monoidal category, we require the tensor product to be bilinear on homomorphisms.*

3. *A braiding on a (strict) monoidal category associates to any pair of objects V, W an isomorphism $c_{V,W} \in \mathrm{Hom}(V \otimes W, W \otimes V)$ such that*

   (i) $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W)$.

   (ii) $c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W})$.

   (iii) $(g \otimes f) \circ c_{V,W} = c_{V,W} \circ (f \otimes g)$.

   Here $U, V, \ldots$ are arbitrary objects and $f \in \mathrm{Hom}(V, V')$, $g \in \mathrm{Hom}(W, W')$ are arbitrary morphisms.

4. *A left duality on a (strict) monoidal category associates to any object V a dual object V^* and morphisms $b_V \in \mathrm{Hom}(\mathbb{I}, V \otimes V^*)$, $d_V \in \mathrm{Hom}(V^* \otimes V, \mathbb{I})$ such that*

(i) $(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.

(ii) $(d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$.

5. *A ribbon category is a strict monoidal category with additional data: a braiding, a twist and a duality. A twist associates to any object V an isomorphism $\theta_V \in \mathrm{Hom}(V, V)$ such that*

   (i) $\theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W)$.

   (ii) $\theta_{V'} \circ f = f \circ \theta_V$.

   (iii) $(\theta_V \otimes \mathrm{id}_{V^*}) \circ b_V = (\mathrm{id}_V \otimes \theta_{V^*}) \circ b_V$.

Lemma 4.1.7.

1. Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to Set$ be a functor. Then F is called representable, if there is an object $X_F \in \mathcal{C}$ such that $F \cong \mathrm{Hom}(X_F, -)$. The object X_F is unique up to unique isomorphism.

2. We need the Yoneda lemma: let $F : \mathcal{C} \to Set$ be any functor and consider for $A \in \mathcal{C}$ the functor $\mathrm{Hom}(A, -)$. Then we have a natural bijection of sets

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A) \ .$$

   In the special case when $F = \mathrm{Hom}(B, -)$, we find

$$\mathrm{Nat}(\mathrm{Hom}(A, -), \mathrm{Hom}(B, -)) \cong \mathrm{Hom}(B, A) \ .$$

3. Let $\mathbb{C}$ be a finitely semisimple $\mathbb{C}$-linear category. Then any additive functor $F : \mathbb{C} \to \mathrm{vect}$ is representable.

Proposition 4.1.8.
A genus 0 non-degenerate $\mathcal{C}$-extended modular functor is equivalent to the structure of a ribbon category on $\mathcal{C}$.

Proof.
We only indicate the ideas and refer to [BK, Section 5.3] for more details. For any n-tuple $(V_1, V_2, \ldots, V_n)$ of objects in $\mathcal{C}$, we introduce the shorthand

$$\langle V_1, V_2, \ldots, V_n\rangle := \tau(S^2, V_1, V_2, \ldots, V_n) \in \mathrm{vect}(\mathbb{C}) \ .$$

- The dual object of an object V is the object representing the functor

$$\begin{array}{rcl} \mathcal{C} & \to & \mathrm{vect} \\ T & \mapsto & \langle V, T\rangle \ . \end{array}$$

- Define a functor $\otimes : \mathcal{C}^{\boxtimes 2} \to \mathbb{C}$ on objects A, B by

$$\langle T, A \otimes B\rangle \cong \langle T, A, B\rangle \ .$$

- The monoidal unit $\mathbb{I}$ is defined as the object that obeys

$$\langle \mathbb{I}, T\rangle \cong \langle T\rangle \ .$$

- To get the associativity constraints $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$, cut the four-punctured sphere in two different ways into two three-punctured spheres. The cutting gives functorial isomorphisms between the corresponding vect-valued functors. Since these are representable with objects $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ respectively, the Yoneda lemma 4.1.7.2 gives an associativity isomorphism.

- The braiding is recovered from a braiding isomorphism

$$\varphi_B : S^2_3 \to S^2_3$$

that maps the three-punctured sphere to itself: the functorial isomorphism

$$\begin{aligned}\langle T, A \otimes B\rangle &\stackrel{\text{def}}{=} \langle T, A, B\rangle \stackrel{\text{def}}{=} \tau(S^2_3, T, A, B)\\ &\stackrel{(\varphi_B)_*}{\to} \tau(S^2_3, T, B, A) \stackrel{\text{def}}{=} \langle T, B, A\rangle \stackrel{\text{def}}{=} \langle T, B \otimes A\rangle\end{aligned}$$

gives, by the Yoneda lemma, an isomorphism $c_{A,B} : A \otimes B \to B \otimes A$.

□

In fact, the representation category of representations of a rational vertex algebra has has the structure of a ribbon category and one more property [H05]: it is a modular tensor category.

Definition 4.1.9
A modular category is a strict monoidal semisimple abelian $\mathbb{C}$-linear ribbon category $\mathcal{C}$ with unit object $\mathbb{I}$ and an additional set of data obeying a system of axioms:

1. *A the set of isomorphism classes of simple objects is finite. The monoidal unit $\mathbb{I}$ is simple. Denote by I a set of representatives containing $\mathbb{I}$.*

2. *The matrix $(s_{i,j}) := (\mathrm{tr}(c_{j,i}c_{i,j}))$ indexed by $i, j \in I$ is invertible.*

One should note that, up to a normalization, this matrix describes the transformation of the zero-point blocks on the torus under the modular group $\mathrm{SL}(2, \mathbb{Z})$.

4.2 The 3-dimensional topological field theory

To every modular category $\mathcal{C}$ there is an associated 3-dimensional topological field theory; such a topological field theory has more structure than a modular functor. The TFT associates a finite dimensional vector space $\mathrm{tft}_{\mathcal{C}}(X)$ (the space of conformal blocks) to each surface X with marked points and additional labels, and an element of $\mathrm{tft}_{\mathcal{C}}(X)$ to each 3-dimensional manifold with a graph of Wilson lines labelled by objects of $\mathcal{C}$ bounding X. We replace in the extended surfaces discs with a point on the boundary by marked arcs. In a $\mathcal{C}$-labelled extended surface, the arcs are labelled by objects of $\mathcal{C}$.

Definition 4.2.1

1. *A cobordism of extended surfaces is a triple $(M, \partial_- M, \partial_+ M)$ such that:*

   (a) *M is a 3-dimensional manifold with boundary containing a ribbon graph. A ribbon graph consists of ribbons, annuli and coupons. Ribbons ends are glued to coupons or are contained in the boundary ∂M.*

(b) $\partial_\pm M$ are disjoint disconnected subsets of the boundary ∂M so that $\partial M = \partial_+ M \cup (-\partial_- M)$. The marked arcs at which the ribbons in M end are given the label of the ribbons whose core is oriented inwards, and the dual label otherwise.

We say that $(M, \partial_- M, \partial_+ M)$ is a cobordism from $\partial_- M$ to $\partial_+ M$.

2. *A cobordism of $\mathcal{C}$-labelled extended surfaces is a cobordism of extended surfaces where ribbons and annuli are labelled by objects in $\mathcal{C}$ and coupons by appropriate morphisms in $\mathcal{C}$.*

Definition 4.2.2
The three-dimensional topological field theory $\mathrm{tft}_\mathcal{C}$ associated to a modular category $\mathcal{C}$ over $\mathbb{C}$ consists of the following data.

(i) *For each $\mathcal{C}$-labelled extended surface X, there is a finite dimensional complex vector space $\mathrm{tft}_\mathcal{C}(X)$, the space of states (or of conformal blocks), such that $\mathrm{tft}_\mathcal{C}(\emptyset) = \mathbb{C}$ and $\mathrm{tft}_\mathcal{C}(X \sqcup Y) = \mathrm{tft}_\mathcal{C}(X) \otimes \mathrm{tft}_\mathcal{C}(Y)$.*

(ii) *To each homeomorphism of $\mathcal{C}$-labelled extended surfaces $f\colon X \to Y$ there is an isomorphism $f_\sharp\colon \mathrm{tft}_\mathcal{C}(X) \to \mathrm{tft}_\mathcal{C}(Y)$.*

(iii) *If $(M, \partial_- M, \partial_+ M)$ is a cobordism of $\mathcal{C}$-labelled extended surfaces, then the TFT associates to it a linear map*

$$\mathrm{tft}_\mathcal{C}(M, \partial_- M, \partial_+ M): \quad \mathrm{tft}_\mathcal{C}(\partial_- M) \to \mathrm{tft}_\mathcal{C}(\partial_+ M)$$

depending linearly on the labels of the coupons.

These data obey the following axioms.

1. *(Naturality) Let $(M, \partial_- M, \partial_+ M)$, $(N, \partial_- N, \partial_+ N)$ be cobordisms of $\mathcal{C}$-labelled extended surfaces. Let $f\colon M \to N$ be a degree one homeomorphism mapping the ribbon graph in M onto the ribbon graph in N, restricting to homeomorphisms $f_\pm\colon \partial_\pm M \to \partial_\pm N$ (preserving the Lagrangian subspaces). Then*

$$(f_+)_\sharp \circ \mathrm{tft}_\mathcal{C}(M, \partial_- M, \partial_+ M) = \mathrm{tft}_\mathcal{C}(N, \partial_- N, \partial_+ N) \circ (f_-)_\sharp$$

2. *(Multiplicativity) If M_1, M_2 are two cobordisms of $\mathcal{C}$-labelled extended surfaces, then under the identification $\mathrm{tft}_\mathcal{C}(\partial_\pm M_1 \sqcup \partial_\pm M_2) = \mathrm{tft}_\mathcal{C}(\partial_\pm M_1) \otimes \mathrm{tft}_\mathcal{C}(\partial_\pm M_2)$ we have $\mathrm{tft}_\mathcal{C}(M_1 \sqcup M_2) = \mathrm{tft}_\mathcal{C}(M_1) \otimes \mathrm{tft}_\mathcal{C}(M_2)$.*

3. *(Functoriality) Suppose a cobordism M is obtained from the disjoint union of M_1 and M_2 by gluing $\partial_+ M_1$ to $\partial_- M_2$ along a degree one homeomorphism $f\colon \partial_+ M_1 \to \partial_- M_2$ preserving marked arcs with their orientation and labels. Then*

$$\mathrm{tft}_\mathcal{C}(M, \partial_- M_1, \partial_+ M_2) = \kappa^m \, \mathrm{tft}_\mathcal{C}(M_2, \partial_- M_2, \partial_+ M_2) \circ f_\sharp \circ \mathrm{tft}_\mathcal{C}(M_1, \partial_- M_1, \partial_+ M_1),$$

for some integer m. (Here κ is a constant associated to the category $\mathcal{C}$.)

4. *(Normalization) Let X be an extended surface. Let the* cylinder over X *be the 3-manifold $X \times [-1, 1]$, with the ribbon graph consisting of the ribbons $z \times [-1, 1]$, where z runs over the marked arcs of X. Their orientation is such that they induce the orientation of the arcs on $X \times \{1\}$. Their core is oriented from 1 to -1. Then*

$$\mathrm{tft}_\mathcal{C}(X \times [-1, 1], X \times \{-1\}, X \times \{1\}) = \mathrm{id}_{\mathrm{tft}_\mathcal{C}(X)}. \qquad (32)$$

Remarks 4.2.3.

1. The homomorphism $\mathrm{tft}_{\mathcal{C}}(M, \partial_- M, \partial_+ M)$ is called the *invariant* of the cobordism of $\mathcal{C}$-labelled extended surfaces $(M, \partial_- M, \partial_+ M)$. By the naturality axiom it is invariant under degree one homeomorphisms that restrict to the identity on the boundary.

2. The TFT gives the system of vector spaces $\mathrm{tft}_{\mathcal{C}}(X)$ the structure of a *modular functor*.

3. The action $f \mapsto f_\sharp$ of homeomorphisms may be expressed in terms of the TFT. Namely, let $f\colon X \to Y$ be a homeomorphism of extended surfaces. Then the 3-manifold obtained by gluing the cylinder over X to the cylinder over Y defines a cobordism (M_f, X, Y). The normalization and functoriality axioms then imply that $f_\sharp = \mathrm{tft}_{\mathcal{C}}(M_f, X, Y)$. Moreover, it can be shown, using the naturality axiom, that if f, g are homotopic in the class of homeomorphism of extended surfaces, then $f_\sharp = g_\sharp$. In particular, if $X = Y$, $f \mapsto f_\sharp$ defines a projective representation of the mapping class group of X.

5 Full conformal field theory

We are now ready to discuss full, local two-dimensional conformal field theories and to combine left and right movers in the sense of Section 1.6. We restrict to the case when X is a compact oriented two-dimensional conformal manifold, but allow for boundaries.

5.1 Decoration data

Observation 5.1.1.
We expect that then the following data have to be specified:

- Whenever a two-manifold X has a boundary, one expects that it is necessary to specify boundary conditions. We take the possible boundary conditions to be the objects of a decoration category $\mathcal{M}$. Morphisms are boundary fields that can change the boundary condition. The composition of morphisms in the category $\mathcal{M}$ captures the operator product of boundary fields.

- Conformal field theories can have topological defect lines. We label the possible types of defect lines by objects in yet another decoration category $\mathcal{D}$.

  There is a natural notion of fusion of defect lines; accordingly, $\mathcal{D}$ will be a tensor category. Also, to take into account the topological nature of defect lines, we assume that the tensor category $\mathcal{D}$ has dualities and that it is even sovereign. In contrast, there is no natural notion of a braiding of defect lines, so $\mathcal{D}$ is, in general, not a ribbon category.

We can now formulate a central insight: the decoration categories can be expressed in terms of Frobenius algebras in the modular tensor category $\mathcal{C}$ obtained from the chiral conformal field theory. The additional structure of a Frobenius algebra comes from a field-theoretic analysis, taking in particular into account the non-degeneracy of the two-point functions of boundary fields on a disk.

Definition 5.1.2

1. *A Frobenius algebra $A = (A, m, \eta, \Delta, \epsilon)$ in $\mathcal{C}$ is an object of $\mathcal{C}$ carrying the structures of a unital associative algebra (A, m, η) and of a counital coassociative coalgebra (A, Δ, ϵ) in $\mathcal{C}$, with the algebra and coalgebra structures satisfying the compatibility requirement that the coproduct $\Delta\colon A \to A \otimes A$ is a morphism of A-bimodules (or, equivalently, that the product $m\colon A \otimes A \to A$ is a morphism of A-bi-comodules).*

2. *A Frobenius algebra is called special, iff the coproduct is a right-inverse to the product – this means in particular that the algebra is separable – and a nonvanishing multiple of the unit η: $\mathbb{I} \to A$ is a right-inverse to the counit ϵ: $A \to \mathbb{I}$.*

3. *There are two isomorphisms $A \to A^\vee$ that are naturally induced by product, counit and duality; A is called symmetric, iff these two isomorphisms coincide.*

We summarize the insight of [FFRS1, FFRS2]:

Proposition 5.1.3.

1. The category of boundary conditions $\mathcal{M}$ is equivalent to the category of left A-modules in $\mathcal{C}$.

2. The category $\mathcal{D}$ labelling defect lines is equivalent to the category of A-bimodules. Types of defect lines separating A_1 and A_2 are described by isomorphism classes of $A_1 - A_2$-bimodules.

3. The monoidal structure capturing operator products products are tensor products over A.

4. Full conformal field theories combining left movers and right movers are in bijection to Frobenius algebras.

We now present a construction of CFT-correlators based on Frobenius algebras. Recall the double covering $\hat{X}$, obtained from the orientation cover that we considered in Section 1.6. We consider it for a smooth two-dimensional manifold; it comes with an orientation reversing involution σ such that the quotient $\hat{X}/\langle\sigma\rangle$ is naturally isomorphic to X, and we have a canonical projection

$$\pi : \quad \hat{X} \mapsto X \cong \hat{X}/\langle\sigma\rangle .$$

The set of fixed points of σ is just the preimage under π of the boundary ∂X.

Given the modular tensor category $\mathcal{C}$, the complex modular functor [BK] provides us with a vector bundle $\mathcal{V}$ with projectively flat connection on $\mathcal{M}_{g,m}$. We recall from Section 1.6 the 'principle of holomorphic factorization'. It states that, first of all, the conformal surface X should be decorated in such a way that the double $\hat{X}$ has the structure of an object in the decorated cobordism category for the topological field theory based on $\mathcal{C}$. It then makes sense to require, secondly, that the correlation function is a certain global section of the restriction of $\mathcal{V}$ to $\mathcal{M}^\sigma_{g,m}$.

At this point, it proves to be convenient to use the equivalence of the complex modular functor and the topological modular functor $\mathrm{tft}_\mathcal{C}$ based on the modular tensor category $\mathcal{C}$ [BK] so as to work in a topological (rather than complex-analytic) category. We are thereby lead to the description of a correlation function on X as a specific vector $\mathrm{C}or(X)$ in the vector space $\mathrm{tft}_\mathcal{C}(\hat{X})$ that is assigned to the double $\hat{X}$ by the topological modular functor $\mathrm{tft}_\mathcal{C}$. These vectors must obey two additional axioms:

- *Covariance*: Given any morphism f: $X \to Y$ in the relevant decorated geometric category $\}_\mathcal{C}$, we demand
$$\mathrm{C}or(Y) = \mathrm{tft}_\mathcal{C}(f)\big(\mathrm{C}or(X)\big) .$$

- *Factorization*: Certain factorization properties must be fulfilled.

We refer to [FFRS1, FFRS2] for a precise formulation of these constraints.

The covariance axiom implies in particular that the vector $\mathrm{C}or(X)$ is invariant under the action of the mapping class group $\mathrm{Map}(X) \cong \mathrm{Map}(\hat{X})^\sigma$. This group, also called the relative modular group, acts genuinely on $\mathrm{tft}_\mathcal{C}(\hat{X})$.

5.2 The TFT construction of RCFTs

To find solutions to the covariance and factorization constraints on the vectors $\mathrm{C}or(X) \in \mathrm{tft}_{\mathcal{C}}(\hat{X})$ we use the three-dimensional topological field theory associated to the modular tensor category $\mathcal{C}$. Thus we look for a (decorated) cobordism $(M_X, \emptyset, \hat{X})$ such that the vector $\mathrm{tft}_{\mathcal{C}}(M_X, \emptyset, \hat{X})1 \in \mathrm{tft}_{\mathcal{C}}(\hat{X})$ is the correlator $\mathrm{C}or(X)$.

The three-manifold M_X should better not introduce any topological information that is not already contained in X. This leads to the idea to use an interval bundle as a "fattening" of the world sheet.

Definition 5.2.1
Given a surface X, possibly with boundary and possibly unorientable, the connecting manifold *is the following oriented three-manifold*

$$M_X := \big(\hat{X} \times [-1,1] \big) \, / \, \langle(\sigma, t \mapsto -t)\rangle \,,$$

where $\hat{X}$ is the orientation cover of X.

The connecting manifold has boundary $\partial M_X \cong \hat{X}$ and contains X as a retract: the embedding ι of X is to the fiber $t{=}0$, the retracting map contracts along the intervals.

The connecting manifold M_X must now be decorated with the help of Frobenius algebras in $\mathcal{C}$ and (bi-)modules over them.

The conformal surface X is decomposed by defect lines (which are allowed to end on ∂X) into various two-dimensional regions. There are two types of one-dimensional structures: boundary components of X and defect lines. Defect lines, in general, form a network; they can be closed or have end points, and in the latter case they can end either on the boundary or in the interior of X. Both one-dimensional structures are partitioned into segments by marked "insertion" points. The end points of defect lines carry insertions, too. Finally, we also allow for insertion points in the interior of two-dimensional regions.

To these geometric structures, data coming from Frobenius algebras are now assigned as follows.

- First, we attach to each two-dimensional region a symmetric special Frobenius algebra, i.e. an object of $\mathrm{Frob}_{\mathcal{C}}$.
- To a segment of a defect line that separates regions with label A and A', respectively, we attach an A-A'-bimodule.
- Similarly, to a boundary segment adjacent to a region labeled by A, we assign a left A-module.
- Finally, zero-dimensional geometric objects are labeled with morphisms of modules or bimodules, respectively.

  Two types of points, the insertion points of field, still deserve more comments: those separating boundary segments on the one hand, and those separating or creating segments of defect lines or appearing in the interior of two-dimensional regions on the other.

  - An insertion point $p \in \partial X$ that separates two boundary segments labeled by objects $M_1, M_2 \in A$-mod has a single preimage under the canonical projection π from $\hat{X}$ to X; to the interval in M_X that joins this preimage to the image $\iota(p)$ of p under the embedding ι of X into M_X, we assign an object U of the category $\mathcal{C}$ of chiral data. To the insertion point itself, we then attach a morphism of A-modules $\mathrm{Hom}_A(M_1 \otimes U, M_2)$.

– An insertion point in the interior of X has two preimages on $\hat{X}$; these two points are connected to $\iota(p)$ by two intervals. To each of these two intervals we assign an object U and V, respectively, of the category $\mathcal{C}$ of chiral data. The orientation is used to attribute the two objects U, V to the two preimages.

  We now first consider an insertion point separating a segment of a defect line labeled by an object $B_1 \in \mathrm{Hom}(A, A')$ from a segment labeled by $B_2 \in \mathrm{Hom}(A, A')$. We then use the left action ρ_l of A and the right action ρ_r of A' on the bimodule B_1 to define a bimodule structure on the object $U \otimes B_1 \otimes V$ of $\mathcal{C}$ by taking the morphisms $(\mathrm{id}_U \otimes \rho_l \otimes \mathrm{id}_V) \circ (c_{U,A}^{-1} \otimes \mathrm{id}_{B_1} \otimes \mathrm{id}_V)$ and $(\mathrm{id}_U \otimes \rho_r \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \mathrm{id}_{B_1} \otimes c_{A',V}^{-1})$ as the action of A and A', respectively, where c denotes the braiding isomorphisms of $\mathcal{C}$. The insertion point separating the defect lines is now labeled by a morphism of A-A'-bimodules in $\mathrm{Hom}(U \otimes B_1 \otimes V, B_2)$.

– To deal with insertion points in the interior of a two-dimensional region labeled by a Frobenius algebra A, we need to invoke one further idea: such a region has to be endowed with (the dual of) a *triangulation* Γ. To each edge of Γ we attach the morphism $\Delta \circ \eta \in \mathrm{Hom}(\mathbb{I}, A \otimes A)$, and to each trivalent vertex of Γ the morphism $\epsilon \circ m \circ (m \otimes \mathrm{id}_A) \in \mathrm{Hom}(A \otimes A \otimes A, \mathbb{I})$.

  Now each of the insertion points p that we still need to discuss is located inside a two-dimensional region labeled by some Frobenius algebra A or creates a defect line. For the first type of points, we choose the triangulation such that an A-ribbon passes through p; to p we then attach a bimodule morphism in $\mathrm{Hom}(U \otimes A \otimes V, A)$, with U and V objects of $\mathcal{C}$ as above. To a point p at which a defect line of type B starts or ends, we attach a bimodule morphism in $\mathrm{Hom}(U \otimes A \otimes V, B)$ and in $\mathrm{Hom}(U \otimes B \otimes V, A)$, respectively.

We have now obtained a complete labelling of a ribbon graph in the connecting manifold M_X with objects and morphisms of the modular tensor category $\mathcal{C}$; in other words, a cobordism from $\emptyset$ to $\hat{X}$. We can apply to it the $\mathrm{tft}_\mathcal{C}$-functor for the tensor category $\mathcal{C}$ to obtain a vector

$$\mathrm{C}or(X) = \mathrm{tft}_\mathcal{C}(M_X)\, 1 \, \in \mathrm{tft}_\mathcal{C}(\hat{X}) \,.$$

This is the prescription for RCFT correlation functions in the TFT approach. It follows from the defining properties of a symmetric special Frobenius algebra that $\mathrm{C}or(X)$ does not depend on the choice of triangulation; for details see [FFRS1].

Remark 5.2.2.
On the basis of this construction one can establish many further results. Let us list some of them, without indicating their proofs:

- One can compute the coefficients of partition functions for bulk fields, boundary fields and defect fields. One can show that they obey all consistency conditions that have been proposed in the literature, in particular integality and modular invariance of torus partition functions.

- One can derive explicit expressions for the coefficients of operator product expansions of bulk, boundary, and defect fields.

- The theory can be extended to unorientable surfaces. This is necessary for string compactifications of type I. Then, additional structure has to be chosen on the Frobenius algebra.

- For arbitrary topology of the surface X the correlators obtained in the TFT construction can be shown to satisfy the covariance and factorization axioms that were stated earlier.

- One can explicitly compute symmetries and Kramers-Wannier dualities of rational conformal field theories.

The TFT approach to the construction of CFT correlation functions represents CFT quantities as invariants of knots and links in three-manifolds, It constitutes a powerful algebraization of many questions that arise in the study of (rational) conformal field theory.

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