Algebraic Topology

Summer term 2016

Christoph Schweigert

Hamburg University

Department of Mathematics

Section Algebra and Number Theory

and

Center for Mathematical Physics

(as of: 01.11.2020)

Contents

Literature:

Some of the literature I used to prepare the course:

- A. Hatcher, *Algebraic Topology*
Cambridge University Press, 2002.
- G. Bredon, *Topology and Geometry.*
Springer Graduate Text in Mathematics 139, Springer, New York, 2010
- R. Stöcker, H. Zieschang, *Algebraische Topologie.*
Teubner, Stuttgart, 1994

The current version of these notes can be found under
`http://www.math.uni-hamburg.de/home/schweigert/skripten/atskript.pdf`
as a pdf file.

Please send comments and corrections to `christoph.schweigert@uni-hamburg.de`!

These notes are based on lectures delivered at the University of Hamburg in the summer terms 2016 and 2018. I would like to thank Birgit Richter for making her notes available to me. The present notes are indeed just a minor extension of the notes for her course. I am grateful to S. Azzali, F. Bartelmann, S. Bunk, V. Koppen, A. Lahtinen, M. Manjunatha, L. Wienke and L. Woike for helpful comments.

1 Homology theory

We recall a few facts about the fundamental group:

- It assigns to any path connected topological space X with a base point $x \in X$ an algebraic object, a group $\pi_1(X, x)$. The assignment is functorial, i.e. for a continuous map $f : X \to Y$ we get a group homomorphism $f_* : \pi(X, x) \to \pi_1(Y, f(x))$. Homotopic maps $f \simeq g$ induce the same maps on the fundamental group, $f_* = g_*$.
- The fundamental group is computable invariant, most notably due to the theorem of Seifert-van Kampen.
- The invariant crucially enters in covering theory: if a topological space X is sufficiently connected, the equivalence classes of path-connected coverings are classified by conjugacy classes of subgroups of $\pi_1(X)$.
- However, for CW complexes, it is insensitive to n-cells with $n \geqslant 3$. As a consequence, it cannot distinguish spheres $\mathbb{S}^n$ for different $n \geqslant 2$.

  A possible remedy is to consider continuous maps $I^n \to M$, with $I = [0, 1]$, up to homotopy relative boundary. But the corresponding homotopy groups $\pi_n(M)$ are difficult to compute, even for spaces as fundamental as spheres. For example, for the 2-sphere $\pi_n(\mathbb{S}^2)$ is non-zero, although the 2-sphere does not have cells in dimensions greater than 2.

Homology is a *computable* algebraic invariant that is sensitive to higher cells as well; but it takes some effort to define it. In particular, we will have rather huge objects in intermediate steps to which we turn now:

1.1 Chain complexes

Homology is defined using algebraic objects called chain complexes.

Definition 1.1.1
A chain complex is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n : C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$.

Let R be an (associative) ring with unit 1_R. A chain complex of R-modules can analogously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

Definition 1.1.2
We fix the following terminology:

- *The homomorphisms d_n are called differentials or boundary operators.*
- *The elements $x \in C_n$ are called n-chains.*
- *Any $x \in C_n$ such that $d_n x = 0$ is called an n-cycle. We denote the group of n-cycles by*

$$Z_n(C) := \ker(d_n) = \{x \in C_n \mid d_n x = 0\}.$$

- *Any $x \in C_n$ of the form $x = d_{n+1} y$ for some $y \in C_{n+1}$ is called an n-boundary.*

$$B_n(C) := \mathrm{Im}(d_{n+1}) = \{d_{n+1} y, y \in C_{n+1}\}.$$

The cycles and boundaries form subgroups of the group of chains. The identity $d_n \circ d_{n+1} = 0$ implies that the image of d_{n+1} is a subgroup of the kernel of d_n and thus

$$B_n(C) \subset Z_n(C).$$

We often drop the subscript n from the boundary maps and just write C_* for the chain complex.

Definition 1.1.3
The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is called the nth homology group of the complex C_*.

We denote by $[c] \in H_n(C)$ the equivalence class of a cycle $c \in Z_n(C)$. If $c, c' \in C_n$ are such that $c - c'$ is a boundary, then c is said to be homologous to c'. This defines an equivalence relation on chains. A complex is called acyclic, if its homology except in degree 0 vanishes.

Examples 1.1.4.

1. Consider the complex with
$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
Here, the only non-zero differential is d_1; let it be the multiplication with $N \in \mathbb{N}$, then
$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and consider differentials
$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$
The homology of this chain complex vanishes in all degrees.

3. Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. The homology of this chain complex equals $\mathbb{Z}$ in all degrees.

We need morphisms of chain complexes:

Definition 1.1.5
Let C_* and D_* be two chain complexes. A chain map $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, i.e., *the diagram*

$$\begin{array}{ccc} C_n & \xrightarrow{d_n^C} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{d_n^D} & D_{n-1} \end{array}$$

commutes for all n.

A chain map f sends cycles to cycles, since

$$d_n^D f_n(c) = f_{n-1}(d_n^C c) = 0 \quad \text{for a cycle } c\ ,$$

and boundaries to boundaries, since

$$f_n(d^C_{n+1}\lambda) = d^D_{n+1} f_{n+1}(\lambda) \ .$$

We therefore obtain an induced map of homology groups

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)[c] = [f_n c]$.

Examples 1.1.6.

1. There is a chain map from the chain complex mentioned in Example 1.1.4.1 to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot N} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_N & \longrightarrow & 0 \end{array}$$

   Note that $(f_0)_*$ is an isomorphism on the zeroth homology group; all homology groups are isomorphic.

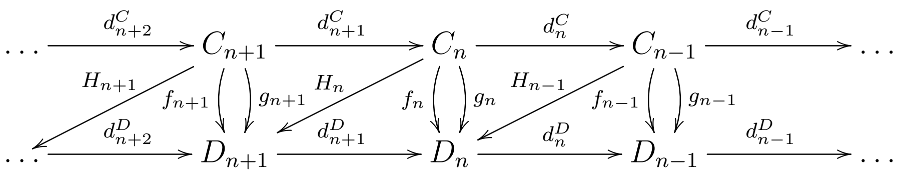
2. Are there chain maps between the complexes from Examples 1.1.4.2. and 1.1.4.3?

Lemma 1.1.7.
If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, then $H_n(g) \circ H_n(f) = H_n(g \circ f)$ for all n.

We next study situations in which two chain maps induce the same map on homology.

Definition 1.1.8
A chain homotopy H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d^D_{n+1} \circ H_n + H_{n-1} \circ d^C_n = f_n - g_n.$$



If such an H exists, then the chain maps f and g are said to be (chain) homotopic. We write $f \simeq g$.

We will see in Section 1.4 geometrically defined examples of chain homotopies.

Proposition 1.1.9.

1. Being chain homotopic is an equivalence relation on chain maps.

2. If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.

Proof.

1. If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each chain map f is homotopic to itself with chain homotopy $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$.

2. We have for every cycle $c \in Z_n(C_*)$:

$$H_n(f)[c] - H_n(g)[c] = [f_n c - g_n c] = [d^D_{n+1} \circ H_n(c)] + [H_{n-1} \circ d^C_n(c)] = 0.$$

Here, the class of the first term vanishes; in the second term $d^C_n c = 0$, since c is a cycle.

□

Definition 1.1.10
Let $f\colon C_* \to D_*$ be a chain map. We call f a chain homotopy equivalence, if there exists a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are said to be chain homotopically equivalent.

Chain homotopically equivalent chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent, cf. Example 1.1.6.1: there is no non-zero morphism of abelian groups $\mathbb{Z}_N \to \mathbb{Z}$.

Definition 1.1.11
If C_* and C'_* are chain complexes, then their direct sum, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d \oplus d'$ given by

$$d_\oplus(c, c') = (dc, d'c').$$

Similarly, if $(C^{(j)}_*, d^{(j)})_{j\in J}$ is a family of chain complexes, then we can define their direct sum as follows:

$$(\bigoplus_{j\in J} C^{(j)}_*)_n := \bigoplus_{j\in J} C^{(j)}_n$$

as abelian groups and the differential $d_\oplus$ is defined via the property that its restriction to the jth summand is $d^{(j)}$.

1.2 Singular homology

In the definition of the fundamental group, we test a topological space X by (homotopy classes of) maps $\mathbb{S}^1 \to X$. In the definition of singular homology, we use maps from higher-dimensional objects, simplices.

Let $v_0, \dots, v_n$ be $n+1$ points in $\mathbb{R}^{n+1}$. Consider the convex hull

$$K(v_0, \dots, v_n) := \{\sum_{i=0}^{n} t_i v_i | \sum_{i=0}^{n} t_i = 1, t_i \geqslant 0\} \subset \mathbb{R}^{n+1}.$$

Definition 1.2.1
If the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, then $K(v_0, \dots, v_n)$ is the simplex generated by $v_0, \dots, v_n$. We denote such a simplex by $\mathrm{simp}(v_0, \dots, v_n)$.

Note that simplex really means "simplex with an ordering of its vertices".

Example 1.2.2.

1. Denote by $e_i \in \mathbb{R}^{n+1}$ the vector that has an entry 1 in coordinate $i+1$ and is zero in all other coordinates. The standard topological n-simplex is $\Delta^n := \mathrm{simp}(e_0, \ldots, e_n)$.

2. The first examples of standard topological simplices are:

   - Δ^0 is the point $e_0 = 1 \in \mathbb{R}$.
   - Δ^1 is the line segment in $\mathbb{R}^2$ between $e_0 = (1,0) \in \mathbb{R}^2$ and $e_1 = (0,1) \in \mathbb{R}^2$.
   - Δ^2 is a triangle in $\mathbb{R}^3$ with vertices e_0, e_1 and e_2 and Δ^3 is homeomorphic to a tetrahedron.

3. The coordinate description of the standard n-simplex in $\mathbb{R}^{n+1}$ is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n} t_i = 1, t_i \geqslant 0\}.$$

   We consider the standard simplex Δ^n as a subset $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \ldots$.

   The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}.

   We describe the boundary by the following $(n+1)$ face maps for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

   The image of d_i^{n-1} in Δ^n is the face that is opposite to the vertex e_i. It is the $(n-1)$-simplex generated by the $n-1$-tuple $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$ of vectors in $\mathbb{R}^{n+1}$.

Lemma 1.2.3.

Concerning the composition of face maps, the following rule holds:

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \qquad \text{for all } 0 \leqslant j < i \leqslant n.$$

Example: face maps for Δ^0 and composition into Δ^2: $d_2 \circ d_0 = d_0 \circ d_1$.

Proof.

Both expressions yield

$$\begin{aligned} d_i^{n-1} \circ d_j^{n-2}(t_0, \ldots, t_{n-2}) &= (t_0, \ldots, t_{j-1}, 0, t_j \ldots, t_{i-2}, 0, t_{i-1}, \ldots, t_{n-2}) \\ &= d_j^{n-1} d_{i-1}^{n-2}(t_0, \ldots, t_{n-2}). \end{aligned}$$

□

Definition 1.2.4

Let X be an arbitrary topological space, $X \neq \emptyset$. A singular n-simplex in X is a continuous map $\alpha \colon \Delta^n \to X$.

Note, that α is just required to be continuous. (It does not make sense to require it to be smooth. We do not require α to be injective either.) In comparison to the definition of the fundamental group, note that we do not identify simplices and we do not fix a base point.

We want to be able to express the idea that the boundary of a 1-simplex, i.e. of an interval, is the the difference of its endpoints. To this end, we have to be able to add and subtract 0-simplices.

We recall some algebraic notions:

Remark 1.2.5.

1. Any abelian group A can be seen as a $\mathbb{Z}$-module with $n.a := \underbrace{a + \ldots + a}_{n-\text{times}}$ for $n \in \mathbb{N}$ and $a \in A$ and $(-n).a := -n.a$. Thus, abelian groups are in bijection with $\mathbb{Z}$-modules. An abelian group A is called free over a subset $B \subset A$, if B is a $\mathbb{Z}$-basis, i.e. if any element $a \in \mathbb{Z}$ can be uniquely written as a $\mathbb{Z}$-linear combination of elements in B.

2. The group $\mathbb{Z}^r$ is free abelian with basis $\{e_1, \ldots, e_r\}$ with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. The group $\mathbb{Z}_2$ is not free, since it does not admit a basis: the vector $1 \in \mathbb{Z}_2$ is not free since $2 \cdot 1 = 0$.

3. A free abelian group F with basis B can be characterized by the following universal property: any map $f \colon B \to A$ of sets into an arbitrary *abelian* group A can be extended uniquely to a group homomorphism $h \colon F \to A$, i.e. $h(b) = f(b)$ for all $b \in B$,
$$\mathrm{Hom}_{Set}(B, A) \cong \mathrm{Hom}_{group}(F, A) \ .$$

4. Any subgroup of a free abelian group F is a free abelian group of smaller rank.

Definition 1.2.6
Let X be a topological space. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the n-th singular chain module of X.

Remarks 1.2.7.

1. Elements of the singular chain group $S_n(X)$ are thus sums $\sum_{i \in I} \lambda_i \alpha_i$ with $\lambda_i \in \mathbb{Z}$ and $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i \colon \Delta^n \to X$ a singular n-simplex. All sums are effectively finite sums.

2. For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \emptyset$: we can always chose a point $x_0 \in X$ and consider the constant map $\kappa_{x_0} \colon \Delta^n \to X$ as a singular n-simplex α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$.

3. By the universal property 1.2.5.3, to define group homomorphisms from $S_n(X)$ to some abelian group, it suffices to define such a map on generators.

Example 1.2.8.
Let X be any topological space. As an example, we compute $S_0(X)$: a continuous map $\alpha \colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i \in I} \lambda_i \alpha_i$ can thus be identified with the formal sum of points $\sum_{i \in I} \lambda_i x_{\alpha_i}$ with $\lambda_i \in \mathbb{Z}$.

Such objects appear in complex analysis: counting the zeroes and poles of a meromorphic function with multiplicities then this gives an element in $S_0(X)$. In algebraic geometry, a divisor is an element in $S_0(X)$.

Definition 1.2.9
Using the face maps $d_i^{n-1} : \Delta^{n-1} \to \Delta^n$ from Example 1.2.3.3, we define a group homomorphism $\partial_i \colon S_n(X) \to S_{n-1}(X)$ on generators by precomposition with the face map
$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$
and call it the ith face of the singular simplex α.

On $S_n(X)$, we thus get by $\mathbb{Z}$-linear extension $\partial_i(\sum_j \lambda_j \alpha_j) = \sum_j \lambda_j (\alpha_j \circ d_i^{n-1})$.

Lemma 1.2.10.
The face maps on $S_n(X)$ satisfy the simplicial relations

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

Proof.
The relation follows immediately from the relation

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \qquad \text{for all } 0 \leqslant j < i \leqslant n.$$

in Lemma 1.2.3. □

Definition 1.2.11
We define the boundary operator on singular chains as $\partial\colon S_n(X) \to S_{n-1}(X)$ as the alternating sum $\partial = \sum_{i=0}^{n}(-1)^i \partial_i$.

Lemma 1.2.12.
The map ∂ *is* a boundary operator, i.e. $\partial \circ \partial = 0$.

Proof.
This is an immediate consequence of the simplicial relations in Lemma 1.2.10

$$\begin{aligned}\partial \circ \partial = (\sum_{j=0}^{n-1}(-1)^j \partial_j) \circ (\sum_{i=0}^{n}(-1)^i \partial_i) &= \sum_i \sum_j (-1)^{i+j} \partial_j \circ \partial_i \\ &= \sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} \partial_j \circ \partial_i + \sum_{0 \leqslant i \leqslant j \leqslant n-1} (-1)^{i+j} \partial_j \circ \partial_i \\ &\overset{1.2.10}{=} \sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} \partial_{i-1} \circ \partial_j + \sum_{0 \leqslant i \leqslant j \leqslant n-1} (-1)^{i+j} \partial_j \circ \partial_i = 0.\end{aligned}$$

□

We therefore obtain for a topological space X a complex of (free) abelian groups,

$$\ldots \to S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \to 0 \ ,$$

the singular chain complex, $S_*(X)$. We abbreviate the group $Z_n(S_*(X))$ of cycles by $Z_n(X)$, the group $B_n(S_*(X))$ of boundaries by $B_n(X)$ and the n-th homology group $H_n(S_*(X))$ by $H_n(X)$.

Definition 1.2.13
For a space X, the abelian group $H_n(X)$ is called the nth singular homology group of X.

Example 1.2.14.
1. Note that all 0-chains are 0-cycles, $Z_0(X) = S_0(X)$.
2. The boundary of a 1-chain $\alpha : \Delta^1 \to X$ is

   $$\partial\alpha = \alpha \circ d_0 - \alpha \circ d_1 = \alpha(e_1) - \alpha(e_0)$$

   which justifies the name “boundary”.

3. To find an example of a 1-cycle, consider a 1-chain $c = \alpha + \beta + \gamma$, where we take singular 1-simplices $\alpha, \beta, \gamma\colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha(e_0)$. Calculate $\partial\alpha = \partial_0\alpha - \partial_1\alpha = \alpha(e_1) - \alpha(e_0)$ and similarly for β and γ to find $\partial c = 0$. This motivates the word "cycle".

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology.

Definition 1.2.15
Let $f\colon X \to Y$ be a continuous map. The map $f_n = S_n(f)\colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha\colon \Delta^n \to X$ by postcomposition

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

Lemma 1.2.16.
For any continuous map $f\colon X \to Y$ we have commuting diagrams

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_n} & S_n(Y) \\ {\scriptstyle\partial^X}\downarrow & & \downarrow{\scriptstyle\partial^Y} \\ S_{n-1}(X) & \xrightarrow{f_{n-1}} & S_{n-1}(Y), \end{array}$$

i.e. $(f_n)_{n\in\mathbb{Z}}$ is a chain map and hence induces by the remarks following Definition 1.1.5, a map $H_n(f)\colon H_n(X) \to H_n(Y)$ of the homology groups.

Proof.
By definition, we have for a singular n-simplex $\alpha : \Delta^n \to X$ by the associativity of the composition of maps

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n}(-1)^i(f \circ \alpha) \circ d_i = \sum_{i=0}^{n}(-1)^i f \circ (\alpha \circ d_i) = f_{n-1}(\partial^X \alpha).$$

□

Remarks 1.2.17.

1. The identity map on X induces the identity map on $H_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g \circ f) = S_n(g) \circ S_n(f)$ and thus $H_n(g \circ f) = H_n(g) \circ H_n(f)$.

2. In categorical language, this says precisely that $S_n(-)$ and $H_n(-)$ are functors from the category of topological spaces and continuous maps into the category of abelian groups. Taking all $S_n(-)$ together turns $S_*(-)$ into a functor from topological spaces and continuous maps into the category of chain complexes of abelian groups with chain maps as morphisms.

3. One implication of Lemma 1.2.16 is that homeomorphic spaces have isomorphic homology groups:
$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$
In Theorem 1.4.7, we will see the stronger statement that homotopic maps induce the same morphism in homology.

Our first (not too exciting) calculation is the following:

Proposition 1.2.18.
The homology groups of a one-point space pt are trivial but in degree zero,
$$H_n(\mathrm{pt}) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof.
For every $n \geqslant 0$ there is precisely one continuous map $\alpha\colon \Delta^n \to \mathrm{pt}$, the constant map. We denote this map by κ_n. Then the boundary of κ_n is
$$\partial\kappa_n = \sum_{i=0}^{n}(-1)^i \kappa_n \circ d_i = \sum_{i=0}^{n}(-1)^i \kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$
For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:
$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \to 0 \ ,$$
cf. Example 1.1.4.2. □

1.3 The homology groups H_0 and H_1

We start with the following observation:

Proposition 1.3.1.
For any topological space X, there is a homomorphism $\varepsilon\colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.

Proof.

- If $X \neq \varnothing$, we have a unique morphism $X \to \mathrm{pt}$ of topological spaces which induces by Lemma 1.2.16 a morphism of chain complexes $S_*(X) \to S_*(\mathrm{pt})$. It maps any 0-simplex $\alpha\colon \Delta^0 \to X$ to
$$\Delta^0 \xrightarrow{\alpha} X \to \mathrm{pt} \ ,$$
the generator of $H_0(\mathrm{pt})$, the constant map $\kappa_0 : \Delta^0 \to \mathrm{pt}$, cf. Proposition 1.2.18.

- It is instructive to show directly that the map
$$\tilde{\varepsilon} : \quad S_0(X) \to \mathbb{Z}$$
with $\tilde{\varepsilon}(\alpha) = 1$ for any generator $\alpha\colon \Delta^0 \to X$, thus $\tilde{\varepsilon}(\sum_{i\in I} \lambda_i \alpha_i) = \sum_{i\in I} \lambda_i$ on $S_0(X)$ gives a well-defined map on homology. (As only finitely many λ_i are non-trivial, this is in fact a finite sum.)

Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i\in I} \nu_i\beta_i$ with $\beta_i\colon \Delta^1 \to X$ and a finite set I. Then we get

$$\partial b = \partial \sum_{i\in I} \nu_i\beta_i = \sum_{i\in I} \nu_i(\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i\in I} \nu_i\beta_i \circ d_0 - \sum_{i\in I} \nu_i\beta_i \circ d_1$$

and hence

$$\tilde{\varepsilon}(c) = \tilde{\varepsilon}(\partial b) = \sum_{i\in I} \nu_i - \sum_{i\in I} \nu_i = 0.$$

□

We said that $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$. In this case, we define ε to be the zero map.

If $X \neq \varnothing$, then any singular 0-simplex $\alpha\colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 1.3.2.
If X is a path-connected, non-empty space, then $\varepsilon\colon H_0(X) \xrightarrow{\cong} \mathbb{Z}$.

Proof.

1. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore, the group homomorphism ε is surjective.
2. For any other point $y \in X$ there is a continuous path $\omega\colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define a singular 1-simplex $\alpha_\omega\colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0)$$

for $t_0 + t_1 = 1$, $0 \leqslant t_0, t_1 \leqslant 1$. Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0,1) - \alpha_\omega(1,0) = \kappa_y - \kappa_x,$$

and the two singular 0-simplices κ_x, κ_y in the path connected space X are homologous. This shows that ε is injective.

□

Note that in the proof, we associated to a continuous path ω in X from x to y a 1-simplex α_ω on X with $\partial\alpha_\omega = \kappa_y - \kappa_x$. In the sequel, we will identify them frequently.

Corollary 1.3.3.
If X is a disjoint union, $X = \bigsqcup_{i\in I} X_i$, such that all X_i are non-empty and path-connected, then

$$H_0(X) \cong \bigoplus_{i\in I} \mathbb{Z}.$$

This gives an interpretation of the zeroth homology group of X: it is the free abelian group generated by the path-components of X.

Proof.
The singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i\in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial\colon S_1(X) \cong \bigoplus_{i\in I} S_1(X_i) \to \bigoplus_{i\in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial\colon S_1(X_i) \to S_0(X_i)$ and the claim follows from Proposition 1.3.2. □

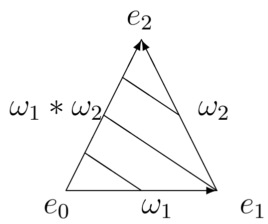
Next, we relate the homology group H_1 to the fundamental group π_1. To this end, we see continuous paths ω in X as 1-simplices α_ω, as in the proof of Proposition 1.3.2.

Lemma 1.3.4.
Let $\omega_1, \omega_2, \omega$ be paths in a topological space X.

1. Constant paths are null-homologous.
2. If $\omega_1(1) = \omega_2(0)$, we can define the concatenation $\omega_1 * \omega_2$ of ω_1 followed by ω_2. Then $\alpha_{\omega_1*\omega_2} - \alpha_{\omega_1} - \alpha_{\omega_2}$ is a boundary.
3. If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0,1\}$, then α_{ω_1} and α_{ω_2} are homologous as singular 1-chains.
4. Any 1-chain of the form $\alpha_{\bar{\omega}*\omega}$ is a boundary. Here, $\bar{\omega}(t) := \omega(1-t)$.

Proof.

1. Denote by c_x the constant path on $x \in X$. Consider the constant singular 2-simplex $\alpha(t_0,t_1,t_2) = x$. Then $\partial\alpha = c_x - c_x + c_x = c_x$.
2. We define a singular 2-simplex $\beta\colon \Delta^2 \to X$ on X as follows.



   We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

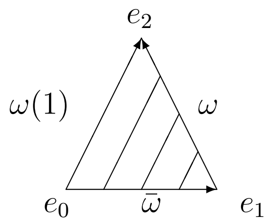
$$\partial\beta = \beta\circ d_0 - \beta\circ d_1 + \beta\circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

3. Let $H\colon [0,1]\times[0,1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0,t) = \omega_1(0) = \omega_2(0)$, we can factor H over the quotient $[0,1]\times[0,1]/\{0\}\times[0,1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h\circ d_0 - h\circ d_1 + h\circ d_2.$$

   The first summand is null-homologous by 1., because it is constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

4. Consider a singular 2-simplex $\gamma\colon \Delta^2 \to X$ as indicated below.



□

Definition 1.3.5
Let X be path-connected and $x \in X$. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class $[\omega]_{\pi_1}$ of a closed path ω to its homology class $[\omega] = [\alpha_\omega]_{H_1}$. This map is called the Hurewicz-homomorphism. [1]

Lemma 1.3.4.3 ensures that h is well-defined and by Lemma 1.3.4.2

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) \overset{1.3.4.2}{=} [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2])$$

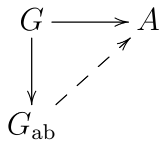
thus h is a group homomorphism. For a closed path ω we have by Lemma 1.3.4.4 that $[\bar{\omega}] = -[\omega]$ in $H_1(X)$.

Recall that the commutator subgroup $[G,G]$ of G is the smallest subgroup of a group G containing all commutators $[g,h] := ghg^{-1}h^{-1}$ for all $g,h \in G$.

It is a normal subgroup of G: If $c \in [G,G]$, then for any $g \in G$ the element $gcg^{-1}c^{-1}$ is a commutator and also by the closure property of subgroups the element $gcg^{-1}c^{-1}c = gcg^{-1}$ is in the commutator subgroup $[G,G]$.

Definition 1.3.6
Let G be an arbitrary group, then its abelianization, G_{ab}, is the quotient group $G/[G,G]$.

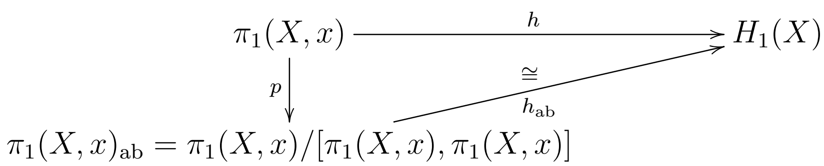
Remark 1.3.7.
The abelianization comes with a projection $G \to G_{\mathrm{ab}}$. It can be characterized by the universal property that any group homomorphism $G \to A$ with A abelian factorizes uniquely as



Proposition 1.3.8.
Let X be a path-connected non-empty space. Since $H_1(X)$ is abelian, the Hurewicz homomorphism factors over the abelianization of $\pi_1(X,x)$. It induces an isomorphism

$$\pi_1(X,x)_{\mathrm{ab}} \cong H_1(X) \ ,$$

i.e.



[1] Witold Hurewicz: 1904–1956.

Proof.

- We construct an inverse ϕ to h_{ab}. To this end, choose as an auxiliary datum for any point $y \in X$ a path u_y from the base point x to y. For the base point x itself, we take u_x to be the constant path on x.

  Let α be an arbitrary singular 1-simplex and let $y_i := \alpha(e_i)$. Define

  $$\tilde{\phi}\colon \quad S_1(X) \quad \to \quad \pi_1(X,x)_{\mathrm{ab}}$$

  on the generator α of $S_1(X)$ as the class of the closed path $\tilde{\phi}(\alpha) = [u_{y_0} * \omega_\alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

- We have to show that $\tilde{\phi}$ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$ a singular 2-simplex. Then
  $$\tilde{\phi}(\partial\beta) = \tilde{\phi}(\beta\circ d_0 - \beta\circ d_1 + \beta\circ d_2) = \tilde{\phi}(\beta\circ d_0)\tilde{\phi}(\beta\circ d_1)^{-1}\tilde{\phi}(\beta\circ d_2).$$
  Abbreviating $\beta\circ d_i$ with α_i, we get as a result

  $$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_0} * \alpha_1 * \bar{u}_{y_2}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha}_1 * \bar{u}_{y_0}].$$

  Here, we have used that the image of $\tilde{\phi}$ is abelian. We can reduce the paths $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha}_1 * \bar{u}_{y_0}]$ but $\alpha_2 * \alpha_0 * \bar{\alpha}_1$ is the closed path tracing the boundary of the singular 2-simplex β and therefore it is null-homotopic in X. Thus $\tilde{\phi}(\partial\beta) = 1$ and $\tilde{\phi}$ passes to a map

  $$\phi\colon H_1(X) \to \pi_1(X,x)_{\mathrm{ab}}\ .$$

- The composition $\phi\circ h_{\mathrm{ab}}$ evaluated on the class of a closed path ω gives

  $$\phi\circ h_{\mathrm{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

  But we chose u_x to be constant, thus $\phi\circ h_{\mathrm{ab}} = \mathrm{id}$.

  If $c = \sum \lambda_i\alpha_i$ is a 1-cycle, then $h_{\mathrm{ab}}\circ\phi(c)$ is of the form $[c + D_{\partial c}]$ where the $D_{\partial c}$-part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$ implies that the summands in $D_{\partial c}$ cancel off and thus $h_{\mathrm{ab}}\circ\phi = \mathrm{id}_{H_1(X)}$.

□

Note that abelianization of an abelian group is the group itself: $G \cong G_{\mathrm{ab}}$. Whenever the fundamental group is abelian, we thus have $H_1(X) \cong \pi_1(X,x)$.

Corollary 1.3.9.
Standard results on the fundamental group π_1 yield explicit results for the following first homology groups:

$$H_1(\mathbb{S}^n) = 0,\ \text{for } n > 1, \quad H_1(\mathbb{S}^1) \cong \mathbb{Z},$$
$$H_1(\underbrace{\mathbb{S}^1\times\ldots\times\mathbb{S}^1}_{n}) \cong \mathbb{Z}^n,$$
$$H_1(\mathbb{S}^1\vee\mathbb{S}^1) \cong (\mathbb{Z}*\mathbb{Z})_{\mathrm{ab}} \cong \mathbb{Z}\oplus\mathbb{Z},$$
$$H_1(\mathbb{R}P^n) \cong \begin{cases}\mathbb{Z}, & \text{if } n = 1,\\ \mathbb{Z}/2\mathbb{Z}, & \text{for } n > 1.\end{cases}$$

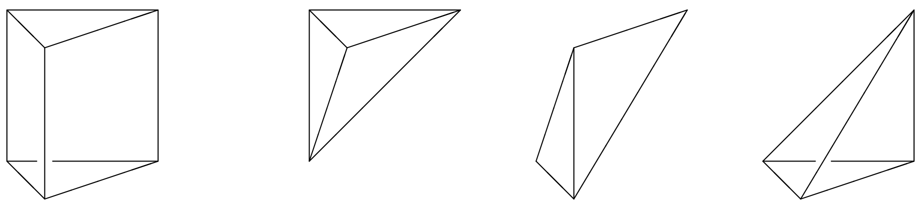
1.4 Homotopy invariance

The main goal of this section is to show that two continuous maps that are homotopic induce identical maps on homology groups.

Observation 1.4.1.

- Let $\alpha\colon \Delta^n \to X$ a singular n-simplex; consider two homotopic maps $f, g : X \to Y$. The homotopy
$$H : X \times [0,1] \to Y$$
from f to g induces a homotopy
$$\Delta^n \times [0,1] \overset{\alpha\times\mathrm{id}}{\to} X \times [0,1] \overset{H}{\to} Y$$
from $f \circ \alpha$ to $g \circ \alpha$. This is a map with codomain $\Delta^n \times [0,1]$, i.e. from a prism over Δ^n. From this geometric homotopy, we want to obtain a chain homotopy from the chain map $S(f)$ to the chain map $S(g)$ of singular chain complexes.

- To that end we have to cut the $(n+1)$-dimensional prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices. In low dimensions this is intuitive:
  - The one-dimensional prism $\Delta^0 \times [0,1]$ is homeomorphic to the standard 1-simplex Δ^1.
  - The two-dimensional prism $\Delta^1 \times [0,1] \cong [0,1]^2$ (which has the shape of a square) can be cut in two triangles, i.e. into two copies of the standard 2-simplex Δ^2.
  - $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedrons, e.g. like



In general, we compose the $(n+1)$-dimensional prism $\Delta^n \times [0,1]$ from $n+1$ copies of the standard simplex Δ^{n+1}:

Definition 1.4.2
For a given $n \in \mathbb{N}_0$, define $n+1$ injections

$$\begin{array}{rcl} p_i\colon \Delta^{n+1} & \to & \Delta^n \times [0,1] \\ (t_0,\dots,t_{n+1}) & \mapsto & ((t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_{n+1}),t_{i+1}+\ldots+t_{n+1}) \end{array}$$

with $i = 0,\dots,n$. These are $(n+1)$-simplices on the prism on the topological space $\Delta^n \times [0,1]$.

Remark 1.4.3.

- The image of the standard basis vectors e_k with $k = 0, 1, \dots n+1$ is

$$p_i(e_k) = \begin{cases} (e_k, 0), & \text{for } 0 \leqslant k \leqslant i, \\ (e_{k-1}, 1), & \text{for } k > i. \end{cases}$$

  For example, in the case $n = 1$, we have

$$p_0 \left| \begin{array}{lll} e_0 & \mapsto & e_0 \\ e_1 & \mapsto & e_0 + e_2 \\ e_2 & \mapsto & e_1 + e_2 \end{array} \right. \qquad\qquad p_1 \left| \begin{array}{lll} e_0 & \mapsto & e_0 \\ e_1 & \mapsto & e_1 \\ e_2 & \mapsto & e_1 + e_2 \end{array} \right.$$

- For all $n \geqslant 0$, we obtain $n+1$ group homomorphisms

$$P_i \colon S_n(X) \to S_{n+1}(X \times [0,1])$$

  for $i = 0, 1, \dots n$ which is defined on a generator $\alpha : \Delta^n \to X$ of $S_n(X)$ via precomposition:

$$P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i : \quad \Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0,1].$$

- For $k = 0, 1$ let $j_k \colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$.

Lemma 1.4.4.

The group homomorphisms P_i satisfy the following relations:

1. $\partial_0 \circ P_0 = S_n(j_1)$ as group homomorphisms $S_n(X) \to S_n(X \times [0,1])$.
2. $\partial_{n+1} \circ P_n = S_n(j_0)$,
3. $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ for $1 \leqslant i \leqslant n$.
4. $$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \text{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \text{for } i \geqslant j+1. \end{cases}$$

Proof.

- For the first point, note that for $\alpha : \Delta^n \to X$, $\partial_0 \circ P_0(\alpha)$ is the singular n-simplex

$$\Delta^n \overset{d_0}{\to} \Delta^{n+1} \overset{p_0}{\to} \Delta^n \times [0,1] \overset{\alpha \times \mathrm{id}}{\to} X \times [0,1] \ .$$

  The composition of the first two maps on Δ^n evaluates to

$$p_0 \circ d_0(t_0, \dots, t_n) = p_0(0, t_0, \dots, t_n) = ((t_0, \dots, t_n), \sum_{i=0}^{n} t_i) = ((t_0, \dots, t_n), 1)$$

  and thus the whole map equals

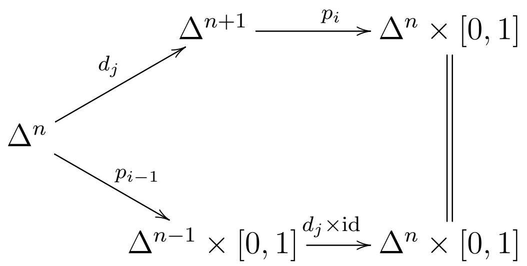
$$S_n(j_1)(\alpha) : \quad \Delta^n \overset{\alpha}{\to} X \overset{j_1}{\to} X \times [0,1]$$

- Similarly, we compute

$$p_n \circ d_{n+1}(t_0, \dots, t_n) = p_n(t_0, \dots, t_n, 0) = ((t_0, \dots, t_n), 0)$$

  and deduce $\partial_{n+1} \circ P_n = S_n(j_0)$.

- For the third identity, one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both give $((t_0, \dots, t_n), \sum_{j=i}^n t_j)$ on $(t_0, \dots, t_n)$.
- For d) in the case $i \geqslant j+1$, consider the following diagram



  Checking coordinates one sees that this diagram commutes: both give $((t_0, \dots, t_{j-1}, 0, \dots t_{i-1} + t_i, \dots t_n), \sum_{j=i}^n t_j)$ on $(t_0, \dots, t_n)$.

  The remaining case follows from a similar observation.

□

Definition 1.4.5
For each $n \geqslant 0$, we define a group homomorphism

$$P\colon \quad S_n(X) \to S_{n+1}(X \times [0,1])$$

as the alternating sum $P = \sum_{i=0}^n (-1)^i P_i$.

Lemma 1.4.6.
The group homomorphisms P provide a chain homotopy between the chain maps $S(j_0), S(j_1) : S_*(X) \to S_*(X \times [0,1])$, i.e. we have for all n

$$\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0) \ .$$

Proof.
We evaluate the left hand side on a singular n simplex $\alpha\colon \Delta^n \to X$ and find from the definitions

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \partial_j P_i \alpha + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n, n+1)$ in the first sum, we are left with by Lemma 1.4.4.1 and 2.

$$S_n(j_1)(\alpha) - S_n(j_0)(\alpha) + \sum_{(i,j) \neq (0,0),(n,n+1)} (-1)^{i+j} \partial_j P_i \alpha + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

Using Lemma 1.4.4 we see that only the first two summands survive: the terms in the first sum with $i = j$ and $i = j-1$ cancel by Lemma 1.4.4.3. The remaining terms cancel by the same mechanism as in the proof of Lemma 1.2.12. □

So, finally we can prove the main result of this section:

Theorem 1.4.7 (Homotopy invariance)**.**
If $f, g\colon X \to Y$ are homotopic maps, then they induce the same map on homology.

Proof.
Let $H\colon X \times [0,1] \to Y$ be a homotopy from f to g, i.e. $H \circ j_0 = f$ and $H \circ j_1 = g$. Set

$$K_n := S_{n+1}(H) \circ P : \quad S_n(X) \xrightarrow{P} S_{n+1}(X \times [0,1]) \xrightarrow{S_{n+1}(H)} S_{n+1}(Y)\ .$$

We claim that $(K_n)_n$ is a chain homotopy between the two chain maps $(S_n(f))_n$ and $(S_n(g))_n$. Note that $H : X \times I \to Y$ induces a chain map $(S_n(H))_n$. Therefore we get

$$\begin{aligned}
\partial \circ \underbrace{S_{n+1}(H) \circ P}_{K_n} + \underbrace{S_n(H) \circ P}_{K_{n-1}} \circ \partial &= S_n(H) \circ \partial \circ P + S_n(H) \circ P \circ \partial \qquad [S_\bullet(H) \text{ is a chain map}] \\
&= S_n(H) \circ (\partial \circ P + P \circ \partial) \\
&\overset{1.4.6}{=} S_n(H) \circ (S_n(j_1) - S_n(j_0)) \\
&= S_n(H \circ j_1) - S_n(H \circ j_0) \\
&= S_n(g) - S_n(f).
\end{aligned}$$

Hence the two chain maps $S(f)$ and $S(g)$ are chain homotopic; by Proposition 1.1.9.2, we have $H_n(g) = H_n(f)$ for all n. □

Corollary 1.4.8.

1. If two spaces X, Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$.
2. In particular, if X is contractible, then

$$H_*(X) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Examples 1.4.9.

1. Since $\mathbb{R}^n$ is contractible for all n, the above corollary implies that its homology is trivial but in degree zero where it consists of the integers.
2. As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic.
3. The zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

1.5 The long exact sequence in homology

In a typical situation, we have a subspace A of a topological space X and might know something about A or X and want to calculate the homology of the other space using that partial information.

Before we can move on to topological applications, we need some algebraic techniques for chain complexes. We need to know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

Definition 1.5.1

1. *Let A, B, C be abelian groups and*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a sequence of homomorphisms. Then this sequence is exact, if the image of f equals the kernel of g.

2. *A sequence*

$$\dots \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i-1}} \dots$$

of homomorphisms of abelian groups (indexed over the integers) is called (long) exact, if it is exact at every A_i, i.e. the image of f_{i+1} equals the kernel of f_i for all i.

3. *An exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence.

Examples 1.5.2.

1. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

2. The sequence

$$0 \longrightarrow U \xrightarrow{\iota} A$$

is exact, iff $\iota \colon U \to A$ is a monomorphism. The sequence

$$B \xrightarrow{\varrho} Q \longrightarrow 0$$

is exact, iff $\varrho \colon B \to Q$ is an epimorphism. Finally, $\Phi : A \to A'$ is an isomorphism, iff the sequence $0 \longrightarrow A \xrightarrow{\phi} A' \longrightarrow 0$ is exact.

3. A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, iff f is injective, the image of f equals the kernel of g and g is an epimorphism.

4. Another equivalent description is to view a long exact sequence as a chain complex with vanishing homology groups. Homology measures the deviation from exactness.

Definition 1.5.3

If A_*, B_*, C_* are chain complexes and $f_* \colon A_* \to B_*$, $g_* \colon B_* \to C_*$ are chain maps, then we call the sequence of chain complexes

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, *if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.*

Thus such an exact sequence of chain complexes is a commuting double ladder

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
\vdots & & \vdots & & \vdots
\end{array}$$

in which every row is exact and where in the columns we have differentials, i.e. $d \circ d = 0$.

Example 1.5.4.
Let p be a prime, then the diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle \cdot p} & & \downarrow{\scriptstyle \cdot p^2} & & \downarrow \\
\mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbb{Z}/p\mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

has exact rows and columns. It is an exact sequence of chain complexes. Here, π denotes the appropriate canonical projection map.

Proposition 1.5.5.
If $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ is a short exact sequence of chain complexes, then there exists a homomorphism $\delta_n \colon H_n(C_*) \to H_{n-1}(A_*)$ for all $n \in \mathbb{Z}$ which is natural, i.e. if

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f} & B_* & \xrightarrow{g} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{f'} & B'_* & \xrightarrow{g'} & C'_* & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain complexes in which the rows are exact, then $H_{n-1}(\alpha) \circ \delta_n = \delta'_n \circ H_n(\gamma)$,

$$\begin{array}{ccc}
H_n(C_*) & \xrightarrow{\delta_n} & H_{n-1}(A_*) \\
{\scriptstyle H_n(\gamma)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(\alpha)} \\
H_n(C'_*) & \xrightarrow{\delta'_n} & H_{n-1}(A'_*)
\end{array}$$

The method of proof is an instance of a diagram chase. The homomorphism δ_n is called connecting homomorphism.

Proof.
We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

We work with the following maps:

$$\begin{array}{ccccc} & & B_n \ni b & \stackrel{g_n}{\longmapsto} & c \in C_n \\ & & \downarrow{\scriptstyle d} & & \\ A_{n-1} \ni a & \stackrel{f_{n-1}}{\longmapsto} & db \in B_{n-1} & \stackrel{g_{n-1}}{\longmapsto} & 0 \end{array}$$

For $c \in C_n$ with $d(c) = 0$, we choose a preimage $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1} db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle. We define $\delta[c] := [a]$.

In order to check that δ is well-defined, we assume that there are two different preimages b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n \tilde{a} = b - b'$. Define $a' := a - d\tilde{a} \in A_{n-1}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = df_n\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of the preimage b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, i.e. take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under the surjective maps g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore the connecting morphism $\delta_n \colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for some $b \in B_n$ with $g_n b = c$ and $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of the connecting homomorphism δ' in the second long exact sequence

$$\delta'[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta' \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$.

□

With this auxiliary result at hand we can now prove the main result in this section:

Proposition 1.5.6 (Long exact sequence in homology)**.**
For any short exact sequence

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\dots \xrightarrow{\delta} H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f)} \dots$$

Proof.

a) Exactness at $H_n(B_*)$:
We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$, because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Since $g_n b$ is a boundary, there exists $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness at B_n allows to find $a \in A_n$ with $f_n a = b - db'$. Now

$$f_{n-1}(da) = df_n(a) = d(b - db') = db = 0$$

since b is a cycle. The map f_{n-1} is injective, thus $da = 0$. Therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$. Thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at $H_n(C_*)$:

Let $[b] \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under the injective map f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of the connecting morphism δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage b of c under g_n, we have by the definition of a

$$d(b - f_n a') = db - df_n a' = db - f_{n-1}a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ under $H_n(g)$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$.

□

1.6 The long exact sequence of a pair of spaces

Let X be a topological space and $A \subset X$ a subspace of X.

Remarks 1.6.1.

1. Consider the inclusion map $i\colon A \to X$, $i(a) = a$. We obtain an induced map of chain complexes $S_n(i)\colon S_n(A) \to S_n(X)$. The inclusion of spaces does not have to yield a monomorphism on homology groups. For instance, we can include $A = \mathbb{S}^1$ into $X = \mathbb{D}^2$. By Corollary 1.4.8.2, since $\mathbb{D}$ is contractible, we know that $H_n(\mathbb{D}) = 0$ for $n \geqslant 1$ and by Corollary 1.3.9 that $H_1(\mathbb{S}^1) = \mathbb{Z}$.

2. Consider the quotient groups $S_n(X, A) := S_n(X)/S_n(A)$. Since $d_n(S_n(A)) \subset S_{n-1}(A)$, the differential induces a well-defined map
$$\begin{array}{rrcl} d_n & S_n(X)/S_n(A) & \to & S_{n-1}(X)/S_{n-1}(A) \\ & c_n + S_n(A) & \mapsto & d_n(c_n) + S_{n-1}(A) \end{array}$$
that squares to zero.

3. Alternatively, $S_n(X, A)$ is isomorphic to the free abelian group generated by all n-simplices $\beta\colon \Delta^n \to X$ whose image is not completely contained in A, i.e. $\beta(\Delta^n) \cap (X \setminus A) \neq \varnothing$.

We consider pairs of spaces (X, A).

Definition 1.6.2
The relative chain complex of the pair (X, A) is
$$S_*(X, A) := S_*(X)/S_*(A)$$
with the differentials described in Remark 1.6.1.2.

Definition 1.6.3

- *Elements in $S_n(X, A)$ are called relative chains in (X, A).*
- *Cycles in $S_n(X, A)$ are chains c with $\partial^X(c)$ a linear combination of generators with image in A. These are called relative cycles.*
- *Boundaries in $S_n(X, A)$ are chains c in X such that $c = \partial^X b + a$ where a is a chain in A. These are called relative boundaries.*

The following facts are immediate from the definition:

1. $S_n(X, \varnothing) \cong S_n(X)$.
2. $S_n(X, X) = 0$.
3. $S_n(X \sqcup X', X') \cong S_n(X)$.

Definition 1.6.4
The relative homology groups of the pair of spaces (X, A) are the homology groups of the relative chain complex $S_*(X, A)$ from Definition 1.6.2:
$$H_n(X, A) := H_n(S_*(X, A)).$$

Theorem 1.6.5 (Long exact sequence for relative homology)**.**

1. For any pair of topological spaces $A \subset X$ we obtain a long exact sequence

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

2. For a map of spaces $f\colon X \to Y$ with $f(A) \subset B \subset Y$, we get an induced map of long exact sequences

$$\begin{array}{ccccccccc}
\ldots \xrightarrow{\delta} & H_n(A) & \xrightarrow{H_n(i)} & H_n(X) & \longrightarrow & H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) & \xrightarrow{H_{n-1}(i)} \ldots \\
& \downarrow{\scriptstyle H_n(f|_A)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_{n-1}(f|_A)} & \\
\ldots \xrightarrow[\delta]{} & H_n(B) & \xrightarrow[H_n(i)]{} & H_n(Y) & \longrightarrow & H_n(Y,B) & \xrightarrow[\delta]{} & H_{n-1}(B) & \xrightarrow[H_{n-1}(i)]{} \ldots
\end{array}$$

A map $f\colon X \to Y$ with $f(A) \subset B$ is denoted by $f\colon (X,A) \to (Y,B)$.

Proof.

1. By definition of the relative chain complex $S_*(X,A)$ the sequence

$$0 \longrightarrow S_*(A) \xrightarrow{S_*(i)} S_*(X) \xrightarrow{\pi} S_*(X,A) \longrightarrow 0$$

is an exact sequence of chain complexes and by Proposition 1.5.6 we obtain the long exact sequence in the first claim.

2. For a map $f : (X,A) \to (Y,B)$ the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_n(A) & \xrightarrow{S_n(i)} & S_n(X) & \xrightarrow{\pi} & S_n(X,A) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle S_n(f|_A)} & & \downarrow{\scriptstyle S_n(f)} & & \downarrow{\scriptstyle S_n(f)/S_n(f|_A)} & & \\
0 & \longrightarrow & S_n(B) & \xrightarrow{S_n(i)} & S_n(Y) & \xrightarrow{\pi} & S_n(Y,B) & \longrightarrow & 0
\end{array}$$

commutes. We now use Proposition 1.5.5.

□

Example 1.6.6.
Consider the embedding

$$\iota : \quad \mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n .$$

We obtain a long exact sequence

$$\ldots \to H_j(\mathbb{S}^{n-1}) \to H_j(\mathbb{D}^n) \to H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \xrightarrow{\delta} H_{j-1}(\mathbb{S}^{n-1}) \to H_{j-1}(\mathbb{D}^n) \to \ldots$$

The disc $\mathbb{D}^n$ is contractible and by Corollary 1.4.8, we have $H_j(\mathbb{D}^n) = 0$ for $j > 0$. From the long exact sequence we get that $\delta\colon H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for $j > 1$ and $n > 1$.

Recall the following definitions:

Definition 1.6.7

1. *A subspace $\iota\colon A \hookrightarrow X$ is a* <u>*weak retract*</u>*, if there is a map $r\colon X \to A$ with $r \circ \iota \simeq \mathrm{id}_A$.*

2. *A subspace $\iota\colon A \hookrightarrow X$ is a deformation retract, if there is a homotopy $R\colon X \times [0,1] \to X$ such that*

   (a) *$R(x,0) = x$ for all $x \in X$,*

   (b) *$R(x,1) \in A$ for all $x \in X$, and*

   (c) *$R(a,1) = a$ for all $a \in A$.*

Any deformation retract is a weak retract: take $r := R(-,1) : X \to A$. Condition (c) then amounts to $r \circ \iota = \mathrm{id}_A$.

Proposition 1.6.8.
If $i\colon A \hookrightarrow X$ is a weak retract, then

$$H_n(X) \cong H_n(A) \oplus H_n(X,A), \quad 0 \leqslant n.$$

Proof.
From the defining identity of a weak retract $r \circ \iota \simeq \mathrm{id}_A$, we get by Theorem 1.4.7 that $H_n(r) \circ H_n(i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$ for all n. Hence $H_n(i)$ is injective for all n. This implies that $0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X)$ is exact. Injectivity of $H_{n-1}(i)$ yields that the image of $\delta\colon H_n(X,A) \to H_{n-1}(A)$ is trivial. Therefore, the long exact sequence of Theorem 1.6.4 decomposes into short exact sequences

$$0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \to 0$$

for all n. As $H_n(r)$ is a left-inverse for $H_n(i)$ we obtain a splitting

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)\ .$$

Indeed, we have a map

$$\begin{array}{rcl} H_n(X) & \to & H_n(A) \oplus H_n(X,A) \\ [c] & \mapsto & ([rc], \pi_*[c]) \end{array}$$

with inverse

$$\begin{array}{rcl} H_n(A) \oplus H_n(X,A) & \to & H_n(X) \\ ([a],[b]) & \mapsto & H_n(i)[a] + [a'] - H_n(i \circ r)[a'] \end{array}$$

for any $[a'] \in H_n(X)$ with $\pi_*[a'] = [b]$. The second map is well-defined: if $[a'']$ is another element with $\pi_*[a''] = [b]$, then $[a' - a'']$ is of the form $H_n(i)[\tilde{a}]$ because this element is in the kernel of π_* and hence $[a' - a''] - H_n(i \circ r)[a' - a''] = H_n(i)[\tilde{a}] - H_n(i \circ r \circ i)[\tilde{a}]$ is trivial. □

Proposition 1.6.9.
For any $\varnothing \neq A \subset X$ such that $A \subset X$ is a deformation retract, then

$$H_n(i)\colon H_n(A) \cong H_n(X), \quad H_n(X,A) \cong 0, \quad 0 \leqslant n.$$

Proof.
Consider the map $r := R(-,1)\colon X \to A$. Then R is a homotopy from id_X to $i \circ r$. The third condition defining a deformation retract can be rewritten as $r \circ i = \mathrm{id}_A$, i.e. r is a retraction. Together, this implies that A and X are homotopically equivalent and by Corollary 1.4.8 $H_n(i)$

is an isomorphism for all $n \geqslant 0$. □

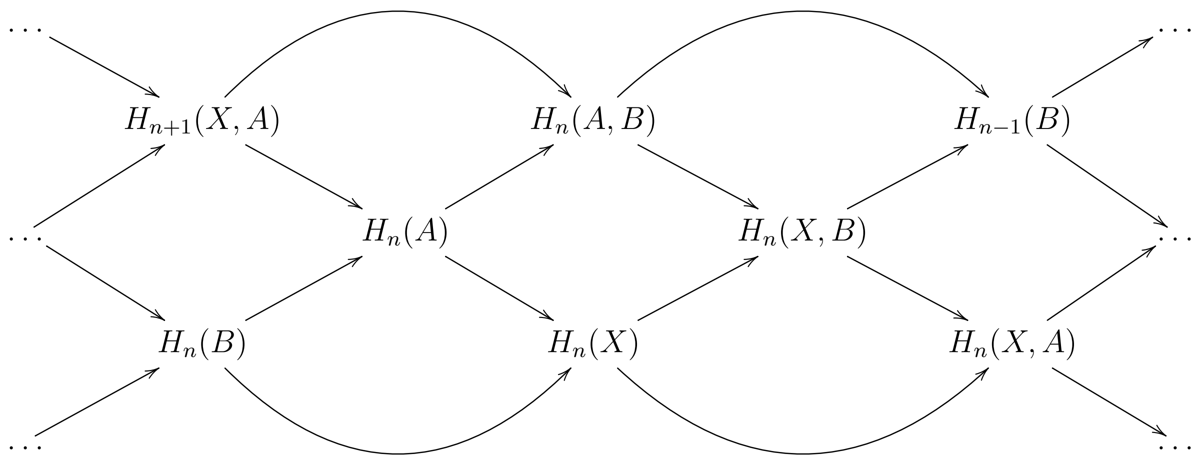
Definition 1.6.10
If X has two subspaces $A, B \subset X$, then (X, A, B) is called a triple, *if $B \subset A \subset X$.*

Any triple gives rise to three pairs of spaces (X, A), (X, B) and (A, B) and accordingly we have three long exact sequences in homology. But there is another long exact sequence:

Proposition 1.6.11.
For any triple (X, A, B), there is a natural long exact sequence

$$\ldots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A, B) \longrightarrow \ldots$$

This sequence is part of the following braided commutative diagram displaying four long exact sequences



In particular, the connecting homomorphism $\delta\colon H_n(X, A) \to H_{n-1}(A, B)$ is the composite $\delta = \pi_*^{(A,B)} \circ \delta^{(X,A)}$.

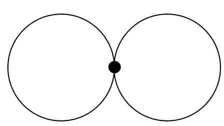
Proof.
Note that $S_n(B) \subset S_n(A) \subset S_n(X)$; by the homomorphism theorem, the sequence

$$0 \longrightarrow S_n(A)/S_n(B) \longrightarrow S_n(X)/S_n(B) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0.$$

is exact. Now apply Proposition 1.5.6 to obtain the long exact sequence. □

1.7 Excision

The aim is to simplify relative homology groups. Let $A \subset X$ be a subspace. Then it is easy to see that $H_*(X, A)$ is not isomorphic to $H_*(X \setminus A)$: Consider the figure eight as X and A as the point connecting the two copies of $\mathbb{S}^1$.



- $X \setminus A$ has two connected components. By Corollary 1.3.3 we have $H_0(X \setminus A) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- Any $x \in X \setminus A$ is a generator for the group of 0-cycles. Since the space X is path connected, it is homologous to the point $a \in A$ and thus vanishes in relative homology. The group $H_0(X, A)$ is trivial.

So if we want to simplify the relative homology group $H_*(X, A)$ by excising something, then we have to be more careful. The first step towards that is to make singular simplices 'smaller'. The technique is called barycentric subdivision; it is a tool that is frequently used.

First, we construct cones. Let $v \in \Delta^p$ and let $\alpha \colon \Delta^n \to \Delta^p$ be a singular n-simplex on Δ^p.

Definition 1.7.1
The cone of $\alpha : \Delta^n \to \Delta^p$ with respect to $v \in \Delta^p$ is the singular $(n+1)$-simplex $K_v(\alpha) \colon \Delta^{n+1} \to \Delta^p$,

$$(t_0, \dots, t_{n+1}) \mapsto \begin{cases} (1 - t_{n+1})\alpha(\frac{t_0}{1-t_{n+1}}, \dots, \frac{t_n}{1-t_{n+1}}) + t_{n+1}v, & t_{n+1} < 1, \\ v, & t_{n+1} = 1. \end{cases}$$

This map is well-defined and continuous. On the standard basis vectors K_v gives $K_v(e_i) = \alpha(e_i)$ for $0 \leqslant i \leqslant n$ but $K_v(e_{n+1}) = v$. Extending K_v linearly gives a map on chain groups

$$K_v \colon S_n(\Delta^p) \to S_{n+1}(\Delta^p).$$

Lemma 1.7.2.
The map K_v satisfies:

1. For $c \in S_0(\Delta^p)$, the boundary of the cone $K_v(c)$ is the 0-chain

$$\partial K_v(c) = \varepsilon(c).\kappa_v - c$$

   with $\kappa_v(e_0) = v$ and ε the augmentation as introduced in Proposition 1.3.1.

2. For $n > 0$ we have that $\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$.

Proof.

1. For a singular 0-simplex $\alpha \colon \Delta^0 \to \Delta^p$ we know that $\varepsilon(\alpha) = 1$ and we calculate

$$\partial K_v(\alpha)(e_0) = K_v(\alpha) \circ d_0(e_0) - K_v(\alpha) \circ d_1(e_0) = K_v(\alpha)(e_1) - K_v(\alpha)(e_0) = v - \alpha(e_0).$$

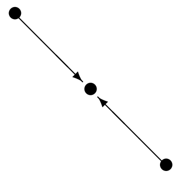
   Extending linearly shows the claim.

2. For $n > 0$ we have to calculate $\partial_i K_v(\alpha)$ and it is straightforward to see that $\partial_{n+1} K_v(\alpha) = \alpha$ and $\partial_i(K_v(\alpha)) = K_v(\partial_i \alpha)$ for all $0 \leqslant i < n + 1$.

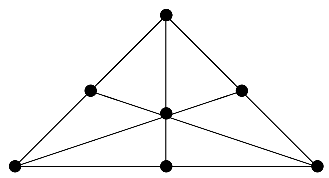
□

Definition 1.7.3
For an n-simple $\alpha \colon \Delta^n \to \Delta^p$ on Δ^p, choose as the additional vertex the barycenter $v(\alpha) = v := \frac{1}{n+1} \sum_{i=0}^n \alpha(e_i)$ of the vertices. The barycentric subdivision $B \colon S_n(\Delta^p) \to S_n(\Delta^p)$ is defined inductively as $B(\alpha) = \alpha$ for $\alpha \in S_0(\Delta^p)$ and $B(\alpha) = (-1)^n K_v(B(\partial\alpha))$ for $n > 0$. For $n \geqslant 1$ this equals $B(\alpha) = \sum_{i=0}^n (-1)^{n+i} K_v(B(\partial_i \alpha))$.

If we take $n = p$ and $\alpha = \mathrm{id}_{\Delta^n}$, then for small n this looks as follows: You cannot subdivide a point any further. For $n = 1$ we get



And for $n = 2$ we get (up to tilting)



Lemma 1.7.4.
The barycentric subdivision is a chain map

$$B: \quad S_*(\Delta^p) \to S_*(\Delta^p) \ .$$

Proof.
We have to show that $\partial B = B\partial$.

- If α is a singular zero chain, then the fact $B\alpha = \alpha$ from the definition implies $\partial B\alpha = \partial\alpha = 0$ and $B\partial\alpha = B(0) = 0$.
- Let $n = 1$. Then by Definition 1.7.3

  $$\partial B\alpha = -\partial K_v B(\partial_0\alpha) + \partial K_v B(\partial_1\alpha).$$

  But the boundary terms are zero chains on which B is the identity, so we get with Lemma 1.7.2.1

  $$-\partial K_v(\partial_0\alpha) + \partial K_v(\partial_1\alpha) \overset{1.7.2.1}{=} -\kappa_v + \partial_0\alpha + \kappa_v - \partial_1\alpha = \partial\alpha = B\partial\alpha.$$

  In the last step, we used that B is the identity on the 0-chain $\partial\alpha$. Note, that the v is $v(\alpha)$, not a $v(\partial_i\alpha)$.
- We prove the claim inductively, so let $\alpha \in S_n(\Delta^p)$. Then

  $$\begin{aligned} \partial B\alpha &\overset{\text{def}}{=} (-1)^n \partial K_v(B\partial\alpha) \\ &\overset{1.7.2.2}{=} (-1)^n((-1)^n B\partial\alpha + K_v\partial B\partial\alpha) \\ &\overset{\text{ind.}}{=} B\partial\alpha + (-1)^n K_v B\partial\partial\alpha = B\partial\alpha. \end{aligned}$$

  Here, the first equality is by definition, the second one follows by Lemma 1.7.2.2 and then we use the induction hypothesis and the fact that $\partial\partial = 0$.

□

Our aim is to show that barycentric subdivision B does not change anything on the level of homology groups and to that end we prove that the chain map $B: \ S_*(\Delta^p) \to S_*(\Delta^p)$ is chain homotopic to the identity.

To this end, we construct $\psi_n \colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ again inductively on generators as

$$\psi_0(\alpha) := 0, \quad \psi_n(\alpha) := (-1)^{n+1} K_v(B\alpha - \alpha - \psi_{n-1}\partial\alpha)$$

with $v := \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$ the barycenter.

Lemma 1.7.5.
The sequence $(\psi_n)_n$ is a chain homotopy from B to the identity on $S_*(\Delta^p)$.

Proof.

- For $n = 0$ we have $\partial\psi_0 = 0$ and this agrees with $B - \mathrm{id}$ in that degree.
- For $n = 1$, we get

$$\partial\psi_1 + \psi_0\partial = \partial\psi_1 \overset{\text{def}}{=} \partial(K_vB - K_v - K_v\psi_0\partial) = \partial K_vB - \partial K_v.$$

  With Lemma 1.7.2.2 we can transform the latter to $B + K_v\partial B - \partial K_v$ and as B is a chain map, this equals $B + K_vB\partial - \partial K_v$. In chain degree one $B\partial$ agrees with ∂, thus this reduces to

$$B + K_v\partial - \partial K_v = B - (\partial K_v - K_v\partial) \overset{1.7.2.2}{=} B - \mathrm{id}.$$

- So, finally we can do the inductive step:

$$\begin{aligned}
\partial\psi_n =&(-1)^{n+1}\partial K_v(B - \mathrm{id} - \psi_{n-1}\partial) \qquad \text{[defn.]}\\
=&(-1)^{n+1}\partial K_vB - (-1)^{n+1}\partial K_v - (-1)^{n+1}\partial K_v\psi_{n-1}\partial\\
=&(-1)^{n+1}((-1)^{n+1}B + K_v\partial B) \qquad \text{[Lemma 1.7.2.2]}\\
&- (-1)^{n+1}((-1)^{n+1}\mathrm{id} + K_v\partial) \qquad \text{[Lemma 1.7.2.2]}\\
&- (-1)^{n+1}((-1)^{n+1}\psi_{n-1}\partial + K_v\partial\psi_{n-1}\partial) \qquad \text{[1.7.2.2]}\\
=&B - \mathrm{id} - \psi_{n-1}\partial + \text{remaining terms}
\end{aligned}$$

  The equation

$$K_v\partial\psi_{n-1}\partial + K_v\psi_{n-2}\partial^2 = K_v\left(\partial\psi_{n-1} + \psi_{n-2}\partial\right)\partial \overset{\text{ind. ass.}}{=} K_vB\partial - K_v\partial$$

  from the inductive assumption ensures that these terms give zero.

□

Definition 1.7.6
A singular n-simplex $\alpha\colon \Delta^n \to \Delta^p$ is called affine, if

$$\alpha(\sum_{i=0}^{n} t_ie_i) = \sum_{i=0}^{n} t_i\alpha(e_i).$$

We abbreviate $v_i := \alpha(e_i)$, so $\alpha(\sum_{i=0}^{n} t_ie_i) = \sum_{i=0}^{n} t_iv_i$ and we call the elements $v_i \in \Delta^p$ the vertices of α.

Definition 1.7.7
Let A be a subset of a metric space (X, d). The diameter of A is

$$\sup\{d(x,y)|x, y \in A\}$$

and we denote it by $\mathrm{diam}(A)$.

Accordingly, the diameter of an affine n-simplex α in Δ^p is the diameter of its image, and we abbreviate that with $\mathrm{diam}(\alpha)$.

Lemma 1.7.8.
For any affine singular n-simplex α every simplex in the chain $B\alpha$ has diameter $\leqslant \frac{n}{n+1}\mathrm{diam}(\alpha)$.

Thus barycentric subdivision of affine simplices decreases the diameter. Either you believe this lemma, or you prove it, or you check Bredon, Proof of Lemma 13.7 (p. 226).

Each simplex in the chain $B\alpha$ is again affine; this allows us to iterate the application of B and get smaller and smaller diameter of individual simplices. Thus, the k-fold iteration, $B^k(\alpha)$, has diameter at most $\left(\frac{n}{n+1}\right)^k \mathrm{diam}(\alpha)$.

In the following we use the easy, but powerful trick to express the singular n-simplex $\alpha : \Delta^n \to X$ as

$$\alpha = \alpha \circ \mathrm{id}_{\Delta^n} = S_n(\alpha)(\mathrm{id}_{\Delta^n}) \ ,$$

i.e. as the image of an n-simplex on Δ^n. This allows us to use the barycentric subdivision for general spaces: note that $\mathrm{id}_{\Delta_n} : \Delta^n \to \Delta^n$ can be seen as an n-simplex on Δ^n. To this simplex, we can apply barycentric subdivision to get a chain $B(\mathrm{id}_{\Delta^n}) \in S_n(\Delta^n)$. Now a singular n-simplex on X is a map $\alpha : \Delta^n \to X$ and thus gives rise to a morphism of abelian groups

$$S_n(\alpha) : S_n(\Delta^n) \to S_n(X) \ .$$

Therefore, $S_n(\alpha)B(\Delta_n) \in S_n(X)$.

Definition 1.7.9

1. *We define $B_n^X : S_n(X) \to S_n(X)$ as*

$$B_n^X(\alpha) := S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}).$$

2. *Similarly, $\psi_n^X : S_n(X) \to S_{n+1}(X)$ is defined as*

$$\psi_n^X(\alpha) := S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Lemma 1.7.10.

1. The maps B^X are natural in X , i.e. for any map $X \xrightarrow{f} Y$ of topological spaces the diagram

$$\begin{array}{ccc} S_*(X) & \xrightarrow{B^X} & S_*(X) \\ {\scriptstyle S_*(f)}\downarrow & & \downarrow{\scriptstyle S_*(f)} \\ S_*(Y) & \xrightarrow{B^Y} & S_*(Y) \end{array}$$

   commutes.

2. The maps

$$B^X : S_*(X) \to S_*(X)$$

   are homotopic to the identity on $S_n(X)$.

Proof.

- Let $f\colon X \to Y$ be a continuous map. We have

$$\begin{aligned} S_n(f)B_n^X(\alpha) =& S_n(f) \circ S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& S_n(f \circ \alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& B_n^Y(f \circ \alpha) = B_n^Y S_n(f)(\alpha). \end{aligned}$$

  In the first step, we used the definition of B_n^X; in the second step the functorality of $S_n(-)$. In the last step, we used the definition of $S_n(f)$. Thus the maps B^X are natural in X.

- The calculation for $\partial\psi_n^X + \psi_{n-1}^X\partial = B_n^X - \mathrm{id}_{S_n(X)}$ uses that α induces a chain map and thus we get

$$\partial\psi_n^X(\alpha) \overset{\text{defn}}{=} \partial \circ S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}) \overset{S \text{ chain map}}{=} S_n(\alpha) \circ \partial \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

  Hence

$$\begin{aligned} (\partial\psi_n^X + \psi_{n-1}^X\partial)(\alpha) &= S_n(\alpha) \circ (\partial \circ \psi_n(\mathrm{id}_{\Delta^n}) + \psi_{n-1} \circ \partial(\mathrm{id}_{\Delta^n})) \\ &\overset{[1.7.5]}{=} S_n(\alpha) \circ (B - \mathrm{id})(\mathrm{id}_{\Delta^n}) = B_n^X(\alpha) - \alpha. \end{aligned}$$

□

Now we consider singular n-chains that are spanned by 'small' singular n-simplices. Here, "smallness' is defined in terms of an open covering.

Definition 1.7.11
Let $\mathfrak{U} = \{U_i, i \in I\}$ be an open covering of X. Then $S_n^{\mathfrak{U}}(X)$ is the free abelian group generated by all singular n-simplices $\alpha\colon \Delta^n \to X$ such that the image of Δ^n under α is contained in one of the open sets $U_i \in \mathfrak{U}$.

Note that $S_n^{\mathfrak{U}}(X)$ is an abelian subgroup of the singular chain group $S_n(X)$. The restriction of the differential of $S_n(X)$ gives a chain complex

$$\ldots \to S_n^{\mathfrak{U}}(X) \to S_{n-1}^{\mathfrak{U}}(X) \to \ldots \ .$$

We denote its homology by $H_n^{\mathfrak{U}}(X)$. As we will see now, these chains suffice to detect everything in singular homology.

Lemma 1.7.12.

1. For any subspace $A \subset X$, the barycentric subdivision of $c \in S_n(A)$ is again in $S_n(A)$, i.e. $B^k(c) \in S_n(A)$.

2. If $c \in S_n(X)$ is a cycle relative $A \subset X$, then $B(c)$ is a cycle relative A as well that is homologous to c relative A.

3. Let $\mathfrak{U}$ be an open covering of X. Then every cycle in $S_n(X)$ is homologous to a cycle in $S_n^{\mathfrak{U}}(X)$.

Proof.

1. This follows at once from the definition of barycentric subdivision.

2. We note that the map $\psi_n : S_n(X) \to S_{n+1}(X)$ maps $\alpha \in S_n(A)$ to $\psi_n(\alpha) \in S_{n+1}(A)$. Now consider for a relative cycle c the equation. cf. 1.7.10.2

$$Bc = c + \psi_{n-1}\partial c + \partial\psi_n c \ .$$

Since c is a relative cycle, $\partial c \in S_{n-1}(A)$ and by part 1, $\psi_{n-1}\partial c \in S_n(A)$. Thus Bc is homologous to c relative A. Its boundary is

$$\partial Bc = \partial c + \partial\psi_{n-1}\partial c \ .$$

Thus Bc is a relative cycle as well.

3. Consider a singular n-chain $\alpha = \sum_{j=1}^{m} \lambda_j \alpha_j \in S_n(X)$ on X and let L_j for $1 \leqslant j \leqslant m$ be the Lebesgue numbers for the m coverings $\{\alpha_j^{-1}(U_i), i \in I\}$ of the simplex Δ^n. Choose k, such that $\left(\frac{n}{n+1}\right)^k \leqslant L_1, \ldots, L_m$. Then $B^k\alpha_1$ up to $B^k\alpha_m$ are all chains in $S_n^{\mathfrak{U}}(X)$. Therefore

$$B^k(\alpha) = \sum_{j=1}^{m} \lambda_j B^k(\alpha_j) =: \alpha' \in S_n^{\mathfrak{U}}(X).$$

From part 2, we know that $B^k\alpha$ is a cycle as well that is homologous to α.

□

We conclude:

Corollary 1.7.13.
For any open covering $\mathfrak{U}$, the injective chain map

$$S_*^{\mathfrak{U}}(X) \hookrightarrow S_*(X)$$

induces an isomorphism in homology, $H_n^{\mathfrak{U}}(X) \cong H_n(X)$.

Proof.
The map on homology is surjective, since for any cycle $c \in S_n(X)$, we find by Lemma 1.7.12.3 a homologous cycle $c' \in S_n^{\mathfrak{U}}(X)$.

The map is injective as well: suppose $c \in S_n^{\mathfrak{U}}(X)$ is a boundary in $S_n(X)$, i.e. $c = \partial e$ with $e \in S_{n+1}(X)$. Find $k \in \mathbb{N}$ such that $B^k(e) \in S_{n+1}^{\mathfrak{U}}(X)$ and

$$B^k(e) - e = \tilde{\psi}_{n-1}(\partial e) + \partial\tilde{\psi}_n(e) = \tilde{\psi}_{n-1}(c) + \partial\tilde{\psi}_n(e) \ .$$

Thus

$$\partial B^k(e) - \partial e = \partial\tilde{\psi}_{n-1}(c)$$

is a boundary in $S_n^{\mathfrak{U}}(X)$. Thus,

$$c = \partial e = \partial(B^k(e) - \tilde{\psi}_{n-1}(c))$$

is a boundary in $S_n^{\mathfrak{U}}(X)$ as well. □

We remark that this isomorphism actually comes from a homotopy equivalence of chain complexes.

With this we get the main result of this section:

Theorem 1.7.14 (Excision)**.**
Let $W \subset A \subset X$ such that $\bar{W} \subset \mathring{A}$. Then the inclusion $i\colon (X \setminus W, A \setminus W) \hookrightarrow (X, A)$ induces an isomorphism of relative homology groups

$$H_n(i)\colon \qquad H_n(X \setminus W, A \setminus W) \cong H_n(X, A)$$

for all $n \geqslant 0$.

Proof.

- We first prove that $H_n(i)$ is surjective.
  Let $c \in S_n(X, A)$ be a relative cycle, i.e. $\partial c \in S_{n-1}(A)$. Consider the open covering $\mathfrak{U} = \{\mathring{A}, X \setminus \bar{W}\} =: \{U, V\}$ of X. Now subdivide and find k such that $c' := B^k c$ is a chain in $S_n^{\mathfrak{U}}(X)$. It follows from Lemma 1.7.12.2 that c' is homologous to c relative A. Decompose $c' = c^U + c^V$ with c^U and c^V being chains on the corresponding open sets. (This decomposition is not unique.)

  The boundary of c' is $\partial c' = \partial B^k c = B^k \partial c$; by assumption this is a chain in $S_{n-1}(A)$. Moreover, we find from the decomposition $c' = c^U + c^V$

  $$\partial c' = \partial c^U + \partial c^V$$

  with $\partial c^U \in S_{n-1}(U) \subset S_{n-1}(A)$. Thus, $\partial c^V = \partial c' - \partial c^U \in S_{n-1}(A)$. Since $\partial c^V \in V$ is supported in $X \setminus \overline{W}$, we have $\partial c^V \in S_{n-1}(A \setminus W)$. Therefore, c^V is a relative cycle in $S_n(X \setminus W, A \setminus W)$.

  In $H_n(X, A)$, we find $[c] = [c'] = [c^U + c^V] = [c^V]$, where we used in the first step Lemma 1.7.12.2. This shows that $H_n(i)[c^V] = [c] \in H_n(X, A)$. Thus $[c^V]$ is a preimage of $[c]$ in $H_n(X \setminus W, A \setminus W)$.

- The injectivity of $H_n(i)$ is shown as follows.
  Assume that there exists $c \in S_n(X \setminus W)$ with $\partial c \in S_{n-1}(A \setminus W)$ such that $H_n(i)[c] = 0$. The last statement means that c is of the form $c = \partial b + a'$ for some $b \in S_{n+1}(X)$ and $a' \in S_n(A)$. We can choose all summands such that they avoid W.

  We write b as $b^U + b^V$ with $b^U \in S_{n+1}(U) \subset S_{n+1}(A)$ and $b^V \in S_{n+1}(V) \subset S_{n+1}(X \setminus W)$. Then

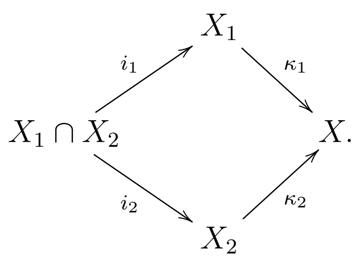
  $$c = \partial b^U + \partial b^V + a'.$$

  Note that a' and ∂b^U are chains in $S_n(A \setminus W)$. So we have written c as a boundary of a chain b^V in $S_{n+1}(X \setminus W)$ plus a chain $a' + \partial b^U$ in $S_n(A \setminus W)$. Thus $[c] = 0 \in H_n(X \setminus W, A \setminus W)$.

□

1.8 Mayer-Vietoris sequence

We consider the following situation: there are subspaces $X_1, X_2 \subset X$ such that X_1 and X_2 are open in X and such that $X = X_1 \cup X_2$. We consider the open covering $\mathfrak{U} = \{X_1, X_2\}$. We need the following maps:



Note that by definition, the sequence of complexes

$$0 \longrightarrow S_*(X_1 \cap X_2) \xrightarrow{(i_1,i_2)} S_*(X_1) \oplus S_*(X_2) \longrightarrow S_*^{\mathfrak{U}}(X) \longrightarrow 0 \tag{1}$$

is exact. Here, the second map is

$$(\alpha_1, \alpha_2) \mapsto \kappa_1(\alpha_1) - \kappa_2(\alpha_2).$$

Note that here the open sets are ordered to define the difference.

Theorem 1.8.1 (Mayer-Vietoris sequence)**.**
There is a long exact sequence

$$\ldots \xrightarrow{\delta} H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

Proof.
The proof follows from the exact sequence (1) of chain complexes by Lemma 1.7.12, because $H_n^{\mathfrak{U}}(X) \cong H_n(X)$, by Corollary 1.7.13 □

Observation 1.8.2.

1. As an application, we calculate the homology groups of spheres. Let $X = \mathbb{S}^m$ and let $X^{\pm} := \mathbb{S}^m \backslash \{\mp e_{m+1}\}$. The subspaces X^+ and X^- are contractible and therefore $H_*(X^{\pm}) = 0$ for all positive $*$.

   The Mayer-Vietoris sequence is as follows

   $$\ldots \xrightarrow{\delta} H_n(X^+ \cap X^-) \longrightarrow H_n(X^+) \oplus H_n(X^-) \longrightarrow H_n(\mathbb{S}^m) \xrightarrow{\delta} H_{n-1}(X^+ \cap X^-) \longrightarrow \ldots$$

   For $n > 1$ we can deduce from $H_n(X^{\pm}) = 0$

   $$H_n(\mathbb{S}^m) \cong H_{n-1}(X^+ \cap X^-) \cong H_{n-1}(\mathbb{S}^{m-1}).$$

   The first map is the connecting homomorphism δ and the second map is $H_{n-1}(i)\colon H_{n-1}(\mathbb{S}^{m-1}) \to H_{n-1}(X^+ \cap X^-)$ where i is the inclusion of $\mathbb{S}^{m-1}$ into $X^+ \cap X^-$ and this inclusion is a homotopy equivalence. Thus define

   $$D := H_{n-1}(i)^{-1} \circ \delta : \quad H_n(\mathbb{S}^m) \to H_{n-1}(\mathbb{S}^{m-1}) \ .$$

   This D is an isomorphism for all $n \geqslant 2$.

   We have to control what is going on in small degrees and dimensions.

2. We know from the Hurewicz isomorphism that $H_1(\mathbb{S}^m)$ is trivial for $m > 1$, cf. Corollary 1.3.9. Here, we show this directly via the Mayer-Vietoris sequence:

   $$\begin{aligned} &\ldots \to 0 \cong H_1(X^+) \oplus H_1(X^-) \to H_1(\mathbb{S}^m) \\ &\xrightarrow{\delta} \mathbb{Z} \cong H_0(X^+ \cap X^-) \to H_0(X^+) \oplus H_0(X^-) \cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

   We have to understand the map in the second line. Let 1 be a base point of $X^+ \cap X^-$. Then the map on H_0 is

   $$[1] \mapsto ([1], [1]).$$

   This map is injective and therefore the connecting homomorphism $\delta\colon H_1(\mathbb{S}^m) \to H_0(X^+ \cap X^-)$ is zero. We find

   $$H_1(\mathbb{S}^m) \cong 0, \quad m > 1.$$

3. We also compute $H_1(\mathbb{S}^1)$ using a Mayer-Vietoris argument and consider the case of $n = 1 = m$. In this case, the intersection $X^+ \cap X^-$ splits into two components. We choose base points $P_+ \in X^+$ and $P_- \in X^-$. Consider the exact sequence
$$0 \longrightarrow H_1(\mathbb{S}^1) \xrightarrow{\delta} H_0(X^+ \cap X^-) \xrightarrow{(H_0(i_1), H_0(i_2))} H_0(X^+) \oplus H_0(X^-) \longrightarrow H_0(\mathbb{S}^1)$$
which gives
$$0 \longrightarrow H_1(\mathbb{S}^1) \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}.$$
The kernel of the last map, the difference of $H_0(\kappa_1)$ and $H_0(\kappa_2)$,
$$H_0(X^+) \oplus H_0(X^-) \to H_0(\mathbb{S}^1)$$
is spanned by $([P_+], [P_-])$ and thus isomorphic to $\mathbb{Z}$. This is the image of $(H_0(i_1), H_0(i_2))$. Therefore, the sequence
$$0 \longrightarrow H_1(\mathbb{S}^1) \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$
is short exact; thus $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ is a free abelian group of rank 1. We already knew this from the Hurewicz isomorphism.

4. We now combine the arguments.
   - For $0 < n < m$ we get by applying D repeatedly,
   $$H_n(\mathbb{S}^m) \xrightarrow{\cong} H_{n-1}(\mathbb{S}^{m-1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^{m-n+1}) \cong \pi_1(\mathbb{S}^{m-n+1}).$$
   and the latter is trivial by 2.
   - Similarly, for $0 < m < n$ we have similarly
   $$H_n(\mathbb{S}^m) \xrightarrow{\cong} H_{n-1}(\mathbb{S}^{m-1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_{n-m+1}(\mathbb{S}^1) \cong 0.$$
   The last claim follows directly by another simple Mayer-Vietoris argument.
   - The remaining case $0 < m = n$ gives a non-vanishing result:
   $$H_n(\mathbb{S}^n) \xrightarrow{\cong} H_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

We can summarize the result as follows.

Proposition 1.8.3.
The homology groups of spheres are:
$$H_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = m = 0, \\ \mathbb{Z}, & n = 0, m > 0, \\ \mathbb{Z}, & n = m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We specify a generator of $H_n(\mathbb{S}^n)$.

Definition 1.8.4
Let $\mu_0 := [P_+] - [P_-] \in H_0(X^+ \cap X^-) \cong H_0(\mathbb{S}^0)$ and let $\mu_1 \in H_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1)$ be given by the degree one map (i.e. the class of the identity on $\mathbb{S}^1$, i.e. the class of the loop $t \mapsto e^{2\pi i t}$).

Define the higher μ_n via the map D from 1.8.2.1 as $D\mu_n = \mu_{n-1}$. Then μ_n is called the fundamental class in $H_n(\mathbb{S}^n)$.

In order to obtain a relative version of the Mayer-Vietoris sequence, we need a tool from homological algebra.

Lemma 1.8.5 (Five-Lemma).
Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

be a commutative diagram of exact sequences. If the four maps f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Proof.
Again, we are chasing diagrams.

- We show that f_3 is injective.
  Assume that there is an $a \in A_3$ with $f_3 a = 0$. Then $\beta_3 f_3 a = f_4 \alpha_3 a = 0$, as well. But f_4 is injective, thus $\alpha_3 a = 0$. Exactness of the top row gives, that there is an $a' \in A_2$ with $\alpha_2 a' = a$. This implies
  $$f_3 \alpha_2 a' = f_3 a = 0 = \beta_2 f_2 a'.$$
  Exactness of the bottom row gives us a $b \in B_1$ with $\beta_1 b = f_2 a'$, but f_1 is an isomorphism so we can lift b to $a_1 \in A_1$ with $f_1 a_1 = b$.

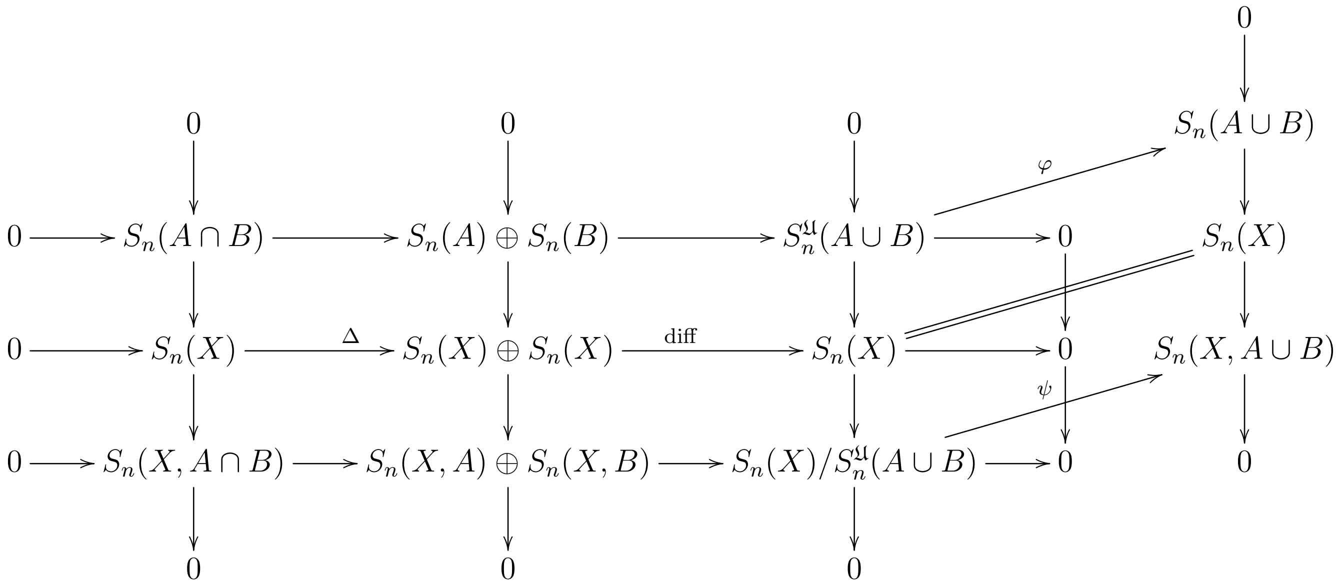
  Thus $f_2 \alpha_1 a_1 = \beta_1 b = f_2 a'$ and as f_2 is injective, this implies that $\alpha_1 a_1 = a'$. So finally we get that $a = \alpha_2 a' = \alpha_2 \alpha_1 a_1$, but the latter is zero, thus $a = 0$.

- For the surjectivity of f_3, assume $b \in B_3$ is given. Move b over to B_4 via β_3 and set $a := f_4^{-1} \beta_3 b$. (Note here, that if $\beta_3 b = 0$ we actually get a shortcut: Then there is a $b_2 \in B_2$ with $\beta_2 b_2 = b$ and thus an $a_2 \in A_2$ with $f_2 a_2 = b_2$. Then $f_3 \alpha_2 a_2 = \beta_2 b_2 = b$.)

  Consider $f_5 \alpha_4 a$. This is equal to $\beta_4 \beta_3 b$ and hence trivial. Therefore $\alpha_4 a = 0$ and thus there is an $a' \in A_3$ with $\alpha_3 a' = a$. Then $b - f_3 a'$ is in the kernel of β_3, because
  $$\beta_3 (b - f_3 a') = \beta_3 b - f_4 \alpha_3 a' = \beta_3 b - f_4 a = 0.$$
  Hence we get a $b_2 \in B_2$ with $\beta_2 b_2 = b - f_3 a'$. Define a_2 as $f_2^{-1}(b_2)$, so $a' + \alpha_2 a_2$ is in A_3 and
  $$f_3(a' + \alpha_2 a_2) = f_3 a' + \beta_2 f_2 a_2 = f_3 a' + \beta_2 b_2 = f_3 a' + b - f_3 a' = b.$$

□

We now consider a relative situation, so let X be a topological space with $A, B \subset X$ open in $A \cup B$ and set $\mathfrak{U} := \{A, B\}$. This is an open covering of $A \cup B \subset X$. The following diagram of exact sequences combines absolute chains with relative ones:



Here, ψ is induced by the inclusion $\varphi\colon S_n^{\mathfrak{U}}(A \cup B) \to S_n(A \cup B)$, Δ denotes the diagonal map and diff is the difference map. It is clear that the first two rows are exact. That the third row is exact follows by a version of the nine-lemma or a direct diagram chase.

Consider the two right-most non-trivial columns in this diagram. Each gives a long exact sequence in homology and we focus on five terms:

$$\begin{array}{ccccccccc}
H_n(S_*^{\mathfrak{U}}(A \cup B)) & \longrightarrow & H_n(X) & \longrightarrow & H_n(S_*(X)/S_*^{\mathfrak{U}}(A \cup B)) & \xrightarrow{\delta} & H_{n-1}(S_*^{\mathfrak{U}}(A \cup B)) & \longrightarrow & H_{n-1}(X) \\
\downarrow{\scriptstyle H_n(\varphi)} & & \| & & \downarrow{\scriptstyle H_n(\psi)} & & \downarrow{\scriptstyle H_{n-1}(\varphi)} & & \| \\
H_n(A \cup B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A \cup B) & \xrightarrow{\delta} & H_{n-1}(A \cup B) & \longrightarrow & H_{n-1}(X)
\end{array}$$

Then by the five-lemma 1.8.5, as $H_n(\varphi)$ and $H_{n-1}(\varphi)$ are isomorphisms by Corollary 1.7.13, so is $H_n(\psi)$. This observation, together with the bottom non-trivial exact row of the first diagram, proves the following

Theorem 1.8.6 (Relative Mayer-Vietoris sequence)**.**
If $A, B \subset X$ are open in $A \cup B$, then the following sequence is exact:

$$\ldots \xrightarrow{\delta} H_n(X, A \cap B) \longrightarrow H_n(X, A) \oplus H_n(X, B) \longrightarrow H_n(X, A \cup B) \xrightarrow{\delta} \quad \ldots$$

1.9 Reduced homology and suspension

For any path-connected space, the zeroth homology is isomorphic to the integers, so this copy of $\mathbb{Z}$ is superfluous information and we want to get rid of it. Let pt denote the one-point topological space. Then for any space X there is a unique continuous map $\varepsilon\colon X \to \text{pt}$.

Definition 1.9.1
We define $\widetilde{H}_n(X) := \ker(H_n(\varepsilon)\colon H_n(X) \to H_n(\text{pt}))$ and call it the reduced nth homology group *of the space X.*

Remarks 1.9.2.

1. Note that $\widetilde{H}_n(X) \cong H_n(X)$ for all positive n.

2. If X is path-connected, then $\widetilde{H}_0(X) = 0$, cf. Proposition 1.3.1.

3. Choose a base point $x \in X$. Then the composition

$$\{x\} \hookrightarrow X \to \{x\}$$

is the identity. Because of $H_n(\mathrm{pt}) \cong H_n(\{x\})$, we get from proposition 1.6.8 about weak retracts

$$\widetilde{H}_n(X) \oplus H_n(\{x\}) \cong H_n(X) \ .$$

The retraction $r\colon X \to \{x\}$ splits the long exact sequence of relative homology for $\{x\} \to X$

$$\dots H_n(\{x\}) \to H_n(X) \to H_n(X, \{x\}) \to \dots$$

and thus we identify reduced homology as relative homology, $\widetilde{H}_n(X) \cong H_n(X, \{x\})$.

4. We can prolong the singular chain complex $S_*(X)$ and consider the chain complex of free abelian groups $\widetilde{S}_*(X)$:

$$\dots \to S_1(X) \to S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

where $\varepsilon(\alpha) = 1$ for every singular 0-simplex α. This is precisely the augmentation we considered in Proposition 1.3.1. Then for all $n \geqslant 0$,

$$\widetilde{H}_*(X) \cong H_*(\widetilde{S}_*(X)).$$

For every continuous map $f\colon X \to Y$ induces a chain map $S_*(f)\colon S_*(X) \to S_*(Y)$; for the evaluation, we have $\varepsilon^Y \circ S_0(f) = \varepsilon^X$. We thus obtain the following result:

Lemma 1.9.3.
The assignment $X \mapsto H_*(\widetilde{S}_*(X))$ is a functor, i.e. for a continuous map $f\colon X \to Y$ we get an induced map $H_*(\widetilde{S}_*(f))\colon H_*(\widetilde{S}_*(X)) \to H_*(\widetilde{S}_*(Y))$ such that the identity on X induces the identity and composition of maps is respected.

As a consequence, $\widetilde{H}_*(-)$ is a functor.

Definition 1.9.4
For $\varnothing \neq A \subset X$ we define

$$\widetilde{H}_n(X, A) := H_n(X, A).$$

Since we identified in Remark 1.9.2.3 reduced homology groups with relative homology groups $H_n(X, \{x\})$, we obtain a reduced version of the Mayer-Vietoris sequence. A similar remark applies to the long exact sequence for a pair of spaces.

Proposition 1.9.5.
For each pair (X, A) of spaces, there is a long exact sequence

$$\dots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \dots$$

and a reduced Mayer-Vietoris sequence, if $X_1 \cap X_2 \neq \emptyset$, which is identical in positive degrees and ends as

$$\dots \tilde{H}_0(X_1 \cap X_2) \to \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \to \tilde{H}_0(X) \to 0$$

Examples 1.9.6.

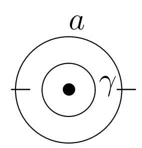
1. Recall that we can express the real projective plane $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:

$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm \mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$

We use the latter definition and set $X = \mathbb{R}P^2$, $A = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $B = \mathring{\mathbb{D}}^2$. Then

$$A \cap B = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$

Thus we know that $H_1(A) \cong \mathbb{Z}$, $H_1(B) \cong 0$ and $H_2(A) = H_2(B) = 0$. We choose generators for $H_1(A)$ and $H_1(A \cap B)$ as follows:



Let a be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. This is the generator for $H_1(A)$. Let γ be the loop that runs along the inner circle in mathematical positive direction. This is the generator for $H_1(A \cap B)$; note that $A \cap B \simeq \mathbb{D} \setminus \{0\}$. Then the inclusion $i_{A\cap B} \colon A \cap B \to A$ induces

$$H_1(i_{A\cap B})[\gamma] = 2[a].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$\widetilde{H}_2(A)\oplus\widetilde{H}_2(B) = 0 \to \widetilde{H}_2(X) \to \widetilde{H}_1(A\cap B) \cong \mathbb{Z} \stackrel{\cdot 2}{\to} \widetilde{H}_1(A) \cong \mathbb{Z} \to \widetilde{H}_1(X) \to \widetilde{H}_0(A\cap B) = 0.$$

On the two copies of the integers, the map is given by multiplication by two and thus we obtain:

$$\begin{aligned} &H_2(\mathbb{R}P^2) \cong \ker(2\cdot \colon \mathbb{Z} \to \mathbb{Z}) = 0, \\ &H_1(\mathbb{R}P^2) \cong \mathrm{coker}(2\cdot \colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\ &H_0(\mathbb{R}P^2) \cong \mathbb{Z}. \end{aligned}$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

2. We can now calculate the homology groups of bouquets of spaces in terms of the homology groups of the single spaces, at least in good cases. Let $(X_i)_{i\in I}$ be a family of topological spaces with chosen basepoints $x_i \in X_i$. Consider the bouquet

$$X = \bigvee_{i\in I} X_i.$$

If the inclusion of x_i into X_i is pathological, we cannot apply the Mayer-Vietoris sequence

However, we get the following:

Proposition 1.9.7.
If there are neighbourhoods U_i of $x_i \in X_i$ together with a deformation of U_i to $\{x_i\}$, then we have for any finite $E \subset I$

$$\widetilde{H}_n(\bigvee_{i\in E} X_i) \cong \bigoplus_{i\in E} \widetilde{H}_n(X_i).$$

In the situation above, we say that the space X_i is well-pointed with respect to the point $x_i \in X_i$.

Proof.
First we consider the case of two bouquet summands. We have $X_1 \vee U_2 \cup U_1 \vee X_2$ as an open covering of $X_1 \vee X_2$. Since $(X_1 \vee U_2) \cap (U_1 \vee X_2) = U_1 \cap U_2$ is contractible, the Mayer-Vietoris sequence then gives that $H_n(X) \cong H_n(X_1 \vee U_2) \oplus H_n(U_1 \vee X_2)$ for $n > 0$. For H_0 we get the exact sequence

$$0 \to \widetilde{H}_0(X_1 \vee U_2) \oplus \widetilde{H}_0(U_1 \vee X_2) \to \tilde{H}_0(X) \to 0.$$

By induction we obtain the case of finitely many bouquet summands. □
We also get

$$\widetilde{H}_n(\bigvee_{i\in I} X_i) \cong \bigoplus_{i\in I} \widetilde{H}_n(X_i)$$

but for this one needs a colimit argument. We postpone that for a while.

We can extend such results to the full relative case. Let $A \subset X$ be a *closed* subspace and assume that A is a strong deformation retract of an open neighbourhood $A \subset U$. Let $\pi\colon X \to X/A$ be the canonical projection and $b = \{A\} \in X/A$ the image of A. Then X/A is well-pointed with respect to the point $b \in X/A$ by the neighborhood $\pi(U)$.

Proposition 1.9.8.
In the situation above

$$H_n(X, A) \cong \widetilde{H}_n(X/A), \quad 0 \leqslant n.$$

Proof.
The canonical projection $\pi : X \to X/A$ induces a homeomorphism of pairs $(X \setminus A, U \setminus A) \cong (X/A \setminus \{b\}, \pi(U) \setminus \{b\})$. Consider the following diagram:

$$\begin{array}{ccccc}
H_n(X,A) & \xrightarrow{\cong} & H_n(X,U) & \xleftarrow{\cong} & H_n(X\setminus A, U\setminus A) \\
\downarrow{\scriptstyle H_n(\pi)} & & & & \cong \downarrow{\scriptstyle H_n(\pi)} \\
H_n(X/A,b) & \xrightarrow{\cong} & H_n(X/A,\pi(U)) & \xleftarrow{\cong} & H_n(X/A\setminus\{b\},\pi(U)\setminus\{b\})
\end{array}$$

The upper and lower left arrows are isomorphisms because A is a deformation retract of U, the isomorphism in the upper right is a consequence of excision, because $A = \bar{A} \subset U$, cf. Theorem 1.7.14. The lower right one follows from excision as well. The right vertical arrow is an isomorphism, because we have a homeomorphism of pairs. □

Definition 1.9.9

1. *The cone of a topological space X is the topological space*

$$CX := X \times [0, 1]/X \times \{0\}\ .$$

2. *The* <u>*suspension*</u> *of a topological space* X *is the topological space*

$$\Sigma X := X \times [0,1]/(x_1,0) \sim (x_2,0) \text{ and } (x_1,1) \sim (x_2,1) \text{ for all } x_1,x_2 \in X \quad .$$

Remarks 1.9.10.

1. The cone over a point p is an interval. The cone over an interval is a triangle, a 2-simplex. The cone over an n-simplex is an $(n+1)$-simplex. The cone over $\mathbb{S}^n$ is a closed $(n+1)$-ball.
2. Note that for any topological space X, the cone CX is contractible to its apex. Thus $\tilde{H}_n(CX) = 0$ for all $n \geqslant 0$. Similarly, for $A \subset X$, we have $CA \subset CX$ and $\tilde{H}_n(CX, CA) = 0$ for all $n \geqslant 0$.
3. The suspension of $\mathbb{S}^n$ is $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$.
4. We have natural embeddings $X \to CX$ with $x \mapsto [x,1]$ and $CX \to \Sigma X$ with $x \mapsto [x, \frac{1}{2}]$. We can see the suspension as two cones, glued together at their bases.

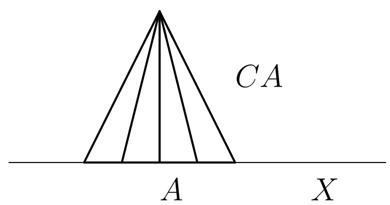
Theorem 1.9.11 (Suspension isomorphism)**.**
Let $A \subset X$ be a closed subspace and assume that A is a deformation retract of an open neighbourhood $A \subset U$. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

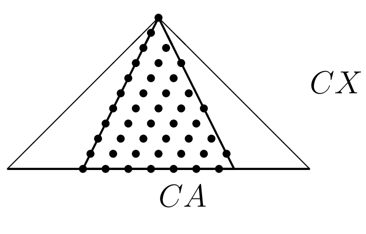
Proof.

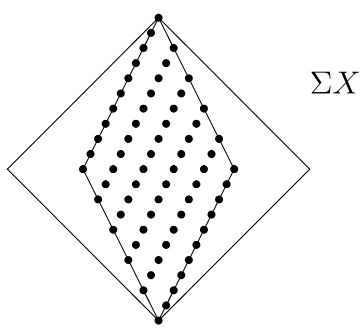
1. We first note two equivalences:

   $X \cup CA/CA \simeq X/A$, where the cone CA is attached to X by identifying $A \subset X$ and the base $A \subset CA$:



   and $CX/(CA \cup X) \simeq \Sigma X/\Sigma A$:





2. Consider the triple $(CX, X \cup CA, CA)$. We obtain from Proposition 1.6.11 the long exact sequence on homology groups

$$\ldots \longrightarrow H_n(CX, CA) \longrightarrow H_n(CX, CA \cup X) \xrightarrow{\delta} \tilde{H}_{n-1}(X \cup CA, CA) \longrightarrow \ldots$$

Since cones are contractible, the connecting morphism δ gives us isomorphisms

$$H_n(CX, CA \cup X) \cong \tilde{H}_{n-1}(X \cup CA, CA)$$

3. Using Proposition 1.9.8 and the equivalences from part 1. of the proof, we compute the right hand side:

$$\tilde{H}_{n-1}(X \cup CA, CA) \overset{1.9.8}{\cong} \tilde{H}_{n-1}(X \cup CA/CA) \overset{1.}{\cong} \tilde{H}_{n-1}(X/A) \overset{1.9.8}{\cong} \tilde{H}_{n-1}(X, A) \ .$$

Similarly, we get for the left hand side

$$\tilde{H}_n(CX, CA \cup X) \overset{1.9.8}{\cong} \tilde{H}_n(CX/CA \cup X) \overset{1.}{\cong} \tilde{H}_n(\Sigma X/\Sigma A) \overset{1.9.8}{\cong} H_n(\Sigma X, \Sigma A).$$

□

Note that the corresponding statement is wrong for homotopy groups. We have $\Sigma\mathbb{S}^2 \cong \mathbb{S}^3$, but $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, whereas $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, so homotopy groups (unlike homology groups) do not satisfy such an easy form of a suspension isomorphism. There is a Freudenthal suspension theorem for homotopy groups, but that is more complicated. For the above case it yields:

$$\pi_{1+3}(\mathbb{S}^3) \cong \pi_{1+4}(\mathbb{S}^4) \cong \ldots =: \pi_1^s$$

where π_1^s denotes the first stable homotopy group of the sphere.

1.10 Mapping degree

Recall that we defined in Definition 1.8.4 fundamental classes $\mu_n \in \tilde{H}_n(\mathbb{S}^n) \cong \mathbb{Z}$ for all $n \geqslant 0$.

Definition 1.10.1
A continuous map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ induces a homomorphism

$$\tilde{H}_n(f)\colon \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$$

and therefore we get

$$\tilde{H}_n(f)\mu_n = \deg(f)\mu_n$$

with $\deg(f) \in \mathbb{Z}$. We call this integer the <u>degree</u> of f.

In the case $n = 1$ we can relate this notion of a mapping degree to the one defined via the fundamental group of the 1-sphere: if we represent the generator of $\pi_1(\mathbb{S}^1, 1)$ as the class given by the loop

$$\omega\colon [0,1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t},$$

then the abelianized Hurewicz map, $h_{\mathrm{ab}}\colon \pi_1(\mathbb{S}^1, 1) \to H_1(\mathbb{S}^1)$, sends by definition 1.8.5 the class of ω precisely to $\mu_1 \in H_1(\mathbb{S}^1)$ and therefore the naturality of h_{ab}

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1,1) & \xrightarrow{\pi_1(f)} & \pi_1(\mathbb{S}^1,1) \\ {\scriptstyle h_{\mathrm{ab}}}\downarrow & & \downarrow{\scriptstyle h_{\mathrm{ab}}} \\ H_1(\mathbb{S}^1) & \xrightarrow{H_1(f)} & H_1(\mathbb{S}^1) \end{array}$$

shows that

$$\deg(f)\mu_1 \overset{\text{def}}{=} H_1(f)\mu_1 \overset{\text{nat}}{=} h_{\text{ab}}(\pi_1(f)[w]) \overset{\text{def}}{=} h_{\text{ab}}(k[w]) = k\mu_1.$$

where k is the degree of f defined via the fundamental group. Thus both notions coincide for $n = 1$.

As we know that the connecting homomorphism in the long exact sequence in relative homology induces an isomorphism between $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$, we can consider degrees of maps $f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ by defining a fundamental class $\bar{\mu}_n := \delta^{-1}\mu_n \in H_n(\mathbb{D}^n, \mathbb{S}^n)$. Then $H_n(f)(\bar{\mu}_n) := \deg(f)\bar{\mu}_n$ gives a well-defined integer $\deg(f) \in \mathbb{Z}$.

The degree of self-maps of $\mathbb{S}^n$ satisfies the following properties:

Proposition 1.10.2.

1. If f is homotopic to g, then $\deg(f) = \deg(g)$.
2. The degree of the identity on $\mathbb{S}^n$ is one.
3. The degree is multiplicative, *i.e.*, $\deg(g \circ f) = \deg(g)\deg(f)$.
4. If f is not surjective, then $\deg(f) = 0$.

Proof.
The first three properties follow directly from the definition of the degree. If f is not surjective, then it is homotopic to a constant map and this has degree zero. □

It is true that the group of (pointed) homotopy classes of self-maps of $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$ and thus the first statement in Proposition 1.10.2 can be upgraded to an 'if and only if', but we will not prove that here.

Recall that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. If $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then the suspension $\Sigma(f)\colon \Sigma\mathbb{S}^n \to \Sigma\mathbb{S}^n$ is given as $\Sigma\mathbb{S}^n \ni [x, t] \mapsto [f(x), t]$.

Lemma 1.10.3.
Suspensions leave the degree invariant, i.e. for $f\colon \mathbb{S}^n \to \mathbb{S}^n$ we have

$$\deg(\Sigma(f)) = \deg(f).$$

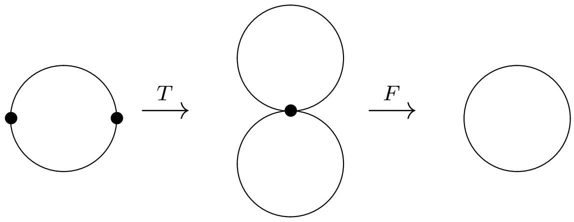
In particular, for every integer $k \in \mathbb{Z}$ there is a continuous map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = k$.

Proof.
The suspension isomorphism of Theorem 1.9.11 is induced by a connecting homomorphism which is functorial by Proposition 1.5.5. Using the isomorphism $H_{n+1}(\mathbb{S}^{n+1}) \cong H_{n+1}(\Sigma\mathbb{S}^n)$, the connecting homomorphism sends $\mu_{n+1} \in H_{n+1}(\mathbb{S}^{n+1})$ to $\pm\mu_n \in \tilde{H}_n(\mathbb{S}^n)$. But then the commutativity of

$$\begin{array}{ccccccc} H_{n+1}(\mathbb{S}^{n+1}) & \xrightarrow{\cong} & H_{n+1}(\Sigma\mathbb{S}^n) & \xrightarrow{H_{n+1}(\Sigma f)} & H_{n+1}(\Sigma\mathbb{S}^n) & \xleftarrow{\cong} & H_{n+1}(\mathbb{S}^{n+1}) \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\ & & \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{H_n(f)} & \tilde{H}_n(\mathbb{S}^n) & & \end{array}$$

ensures that $\pm\deg(f)\mu_n = \pm\deg(\Sigma f)\mu_n$ with the same sign. □

For the degree of a based self-map of $\mathbb{S}^1$ one has an additivity relation $\deg(\omega'' \star \omega') = \deg\omega'' + \deg\omega'$ with respect to concatenation of paths. We can generalize this to higher dimensions. Consider the *pinch map* $T\colon \mathbb{S}^n \to \mathbb{S}^n/\mathbb{S}^{n-1} \simeq \mathbb{S}^n \vee \mathbb{S}^n$ and the *fold map* $F\colon \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$. Here, F is induced by the identity of $\mathbb{S}^n$.



Proposition 1.10.4.
For $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ based, we have

$$\deg(F \circ (f \vee g) \circ T) = \deg(f) + \deg(g).$$

Proof.
The map $H_n(T)$ sends μ_n to $(\mu_n, \mu_n) \in \tilde{H}_n\mathbb{S}^n \oplus \tilde{H}_n\mathbb{S}^n \cong \tilde{H}_n(\mathbb{S}^n \vee \mathbb{S}^n)$. Under this isomorphism, the map $H_n(f \vee g)$ corresponds to $(\mu_n, \mu_n) \mapsto (\tilde{H}_n(f)\mu_n, \tilde{H}_n(g)\mu_n)$ and this yields $(\deg(f)\mu_n, \deg(g)\mu_n)$ which under the fold map is sent to the sum. $\square$

We use the mapping degree to show some geometric properties of self-maps of spheres.

Proposition 1.10.5.
Let $f^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map

$$(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n).$$

Then $f^{(n)}$ has degree -1.

Proof.
We prove the claim by induction. μ_0 was by definition 1.8.4 the difference class $[+1] - [-1]$, and

$$f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

We defined μ_n in such a way that $D\mu_n = \mu_{n-1}$. Therefore, as D is obtained from a connecting homomorphism and thus by proposition 1.5.5 natural,

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{(n-1)})\mu_{n-1} = D^{-1}(-\mu_{n-1}) = -\mu_n.$$

$\square$

Corollary 1.10.6.
The antipodal map

$$\begin{array}{rccc} A\colon & \mathbb{S}^n & \to & \mathbb{S}^n \\ & x & \mapsto & -x \end{array}$$

has degree $(-1)^{n+1}$.

Proof.
Let $f_i^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. As in Proposition 1.10.5, one shows that the degree of $f_i^{(n)}$ is -1. As $A = f_n^{(n)} \circ \ldots \circ f_0^{(n)}$, the claim follows from Proposition 1.10.2.3. □

In particular, for even n, the antipodal map cannot be homotopic to the identity.

Proposition 1.10.7.
Let $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ with $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$, then f is homotopic to $A \circ g$, with A the antipodal map. In particular,

$$\deg(f) = \deg(A \circ g) = \deg(A) \cdot \deg(g) = (-1)^{n+1}\deg(g).$$

Proof.
By assumption, for all $x \in \mathbb{S}^n$ the segment $t \mapsto (1-t)f(x) - tg(x)$ does not pass through the origin for $0 \leqslant t \leqslant 1$. Thus the homotopy

$$H(x,t) = \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

with values in $\mathbb{S}^n$ connects f to $-g = A \circ g$. □

Corollary 1.10.8.
For any $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = 0$ there exists a point $x_+ \in \mathbb{S}^n$ with $f(x_+) = x_+$ and a point x_- with $f(x_-) = -x_-$.

Proof.
If $f(x) \neq x = \mathrm{id}(x)$ for all x, then by Proposition 1.10.7, f is homotopic to $A \circ \mathrm{id} = A$. Thus $\deg(f) = \deg(A) \neq 0$. If $f(x) \neq -x$ for all x, then f is homotopic to $A \circ (-\mathrm{id})$ and thus $\deg(f) = (-1)^{n+1}\deg(A) \neq 0$. □

Corollary 1.10.9.
If n is even, then for any continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$, there is an $x \in \mathbb{S}^n$ with $f(x) = x$ or $f(x) = -x$.

Proof.
Because n is even, $\deg(A) = -1$ If $f(x) \neq x$ for all $x \in \mathbb{S}^n$, by the argument given in the proof of Corollary 1.10.8, we have $\deg(f) = \deg(A) = -1$. If $f(x) \neq -x$ for all $x \in \mathbb{S}^n$, then $\deg(f) = \deg(A)\deg(-\mathrm{id}_{|\mathbb{S}^n}) = 1$. Both at the same time is impossible. □

Finally, we can say the following about hairstyles of hedgehogs of arbitrary even dimension:

Proposition 1.10.10 (Hairy Ball theorem)**.**
Any tangential vector field on an even-dimensional sphere $\mathbb{S}^{2k}$ vanishes in at least one point.

Proof.
Recall that we can describe the tangent space at a point $x \in \mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ as

$$T_x(\mathbb{S}^{2k}) = \{y \in \mathbb{R}^{2k+1} | \langle x, y\rangle = 0\}.$$

Assume that V is a tangential vector field which does nowhere vanish, i.e. $V(x) \neq 0$ for all $x \in \mathbb{S}^{2k}$ and $V(x) \in T_x(\mathbb{S}^{2k})$ for all x. Consider the continuous map

$$\begin{array}{rccc} f: & \mathbb{S}^{2k} & \to & \mathbb{S}^{2k} \\ & x & \mapsto & \frac{V(x)}{||V(x)||} \end{array}$$

Assume $f(x) = x$, which amounts to $V(x) = ||V(x)||x$. But this means that $V(x)$ points into the direction of x and thus it cannot be tangential. Thus $f(x) \neq x$ for all $x \in \mathbb{S}^{2k}$. Similarly, $f(x) = -x$ yields the same contradiction. Thus the existence of such a V is in contradiction to Corollary 1.10.9. □

1.11 CW complexes

Definition 1.11.1
A topological space X is called an n-cell, if X is homeomorphic to $\mathbb{R}^n$. The number n is called the dimension of the cell.

Examples 1.11.2.

1. Every point is a zero cell. The spaces $\mathring{\mathbb{D}}^n \cong \mathbb{R}^n \cong \mathbb{S}^n \setminus N$ are n-cells.

2. Note that an n-cell cannot be an m-cell for $n \neq m$, because $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$. This follows, since $\mathbb{R}^n \cong \mathbb{R}^m$ would imply

$$\mathbb{S}^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\} \simeq \mathbb{S}^{m-1} ,$$

but $\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ for all n and $\tilde{H}_{n-1}(\mathbb{S}^{m-1}) = 0$ for $n \neq m$. Hence the dimension of a cell is well-defined.

Definition 1.11.3
A cell decomposition of a space X is a decomposition of X into subspaces, each of which is a cell of some dimension, i.e.,

$$X = \bigsqcup_{i\in I} X_i, \quad X_i \cong \mathbb{R}^{n_i}.$$

Here, this decomposition is meant as a set, not as a topological space.

Examples 1.11.4.

1. The boundary of a 3-dimensional cube has a cell decomposition into 8 points, 12 open edges, and 6 open faces.

2. The standard 3-simplex can be decomposed into 4 zero-cells, six 1-cells, four 2-cells, and a 3-cell.

3. The n-dimensional sphere (for $n > 0$) has a cell decomposition into the north pole and its complement, thus into a single zero-cell and n-cell.

Definition 1.11.5
A topological Hausdorff space X together with a cell decomposition is called a CW complex, if it satisfies the following conditions:

(a) *[Characteristic maps] For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to the interior $\mathring{\mathbb{D}}^n$ is a homeomorphism*

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \colon \mathring{\mathbb{D}}^n \xrightarrow{\cong} \sigma$$

and such that Φ_σ maps $\mathbb{S}^{n-1} \cong \partial\mathbb{D}^n$ to the union of cells of dimension at most $n-1$.

(b) *[Closure finiteness] For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with only finitely many cells of X.*

(c) *[Weak topology] A subset $A \subset X$ is closed, if and only if $A \cap \bar{\sigma} \subset \bar{\sigma}$ is closed for all cells σ in X.*

Remarks 1.11.6.

1. The map Φ_σ as in (a) is called a characteristic map of the cell σ. Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ to the boundary $\partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$ is called an attaching map.

2. Property (b) is the closure finite condition: the closure of every cell is contained in finitely many cells. This is the 'C' in CW.

3. Since $\bar{\sigma}$ is closed in X, condition (c) is equivalent to requiring that $A \cap \bar{\sigma}$ is closed in X. Condition (c) can be replaced by the equivalent axiom that a subset $A \subset X$ is open, if and only if $A \cap \bar{\sigma}$ is open in $\bar{\sigma}$ for all cells σ in X.

4. If X is a CW complex with only finitely many cells, then we call X finite. Conditions (b) and (c) are then automatically fulfilled.

5. Every non-empty CW complex must contain at least one zero cell. Indeed, if $n > 0$ would be the lowest dimension of a cell, its boundary $\mathbb{S}^{n-1}$ could not be taken into cells of dimension at most $n-1$.

6. It follows from axiom (a) that for every n-cell σ, we have $\overline{\sigma} = \Phi_\sigma(\mathbb{D}^n)$.

   Proof: From the general inclusion $f(\overline{B}) \subset \overline{f(B)}$ for continuous maps, we conclude

   $$\overline{\sigma} = \overline{\Phi_\sigma(\mathring{\mathbb{D}}^n)} \supset \Phi_\sigma(\mathbb{D}^n) \supset \sigma \ .$$

   As a compact subspace of a Hausdorff space, $\Phi_\sigma(\mathbb{D}^n)$ is closed; since it lies between σ and $\overline{\sigma}$, we conclude $\Phi_\sigma(\mathbb{D}^n) = \overline{\sigma}$. In particular the closure $\overline{\sigma}$ is compact in X as the continuous image of the compact set $\mathbb{D}^n$.

7. It follows that $\bar{\sigma} \setminus \sigma$ for an n-cell σ is contained in the union of cells of dimension at most $n-1$.

8. Every finite CW complex is compact, since it is the union of finitely many compact subspaces $\Phi_\sigma(\mathbb{D}^n)$.

Examples 1.11.7.

1. The CW structures on a fixed topological space are not unique. For example, $\mathbb{S}^2$ with the CW structure from the cell-decomposition $\mathbb{S}^2 \setminus \{N\} \sqcup \{N\}$ has a single 0-cell consisting of the north pole and one 2-cell. Projections of a tetrahedron, cube, octahedron or even less regular bodies to the sphere provide other CW structures.

2. Consider the following spaces with cell decomposition:

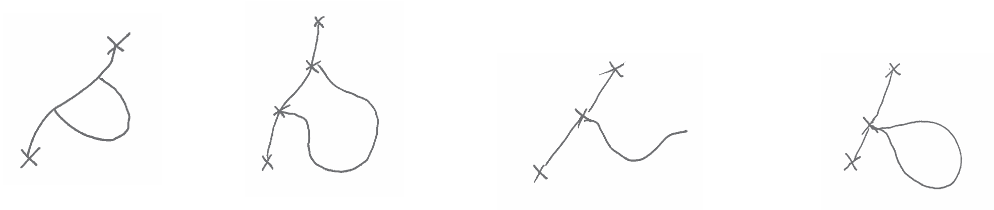


Figure 1 has two 0-cells and two 1-cells. The cell boundary of one of the 1-cells is not contained in 0-cells, cf. Remark 1.11.6.7. Hence axiom (a) is violated. It is satisfied in figure 2, where we have four 1-cells and four 0-cells. Figure 3 with three 1-cells and three 0-cells is not a CW complex, since the cell closure of one of the 1-cells is not compact. Figure 4 is again a CW complex.

3. Consider the topological space $X = X_1 \cup X_2 \subset \mathbb{R}^2$ with

$$X_1 := \{(x, \sin\frac{1}{x}) \,|\, 0 < x < 1\} \subset \mathbb{R}^2 \quad X_2 := \{(0, y) \,|\, -1 \leqslant y \leqslant 1\}$$

with the topology induced from $\mathbb{R}^2$. We consider a cell decomposition with $(0, \pm 1)$ as 0-cells and $\mathring{X}_2$ and X_1 as 1-cells. Here the axiom (a) is violated, since the boundary of $\mathring{X}_1$ is not in the 0-skeleton. (This space is indeed not CW decomposable.)

4. Consider the disc with the following two different cell decompositions:

   - The center $0 \in \mathbb{D}^2$ and any point on the boundary are declared to be a 0-cell. Every radius is a 1-cell. Axioms (a) and (b) are satisfied, but axiom (c) is not: take an open interval on the boundary. Then all intersections with all closures of cells are closed, but it is not a closed subspace of $\mathbb{D}^2$.
   - Any point in the boundary is a 0-cell, the only 2-cell is $\mathring{\mathbb{D}}^2$. Axiom (b) is not satisfied, since the closure of the two-cell has a non-trivial intersection with infinitely many 0-cells. But axioms (a) and (c) hold.

5. The unit interval $[0, 1]$ has a CW structure with two zero cells and one 1-cell. But for instance the decomposition $\sigma_0^0 = \{0\}$, $\sigma_k^0 = \{\frac{1}{k}\}, k > 0$ and $\sigma_k^1 = (\frac{1}{k+1}, \frac{1}{k})$ does not give a CW structure on $[0, 1]$. Consider the following countable subset $A \subset [0, 1]$

$$A := \left\{ \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right) |k \in \mathbb{N} \right\}.$$

Then $A \cap \bar{\sigma}_k^1$ is precisely the point $\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$. This is closed, but the subset A is not closed in $[0, 1]$, since it does not contain the limit point 0 of A.

Remark 1.11.8.

- Historically, the notion of a simplicial complex plays an important role: a set K of simplices in $\mathbb{R}^n$ is called a <u>simplicial complex</u> or <u>polyhedron</u>, if the following conditions are satisfied:

  (a) If K contains a simplex, it contains all faces of this simplex.

  (b) The intersection of two simplices of K is either empty or a common face.

  (c) K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighborhood that intersects only finitely many simplices of K.

A simplicial map is a map that takes any k-simplex affinely into a k'-simplex with $k' \leqslant k$.

- The subspace $|K| := \cup_{s \in K} s \subset \mathbb{R}^n$ is called the topological space underlying the complex K. Simplicial homology can be defined for a simplicial complex; it depends only on $|K|$.

  Simplicial homology has various disadvantages: [2] for example, $\mathbb{S}^2$ can be written as a simplicial complex with 14 simplices only (obtained from the projection of the tetrahedron to the sphere), but as a CW complex with a 0-cell and a 2-cell only. A 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$ can be written as a CW complex with 4 cells, but the smallest simplicial complex has 42 cells.

Definition 1.11.9

1. *The union $X^n := \bigcup_{\sigma \subset X, \dim(\sigma) \leqslant n} \sigma$ of cells of dimension at most n is called the n-skeleton of X.*

2. *If we have $X = X^n$, but $X^{n-1} \subsetneq X$, then we say that X is n-dimensional,* i.e., $\dim(X) = n$.

3. *A subset $Y \subset X$ of a CW complex X is called a subcomplex (sub-CW complex), if it has a cell decomposition by cells of X and if for any cell $\sigma \subset Y$, also its closure $\overline{\sigma}$ in X is contained in Y, i.e. $\bar{\sigma} \subset Y$.*

4. *For any subcomplex $Y \subset X$, (X, Y) is called a CW pair.*

We characterize subcomplexes:

Lemma 1.11.10.
Let X be a CW complex and $Y \subset X$ be a subspace, together with a cell decomposition by a subset of cells of X. Then the following conditions are equivalent:

1. Y is a subcomplex, i.e. for any cell $\sigma \subset Y$, the closure $\overline{\sigma}$ in X is contained in Y.

2. Y is closed in X.

3. The cell decomposition (with the cells of X) endows Y with the structure of a CW complex.

Proof.

2⇒ 1 is trivial: $\overline{\sigma} \subset \overline{Y} = Y$. (Here the bar denotes closure in X, of course.)

1⇒ 2 The topology of X is such by axiom “W” that Y is closed, if and only if $Y \cap \overline{\sigma}$ is closed in X for all cells σ in X. Since X is closure finite, $\overline{\sigma}$ hits only finitely many cells of X. Since Y is the union of cells of X, only finitely many of these cells appear in

$$\overline{\sigma} \cap Y = \overline{\sigma} \cap (\sigma_1 \cup \ldots \cup \sigma_r)$$

with σ_i cells of Y. By 1., $\overline{\sigma}_i \subset Y$, thus

$$\overline{\sigma} \cap Y = \overline{\sigma} \cap (\overline{\sigma}_1 \cup \ldots \cup \overline{\sigma}_r)$$

The intersection of finite unions of closed subsets of X is closed in X, thus this is closed in X.

---

[2] “Computing homology with simplicial chains is like computing integrals $\int_a^b f(x)dx$ with approximating Riemann sums.” (Dold, Lectures in algebraic topology, 1972)

3⇒ 1 For any cell $\sigma \subset Y$, a characteristic map Φ'_σ for Y exists by 3. It is also characteristic for X. Remark 1.11.6.6 that $\overline{\sigma} = \Phi'_\sigma(\mathbb{D})$ now implies that the closure of σ in Y agrees with the closure of σ in X.

1,2⇒3 1 implies that a characteristic map for a cell $\sigma \subset X$ relative to the complex X is also characteristic relative to Y. This implies the existence of characteristic maps for Y. Closure finiteness for Y is immediate from the one for X. Thus axioms (a) and (b) hold.

We still have to show that for $A \subset Y$ the condition that $A \cap \overline{\sigma}^Y$ is closed in Y for all cells $\sigma \subset Y$ implies that A is closed in Y. (Here $\overline{\sigma}^Y$ is the closure of σ in Y.) By 2), a set is closed in Y, if and only if it is closed in X.

It follows that the closure of each cell $\sigma \subset Y$ in Y agrees with the closure of σ in X. It also follows that it is enough to show that $A \cap \overline{\sigma}^X$ is closed in X for all cells $\sigma \subset X$. Closure finiteness implies

$$\overline{\sigma} \cap A = \overline{\sigma} \cap (\sigma_1 \cup \ldots \cup \sigma_r) \cap A$$

where we can assume that σ_i are all cells in Y. Now

$$\overline{\sigma} \cap A = \overline{\sigma} \cap (\overline{\sigma}_1 \cup \ldots \cup \overline{\sigma}_r) \cap A$$

and by assumption $A \cap \overline{\sigma}_i$ is closed in X for all i. Thus $A \cap \overline{\sigma}$ is closed in X.

□

Corollary 1.11.11.

1. Arbitrary intersections and arbitrary unions of subcomplexes are again subcomplexes.
2. The skeleton X^n is a subcomplex.
3. Every union of n-cells in X with X^{n-1} forms a subcomplex.
4. Every cell lies in a finite subcomplex.

Proof.

1. The subcomplexes are closed in X by Lemma 1.11.10, hence their intersection is closed and by Lemma 1.11.10 a subcomplex. The statement about the union follows directly from the definition of a subcomplex.

2. and 3. follow from the observation that for an n-cell σ we have that $\overline{\sigma} = (\overline{\sigma} \setminus \sigma) \cup \sigma$ is contained in $X^{n-1} \cup \sigma$.

4. Induction on the dimension of the cell; then use closure finiteness and $\overline{\sigma} = \Phi_\sigma(\mathbb{D}^n)$.

□

We want to understand the topology of CW complexes.

Remarks 1.11.12.

1. Cells do not have to be open in X. For example, in the CW structure on $[0, 1]$ with two zero cells 0 and 1, the 0-cells are not open in $[0, 1]$.

2. If X is a CW complex and σ is an n-cell, then σ is open in the n-skeleton X^n. Indeed, for $x \in \sigma$, choose a neighborhood U that is open in σ. The intersection $U \cap \overline{\sigma'}$ for any other cell σ' is empty, unless $\sigma' = \sigma$. (Since there are no cells of higher dimension, only the boundary of σ intersects other cells.) By the weak topology, then U is also open in X.

   The n-skeleton X^n is by corollary 1.11.11.2 a subcomplex and thus by lemma 1.11.10.2 closed in X.

3. We can replace condition (c), that A is closed (resp. open) in X, if and only if the intersection of A with $\bar{\sigma}$ is closed (resp. open) in $\overline{\sigma}$ for any cell σ, by the equivalent condition that A is closed (resp. open) in X if and only if $A \cap X^n$ is closed (resp. open) in X^n for all $n \geqslant 0$.

4. A CW-complex X is the direct limit of its skeleta, $\lim_{\longrightarrow} X^n$. Recall that a direct limit of a of a directed system of topological spaces $(X^n)_{n\in\mathbb{N}}$ that is an ascending system of subspaces $X^0 \subset X^1 \subset \ldots$ is the union $X = \bigcup_{n\geqslant 0} X^n = \sqcup X^n/\sim$ with the quotient topology. Thus a CW complex has the final ("weak") topology. This is the 'W'

   Such a direct limit has the following universal property: for any system of maps $(f_n \colon X^n \to Z)_{n\geqslant 0}$ such that $f_{n+1}|_{X^n} = f_n$ there is a uniquely determined continuous map $f \colon X \to Z$ such that $f|_{X^n} = f_n$. This is important for constructing maps out of CW complexes by extending maps recursively on n-cells.

   Using the universal property of the sum of topological spaces, the characteristic maps of all cells combine into a single map

   $$\Phi : \qquad \sqcup_\sigma \mathbb{D}^{n_\sigma} \to X$$

   which endows X with the quotient topology.

Definition 1.11.13
Let X and Y be CW complexes. A continuous map $f \colon X \to Y$ is called <u>cellular</u>*, if $f(X^n) \subset Y^n$ for all $n \geqslant 0$.*

The category of CW complexes together with cellular maps is rather flexible. Most of the classical constructions do not lead out of it: for example, CW complexes nicely behave with respect to collapsing subspaces to points. If X is a CW complex and $A \subset X$ a subcomplex, the cell decomposition of X/A consisting the zero-cell A and the cells of $X \setminus A$ is again a CW decomposition. Thus X/A is a CW complex in a canonical way.

However, one has to be careful with respect to products:

Proposition 1.11.14.
If X and Y are CW complexes, then $X \times Y$ is a CW complex, if one of the factors is locally compact.

Proof.
As products of cells are cells, $X \times Y$ inherits a cell decomposition from its factors. Characteristic maps are products of the characteristic maps for the factors. Closure finitenss follows from $\overline{\sigma \times \tau} = \overline{\sigma} \times \overline{\tau}$. We need to ensure that $X \times Y$ carries the weak topology.

To this end, we need a few auxiliary facts:

- For two spaces U, V, let $C(U, V)$ be the set of all continuous maps from U to V. The topology of $C(U, V)$ is generated (under finite intersections and arbitrary unions) by the sets $V(K, O) := \{f \in C(U, V) | f(K) \subset O\}$ for compact $K \subset U$ and open $O \subset V$. This is called the compact-open topology. (If U is compact and (V, d) is a metric space, the compact-open topology is the one of the metric of uniform convergence,

$$d(f, g) := \sup_{u \in U} d(f(u), g(u)) ,$$

see [Laures-Szymik, p. 72].)

- If Z is locally compact and all spaces are Hausdorff, there is a homeomorphism

$$C(X \times Z, W) \cong C(X, C(Z, W)) \qquad (*)$$

of topological spaces. Here, for $f : X \times Z \to W$ we consider for any given $x \in X$ the map

$$\begin{array}{rcl} f^{\#}(x) : Z & \to & W \\ z & \mapsto & f(x, z) \end{array}$$

This yieds a continous map $f^{\#} : x \mapsto f^{\#}(x)$. The homeomorphism $(*)$ sends f to $f^{\#} \in C(X, C(Z, W))$.

- Using these facts, we show the following Lemma:
Let X, Y and Z be topological spaces satisfying the Hausdorff condition and suppose that $\pi \colon X \to Y$ gives Y the quotient topology and that Z is locally compact. Then

$$\pi \times \mathrm{id} \colon X \times Z \to Y \times Z$$

gives $Y \times Z$ the quotient topology.

We have to show that $Y \times Z$ has the universal property of a quotient space. Hence suppose that $g \colon Y \times Z \to W$ is a map of sets and assume that the composition $g \circ (\pi \times \mathrm{id}) : X \times Z \to Y \times Z \to W$ is continuous.

Under the adjunction $(*)$, the map $g \circ (\pi \times \mathrm{id})$ corresponds to the composite

$$\tilde{g} \colon X \xrightarrow{\pi} Y \xrightarrow{g^{\#}} C(Z, W).$$

which is continuous as the image under the adjunction $(*)$. Since Y carries the quotient topology, the map $g^{\#}$ is continuous and hence, again by $(*)$, the map $g : Y \times Z \to W$ is continuous, too.

- With the help of this result we consider the characteristic maps of X and Y,

$$\begin{array}{l} \Phi_\sigma \colon \mathbb{D}^{n_\sigma} \to X, \text{ for } \sigma \text{ an } n_\sigma\text{-cell in } X \\ \Psi_\tau \colon \mathbb{D}^{m_\tau} \to Y, \text{ for } \tau \text{ an } m_\tau\text{-cell in } Y. \end{array}$$

We use the product of topological spaces to combine these maps to a single map and write $X \times Y$ as a target of a map

$$\Phi \times \Psi \colon \quad (\bigsqcup_\sigma \mathbb{D}^{n_\sigma}) \times (\bigsqcup_\tau \mathbb{D}^{m_\tau}) \to X \times Y.$$

To establish that $X \times Y$ has the weak topology, we can show that $X \times Y$ carries the quotient topology with respect to this map. We know that each $\mathbb{D}^{n_\sigma}$ is locally compact, thus so is the disjoint union of closed discs. The map $\mathrm{id}_{\bigsqcup \mathbb{D}^{n_\sigma}} \times \Psi$ gives $(\bigsqcup \mathbb{D}^{n_\sigma}) \times Y$ the quotient topology and by assumption Y is locally compact and therefore, by the result of the previous point, $\Phi \times \mathrm{id}_Y$ induces the quotient topology on $X \times Y$.

□

Lemma 1.11.15.
If D is a subset of a CW complex X and D intersects each cell in at most one point, then D is discrete.

Proof.
Let S be an arbitrary subset of D. It suffices to show that S is closed. The closure $\overline{\sigma}$ of any cell σ of X is covered by finitely many cells. Hence $S \cap \bar{\sigma}$ is finite. Since X is by definition Hausdorff (thus T_1), $S \cap \bar{\sigma}$ is closed in $\bar{\sigma}$. Since this holds for all cells σ, the weak topology guarantees that S is closed in X. □

Corollary 1.11.16.
Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact, if and only if it is a finite CW complex.
3. The space X is locally compact, if and only if it is locally finite, i.e. every point has a neighborhood that is contained in finitely many cells.

Proof.

- We show that 1. implies one implication in 2. If X is compact, then by 1. it is contained in a finite union of cells. The converse was shown in Remark 1.11.6.8.
- It is clear that 2. implies 3.
- Thus we only prove 1: consider the intersections $K \cap \sigma$ with all cells σ and choose a point p_σ in every non-empty intersection. Then $D := \{p_\sigma | \sigma \text{ a cell in } X\}$ is discrete by Lemma 1.11.15. It is also compact and therefore finite.

□

Corollary 1.11.17.
If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton.

For the proof just note that the image $f(K)$ is compact in X and apply 1.11.16.1.

Proposition 1.11.18.
Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.

Proof.

- Consider first the case when $X = \mathbb{D}^n$ and $A = \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. For $r\colon \mathbb{D}^n \times [0,1] \to \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times [0,1]$ we can choose the standard retraction of a cylinder onto its bottom and sides, cf. Figure VII-6 in Bredon, p. 451.

- We inductively construct retractions

$$\rho_r : \quad X \times \{0\} \cup (A \times I \cup X^r \times I) \to X \times \{0\} \cup A \times [0,1] \ ,$$

where ρ_{r+1} extends ρ_r. Suppose that ρ_{r-1} is given. Then extending to an r-cell of X amounts to extending on $\mathbb{D}^r \times [0,1]$ along $\mathbb{D}^r \times \{0\} \cup \mathbb{S}^{r-1} \times [0,1]$. As we have seen, this can be done.

These maps for all r-cells fit together to a map on the r-skeleton $(X \times [0,1])^r$ which, by the weak topology, fit together to a retract $X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$.

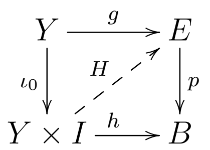
□

Definition 1.11.19

1. *A map $p : E \to B$ has the* homotopy lifting property*, if for any space Y and any homotopy*

$$h : Y \times [0,1] \to B$$

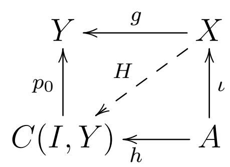
and any map $g : Y \to E$ such that $p \circ g = h_0$ there exists a map $H : Y \times I \to E$ with $p \circ H = h$ such that $H(y,0) = g(y)$ for all $y \in Y$. As a diagram:



Then the continuous map $p : E \to B$ is called a fibration.

2. *A map $\iota : A \to X$ has the* homotopy extension property*, if for any space Y and any map $g : X \to Y$ and $h \colon A \times [0,1] \to Y$ a homotopy such that $h|_{A \times \{0\}} = g \circ \iota$, then there is an extension of h to $H : X \times [0,1] \to Y$, compatible with g and h.*

As a diagram:



(Note that by adjunction $(*)$ a map $A \to C(I,Y)$ amounts to a homotopy $A \times I \to Y$.) Then $\iota : A \to X$ is called a cofibration.

The property in Proposition 1.11.18 implies that any subcomplex of a CW complex has the homotopy extension property. Indeed, two maps

$$g : \ X \to Y \quad \text{and} \quad h : \ A \times [0,1] \to Y$$

such that $h|_{A\times\{0\}} = g$ can be combined to a single map

$$\tilde{g} : \quad X \times \{0\} \cup A \times [0,1] \to Y \ ;$$

a retraction $r : \ X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ provides the homotopy extension $\tilde{g} \circ r : X \times I \to Y$.

In the following we collect some facts about the topology of CW complexes that we do not prove:

Lemma 1.11.20.

1. For any subcomplex $A \subset X$, there is an open neighborhood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (and as well in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.

2. Every CW complex is paracompact, locally path-connected and locally contractible. (A topological space X is paracompact, if every open cover has a locally finite open refinement.)

3. Every CW complex is semi-locally 1-connected, hence possesses a universal covering space which has a natural structure of a CW complex.

Lemma 1.11.21.
For the skeleta of a CW complex X, the following homeomorphisms hold:

1. $$X^n \setminus X^{n-1} = \bigsqcup_{\sigma \text{ an } n\text{-cell}} \sigma \cong \bigsqcup_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2. $$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n.$$

Proof.
The first claim follows directly from the definition of a CW complex. For the second claim note that the characteristic maps send the boundary $\partial \mathbb{D}^n$ to the $(n-1)$-skeleton and hence for every n-cell in X we get a copy of $\mathbb{S}^n$ in the quotient X^n/X^{n-1}. □

Example 1.11.22.
Consider the two-dimensional CW complex given by the hollow cube W^2. Then $W^2/W^1 \cong \bigvee_{i=1}^6 \mathbb{S}^2$, a bouquet of 6 two-dimensional spheres.

1.12 Cellular homology

In the following, X will always be a CW complex.

Lemma 1.12.1.
For the relative homology of the skeleta, we have $H_q(X^n, X^{n-1}) = 0$ for all $q \neq n \geqslant 1$.

Proof.
Using the identification of relative homology and reduced homology of the quotient gives

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

The first isomorphism uses Lemma 1.11.20.1 and Proposition 1.9.8. The last isomorphism uses Lemma 1.11.21.2 and Proposition 1.9.7. □

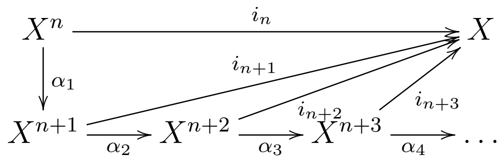
Lemma 1.12.2.
Consider the inclusion $i_n \colon X^n \to X$ of the n-skeleton X^n into X.

1. The induced map $H_n(i_n)\colon H_n(X^n) \to H_n(X)$ is surjective.
2. On the $(n+1)$-skeleton we get an isomorphism

$$H_n(i_{n+1})\colon H_n(X^{n+1}) \cong H_n(X).$$

Proof.

- Using the inclusion of skeleta, we can factor $i_n : X^n \to X$ as



  The map $H_n(\alpha_1)\colon H_n(X^n) \to H_n(X^{n+1})$ is surjective, because Lemma 1.12.1 asserts that $H_n(X^{n+1}, X^n) = 0$. For $i > 1$ we have the following piece of the long exact sequence of the pair (X^{n+i}, X^{n+i-1})

$$0 \cong H_{n+1}(X^{n+i}, X^{n+i-1}) \longrightarrow H_n(X^{n+i-1}) \xrightarrow{H_n(\alpha_i)} H_n(X^{n+i}) \longrightarrow H_n(X^{n+i}, X^{n+i-1}) \cong 0.$$

  Therefore $H_n(\alpha_i)$ is an isomorphism in this range. If the complex X is finite-dimensional, this already proves both claims.

- To deal with the general case, observe that every singular simplex in X, as the continuous image of the compact standard simplex, has compact image which, by Corollary 1.11.17 is contained in one of the skeleta X^n. Let $a \in S_n(X)$ be a chain, $a = \sum_{i=1}^m \lambda_i \beta_i$. Then we can find an M such that the images of all the β_i's are contained in the skeleton X^M, say for $M = n + q$. Therefore $[a] \in H_n(X)$ can be written as $H_n(i_M)[b]$, for some class $[b] \in H_n(X^M)$.

  Now $\alpha_q \circ \ldots \circ \alpha_1$ induces a surjective map in homology, hence $[b]$ can be written as $H_n(\alpha_q) \circ \ldots \circ H_n(\alpha_1)[c]$ for some $[c] \in H_n(X^n)$. This implies

$$[a] = H_n(i_M) \circ H_n(\alpha_q) \circ \ldots \circ H_n(\alpha_1)[c] = H_n(i_n)[c]$$

  thus $H_n(i_n)$ is surjective, showing the first assertion.

- Since $H_n(i_n) = H_n(i_{n+1}) \circ H_n(\alpha_1)$ and $H_n(i_n)$ is surjective by the preceding assertion, it is clear that $H_n(i_{n+1})$ is surjective as well.

  Suppose that $H_n(i_{n+1})[a] = 0$ for some n-chain a. Then there exists an $n+1$-chain β such that $S_n(i_{n+1}a) = \partial b$. Using the same argument, there exists $M = n + q$ such that β can be defined in terms of the M-skeleton. Thus $S_n(\alpha_{n+q-1}) \circ \ldots \circ S_n(\alpha_{n+1})(a) = \partial b$ and thus $H_n(\alpha_{n+q-1}) \circ \ldots \circ H_n(\alpha_{n+1})([a]) = 0$. But all maps are isomorphisms, thus $[a] = 0$.

□

Corollary 1.12.3.
For CW complexes X, Y we have

1. If the n-skeleta X^n and Y^n are homeomorphic, then $H_q(X) \cong H_q(Y)$, for all $q < n$.

2. If X has no q-cells, then $H_q(X) \cong 0$.

3. In particular, if q exceeds the dimension of X, then $H_q(X) \cong 0$.

Proof.

1. The first claim is a direct consequence of Lemma 1.12.2 which asserts that $H_n(X^{n+1}) \cong H_n(X)$.

2. By assumption in 2. $X^{q-1} = X^q$, therefore we have $H_q(X^{q-1}) \cong H_q(X^q)$ and the latter surjects by Lemma 1.12.2.2 onto $H_q(X)$. Hence 2. is reduced to the statement in 3, applied to X^{q-1}.

3. We show that $H_n(X^r) \cong 0$ for $n > r$. Consider the long exact sequence o relative homology

$$\to H_{n+1}(X^i, X^{i-1}) \to H_n(X^{i-1}) \to H_n(X^i) \to H_n(X^i, X^{i-1}) \to \dots$$

For $i < n$, the the adjacent relative groups $H_n(X^i, X^{i-1})$ are trivial by Lemma 1.12.1. In this way, we get a chain of isomorphisms

$$H_n(X^r) \cong H_n(X^{r-1}) \cong \dots \cong H_n(X^0) \ .$$

□

Observation 1.12.4.

1. Let X be a CW complex. Note that by the proof of Lemma 1.12.1

$$C_n(X) := H_n(X^n, X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_n(\mathbb{S}^n) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}$$

is a free abelian group. For $n < 0$, we let $C_n(X)$ be trivial.

2. If X has only finitely many n-cells, then the abelian group $C_n(X)$ is finitely generated.

If X is a finite CW complex, then $C_*(X)$ is finitely generated as a chain complex, i.e. $C_n(X)$ is only non-trivial in finitely many degrees n, and in these degrees, $C_n(X)$ is finitely generated.

3. Consider the map

$$d\colon H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where δ is the connecting morphism in the long exact sequence in relative homology from Theorem 1.6.5 for the pair $X^{n-1} \subset X^n$ and ϱ is the map induced by the projection map $S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2})$.

The map d is a boundary operator: the composition d^2 is $\varrho \circ \delta \circ \varrho \circ \delta$, but

$$\delta \circ \varrho : H_{n-1}(X^{n-1}) \xrightarrow{\rho} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\delta} H_{n-2}(X^{n-2})$$

is a composition in a long exact sequence and thus vanishes.

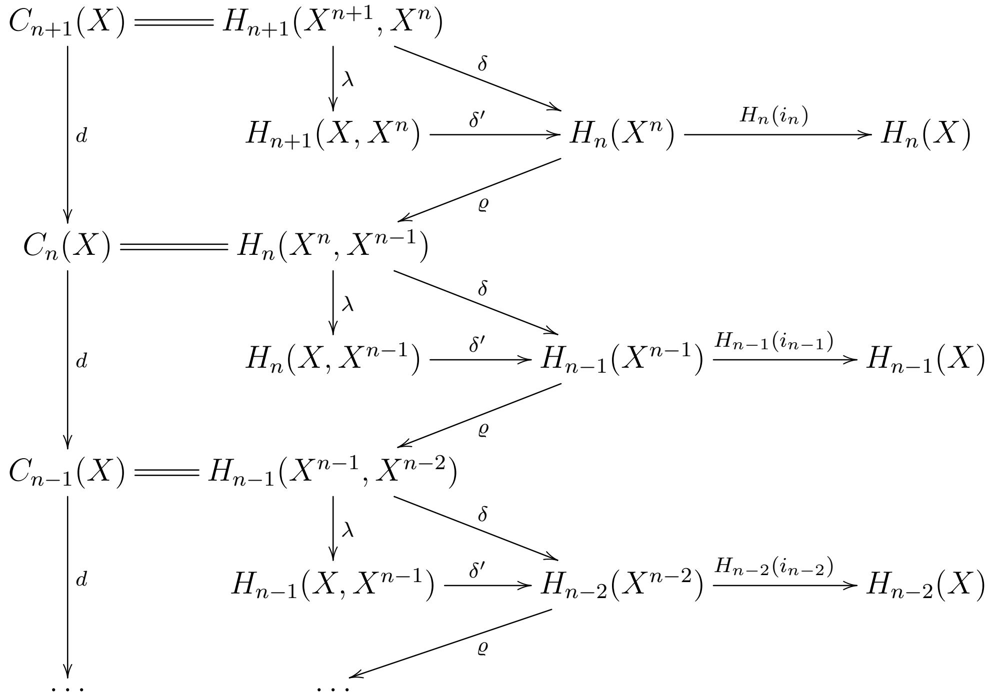
Definition 1.12.5
Let X be a CW complex. The cellular chain complex $C_*(X)$ consists of the free abelian groups $C_n(X) := H_n(X^n, X^{n-1})$ with boundary operator

$$d\colon H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is the map induced by the projection map $S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2})$.

Theorem 1.12.6 (Comparison of cellular and singular homology)**.**
For every CW complex X, there is an isomorphism $\Upsilon\colon H_*(C_*(X), d) \cong H_*(X)$ relating cellular and singular homology.

Proof.
Consider the diagram



1. The fact that $H_k(X^{k-1}) \cong 0$ for all k by Corollary 1.12.3, combined with the long exact sequence
$$0 = H_{n-1}(X^{n-2}) \to H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$
implies that all occurring ϱ-maps are injective.

2. For every $a \in H_n(X^n)$ the element $\varrho(a) \in H_n(X^n, X^{n-1}) = C_n(X)$ is a cycle for d:
$$d\varrho(a) = \varrho\delta\varrho(a) = 0 \ ,$$
since $\delta \circ \varrho = 0$, cf. 1.12.4.3.

3. Conversely, let $c \in C_n(X)$ be a d-cycle, thus $0 = dc = \varrho\delta c$. As ϱ is injective by 1., we obtain $\delta c = 0$. Exactness of the long exact sequence yields that $c = \varrho(a)$ for some $a \in H_n(X^n)$. Hence, ϱ induces an isomorphism
$$H_n(X^n) \cong \ker(d\colon C_n(X) \to C_{n-1}(X)) \ .$$
We have thus expressed the cycles of the cellular complex in terms of the simplical homology group $H_n(X^n)$.

4. We define $\tilde{\Upsilon}\colon \ker(d) \to H_n(X)$ as $\tilde{\Upsilon}[c] = H_n(i_n)(a)$ for $c = \varrho(a)$ with $a \in H_n(X^n)$ and $H_n(i_n)\colon H_n(X^n) \to H_n(X)$.

5. The map $\tilde{\Upsilon}$ is surjective, because $H_n(i_n)$ is surjective by Lemma 1.12.2.1.

6. In the diagram, the triangles commute, i.e. $\delta = \delta' \circ \lambda$, where $\lambda : H_{n+1}(X^{n+1}, X^n) \to H(X, X^n)$ comes from the triple (X, X^{n+1}, X^n).

7. Consider the long exact sequence for the pair (X, X^{n+1}):

$$H_{n+1}(X^{n+1}) \twoheadrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, X^{n+1}) \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X) \ .$$

   The surjectivity of the first morphism is Lemma 1.12.2.1. The last morphism is an isomorphism by Lemma 1.12.2.2. The second morphism is zero, hence the morphism $H_{n+1}(X, X^{n+1}) \to H_n(X^{n+1})$ is injective. Its image is the kernel of an isomorphism and thus zero. This tells us that $H_{n+1}(X, X^{n+1}) = 0$.

   Now consider the triple (X, X^{n+1}, X^n) which by Proposition 1.6.11 yields the exact sequence

$$H_{n+1}(X^{n+1}, X^n) \to H_{n+1}(X, X^n) \to H_{n+1}(X, X^{n+1}) = 0$$

   which implies that $\lambda : H_{n+1}(X^{n+1}, X^n) \to H_{n+1}(X, X^n)$ is surjective.

8. Using this we obtain

$$\mathrm{im}(\delta) \overset{7.}{=} \mathrm{im}(\delta') \overset{\text{les}}{=} \ker(H_n(i_n)) \ ,$$

   where 'les' indicates that we used the long exact sequence. As $d = \varrho \circ \delta$, the injective map ϱ induces an isomorphism between the image of d and the image of δ. Thus

$$\mathrm{im} d \cong \mathrm{im}\delta = \ker H_n(i_n) \ .$$

9. Taking all facts into account we get that ϱ induces an isomorphism

$$\frac{\ker(d\colon C_n(X) \to C_{n-1}(X))}{\mathrm{im}(d\colon C_{n+1}(X) \to C_n(X))} \cong \frac{H_n(X^n)}{\ker(H_n(i_n))}$$

   The numerator is 3., the denominator is 8. and the injectivity of ϱ, cf. 1. But for any n, the sequence

$$0 \longrightarrow \ker H_n(i_n) \longrightarrow H_n(X^n) \longrightarrow \mathrm{im}(H_n(i_n)) \longrightarrow 0$$

   is exact and therefore

$$H_n(X^n)/\ker(H_n(i_n)) \cong \mathrm{im} H_n(i_n) \cong H_n(X) \ ,$$

   where the last isomorphism comes from the surjectivity in Lemma 1.12.2.1.

□

The differential of the cellular complex is very explicitly computable as well:

Proposition 1.12.7 (Cellular Boundary Formula).
Let X be a CW complex. Identify by Observation 1.12.4.1 cells σ^n_α with the generators of the cellular chain group $C_n(X)$. Denote by $d_{\alpha\beta} \in \mathbb{Z}$ the degree of the map

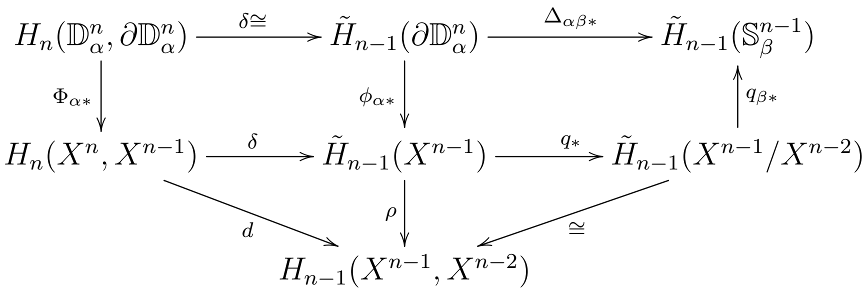
$$\Delta_{\alpha\beta} : \quad S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$$

that is the composition of the attaching map of the n-cell σ_n^α with the quotient map collapsing the complement $X^{n-1} \setminus \sigma_\beta^{n-1}$ of a given $(n-1)$-cell σ_β^{n-1} to a point. Then the differential of the cellular chain complex is

$$d(\sigma_\alpha^n) = \sum_\beta d_{\alpha\beta} \sigma_\beta^{n-1} \ .$$

(This is a finite sum, since the attaching map of σ_α^n has compact image and thus only meets finitely many cells σ_β^{n-1}.)

Proof.
Consider the commuting diagram:



where

- $\Phi_\alpha : \mathbb{D}^n \to X^n$ is the characteristic map for the cell σ_α^n and $\phi_\alpha : \partial\mathbb{D}_\alpha^n \to X^{n-1}$ the attaching map.
- $q : X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map.
- $q_\beta : X^{n-1}/X^{n-2} \to \mathbb{S}_\beta^{n-1}$ collapses the complement $X^{n-1} \setminus \sigma_\beta^{n-1}$ of the cell σ_β^{n-1} to a point. The resulting quotient sphere is identified with $\mathbb{D}_\beta^{n-1}/\partial\mathbb{D}_\beta^{n-1}$ via the characteristic map Φ_β.
- Finally, $\Delta_{\alpha\beta} := q_\beta \circ q \circ \phi_\alpha$ is defined as the composition of the attaching map ϕ_α of the cell σ_α^n, following by collapsing the complement of σ_β^{n-1} in X^{n-1}.

The characteristic map $\Phi_{\alpha*}$ takes a fundamental class $\overline{\mu}_n \in H_n(\mathbb{D}_\alpha^n, \partial\mathbb{D}_\alpha^n)$, cf. comments before Proposition 1.10.2, to a generator e_α^n of the summand in $H_n(X^n, X^{n-1})$ corresponding to the cell σ_α^n. The commutativity of the left part of the diagram implies that $d(e_\alpha^n) = \rho\phi_{\alpha*}\delta\overline{\mu}_n = \rho \circ \phi_{\alpha*}(\mu_{n-1})$.

In terms of the canonical basis of $H_{n-1}(X^{n-1}, X^{n-2})$, the map $q_{\beta*}$ is the projection on the $\mathbb{Z}$-summand corresponding to σ_β^{n-1}. The commutativity of the diagram now shows the claim. □

Examples 1.12.8 (Projective Spaces)**.**
Let K be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$; set $m := \dim_\mathbb{R} K$. The multiplicative group $K^* := K \setminus \{0\}$ acts on the vector space K^{n+1} via scalar multiplication,

$$K^* \times K^{n+1} \setminus \{0\} \to K^{n+1} \setminus \{0\}, \quad (\lambda, v) \mapsto \lambda v.$$

We define $KP^n = (K^{n+1} \setminus \{0\})/K^*$ and we denote the equivalence class of $(x_0, \dots, x_n)$ in KP^n by $[x_0 : \dots : x_n]$. The $n+1$-tuple $(x_0, \dots, x_n)$ is called the homogeneous coordinates of the point $[x_0 : \dots : x_n] \in KP^n$.

We define subsets for $0 \leqslant i \leqslant n$

$$X_i := \{[x_0 : \ldots : x_n] | x_i \neq 0, x_{i+1} = \ldots = x_n = 0\} \subset KP^n$$

and consider the map

$$\xi_i \colon X_i \to K^i, \quad \xi_i[x_0 : \ldots : x_n] = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}).$$

The map ξ_i is a homeomorphism; thus X_i is a cell of KP^n of real dimension $i\dim_{\mathbb{R}}(K) = im$. We can write KP^n as $X_0 \sqcup \ldots \sqcup X_n$ and we have characteristic maps $\Phi_i \colon \mathbb{D}^{mi} \to KP^n$ as

$$\Phi_i(y) = \Phi_i(y_0, \ldots, y_{i-1}) = [y_0 : \ldots : y_{i-1} : 1 - ||y|| : 0 : \ldots : 0]$$

with $X_i = \Phi_i(\mathring{\mathbb{D}}^{mi})$. This defines a structure of a CW complex on KP^n.

1. First, consider the case $K = \mathbb{C}$. Here, we have a cell in each even dimension $0, 2, 4, \ldots, 2n$ for $\mathbb{C}P^n$. Therefore the cellular chain complex is

$$C_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 2i, 0 \leqslant i \leqslant n, \\ 0 & k = 2i-1 \text{ or } k > 2n. \end{cases}$$

   The boundary operator is zero in each degree and thus

$$H_*(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & * = 2i, 0 \leqslant * \leqslant 2n, \\ 0, & \text{otherwise.} \end{cases}$$

2. The case of the quaternions, $K = \mathbb{H}$, is similar. Here the cells are in degrees congruent to zero modulo four, thus

$$H_*(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & * = 4i, 0 \leqslant * \leqslant 4n, \\ 0, & \text{otherwise.} \end{cases}$$

3. Non-trivial boundary operators occur in the case of real projective space, $\mathbb{R}P^n$. Here, we have a cell in each dimension up to n and thus the homology of $\mathbb{R}P^n$ is the homology of the chain complex

$$0 \to C_n \cong \mathbb{Z} \xrightarrow{d} C_{n-1} \cong \mathbb{Z} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \cong \mathbb{Z}.$$

   We first consider the case of $\mathbb{R}P^2$, which we write as a CW-complex with one 0-,1- and 2-cell. The 1-cell is attached to the 0-cell to form a circle. Thus $d[a] = 0$. The 2-cell, a disc, is attached to the circle using a map of degree ± 2, where the sign is undetermined, since we did not fix orientations.

   Thus the complex becomes

$$0 \to \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   and we read off the homology groups we computed in Example 1.9.6.1:

$$H_0(\mathbb{R}P^2) = \mathbb{Z}, \quad H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad H_2(\mathbb{R}P^2) = 0\ .$$

4. In general, $X = \mathbb{R}P^n$ has, for $0 \leqslant k \leqslant n$ the k-skeleton $X^k = \mathbb{R}P^k$. The attaching map of the single k-cell is
$$\phi : \partial \mathbb{D}^k \cong \mathbb{S}^{k-1} \to \mathbb{R}P^{k-1} = \mathbb{S}^{k-1}/A \ ,$$
where A is the antipodal map. We have to compute the degree of the composition
$$\begin{array}{ccc} \mathbb{S}^{k-1} & \xrightarrow{\phi} & \mathbb{S}^{k-1}/\pm\,\mathrm{id} = \mathbb{R}P^{k-1} \\ & \searrow^{\bar{\phi}_k} & \downarrow \pi \\ & & \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong \mathbb{S}^{k-1} \end{array}$$
By construction $\bar{\phi}_k \circ A = \bar{\phi}_k$, with A the antipodal map, and thus
$$\deg(\bar{\phi}_k) = \deg(\bar{\phi}_k \circ A) = (-1)^k \deg(\bar{\phi}_k)$$
and hence the degree of $\bar{\phi}_k$ is trivial for odd k. The complement $\mathbb{S}^{k-1} \setminus \mathbb{S}^{k-2}$ has two components X_+, X_- and A exchanges these two components. The map $\bar{\phi}_k$ sends X_+ and X_- to $[X_+]$. Therefore the degree of $\bar{\phi}_k$ is
$$\deg(\bar{\phi}_k) = \deg(F \circ (\mathrm{id} \vee A) \circ T) \overset{1.10.4}{=} \deg(\mathrm{id}) + \deg(A) = 1 + (-1)^k.$$
and d is either zero or two. For the cellular complex, we find
$$\ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \ .$$
Thus, depending on n we get
$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k < n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$
for n even.

For odd dimensions n we get
$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$
Note that $\mathbb{R}P^1 \cong \mathbb{S}^1$ and $\mathbb{R}P^3 \cong SO(3)$.

1.13 Homology with coefficients

Let G be an arbitrary abelian group.

Definition 1.13.1

The singular chain complex of a topological space X with coefficients in G, $S_*(X; G)$, has as elements in $S_n(X; G)$ finite sums of the form $\sum_{i=1}^N g_i \alpha_i$ with g_i in G and $\alpha_i \colon \Delta^n \to X$ a singular n-simplex. Addition in $S_n(X; G)$ is given by
$$\sum_{i=1}^N g_i \alpha_i + \sum_{i=1}^N h_i \alpha_i = \sum_{i=1}^N (g_i + h_i) \alpha_i.$$

The nth (singular) homology group of X with coefficients in G is

$$H_n(X;G) := H_n(S_*(X;G))$$

where the boundary operator $\partial\colon S_n(X;G) \to S_{n-1}(X;G)$ is given by

$$\partial(\sum_{i=1}^{N} g_i\alpha_i) = \sum_{j=0}^{n}(-1)^j(\sum_{i=1}^{N} g_i(\alpha_i \circ d_j)) \ .$$

We use a similar definition for cellular homology of a CW complex X with coefficients in G. Recall from Observation 1.12.4 that the chain groups are $C_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}$.

Definition 1.13.2
We write $c \in C_n(X;G)$ as $c = \sum_{i=1}^{N} g_i\sigma_i \in \bigoplus_{\sigma \text{ an } n\text{-cell}} G$ and let the boundary operator $\tilde{d}$ be defined by $\tilde{d}c = \sum_{i=1}^{N} g_i d(\sigma_i)$ where $d\colon C_n(X) \to C_{n-1}(X)$ is the boundary in the cellular chain complex of X defined in Observation 1.12.4.

We can transfer Theorem 1.12.6 to the case of homology with coefficients:

$$H_n(X;G) \cong H_n(C_*(X;G), \tilde{d})$$

for every CW complex X and therefore we denote the latter by $H_n(X;G)$ as well.
Note that $H_n(X;\mathbb{Z}) = H_n(X)$ for every space X.

Example 1.13.3.
In the case $X = \mathbb{R}P^2$, we see that coefficients really make a difference.

- Recall from Example 1.9.6 that for coefficients $G = \mathbb{Z}$ we had that $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\mathbb{R}P^2) = 0$.
- However, for coefficients $G = \mathbb{Z}/2\mathbb{Z}$ the cellular chain complex looks rather different:

  $$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  and therefore $H_i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \leqslant i \leqslant 2$.
- For rational coefficients, we we consider $H_*(\mathbb{R}P^2;\mathbb{Q})$. We obtain the cellular complex

  $$0 \longrightarrow \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0$$

  But here, multiplication by 2 is an isomorphism and we get $H_0(\mathbb{R}P^2;\mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{R}P^2;\mathbb{Q}) = \mathbb{Q}/2\mathbb{Q} = 0$ and $H_2(\mathbb{R}P^2;\mathbb{Q}) = 0$.

1.14 Tensor products and the universal coefficient theorem

We need to clarify whether homology $H_*(X,G)$ with coefficients in an abelian group G is computable from singular homology $H_*(X)$ and the group G. To see that this can indeed be done, we need a few more algebraic facts.

Definition 1.14.1
Let A and B be abelian groups. The tensor product $A \otimes B$ of A and B is the quotient of the free abelian group generated by the set $A \times B$ by the subgroup generated by

(a) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,

(b) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$

for $a_1, a_1, a \in A$ *and* $b_1, b_2, b \in B$.

We denote the equivalence class of (a, b) *in* $A \otimes B$ *by* $a \otimes b$.

Remarks 1.14.2.

1. The relations (a) and (b) imply that $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$ for any integer $\lambda \in \mathbb{Z}$ and $a \in A$, $b \in B$.

2. Elements of the abelian group $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^n \lambda_i a_i \otimes b_i$.

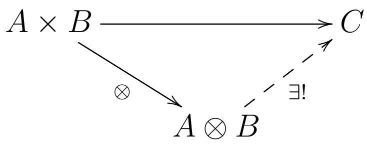
3. The group $A \otimes B$ is generated by elements $a \otimes b$ with $a \in A$ and $b \in B$.

4. The tensor product is symmetric up to isomorphism and the isomorphism $A \otimes B \cong B \otimes A$ is given by
$$\sum_{i=1}^n \lambda_i a_i \otimes b_i \mapsto \sum_{i=1}^n \lambda_i b_i \otimes a_i.$$

5. The tensor product is associative up to isomorphism:
$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
for all abelian groups A, B, C.

6. For homomorphisms $f \colon A \to A'$ and $g \colon B \to B'$ we get an induced homomorphism
$$f \otimes g \colon A \otimes B \to A' \otimes B'$$
which is given by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ on generators.

7. The tensor product has the following universal property. For abelian groups A, B, C, the bilinear maps from $A \times B$ to any abelian group C are in bijection to linear maps from $A \otimes B$ to C,



8. We have already encountered tensor products in Section 1.13: we have group isomorphisms for cellular and singular homology with values in an abelian group G:
$$S_n(X) \otimes G \cong S_n(X; G) \qquad \text{and} \qquad C_n(X) \otimes G \cong C_n(X; G)\ .$$

We collect the following properties of tensor products:

Remarks 1.14.3.

1. For every abelian group A, we have isomorphisms
$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A$$
with $a \otimes n \mapsto n.a$ and inverse $a \mapsto a \otimes 1$.

2. For every abelian group A, we have

$$A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA.$$

Here, $nA = \{na | a \in A\}$ is a subgroup of A for any abelian group A. The isomorphism is given by

$$a \otimes \bar{i} \mapsto \overline{ia}$$

where $\bar{i}$ denotes an equivalence class of $i \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$ and $\bar{ia}$ the class of $ia \in A$ in A/nA.

3. If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence of abelian groups, then for an arbitrary abelian group D, the sequence

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha\otimes \mathrm{id}} B \otimes D \xrightarrow{\beta\otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is not necessarily exact. For example, the sequence

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

is exact, but tensoring with $\mathbb{Z}/2\mathbb{Z}$ yields

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 0$$

is not, because $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Q}/2\mathbb{Q} \cong 0$ and tensoring yields

$$0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to 0$$

which is obviously not exact.

Lemma 1.14.4.

1. For every abelian group D, $(-) \otimes D$ is right exact, i.e. if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence, then

$$A \otimes D \xrightarrow{\alpha\otimes \mathrm{id}} B \otimes D \xrightarrow{\beta\otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact.

2. If the exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a split short exact sequence, then

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha\otimes \mathrm{id}} B \otimes D \xrightarrow{\beta\otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact for any abelian group D.

Proof.
Exercise. □

Suppose, tensoring with the abelian group D would be exact. Then, we could tensor the exact sequence of abelian groups

$$0 \to B_n(X) \to Z_n(X) \to H_n(X) \to 0$$

with the abelian group G and get the isomorphism between

$$H_n(X; G) \stackrel{\text{def}}{=} H_n(S_*(X) \otimes G) \stackrel{\text{def}}{=} Z_n(X) \otimes G / B_n(X) \otimes G$$

and

$$H_n(X) \otimes G = H_n(S_*(X)) \otimes G \ .$$

A consequence of the failure of the functor $(-) \otimes D$ to be exact on the left hand side is that this isomorphism is, in general, wrong.

Definition 1.14.5
Let A be an abelian group. A short exact sequence $0 \to R \longrightarrow F \longrightarrow A \to 0$ with F a free abelian group is called a free resolution of A.

Note that in the situation above R is also free abelian, because it can be identified with a subgroup of the free abelian group F.

Example 1.14.6.
For every $n \geqslant 1$, the sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Proposition 1.14.7.
Every abelian group possesses a free resolution.

The resolution that we will construct in the proof is called the standard resolution of A.

Proof.
Let F be the free abelian group generated by the elements of the underlying set of A. We denote by y_a the basis element in F corresponding to $a \in A$. Define a homomorphism

$$\begin{array}{rcl} p\colon F & \to & A \\ \sum_{a \in A} \lambda_a y_a & \mapsto & \sum_{a\in A} \lambda_a a. \end{array}$$

Here, $\lambda_a \in \mathbb{Z}$ and this integer is non-zero for only finitely many $a \in A$. By construction, p is an epimorphism. We set R to be the kernel of p. Since R is a subgroup of a free abelian group and thus a free abelian group as well, we obtain the desired free resolution of A. □

Definition 1.14.8
For two abelian groups A and B and for $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ the standard resolution of A we define

$$\mathrm{Tor}(A, B) := \ker(i \otimes \mathrm{id}\colon R \otimes B \to F \otimes B).$$

In general, $i \otimes \mathrm{id}$ is not injective, thus $\mathrm{Tor}(A, B)$ is in general not trivial. Unfortunately, the standard resolution constructed in Proposition 1.14.7 is typically very large. We show that we can calculate $\mathrm{Tor}(A, B)$ via an arbitrary free resolution of A. To that end, we prove the following result.

Proposition 1.14.9.
For every homomorphism $f\colon A \to B$ of abelian groups and for free resolutions $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ and $0 \to R' \xrightarrow{i'} F' \longrightarrow B \to 0$ we have:

1. There exist homomorphisms $g\colon F \to F'$ and $h\colon R \to R'$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{i} & F & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & R' & \xrightarrow{i'} & F' & \xrightarrow{p'} & B & \longrightarrow & 0 \end{array}$$

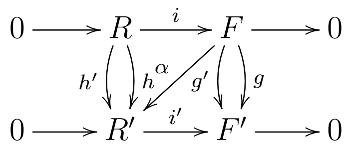
commutes.

If g', h' are other homomorphisms with this property, then there is a group homomorphism $\alpha\colon F \to R'$ with $i' \circ \alpha = g - g'$ and $\alpha \circ i = h - h'$.

2. For every abelian group D, the map $h \otimes \mathrm{id}\colon R \otimes D \to R' \otimes D$ maps the kernel of $i \otimes \mathrm{id}$ to the kernel of $i' \otimes \mathrm{id}$. The restriction $h \otimes \mathrm{id}|_{\ker(i \otimes \mathrm{id})}$ is independent of the choice of g and h. We denote this map by $\varphi(f, R \to F, R' \to F')$.

3. For a homomorphism $f'\colon B \to C$ the map $\varphi(f' \circ f, R \to F, R'' \to F'')$ is equal to the composition $\varphi(f', R' \to F', R'' \to F'') \circ \varphi(f, R \to F, R' \to F')$.

Note that we can view the morphism α in 1.14.9.1 as a chain homotopy between the chain maps g, h and g', h' of free chain complexes



Proof.

- To show 1., let $\{x_i\}$ be a basis of F and choose $y_i \in F'$ such that $p'(y_i) = fp(x_i)$. This is posisble, since p' is surjective. We define $g\colon F \to F'$ on this basis by $g(x_i) = y_i$. Thus $p' \circ g(x_i) = p'(y_i) = fp(x_i)$. For every $r \in R$ we find $p' \circ g(i(r)) = f \circ p \circ i(r) = 0$. Therefore $g(i(r))$ is contained in the kernel of p' which is equal to the image of i'. In order to define h we use the injectivity of i', thus $h(r)$ is the unique preimage of $g(i(r))$ under i'. This shows the first claim in 1.

- Given h, h' and g, g' as in 1., we get for $x \in F$ that $g(x) - g'(x)$ is in the kernel of p' which is the image of i'. Define α as $(i')^{-1}(g - g')$. Then by construction $i'\alpha = g - g'$ and
$$i'(h - h') = (g - g')i = i'\alpha i \ .$$
Here, we first used that the square commutes and then the equation $i'\alpha = g - g'$. As i' is injective, this yields the second relation $h - h' = \alpha i$.

- For 2., we consider an element $z \in \ker(i \otimes \mathrm{id}) \subset R \otimes D$. Then
$$(i' \otimes \mathrm{id}) \circ (h \otimes \mathrm{id})(z) = (g \otimes \mathrm{id}) \circ (i \otimes \mathrm{id})(z) = 0$$
and thus $(h \otimes \mathrm{id})(z)$ is in the kernel of $(i' \otimes \mathrm{id})$. If h' is any other map satisfying the properties, then we find α as in 1. and compute
$$(h' \otimes \mathrm{id})(z) - (h \otimes \mathrm{id})(z) = ((h' - h) \otimes \mathrm{id})(z) = ((\alpha \circ i) \otimes \mathrm{id})(z) = (\alpha \otimes \mathrm{id})(i \otimes \mathrm{id})(z) = 0.$$

- The uniqueness in 2. implies 3.

□

Corollary 1.14.10.

1. For every free resolution $0 \to R' \xrightarrow{i'} F' \longrightarrow A \to 0$ and any abelian group D, we get a unique isomorphism
$$\varphi(\mathrm{id}_A, R' \to F', R \to F)\colon \ker(i' \otimes \mathrm{id}) \to \mathrm{Tor}(A, D).$$
Thus we can calculate $\mathrm{Tor}(A, D)$ with every free resolution of A.

2. Tor is functorial: if $A \stackrel{f}{\to} A'$ and $B \stackrel{g}{\to} B'$ are morphisms of abelian groups, we have morphisms
$$\mathrm{Tor}(f,g): \qquad \mathrm{Tor}(A,B) \to \mathrm{Tor}(A',B') \ .$$

Proof.
For the second statement, note that given a free resolution $0 \to R \stackrel{\iota}{\to} F \to A$ of A, the morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R\otimes B & \longrightarrow & F\otimes B & \longrightarrow & A\otimes B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \\ 0 & \longrightarrow & R\otimes B' & \longrightarrow & F\otimes B' & \longrightarrow & A\otimes B' & \longrightarrow & 0 \end{array}$$

induces a morphism $\ker(\iota \otimes \mathrm{id}_B) \to \ker(\iota \otimes \mathrm{id}_{B'})$. □

Examples 1.14.11.

1. We compute $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D)$ for any abelian group D using the free resolution $0 \to \mathbb{Z} \stackrel{n}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$. By Definition 1.14.8 and by Corollary 1.14.10, we have
$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \ker(n \otimes \mathrm{id}\colon \mathbb{Z}\otimes D \to \mathbb{Z}\otimes D).$$
As $\mathbb{Z}\otimes D \cong D$ and as $n \otimes \mathrm{id}$ induces the multiplication by n, we get
$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \{d \in D | nd = 0\}$ for all $n \geqslant 1$. We thus get the elements in D that are n-torsion. For this reason, Tor is sometimes called torsion product.
2. From the first example we obtain $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$, because the n-torsion subgroup in $\mathbb{Z}/m\mathbb{Z}$ is $\mathbb{Z}/\gcd(m,n)\mathbb{Z}$.
3. For A free abelian, $\mathrm{Tor}(A, D) \cong 0$ for arbitrary D. To see this, note that $0 \to 0 \to A \stackrel{\mathrm{id}}{\longrightarrow} A \to 0$ is a free resolution of A. The kernel is a subgroup of $0 \otimes D = 0$ and hence trivial.
4. For two abelian groups A_1, A_2, D there is an isomorphism
$$\mathrm{Tor}(A_1 \oplus A_2, D) \cong \mathrm{Tor}(A_1, D) \oplus \mathrm{Tor}(A_2, D).$$
Consider two free resolutions
$$0 \to R_i \to F_i \to A_i \to 0, i = 1, 2.$$
Their direct sum
$$0 \to R_1 \oplus R_2 \to F_1 \oplus F_2 \to A_1 \oplus A_2 \to 0$$
is a free resolution of $A_1 \oplus A_2$ with
$$\ker((i_1 \oplus i_2) \otimes \mathrm{id}) = \ker(i_1 \otimes \mathrm{id}) \oplus \ker(i_2 \otimes \mathrm{id}).$$

We extend the definition of tensor products to chain complexes of abelian groups:

Definition 1.14.12
Are (C_*, d) and (C'_*, d') two chain complexes, then $(C_* \otimes C'_*, d_\otimes)$ is the chain complex with
$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$
and with $d_\otimes(c_p \otimes c'_q) = (dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q$.

Lemma 1.14.13.
The map $d_\otimes$ is a differential.

Proof.
The composition is

$$d_\otimes((dc_p)\otimes c'_q + (-1)^p c_p \otimes d'c'_q) = 0 + (-1)^{p-1}(dc_p)\otimes(d'c'_q) + (-1)^p(dc_p)\otimes(d'c'_q) + 0 = 0.$$

□

Remarks 1.14.14.

1. Let G be an abelian group, then let C_G be the chain complex that is concentrated in degree 0, i.e. with
$$(C_G)_n = \begin{cases} G, & n=0, \\ 0, & n\neq 0. \end{cases}$$
Then for every chain complex (C_*, d), the tensor product is
$$(C_*\otimes C_G)_n = C_n \otimes G, \quad d_\otimes = d\otimes \mathrm{id}.$$
In particular, for every topological space X,
$$S_*(X)\otimes C_G \cong S_*(X)\otimes G = S_*(X;G)\ .$$
This allows us to identify the singular chain complex with values in the abelian group G with a tensor product of chain complexes. Similarly, for a CW complex X, we get $C_*(X;G) = C_*(X)\otimes C_G$ for cellular homology with values in G.
2. For every pair of spaces (X, A), we therefore introduce the chain complex
$$S_*(X,A;G) := S_*(X,A)\otimes C_G.$$
3. A map $f\colon (C_*, d)\to(D_*, d_D)$ induces a map of chain complexes
$$f\otimes\mathrm{id}\colon C_*\otimes C'_*\to D_*\otimes C'_*.$$
In particular, for every continuous (cellular) map we get induced maps on singular (cellular) homology with coefficients.
4. Note that, in generalization of proposition 1.2.18 we have $H_*(\mathrm{pt};G)\cong\begin{cases} G, & *=0 \\ 0, & *\neq 0. \end{cases}$

Definition 1.14.15
A chain complex C_* is called <u>*free*</u>*, if the chain group C_n is a free abelian group for all $n\in\mathbb{Z}$.*

For example, the chain complexes $S_*(X,A)$ and $C_*(X)$ are free.

Theorem 1.14.16 (Universal coefficient theorem (algebraic version))**.**
Let C_* be a *free* chain complex and G an abelian group, then for all $n\in\mathbb{Z}$ we have a split short exact sequence

$$0\to H_n(C_*)\otimes G\to H_n(C_*\otimes G)\to \mathrm{Tor}(H_{n-1}(C_*),G)\to 0\ .$$

In particular
$$H_n(C_*\otimes G)\cong H_n(C_*)\otimes G\oplus\mathrm{Tor}(H_{n-1}(C_*),G).$$

This theorem will be a consequence of the more general Theorem 1.14.19. Applying Theorem 1.14.16 to the singular chain complex $C_* := S_*(X)$ of a topological space X, cf. Remark 1.14.14.1, we obtain the following:

Theorem 1.14.17 (Universal coefficient theorem (topological version))**.**
For every space X, there is a split short exact sequence

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0 \ .$$

Therefore, we get an isomorphism

$$H_n(X; G) \cong H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G) \ .$$

Example 1.14.18.
For the real projective space $X = \mathbb{R}P^2$, we obtain

$$H_n(\mathbb{R}P^2; G) \cong H_n(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{n-1}(\mathbb{R}P^2), G) \ .$$

Recalling from Example 1.9.6.1 and Example 1.12.8

$$H_0(\mathbb{R}P^2) = \mathbb{Z}\,, \quad H_1(\mathbb{R}P^2) = \mathbb{Z}_2 \quad \text{and} \quad H_2(\mathbb{R}P^2) = 0$$

we find

$$H_0(\mathbb{R}P^2; G) \cong H_0(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{-1}(\mathbb{R}P^2), G) \cong \mathbb{Z} \otimes G \cong G,$$
$$H_1(\mathbb{R}P^2; G) \cong H_1(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_0(\mathbb{R}P^2), G) \cong G/2G \oplus 0 \cong G/2G,$$

and

$$H_2(\mathbb{R}P^2; G) \cong H_2(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_1(\mathbb{R}P^2), G) \cong \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, G).$$

This reproduces the findings in example 1.13.3.

The universal coefficient theorems 1.14.16 and 1.14.17 are both corollaries of the following more general statement.

Theorem 1.14.19 (Künneth formula)**.**
For a *free* chain complex C_* and a chain complex C'_* we have the following split exact sequence for every integer n

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \xrightarrow{\lambda} H_n(C_* \otimes C'_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \longrightarrow 0,$$

i.e.

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \ .$$

The map $\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \to H_n(C_* \otimes C'_*)$ in the theorem is given on the (p, q)-summand by

$$\lambda([c_p] \otimes [c'_q]) := [c_p \otimes c'_q]$$

for $c_p \in C_p$ and $c'_q \in C'_q$. By the definition of the tensor product of complexes, this map is well-defined.

Lemma 1.14.20.
For any *free* chain complex C_* with *trivial differential* and an arbitrary chain complex, C'_*, λ is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \cong H_n(C_* \otimes C'_*).$$

Proof.

- We abbreviate the subgroup of cycles in C'_q with Z'_q and the subgroup of boundaries in C'_q with B'_q and use analogous abbreviations for the complex C_*. By definition $0 \to Z'_q \longrightarrow C'_q \xrightarrow{d} B'_{q-1} \to 0$ is a short exact sequence. Since $Z_p = C_p$, the group Z_p is free so that tensoring $Z_p \otimes (-)$ is exact by Remark 1.14.11.3. Thus

$$0 \to Z_p \otimes Z'_q \longrightarrow Z_p \otimes C'_q \longrightarrow Z_p \otimes B'_{q-1} \to 0$$

  is a short exact sequence. This implies that $Z_p \otimes Z'_q$ is the subgroup of cycles in $Z_p \otimes C'_q = C_p \otimes C'_q$. Summation over $p + q = n$ yields that the n-cycles in the complex $C_* \otimes C'_*$ are

$$Z_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes Z'_q$$

  and the n-boundaries are given by

$$B_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes B'_q \ .$$

- The sequence

$$0 \to B'_q \longrightarrow Z'_q \longrightarrow H_q(C'_*) \to 0$$

  is exact by definition. Tensoring with Z_p is exact, since Z_p is free. Tensoring and then summing over $p + q = n$ yields the exact sequence

$$0 \to \bigoplus_{p+q=n} Z_p \otimes B'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes Z'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) \to 0$$

  The identification of $Z_n(C_* \otimes C'_*)$ and $B_n(C_* \otimes C'_*)$ in the previous part of the proof implies that the right-most term is isomorphic to the nth homology group of the complex $C_* \otimes C'_*$ and therefore

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

□

Lemma 1.14.21.
Let C_* be a chain complex of abelian groups. Then there exists a free chain complex F_* and a chain map $\varphi : F_* \to C_*$ which induces an isomorphism in homology, $\varphi_* : H(F_*) \xrightarrow{\sim} H(C_*)$.

Proof.
We already know from the exercises that there exists a free chain complex F_* whose homology is isomorphic to the homology of C_*. Fix an isomorphism $\psi_* : H_*(F_*) \to H_*(C_*)$.

Consider the diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n(F_*) & \longrightarrow & Z_n(F_*) & \longrightarrow & H_n(F_*) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \varphi_n^1} & & \downarrow{\scriptstyle \psi_n} & & \\ 0 & \longrightarrow & B_n(C_*) & \longrightarrow & Z_n(C_*) & \longrightarrow & H_n(C_*) & \longrightarrow & 0 \end{array}$$

where we use the fact that $Z_n(F_*)$ is free to lift ψ_n to a map φ_n^1 which induces by restriction a map θ_n. Since $B_{n-1}(F_*)$ is free, the surjection $C_n(F_*) \xrightarrow{d} B_{n-1}(F_*)$ implies that there is a direct sum decomposition $C_n(F_*) = Z_n \oplus Y_n$ such that $d|_{Y_n} : Y_n \xrightarrow{\sim} B_{n-1}(F_*)$. We use the fact that Y_n is free to lift

$$\begin{array}{ccc} Y_n & \longrightarrow & B_{n-1}(F_*) \\ {\scriptstyle \varphi_n^2} \downarrow & & \downarrow {\scriptstyle \theta_{n-1}} \\ C_n & \xrightarrow{d^C} & B_{n-1}(C_*) \end{array}$$

Then

$$\varphi_n^1 \oplus \varphi_n^2 : \quad C_n(F_*) = Z_n \oplus Y_n \to C_n$$

is the chain map inducing ψ in homology.

□

Proof. **of Theorem 1.14.19**

- We consider again the short exact sequence $0 \to Z_p \longrightarrow C_p \xrightarrow{d} B_{p-1} \to 0$. Since B_{p-1} is free, this sequence is split. Tensoring it with C'_q gives, by Lemma 1.14.4.2, an exact sequence. Summing over $p+q=n$ gives the short exact sequence

$$0 \to \bigoplus_{p+q=n} Z_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} C_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} B_{p-1} \otimes C'_q \to 0 \qquad (*)$$

- We define two free chain complexes Z_* and D_* with trivial differential and chain groups

$$(Z_*)_p = Z_p \quad \text{and} \quad (D_*)_p = B_{p-1} \ .$$

  Then the exact sequence $(*)$ can be interpreted as a short exact sequence of complexes. This gives a long exact sequence

$$\ldots \to H_{n+1}(D_*\otimes C'_*) \xrightarrow{\delta_{n+1}} H_n(Z_*\otimes C'_*) \to H_n(C_*\otimes C'_*) \to H_n(D_*\otimes C'_*) \xrightarrow{\delta_n} H_{n-1}(Z_*\otimes C'_*) \to \ldots$$

  Lemma 1.14.20 gives us a description of $H_*(D_* \otimes C'_*)$ and $H_*(Z_* \otimes C'_*)$ and therefore we can consider δ_{n+1} as a map

$$\begin{aligned} \delta_{n+1}\colon \quad & \textstyle\bigoplus_{p+q=n+1} H_p(D_*) \otimes H_q(C'_*) = \bigoplus_{p+q=n+1} B_{p-1} \otimes H_q(C'_*) \xrightarrow{j\otimes \mathrm{id}} \\ & \textstyle\bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(Z_*) \otimes H_q(C'_*) \end{aligned}$$

  with $j\colon B_p \hookrightarrow Z_p$.

- We can cut the long exact sequence in homology in short exact pieces and obtain that all sequences

$$0 \to \mathrm{coker}(\delta_{n+1}) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \ker(\delta_n) \to 0$$

  are exact. The cokernel of δ_{n+1} is isomorphic to $\bigoplus_{p+q=n} (Z_p/B_p) \otimes H_q(C'_*)$ because the tensor functor is right exact, thus

$$\mathrm{coker}(\delta_{n+1}) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

As $0 \to B_p \longrightarrow Z_p \longrightarrow H_p(C_*) \to 0$ is a free resolution of the homology group $H_p(C_*)$, we obtain that

$$\mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(j \otimes \mathrm{id}\colon B_p \otimes H_q(C'_*) \to Z_p \otimes H_q(C'_*))$$

and therefore

$$\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(\delta_n)$$

which proves the exactness of the Künneth sequence.

- We will first prove that the Künneth sequence is split in the case where both chain complexes, C_* and C'_*, are free. In that case the sequences

$$0 \to Z_p \to C_p \to B_{p-1} \to 0, \quad 0 \to Z'_q \to C'_q \to B'_{q-1} \to 0$$

are split and we chose retractions $r\colon C_p \to Z_p$ and $r'\colon C'_q \to Z'_q$. Consider the two compositions

$$\tilde{r} : C_p \longrightarrow Z_p \twoheadrightarrow H_p(C_*), \quad \tilde{r}' : C'_q \longrightarrow Z'_q \twoheadrightarrow H_q(C'_*)$$

and view $H_*(C_*)$ and $H_*(C'_*)$ as chain complexes with trivial differential. Then these compositions yield a chain map

$$C_* \otimes C'_* \xrightarrow{\tilde{r}\otimes\tilde{r}'} H_*(C_*) \otimes H_*(C'_*)$$

which on homology is

$$H_n(C_* \otimes C'_*) \longrightarrow H_n(H_*(C_*) \otimes H_*(C'_*)) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

This map gives the desired splitting.

If the complex C'_* is not free, chose by the preceding lemma a free chain complex F'_*, together with a chain maps

$$\psi' : F' \to C'$$

inducing isomorphism in homology. The naturality of the Künneth exact sequence gives a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(F'_*) & \xrightarrow{\lambda} & H_n(C_* \otimes F'_*) & \longrightarrow & \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(F'_*)) & \longrightarrow 0, \\
& \downarrow \mathrm{id}_*\otimes\psi'_* & & \downarrow (\mathrm{id}\otimes\psi')_* & & \downarrow \mathrm{Tor}(\mathrm{id}_*,\psi'_*) & \\
0 \longrightarrow & \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) & \xrightarrow{\lambda} & H_n(C_* \otimes C'_*) & \longrightarrow & \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) & \longrightarrow 0,
\end{array}$$

Since id_* and ψ'_* are isomorphisms, so are $\mathrm{id}_* \otimes \psi'_*$ and $\mathrm{Tor}(\mathrm{id}_*, \psi'_*)$. Thus $(\mathrm{id} \otimes \psi')_*$ is an isomorphism and the two exact sequences are isomorphic. Hence both are split.

□

In the cases we are interested in (singular or cellular chains), the complexes will be free. The splitting of the Künneth sequence is *not* natural. We have chosen a splitting of the short exact sequences in the proof and usually, there is no canonical choice.

1.15 The topological Künneth formula

Let X and Y be topological spaces. The Künneth sequence for the singular chain complexes $C_* = S_*(X)$ and $C'_* = S_*(Y)$ of two topological spaces states that

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(S_*(X) \otimes S_*(Y)) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

is exact. We will give a geometric meaning to the group $H_n(S_*(X) \otimes S_*(Y))$ by showing that it is actually isomorphic to $H_n(X \times Y)$.

Lemma 1.15.1.
There is a homomorphism of chain complexes $\times\colon S_p(X) \otimes S_q(Y) \longrightarrow S_{p+q}(X \times Y)$ for all $p, q \geqslant 0$ with the following properties.

1. For all points $x_0 \in X$, viewed as zero chains, and for any singular q-simplex $\beta\colon \Delta^q \to Y$ on Y, the product is the following q-simplex on $X \times Y$:

   $$(x_0 \times \beta)(t_0, \ldots, t_q) = (x_0, \beta(t_0, \ldots, t_q))$$

   Analogously, for all $y_0 \in Y$ and any singular p-simplex $\alpha\colon \Delta^p \to X$ on X, we require

   $$(\alpha \times y_0)(t_0, \ldots, t_p) = (\alpha(t_0, \ldots, t_p), y_0) \in X \times Y\ .$$

2. The map $\times$ is natural in X and Y: for $f\colon X \to X'$ and $g\colon Y \to Y'$

   $$S_{p+q}(f, g) \circ (\alpha \times \beta) = (S_p(f) \circ \alpha) \times (S_q(g) \circ \beta).$$

3. The Leibniz rule holds:

   $$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

The map $\times$ is called the homology cross product.

Proof.
For p or q equal to zero, we define $\times$ as dictated by property (1). Therefore we can assume that $p, q \geqslant 1$. The method of proof that we will apply here is called *method of acyclic models*.

- Consider first the specific topological spaces $X = \Delta^p$, $Y = \Delta^q$ with the specific simplices $\alpha = \mathrm{id}_{\Delta^p}$, and $\beta = \mathrm{id}_{\Delta^q}$. If the homology cross product $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}$ of these simplices were already defined, then property (3), the Leibniz rule, would force

  $$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times \partial(\mathrm{id}_{\Delta^q}) =: R \in S_{p+q-1}(\Delta^p \times \Delta^q).$$

  For the boundary of this element R, we get

  $$\partial R = \partial^2(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^{p-1} \partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^p \partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^{2p-1} \mathrm{id}_{\Delta^p} \times \partial^2(\mathrm{id}_{\Delta^q}) = 0$$

  so R is a cycle. But $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ because $p+q-1 \geqslant 1$ and the space $\Delta^p \times \Delta^q$ is contractible and therefore the complex $S_*(\Delta^p \times \Delta^q)$ is acyclic, i.e. all its homology except in degree zero vanishes. Thus R has to be a boundary, so there exists $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$.

  We pick one such c and define the homology cross product as

  $$\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} := c\ .$$

- Now let X and Y be arbitrary spaces and $\alpha\colon \Delta^p \to X$, $\beta\colon \Delta^q \to Y$ arbitrary simplices. Then $S_p(\alpha)(\mathrm{id}_{\Delta^p}) = \alpha$ and $S_q(\beta)(\mathrm{id}_{\Delta^q}) = \beta$ and therefore binaturality (2) dictates

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) = S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}).$$

By construction, this definition satisfies all desired properties.

□

Note that for spaces X, Y with trivial homology in positive degrees, the Künneth Theorem 1.14.19 yields that $H_n(S_*(X) \otimes S_*(Y)) = 0$ for positive n.

Lemma 1.15.2.
Suppose that C_* and C'_* are two chain complexes which are trivial in negative degrees and such that C_n is free abelian for all n and $H_n(C'_*) = 0$ for all positive n. Then we have

1. Any two chain maps $f_*, g_*\colon C_* \to C'_*$ which agree in degree zero, $f_0 = g_0$, are chain homotopic.

2. If $f_0\colon C_0 \to C'_0$ is a homomorphism with $f_0(\partial C_1) \subset \partial C'_1$ then there is a chain map $f_*\colon C_* \to C'_*$ extending f_0.

Proof.

1. We will define a map $H_n\colon C_n \to C'_{n+1}$ for all $n \geqslant 0$ with $\partial H_n + H_{n-1}\partial = f_n - g_n$ inductively. For $n = 0$ we can take $H_0 = 0$, because $f_0 = g_0$ by assumption. Assume that we have found H_k for $k \leqslant n-1$. Let $\{x_i\}$ be a basis of the free abelian group C_n and define
$$y_i := f_n(x_i) - g_n(x_i) - H_{n-1}\partial(x_i) \in C'_n.$$
Then
$$\begin{aligned}\partial y_i =& \partial f_n(x_i) - \partial g_n(x_i) - \partial H_{n-1}\partial(x_i)\\ =&\partial f_n(x_i) - \partial g_n(x_i) - H_{n-2}\partial^2(x_i) - f_{n-1}\partial(x_i) + g_{n-1}\partial(x_i)\\ =&0.\end{aligned}$$
But the complex C'_* is acyclic by assumption. Therefore, y_i has to be a boundary and we define $H_n(x_i) = z_i$ by choosing some z_i such that $\partial z_i = y_i$. Using this definition of $H_n(x_i)$ and then the definition of y_i, we find
$$(\partial H_n + H_{n-1}\partial)(x_i) = y_i + H_{n-1}\partial(x_i) = f_n(x_i) - g_n(x_i).$$

2. To show the second assertion, we define $f_n\colon C_n \to C'_n$ inductively such that $\partial f_n = f_{n-1}\partial$ holds. Assume that $\{x_i\}$ is a basis of C_n. Then $f_{n-1}\partial(x_i)$ is a cycle and thus there exists y_i with $\partial y_i = f_{n-1}\partial(x_i)$, due to the acyclicity of C'_*. We define $f_n(x_i) := y_i$. Then
$$\partial f_n(x_i) = \partial y_i = f_{n-1}\partial(x_i)$$
so that (f_n) is a chain map.

□

We next have to show a uniqueness statement for the chain map constructed in Lemma 1.15.1.

Proposition 1.15.3.
Any two binatural families of chain maps $f_{X,Y}, g_{X,Y}$ from $S_*(X) \otimes S_*(Y)$ to $S_*(X \times Y)$ which agree in degree zero and send the zero chain $x_0 \otimes y_0 \in (S_*(X) \otimes S_*(Y))_0 = S_0(X) \otimes S_0(Y)$ to $(x_0, y_0) \in S_0(X \times Y)$ are chain homotopic.

Proof.

- First we deal with the case $X = \Delta^p$ and $Y = \Delta^q$ for $p, q \geqslant 0$. If $f, g\colon S_*(\Delta^p) \otimes S_*(\Delta^q) \longrightarrow S_*(\Delta^p \times \Delta^q)$ are two chain maps then the complex $S_*(\Delta^p) \otimes S_*(\Delta^q)$ is free abelian and the complex $S_*(\Delta^p \times \Delta^q)$ is acyclic, so we can apply Lemma 1.15.2 and get a chain homotopy $(H_n)_n$,
$$H_n\colon (S_*(\Delta^p) \otimes S_*(\Delta^q))_n \longrightarrow S_{n+1}(\Delta^p \times \Delta^q)$$
with $\partial H_n + H_{n-1}\partial = f_n - g_n$.

- Note that for arbitrary topological spaces X and Y binaturality of f and g implies
$$f_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ f_{\Delta^p,\Delta^q} \quad \text{and} \quad g_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ g_{\Delta^p,\Delta^q}$$
for all singular simplices $\alpha\colon \Delta^p \to X$, $\beta\colon \Delta^q \to Y$.

  We define
$$\begin{aligned} H_n\colon (S_*(X) \otimes S_*(Y))_n &\to S_{n+1}(X \times Y) \\ \alpha \otimes \beta &\mapsto S_{n+1}(\alpha, \beta) \circ H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}). \end{aligned}$$

  This is well-defined and by construction:
$$\begin{aligned} \partial H_n(\alpha \otimes \beta) &= \partial S_{n+1}(\alpha, \beta) \circ H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \qquad \text{[Definition]} \\ &= S_n(\alpha, \beta)\partial H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \qquad [S_*(\alpha, \beta) \text{ is a chain map}] \\ &= S_n(\alpha, \beta) \circ (-H_{n-1}\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) + f_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) - g_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})) \\ &= f_n(\alpha \otimes \beta) - g_n(\alpha \otimes \beta) - H_{n-1}\partial(\alpha \otimes \beta). \end{aligned}$$

  For the last step, observe that
$$\partial_i(\alpha) = \alpha \circ d_i = S_p(\alpha)(\mathrm{id}_{\Delta^p} \circ d_i)$$
implies $\partial\alpha = S_p(\alpha)(\mathrm{id}_{\Delta^p} \circ \partial)$ and thus
$$\begin{aligned} S_n(\alpha, \beta) \circ H_{n-1}\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) &= S_n(\alpha, \beta) \circ H_{n-1}\left(\mathrm{id}_{\Delta^p} \circ \partial \otimes \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q} \circ \partial\right) \\ &= H_{n-1}(\alpha \otimes \beta) \circ (\partial \otimes \mathrm{id} + (-1)^p \mathrm{id} \otimes \partial) \\ &= H_{n-1}\partial(\alpha \otimes \beta) \end{aligned}$$
where we used the definition of the differential and the definition of H_{n-1}.

□

We finally need to ensure the existence of a homotopy inverse.

Proposition 1.15.4.

1. There is a chain map $S_*(X \times Y) \longrightarrow S_*(X) \otimes S_*(Y)$ for all spaces X and Y such that this map is natural in X and Y and such that in degree zero this map sends (x_0, y_0) to $x_0 \otimes y_0$ for all $x_0 \in X$ and $y_0 \in Y$.

2. Any two such maps are chain homotopic.

Proof.

- For the first assertion, let $X = \Delta^n = Y$ for $n \geqslant 0$ and set $C_* = S_*(\Delta^n \times \Delta^n)$ and $C'_* = S_*(\Delta^n) \otimes S_*(\Delta^n)$. Set $f_0 \colon C_0 \to C'_0$ as dictated by the condition. Then by Lemma 1.15.2 there is a chain map $(f_m)_m$, $f_m \colon S_m(\Delta^n \times \Delta^n) \to (S_*(\Delta^n) \otimes S_*(\Delta^n))_m$. For a singular simplex $\alpha \colon \Delta^n \to X \times Y$, we define

$$\tilde{f}_n(\alpha) := (S_*(p_1 \circ \alpha)) \otimes S_*((p_2 \circ \alpha)) \circ f_n(\Delta_{\Delta^n}).$$

Here, $\Delta_{\Delta^n} \colon \Delta^n \longrightarrow \Delta^n \times \Delta^n$ is the diagonal map viewed as a singular n-simplex $\Delta_{\Delta^n} \in S_n(\Delta^n \times \Delta^n)$ and the p_i are the projection maps $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$:

$$\begin{array}{ccc} S_n(\Delta^n \times \Delta^n) & \xrightarrow{f_n} & (S_*(\Delta^n) \otimes S_*(\Delta^n))_n \\ & & \big\downarrow {\scriptstyle S_*(\alpha) \otimes S_*(\alpha)} \\ & & (S_*(X \times Y) \otimes S_*(X \times Y))_n \\ & & \big\downarrow {\scriptstyle S_*(p_1) \otimes S_*(p_2)} \\ & & (S_*(X) \otimes S_*(Y))_n. \end{array}$$

- The second assertion follows as in the proof of Proposition 1.15.3.

□

Theorem 1.15.5 (Eilenberg-Zilber)**.**
The homology cross product $\times \colon S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$ is a homotopy equivalence of chain complexes.

Proof.
Let f be any natural chain map $S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ from Proposition 1.15.4 with $f_0(x_0, y_0) = x_0 \otimes y_0$ for any pair of points. Then the composition

$$f \circ (- \times -) \colon S_*(X) \otimes S_*(Y) \to S_*(X) \otimes S_*(Y)$$

is a chain map that sends $x_0 \otimes y_0$ to itself. Using Lemma 1.15.2 for $X = \Delta^p$ and $Y = \Delta^q$ and then extending by binaturality again, we get that the identity and $f \circ (- \times -)$ are homotopic. Similarly we get that the composition $(- \times -) \circ f$ is homotopic to the identity. □

Corollary 1.15.6 (Topological Künneth formula)**.**
For any pair of spaces X and Y the following sequence is split short exact

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

The sequence is natural in X and Y, but the splitting is not.

Examples 1.15.7.

1. For the n-torus $T^n = (\mathbb{S}^1)^n$ we get inductively
$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}} \ ;$$
all Tor-groups vanish, since all homology groups are free.

2. For a space of the form $X \times \mathbb{S}^n$ we obtain
$$H_q(X \times \mathbb{S}^n) \cong H_q(X) \oplus H_{q-n}(X).$$

Remarks 1.15.8.

1. There is also a *relative version of the Künneth formula*. The homology cross product in its relative form is a map
$$\times\colon S_p(X,A) \otimes S_q(Y,B) \longrightarrow S_{p+q}(X \times Y, A \times Y \cup X \times B)$$
and the corresponding relative homology appears in a topological Künneth formula.

2. In particular for A and B a point we get a *reduced Künneth formula* which is based on a homology cross product
$$\tilde{S}_p(X) \otimes \tilde{S}_q(Y) \longrightarrow S_{p+q}(X \times Y, X \vee Y)$$
and in good cases (see Proposition 1.9.8) the relevant relative homology $\tilde{H}_n(X\times Y, X\vee Y)$ is isomorphic to the homology $\tilde{H}_{p+q}(X \wedge Y)$ of the smash product $X \wedge Y = X \times Y / X \vee Y$.

2 Singular cohomology

2.1 Definition of singular cohomology

Definition 2.1.1
A cochain complex of abelian groups is a sequence $(C^n)_{n\in\mathbb{Z}}$ of abelian groups C^n together with homomorphisms $\delta\colon C^n \to C^{n+1}$ increasing the degree such that $\delta^2 = 0$. The map δ is called coboundary operator. The group
$$H^n(C^*) = \frac{\ker(\delta\colon C^n \to C^{n+1})}{\operatorname{im}(\delta\colon C^{n-1} \to C^n)}$$
is the nth cohomology group of C^*.

If (C_*, d) is a chain complex, we can define $D^n := C_{-n}$ and this is a cochain complex because the fact that d lowers degree by one gives a map $d\colon C_{-n} = D^n \to C_{-n-1} = D^{n+1}$ increasing the degree. We therefore do not need a theory of cochain complexes; it is just convenient to switch to cochain notation.

Definition 2.1.2

For two cochain complexes (C^*, δ) and $(\tilde{C}^*, \tilde{\delta})$ a map of cochain complexes from C^* to $\tilde{C}^*$ is a sequence of homomorphisms $f^n \colon C^n \to \tilde{C}^n$ such that $f^{n+1} \circ \delta = \tilde{\delta} \circ f^n$ for all n.

$$\begin{array}{ccc} C^{n+1} & \xrightarrow{f^{n+1}} & \tilde{C}^{n+1} \\ \delta \uparrow & & \uparrow \tilde{\delta} \\ C^n & \xrightarrow{f^n} & \tilde{C}^n. \end{array}$$

Maps of cochain complexes induce maps on cohomology. In particular, we get for a short exact sequence of cochain complexes a long exact sequence with functorial connecting homomorphisms, as a consequence of Proposition 1.5.5.

Definition 2.1.3

1. *Let G be any abelian group and X a topological space. Then the abelian group*
$$S^n(X;G) := \mathrm{Hom}(S_n(X), G)$$
is called the nth cochain group of X with coefficients in G. In the special case $G = \mathbb{Z}$, we call $S^n(X) := \mathrm{Hom}(S_n(X), \mathbb{Z})$ the nth singular cochain group of X.

2. *The dual $\delta = \partial^* = \mathrm{Hom}(\partial, \mathrm{id}_G)$ of the boundary operator ∂ endows these groups with the structure of a cochain complex.*

3. *The quotient group*
$$H^n(X;G) = \frac{\ker(\delta \colon S^n(X;G) \to S^{n+1}(X;G))}{\mathrm{im}(\delta \colon S^{n-1}(X;G) \to S^n(X;G))}$$
is the nth cohomology group of X with coefficients in G.

Remarks 2.1.4.

1. Explicitly, the differential on a G-valued singular n-cochain $\varphi \colon S_n(X) \to G$ is given by precomposition:
$$\begin{array}{rrcl} \delta(\varphi): & S_{n+1}(X) & \to & G \\ & \alpha & \mapsto & \varphi(\partial\alpha) \end{array}$$

2. We evaluate $\delta^2(\varphi)$ on a singular $(n+2)$-simplex $\beta \colon \Delta^{n+2} \to X$:
$$\delta^2(\varphi)(\beta) = (\delta\varphi)(\partial\beta) = \varphi(\partial^2\beta) = 0 \ .$$
Thus $\delta^2 = 0$ and we indeed have a cochain complex.

3. For a continuous map $f \colon X \to Y$, denote the induced map $S_*(f)$ of singular chains by f_*. Then the dual map
$$S^*(f) = f^* \colon S^*(Y;G) \to S^*(X;G)$$
is defined as usual by precomposition: for $\varphi \in S^*(Y;G)$ and $\alpha \in S_*(X)$,
$$f^*(\varphi)(\alpha) = \varphi(f_*\alpha) \in G.$$
This is indeed a map $S^*(Y;G) \to S^*(X;G)$ of cochain complexes:
$$\langle \delta \circ f^*\varphi, \alpha \rangle = \langle f^*\varphi, \partial\alpha \rangle = \langle \varphi, f_*\partial\alpha \rangle = \langle \varphi, \partial f_*\alpha \rangle = \langle \delta\varphi, f_*\alpha \rangle = \langle f^*\delta\varphi, \alpha \rangle$$
for all chains $\alpha \in S_*(X)$ and cochains $\varphi \in S^*(Y;G)$ and where we write $\langle \psi, \alpha \rangle := \psi(\alpha)$.

4. Note that
$$\langle (f^* \circ g^*)\varphi, \alpha\rangle = \langle g^*\varphi, f\alpha\rangle = \langle \varphi, gf\alpha\rangle = \langle (g\circ f)^*\varphi, \alpha\rangle \ .$$
Thus $S^n(-;G)$ and $H^n(-;G)$ are contravariant functors from the category of topological spaces and continuous maps to the category of abelian groups.

Definition 2.1.5

1. *For two abelian groups A and G, and $\varphi \in \mathrm{Hom}(A, G)$, $a \in A$ the* <u>*Kronecker pairing*</u> *is the G-valued evaluation*
$$\langle -, -\rangle \colon \mathrm{Hom}(A, G) \otimes A \longrightarrow G, \quad \langle \varphi, a\rangle = \varphi(a) \in G.$$
2. *For a homomorphism $f \colon B \to A$ and $\varphi \in \mathrm{Hom}(A, G)$, we have $f^*(\varphi) \in \mathrm{Hom}(B, G)$. On $b \in B$, this takes the value*
$$\langle f^*\varphi, b\rangle = \langle \varphi, fb\rangle = \varphi(f(b)) \in G \ .$$
3. *For a chain complex C_* of abelian groups and the cochain complex $C^n := \mathrm{Hom}(C_n, G)$, we define a pairing with values in G:*
$$\langle -, -\rangle \colon C^n \otimes C_n \to G, \varphi \otimes a \mapsto \langle \varphi, a\rangle = \varphi(a).$$
4. *In particular, for $A = S_n(X)$ a singular chain group and $S^n(X, G) = \mathrm{Hom}(S_n(X), G)$, we get a Kronecker pairing with values in G*
$$\langle -, -\rangle \colon \quad S^n(X; G) \otimes S_n(X) \to G.$$
5. *For $\partial \colon S_{n+1}(X) \to S_n(X)$ and $a \in S_{n+1}(X)$ we get*
$$\langle \delta\varphi, a\rangle = \langle \varphi, \partial a\rangle = \varphi(\partial(a)).$$

Lemma 2.1.6.
Let C_* be a complex of abelian groups and $C^n := \mathrm{Hom}(C_n, G)$ for some abelian group G. The Kronecker pairing $\langle -, -\rangle \colon C^n \otimes C_n \to G$ induces a well-defined pairing on the level of cohomology and homology, i.e. we obtain an induced map
$$\langle -, -\rangle \colon H^n(C^*) \otimes H_n(C_*) \to G \ .$$

Proof.
Let φ be a cocycle, $\delta\varphi = 0$. Then
$$\langle \varphi, a + \partial b\rangle = \langle \varphi, a\rangle + \langle \varphi, \partial b\rangle = \langle \varphi, a\rangle + \langle \delta\varphi, b\rangle = \langle \varphi, a\rangle.$$
Thus $\langle \varphi, -\rangle$ descends to homology. Assume that φ is a coboundary, $\varphi = \delta\psi$ and a is a cycle, $\partial a = 0$. Then we get
$$\langle \varphi, a\rangle = \langle \delta\psi, a\rangle = \langle \psi, \partial a\rangle = 0.$$
Therefore $\langle -, -\rangle$ induces a well-defined G-valued pairing on $H_n(C_*)$ and $H^n(C^*)$. □

Changing perspective, this pairing induces a map
$$\kappa \colon H^n(C^*) \longrightarrow \mathrm{Hom}(H_n(C_*), G)$$
via $\kappa[\varphi][a] := \langle \varphi, a\rangle$. How much of the cohomology $H^n(C^*)$ does the map κ see, i.e. is it surjective, does it have a kernel?

2.2 Universal coefficient theorem for cohomology

Dual to Tor which was defined using the tensor product $(-)\otimes(-)$, we consider a corresponding construction for the functor $\mathrm{Hom}(-,-)$. Let R be a ring; for a short exact sequence of R-modules

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

given an R-module G, the sequence of abelian groups

$$0 \to \mathrm{Hom}(C,G) \xrightarrow{\beta^*} \mathrm{Hom}(B,G) \xrightarrow{\alpha^*} \mathrm{Hom}(A,G) \to 0$$

does not have to be exact. A problem can arise with respect to the surjectivity at the end.

As an example, consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$ for a natural number $n > 1$. Then the sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{n\cdot} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$$

is exact, but multiplication by n is not surjective, so we cannot prolong this sequence to the right with a zero.

Definition 2.2.1
Let A and G be abelian groups. For a free resolution $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ of A, we call $\mathrm{Ext}(A,G)$ *the cokernel of* $\mathrm{Hom}(i,G)\colon \mathrm{Hom}(F,G) \xrightarrow{\iota^*} \mathrm{Hom}(R,G)$.

Remarks 2.2.2.

1. Ext comes from 'extension', because one can describe $\mathrm{Ext}(A,G)$ in terms of extensions of abelian groups.

2. As for Tor it is true that $\mathrm{Ext}(A,G)$ is independent of the free resolution of A.

3. Note that $\mathrm{Ext}(A,G)$ is covariant in G and contravariant in A: for homomorphisms $f\colon A \to B$ and $g\colon G \to H$ we get morphisms of abelian groups

$$f^*\colon \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G), \qquad g_*\colon \mathrm{Ext}(A,G) \to \mathrm{Ext}(A,H).$$

4. For any family of abelian groups $(G_i, i \in I)$

$$\mathrm{Ext}(A, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Ext}(A,G_i)$$

and

$$\mathrm{Ext}(\bigoplus_{i\in I} G_i, B) \cong \prod_{i\in I} \mathrm{Ext}(G_i,B).$$

5. The group $\mathrm{Ext}(A,G)$ is trivial, if A is free abelian. In this case, chose $R = 0$ and $F = A$. The free resolution $0 \to A \xrightarrow{\cong} A \to 0$ gives $\mathrm{Ext}(A,G) = \mathrm{coker}(\mathrm{Hom}(A,G) \xrightarrow{0} \mathrm{Hom}(0,G)) = 0$.

6. We compute

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},G) \cong G/nG.$$

To this end, we use the free resolution $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ and have to compute the cokernel of

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z},G) & \to & \mathrm{Hom}(\mathbb{Z},G) \\ \varphi & \mapsto & \varphi(n.-) \end{array}$$

Identifying $\mathrm{Hom}(\mathbb{Z},G) \cong G$ via $\varphi \mapsto \varphi(1)$, the right hand side is identified with $\varphi(n.1) = n.\varphi(1)$.

7. In particular, $\operatorname{Ext}(A,G)$ is trivial, if G is divisible, i.e. for all $g \in G$ and $n \in \mathbb{Z}\setminus\{0\}$ there exists $t \in G$ with $g = nt$. For example this holds if G is isomorphic to one of the groups $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$, or $\mathbb{C}$.

8. For natural numbers n and m

$$\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}.$$

To state the two main results of this section, we need two simple observations:

Lemma 2.2.3.
Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ be a split short exact sequence of abelian groups. For any abelian group G, the sequence

$$0 \to \operatorname{Hom}(C,G) \xrightarrow{\pi^*} \operatorname{Hom}(B,G) \xrightarrow{\iota^*} \operatorname{Hom}(A,G) \to 0$$

is split exact.

Proof.
The sequence is exact at $\operatorname{Hom}(C,G)$ and $\operatorname{Hom}(B,G)$ in any case, cf. one of the next exercises. Chose a retract $r : B \to A$ for ι, i.e. $r \circ \iota = \mathrm{id}_A$. Then

$$\begin{array}{ccc} \operatorname{Hom}(A,G) & \to & \operatorname{Hom}(B,G) \\ \varphi & \mapsto & \varphi \circ r \end{array}$$

is a section of ι^*. □

Theorem 2.2.4 (Universal coefficient theorem for cochain complexes)**.**
Let G be an abelian group. For every *free* chain complex C_* and $C^* := \operatorname{Hom}(C_*, G)$ the following sequence is exact and splits

$$0 \to \operatorname{Ext}(H_{n-1}(C_*), G) \longrightarrow H^n(C^*) \xrightarrow{\kappa} \operatorname{Hom}(H_n(C_*), G) \to 0.$$

We specify to the singular chain complex, $C_n = S_n(X)$ for a topological space X, which has free chain groups.

Theorem 2.2.5 (Universal coefficient theorem for singular cohomology)**.**
Let X be an arbitrary space. Then the sequence

$$0 \to \operatorname{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \xrightarrow{\kappa} \operatorname{Hom}(H_n(X), G) \to 0$$

is split exact, with κ as defined after Lemma 2.1.6.

Proof. **of Theorem 2.2.4**

- Let C_* be a free chain complex and $C^* := \operatorname{Hom}(C_*, G)$. Then the sequence $0 \to Z_n \longrightarrow C_n \xrightarrow{\partial} B_{n-1} \to 0$ is split exact, since B_n is free as a subgroup of the free group C_n and thus a section of ∂ can be constructed. By lemma 2.2.3, the G-dual sequence

$$0 \to B^{n-1} \longrightarrow C^n \longrightarrow Z^n \to 0$$

is short exact. It gives a short exact sequence of cochain complexes, where we view B^* and Z^* as cochain complexes with trivial differential. This yields a long exact sequence on the level of cohomology groups

$$\ldots \longrightarrow Z^{n-1} \xrightarrow{\partial} B^{n-1} \longrightarrow H^n(C^*) \longrightarrow Z^n \xrightarrow{\partial} B^n \longrightarrow \ldots \qquad (*)$$

Here, ∂ denotes the connecting homomorphism in the cohomological case. By the very definition of the connecting homomorphism we get that ∂ is the dual of the inclusion $i_n \colon B_n \subset Z_n$, $\partial = i_n^*$:

$$\begin{array}{ccc} & C^n & \longrightarrow Z^n \ni \varphi \\ & \downarrow{\scriptstyle \delta} & \\ B^n \xrightarrow[\delta]{} & C^{n+1} & \end{array}$$

A preimage $\psi \in C^n$ of $\varphi \in Z^n$ is any morphism $\psi : C_n \to G$ that restricts to φ on the subgroup Z_n of cycles. It is mapped to $\psi \circ d \in C^{n+1}$. Here, only the value of ψ on boundaries matters, we can thus replace $\psi \circ d = \varphi \circ d$. We are looking for $\tilde{\varphi} : B_n \to G$ such that $\tilde{\varphi} \circ d = \varphi \circ d$. This is achieved by the restriction of φ to the boundaries B_n.

- We cut the long exact sequence $(*)$ into the short one

  $$0 \to \operatorname{coker}(i_{n-1}^*) \longrightarrow H^n(C^*) \longrightarrow \ker(i_n^*) \to 0$$

  and hence we have to compute the kernel and the cokernel of $i_n^* : \operatorname{Hom}(Z_n, G) \to \operatorname{Hom}(B_n, G)$.

- The exact sequence obtained from applying the left exact Hom-functor to the short exact sequence $0 \to B_n \overset{\iota_n}{\to} Z_n \overset{\pi}{\to} H_n(C_*) \to 0$

  $$0 \to \operatorname{Hom}(H_n(C_*), G) \xrightarrow{\pi^*} \operatorname{Hom}(Z_n, G) \xrightarrow{i_n^*} \operatorname{Hom}(B_n, G)$$

  tells us that the kernel of i_n^* is the image of π^* and due to the injectivity of π^* this is isomorphic to $\operatorname{Hom}(H_n(C_*), G)$.

- The sequence

  $$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_*) \to 0$$

  is a free resolution of the homology group $H_{n-1}(C_*)$ and therefore the cokernel of i_{n-1}^* by Definition 2.2.1 equals $\operatorname{Ext}(H_{n-1}(C_*), G)$.

□

Examples 2.2.6.

1. We know from Example 1.12.8.2 that the homology of complex porjective space $\mathbb{C}P^n$ is free with

   $$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & 0 \leqslant k \leqslant 2n, k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

   For free groups, the Ext-groups vanish by 2.2.2.5 and thus $H^k(\mathbb{C}P^n) \cong \operatorname{Hom}(H_k(\mathbb{C}P^n), \mathbb{Z})$. The cohomology is in this example given by the $\mathbb{Z}$-dual of the homology.

2. Recall from Proposition 1.8.3 that

   $$H^m(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & m = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

   for $n \geqslant 1$. For later use we fix a class $\nu_n \in H^n(\mathbb{S}^n)$ with $\langle \nu_n, \mu_n \rangle = 1$.

2.3 Axiomatic description of a cohomology theory

Before we give an axiomatic description of singular cohomology, we establish some consequences of some of the results we proved for singular homology.

Remarks 2.3.1.

1. For a chain map $f\colon C_* \to C'_*$ (such as the barycentric subdivision) the G-dual map
$$f^* = \mathrm{Hom}(f, G)\colon \mathrm{Hom}(C'_*, G) \longrightarrow \mathrm{Hom}(C_*, G)$$
is a map of cochain complexes.

2. If $(H_n\colon C_n \to C'_{n+1})_n$ is a chain homotopy, then the G-dual
$$(H^n := \mathrm{Hom}(H_n, G)\colon \mathrm{Hom}(C'_{n+1}, G) \to \mathrm{Hom}(C_n, G))_n$$
is a cochain homotopy. Thus if $\partial H_n + H_{n-1}\partial = f_n - g_n$, then $H^n\delta + \delta H^{n-1} = f^n - g^n$.

3. We have seen in Lemma 2.2.3 that for a split exact sequence $0 \to B_1 \longrightarrow B_2 \longrightarrow B_3 \to 0$ the dual sequence $0 \to \mathrm{Hom}(B_3, G) \longrightarrow \mathrm{Hom}(B_2, G) \longrightarrow \mathrm{Hom}(B_1, G) \to 0$ is exact. For instance, if A is a subspace of X, then the short exact sequence of chain complexes
$$0 \to S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \to 0$$
is split. To see this, we define a retraction $r_n\colon S_n(X) \to S_n(A)$ on a generator $\alpha\colon \Delta^n \to X$ via
$$r_n(\alpha) = \begin{cases} \alpha, & \text{if } \alpha(\Delta^n) \subset A, \\ 0, & \text{otherwise.} \end{cases}$$
Therefore $0 \to S^*(X, A) \longrightarrow S^*(X) \longrightarrow S^*(A) \to 0$ is a short exact sequence of cochain complexes and gives rise to a long exact sequence in cohomology.

With the help of these facts we can show that singular cohomology satisfies the (Eilenberg-Steenrod) axioms of a cohomology theory:

1. The assignment $(X, A) \mapsto H^n(X, A)$ is a contravariant functor from the category of pairs of topological spaces to the category of abelian groups.

2. If $f, g\colon (X, A) \to (Y, B)$ are two homotopic maps of pairs of topological spaces, then $H^n(f) = H^n(g)\colon H^n(Y, B) \to H^n(X, A)$.

3. For any subspace $A \subset X$ there is a natural connecting homomorphism $\partial\colon H^n(A) \to H^{n+1}(X, A)$ increasing the degree.

4. For any subspace $A \subset X$ we get a long exact sequence
$$\ldots \xrightarrow{\partial} H^n(X, A) \longrightarrow H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial} \ldots$$

5. Excision holds, i.e. for $W \subset \bar{W} \subset \mathring{A} \subset A \subset X$
$$H^n(i)\colon H^n(X, A) \cong H^n(X \setminus W, A \setminus W), \text{ for all } n \geqslant 0.$$

6. For the one-point space pt, we have

$$H^n(\mathrm{pt}) \cong \begin{cases} G, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

   This is called the axiom about the coefficients or the <u>dimension axiom</u>.

7. Singular cohomology is additive under disjoint union:

$$H^n(\bigsqcup_{i \in I} X_i) \cong \prod_{i \in I} H^n(X_i).$$

For singular cohomology with coefficients in G we have an analogous set of axioms. There are important so-called *generalized cohomology theories* like topological K-theory or cobordism theories that satisfy all axioms but the dimension axiom. (For K-theory on a point, we get the integers in every even degree.)

2.4 Cap product

The rough idea of the cap product is to evaluate a (relative) cochain of smaller or equal degree on a piece of a (relative) chain to get a relative chain of smaller degree. (This is a partial evaluation of cochains on chains.)

Definition 2.4.1
Let $a\colon \Delta^n \to X$ be a singular n-simplex on X and let $0 \leqslant q \leqslant n$.

- *The $(n-q)$-dimensional <u>front face</u> of the singular simplex a on X is the $(n-q)$-dimensional singular simplex on X*

$$F(a) = F^{n-q}(a) = a \circ i\colon \Delta^{n-q} \hookrightarrow \Delta^n \xrightarrow{a} X$$

  where i is the inclusion $i\colon \Delta^{n-q} \hookrightarrow \Delta^n$ with $i(e_j) = e_j$ for $0 \leqslant j \leqslant n-q$.

- *The q-dimensional <u>back</u> or <u>rear face</u> of of the singular simplex a is the q-simplex*

$$R(a) = R^q(a) = a \circ r\colon \Delta^q \hookrightarrow \Delta^n \xrightarrow{a} X$$

  where $r\colon \Delta^q \hookrightarrow \Delta^n$ is the inclusion with $r(e_0) = e_{n-q}, \ldots, r(e_q) = e_n$, i.e., *$r(e_i) = e_{n-(q-i)}$.*

Definition 2.4.2
Let $0 \leqslant q \leqslant n$. Let R be an associative ring with unit. We define

$$\cap\colon S^q(X,A;R) \otimes S_n(X,A;R) = \mathrm{Hom}(S_q(X,A),R) \otimes S_n(X,A) \otimes R \longrightarrow S_{n-q}(X) \otimes R = S_{n-q}(X;R)$$

as

$$\alpha \cap (a \otimes r) := F^{n-q}(a) \otimes \langle \alpha, R^q(a) \rangle \cdot r.$$

Remarks 2.4.3.

1. The map $\cap$ is well-defined for relative (co-)homology for the pair (X, A): for $a = a' \in S_n(X, A)$, i.e. $a = a' + b$ with $\mathrm{im}(b) \subset A$ we get

$$\alpha \cap (a \otimes r) = \alpha \cap ((a' + b) \otimes r) = \alpha \cap (a' \otimes r) + F(b) \otimes \langle \alpha, R(b) \rangle r.$$

   The image of $R(b)$ is contained in $A \subset X$, but $\alpha \in \mathrm{Hom}(S_q(X, A), R)$, thus $\alpha \colon S_q(X) \to R$ with $\alpha|_{S_q(A)} = 0$ and $\langle \alpha, R(b) \rangle = 0$.

2. We can express the $(n-q)$-dimensional front face of a in terms of the face maps from Definition 1.2.9 as

$$F^{n-q}(a) = \partial_{n-q+1} \circ \ldots \circ \partial_n(a).$$

   Similarly,

$$R^q(a) = \partial_0 \circ \ldots \circ \partial_0(a) = \partial_0^{n-q} a \ .$$

3. There is a more general version of the cap product. Suppose that there is a pairing of abelian groups

$$G \otimes G' \to G'' \ ;$$

   then we can define

$$\cap \colon S^q(X, A; G) \otimes S_n(X, A; G') \to S_{n-q}(X; G'').$$

Proposition 2.4.4.

1. The Leibniz formula holds for the cap product: for $\alpha \in S^q(X, A)$, we have

$$\partial(\alpha \cap (a \otimes r)) = (\delta\alpha) \cap (a \otimes r) + (-1)^q \alpha \cap (\partial a \otimes r).$$

2. Naturality: for a map of pairs of spaces $f \colon (X, A) \to (Y, B)$, we have a map $f_* \colon S_*(X, A) \to S_*(Y, B)$ of chain complexes and a map $f^* \colon S^*(Y, B) \to S^*(X, A)$ cochain complexes. Given $a \otimes r \in S_n(X, A) \otimes R = S_n(X, A; R)$ and $\beta \in \mathrm{Hom}(S_q(Y, B), R)$, we have

$$f_*(f^*(\beta) \cap (a \otimes r)) = \beta \cap (f_*(a) \otimes r) \ .$$

For the proof, we suppress the tensor product with R. It just adds to notational complexity.

Proof.

1. For the first claim we calculate the left hand side:

$$\begin{aligned}
\partial(\alpha \cap a) &= \partial(F^{n-q}(a) \otimes \langle \alpha, R^q(a) \rangle) \\
&= \partial(F^{n-q}(a)) \otimes \langle \alpha, R^q(a) \rangle \\
&= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q+1} \circ \ldots \circ \partial_n(a)) \otimes \langle \alpha, \partial_0^{n-q}(a) \rangle
\end{aligned}$$

   This has to be compared to the two terms on the right hand side:

$$\begin{aligned}
(\delta\alpha) \cap a &= F^{n-q-1}(a) \otimes \langle \delta\alpha, R^{q+1}(a) \rangle \\
&= F^{n-q-1}(a) \otimes \langle \alpha, \partial R^{q+1}(a) \rangle \\
&\overset{2.4.3.2}{=} \sum_{i=0}^{q} (-1)^i \partial_{n-q} \circ \ldots \circ \partial_n(a) \otimes \langle \alpha, \partial_i \partial_0^{n-(q+1)}(a) \rangle.
\end{aligned}$$

and, noting that $\partial_j a$ is a $(n-1)$-chain

$$\begin{aligned}
\alpha \cap \partial a &= \sum_{j=0}^{n} (-1)^j \alpha \cap \partial_j a \\
&= \sum_{j=0}^{n} (-1)^j F^{n-1-q}(\partial_j a) \otimes \langle \alpha, R^q(\partial_j a)\rangle \\
&= \sum_{j=0}^{n} (-1)^j F^{n-1-q}(\partial_j a) \otimes \langle \alpha, R(\partial_j(a))\rangle \\
&\overset{2.4.3.2}{=} \sum_{j=0}^{n} (-1)^j \partial_{n-q} \circ \ldots \circ \partial_{n-1} \circ \partial_j a \otimes \langle \alpha, \partial_0^{(n-1)-q} \partial_j a\rangle .
\end{aligned}$$

In order to get the result, use the simplicial relations $\partial_j \partial_i = \partial_{i-1}\partial_j$ for $0 \leqslant j < i \leqslant n$.

2. For the claim about naturality, we note that $f_* R = R f_*$ and $f_* F = F f_*$ and plug in the definitions to obtain

$$\begin{aligned}
f_*(f^*(\beta) \cap a) &= f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) \\
&= f_*(F(a) \otimes \langle \beta, f_* R(a)\rangle) \\
&= F(f_*(a)) \otimes \langle \beta, R(f_*(a))\rangle) \\
&= \beta \cap f_*(a).
\end{aligned}$$

□

Remark 2.4.5.
From the Leibniz formula, we conclude the following properties of the cap product:

- A cocycle cap a cycle is a cycle.
  Indeed, for a cycle c with $\partial c = 0$ and for a q-cocycle e with $\delta e = 0$, we find

$$\partial(e \cap c) = (\delta e) \cap c + (-1)^q e \cap (\partial c) = 0 \ .$$

- A cocycle cap a boundary is a boundary.
  Indeed, for a q-cocycle e with $\delta e = 0$ and a boundary $b = \partial c$, we find from the Leibniz rule

$$\partial(e \cap (-1)^q c) = (\delta e) \cap (-1)^q c + e \cap b = e \cap b$$

  so that the cap product $e \cap b$ is a boundary.

- A coboundary cap a cycle is a boundary.

Therefore we obtain the following result:

Proposition 2.4.6.
The cap product induces a map

$$\cap \colon H^q(X, A; R) \otimes H_n(X, A; R) \longrightarrow H_{n-q}(X; R)$$

via

$$[\alpha] \cap [a] := [F(a) \otimes \langle \alpha, R(a)\rangle]$$

Examples 2.4.7.

1. Let R be a ring and consider $1 \in S^0(X;R)$. This is the cochain with value $1(a) = 1 \in R$ for all 0-simplices $a\colon \Delta^0 \to X$. We claim that the cap product obeys $1 \cap a = a$ for any singular simplex $a : \Delta^n \to X$. Indeed, we have $q = 0$ and thus $F(a) = a$. For the rear face, we have $R(a)(e_0) = a(e_n)$. Therefore, $1 \cap a = a \otimes \langle 1, a(e_n)\rangle = a \otimes 1$ and we identify the latter with the n-simplex a.

2. For a space X and a cochain $\alpha \in S^n(X;R)$ and a chain $a \in S_n(X)$ of same degree, we have $q = n$ and thus $F(a)(e_0) = a(e_0) \in X$ and $R(a) = a$. We find

$$\alpha \cap a = a(e_0) \otimes \langle \alpha, a\rangle \ .$$

If X is path-connected, then $[a(e_0)] \in H_0(X) \cong \mathbb{Z}$ is a generator which we identify with $1 \in \mathbb{Z}$. In this sense, the cap product $\alpha \cap a$ generalizes the Kronecker pairing of $S^n(X;G)$ and $S_n(X)$ with values in G, cf. Definition 2.1.5.4.

3. There is also a version of the cap product of the form

$$\cap\colon H^q(X;R) \otimes H_n(X,A;R) \longrightarrow H_{n-q}(X,A;R).$$

Remark 2.4.8.
We explain the notation $\cap$. We take a 2-torus T^2. Its first homology is $H_1(T^2) \cong \mathbb{Z}^2$ and generated by the class of a meridian $b \subset T^2$ and of a longitude $a \subset T^2$. The second homology $H_2(T^2) \cong \mathbb{Z}$ is generated by the class of the singular 2-simplex $\sigma : \Delta^2 \to T^2$ that maps the boundary $\partial\Delta_2$ to $ab(a^{-1}b^{-1})$. We find $F^1(\sigma) = a$ and $R^1(\sigma) = b$.

We consider the class $\beta \in H^1(T^2) \cong \mathrm{Hom}(H_1(T^2),\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ dual to $[b] \in H_1(T^2)$. Then $\beta \cap \sigma = F^1(\sigma) \otimes \langle \beta, R^1(\sigma)\rangle = a$ can be represented by the longitude a which is transversal to the meridian b.

2.5 Cup product on cohomology

In the following, let R be a commutative ring with unit. We will consider homology and cohomology with coefficients in R, but we will suppress the ring R in our notation, so $H_n(X,A)$ will stand for $H_n(X,A;R)$ and similarly $S_n(X,A)$ is $S_n(X,A;R)$. We will use analogous abbreviations for cochains and cohomology. Sometimes, if we have to be explicit, we denote the multiplication in R by μ.

We recall from Proposition 1.15.4 that an Eilenberg-Zilber map is a homotopy equivalence of chain complexes

$$\mathrm{EZ} : \qquad S_*(X\times Y; X\times B \cup A\times Y) \longrightarrow S_*(X,A)\otimes S_*(Y,B)$$

We use this structure in a first step to associate to a pair, consisting of a cochain on X and of a cochain on Y a cochain on the product $X \times Y$:

Definition 2.5.1
For a (relative) cochain $\alpha \in S^p(X,A)$ on X and a (relative) cochain $\beta \in S^q(Y,B)$ on Y the cohomology cross product $\alpha\times\beta$ or external cup product is the (relative) $(p+q)$-cochain on $X\times Y$

$$\alpha\times\beta := \mu\circ(\alpha\otimes\beta)\circ\mathrm{EZ} \in S^{p+q}(X\times Y, X\times B\cup A\times Y)$$

Thus

$$\begin{array}{ccccc}
S_n(X\times Y; X\times B\cup A\times Y) & & & & \\
\downarrow \scriptstyle{EZ} & \searrow^{\alpha\times\beta} & & & \\
\bigoplus_{p'+q'=n} S_{p'}(X,A)\otimes S_{q'}(Y,B) & & & & \\
\downarrow & & & & \\
S_p(X,A)\otimes S_q(Y,B) & \xrightarrow[\alpha\otimes\beta]{} & R\otimes R & \xrightarrow[\mu]{} & R
\end{array}$$

Remarks 2.5.2.

1. Since the Eilenberg-Zilber map is natural, the cohomology cross product is natural, i.e. for maps of pairs of spaces $f\colon (X,A)\to(X',A')$, $g\colon (Y,B)\to(Y',B')$ we have
$$(f,g)^*(\alpha\times\beta) = (f^*\alpha)\times(g^*\beta).$$
2. For the Kronecker pairing and for cohomology classes $\alpha\in H^p(X,A)$ and $\beta\in H^q(Y,B)$ and homology classes $a\in H_p(X,A)$ and $b\in H_q(Y,B)$, we have by definition of $\alpha\times\beta$
$$\langle\alpha\times\beta, a\times b\rangle = \langle\alpha,a\rangle\langle\beta,b\rangle.$$
3. For $1\in R$ and thus $1_X\in S^0(X,A)$ and $1_Y\in S^0(Y,B)$
$$1_X\times\beta = p_2^*(\beta),\quad \alpha\times 1_Y = p_1^*(\alpha)$$
where p_i $(i=1,2)$ denotes the projection onto the ith factor in $X\times Y$. Indeed,
$$\langle 1\times\beta, a\times b\rangle = \langle 1,a\rangle\cdot\langle\beta,b\rangle = \langle\beta,b\rangle$$
and
$$p_2^*\beta(a,b) = \beta\circ p_2(a,b) = \beta(b)\ .$$

We next use the cohomology cross product in order to obtain a multiplication on the graded abelian group $H^*(X,G)$. Consider the diagonal map

$$\begin{array}{rcl}\Delta\colon X & \to & X\times X\\ x & \mapsto & (x,x)\end{array}$$

as a map of pairs
$$\Delta:\quad (X, A\cup B)\to(X\times X, X\times B\cup A\times X)\ .$$

Definition 2.5.3
For $\alpha\in H^p(X,A)$ and $\beta\in H^q(X,B)$ we define the cup product of α and β as
$$\alpha\cup\beta = \Delta^*(\alpha\times\beta).$$

$$\begin{array}{ccc}
H^p(X,A)\otimes H^q(X,B) & \xrightarrow{\times} & H^{p+q}(X\times X, X\times B\cup A\times X)\\
 & \searrow_{\cup} & \downarrow \scriptstyle{\Delta^*}\\
 & & H^{p+q}(X, A\cup B)
\end{array}$$

Conversely, we can express the cohomology cross product via the cup product:

Proposition 2.5.4.
Let X and Y be topological spaces. Consider the projections

$$p_1 : \; X \times Y \to X \qquad \text{and} \qquad p_2 : \; X \times Y \to Y \; .$$

Let $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$ be cohomology classes. Then the external cup product satisfies

$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta) \; ,$$

where the cup product is on the product space $X \times Y$

Proof.
Since by Remark 2.5.2.3, we have $p_1^*(\alpha) = \alpha \times 1_Y$ and $p_2^*(\beta) = 1_X \times \beta$, we find

$$p_1^*(\alpha) \cup p_2^*(\beta) = (\alpha \times 1_Y) \cup (1_X \times \beta).$$

Here, $\alpha \times 1$ and $1 \times \beta$ live in the cohomology of $X \times Y$ and the cup product is to be taken on the product space $X \times Y$. By Definition 2.5.3, the cup product is the pull-back of the cross product by the diagonal. Here, $\Delta_{X \times Y} \colon X \times Y \to X \times Y \times X \times Y$. Therefore, the above is equal to

$$\Delta_{X \times Y}^*((\alpha \times 1_Y) \times (1_X \times \beta)) = \alpha \times \beta.$$

□

In the definition of the cup product, the map

$$S_*(X) \overset{\Delta_*}{\to} S_*(X \times X) \overset{\mathrm{EZ}}{\to} S_*(X) \otimes S_*(X)$$

enters. The Eilenberg-Zilber map was unique up to homotopy. We will get a simple explicit formula of the cup product by choosing a simple morphism of complexes that is still homotopy equivalent.

Definition 2.5.5
A <u>diagonal approximation</u> is a natural chain map $D \colon S_*(X) \longrightarrow S_*(X) \otimes S_*(X)$ such that $D(x) = x \otimes x$ for all 0-chains $x \in S_0(X)$.

With the method of acyclic models, cf. Section 1.15, one can prove:

Proposition 2.5.6.
Any two natural diagonal approximations are chain homotopic.

Definition 2.5.7
The <u>Alexander-Whitney map</u> is the diagonal approximation

$$\mathrm{AW}(a) = \sum_{p+q=n} F^p(a) \otimes R^q(a) \in S_*(X) \otimes S_*(X)$$

for $a \in S_n(X)$.

Remarks 2.5.8.

1. It is obvious that AW is a natural chain map and this map yields a cup product for $\alpha \in H^p(X,A)$ and $\beta \in H^q(X,B)$

$$\begin{aligned}(\alpha \cup \beta)(a) = \mu \circ (\alpha \otimes \beta)\mathrm{AW}(a) &= \mu \circ (\alpha \otimes \beta) \textstyle\sum_{p'+q'=n}(F^{p'}(a) \otimes R^{q'}(a)) \\ &= (-1)^{pq}\alpha(F^p(a))\beta(R^q(a)) \ .\end{aligned}$$

2. From the formula, we see that $\cup$ is associative and distributive on cochain level and not just on the level of cohomology groups. Also a graded Leibniz rule immediately follows: for $\alpha \in H^p(X)$ and $\beta \in H^*(X)$, we have

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^p \alpha \cup \delta\beta \ .$$

   It implies that the cup product is well-defined in cohomology.

3. But note that it does not give a (graded) commutative product on singular cochains. (The cup product is homotopy commutative and in fact it is homotopy commutative up to coherent homotopies: it is an E_∞-*algebra*.)

Proposition 2.5.9.
Let X be a topological space and α, β, γ be cohomology classes on X. The cup product satisfies

1. Associativity:
$$\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma.$$

2. (Graded) commutativity:
$$\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha.$$

3. Compatibility with the connecting homomorphism $\partial\colon H^*(A) \to H^{*+1}(X,A)$ in relative cohomology and $\iota : A \to X$, we find for $\alpha \in H^*(A)$ and $\beta \in H^*(X)$:

$$\partial(\alpha \cup i^*\beta) = (\partial\alpha) \cup \beta \in H^*(X,A).$$

4. Naturality: For $f\colon X \to Y$ and $\alpha, \beta \in H^*(Y)$:

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

Proof.
Associativity and distributivity have already been discussed. Naturality follows from the naturality of the external cup product. Graded commutativity follows from an explicit chain homotopy that is constructed in [Hatcher, Theorem 3.14]. □

Using the relation
$$\alpha \times \beta = p_1^*\alpha \cup p_2^*\beta$$
from Proposition 2.5.4 that expreses the external cup product in terms of the cup product on the product space $X \times Y$, we conclude:

Corollary 2.5.10.

1. The cohomology cross product is associative

$$\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$$

   on the level of cohomology groups.

2. It satisfies a graded version of commutativity. The twist map $\tau\colon X \times Y \to Y \times X$ yields on cohomology
$$\alpha \times \beta = (-1)^{|\alpha||\beta|}\tau^*(\beta \times \alpha).$$

3. The Leibniz formula holds,
$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|}\alpha \times (\delta\beta).$$
Here, $|\alpha|$ denotes the degree of α.

Proposition 2.5.11.

1. For all pairs of spaces (X, A) the cohomology groups $H^*(X, A; R)$ have a structure of a graded commutative ring with unit $1 \in H^0(X, A; R)$.

2. The graded ring $H^*(X, A; R)$ acts on the graded group $H_*(X, A; R)$ via the cap product
$$H^*(X, A; R) \otimes H_*(X, A; R) \ni \alpha \otimes a \mapsto \alpha \cap a,$$
i.e. $1 \cap a = a$, $(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a)$. Thus $H_*(X, A; R)$ is a graded module over the graded ring $H^*(X, A; R)$.

Examples 2.5.12.
Many cup products are trivial for degree reasons.

1. Let $\mathbb{S}^n$ be a sphere of dimension $n \geqslant 1$. We know from Example 2.2.6.2 that $H^0(\mathbb{S}^n) \cong \mathbb{Z} \cong H^n(\mathbb{S}^n)$ and the cohomology is trivial in all other degrees. We have $1 \in H^0(\mathbb{S}^n)$ and $\nu_n \in H^n(\mathbb{S}^n)$. We know that
$$1 \cup \nu_n = \nu_n = \nu_n \cup 1 \quad \text{and} \quad 1 \cup 1 = 1$$
but $\nu_n \cup \nu_n \in H^{2n}(\mathbb{S}^n) = 0$ and thus vanishes. Thus, $H^*(\mathbb{S}^n)$ has the structure of a so-called graded exterior algebra with one generator ν_n in degree n, $H^*(\mathbb{S}^n) \cong \Lambda_{\mathbb{Z}}(\nu_n)$.

2. More generally, if X is a CW complex of finite dimension, then $\alpha \cup \beta = 0$ for all α, β for $|\alpha| + |\beta|$ big enough.

3. In particular, $H^*(X)$ often has nilpotent elements: if
$$\alpha^r := \underbrace{\alpha \cup \ldots \cup \alpha}_{r} = 0 \ ,$$
then commutativity implies $(\alpha \cup \beta)^r = \pm\alpha^r \cup \beta^r = 0$ for any $\beta \in H^*(X)$.

4. Assume that $\alpha \in H^p(X; R)$ with p odd. Then
$$\alpha^2 = (-1)^{p^2}\alpha^2 = -\alpha^2 \ ,$$
where we first used the graded commutativity 2.5.9.2. Therefore $2\alpha^2 = 0$ and if R is a field of characteristic not equal to 2 or if R is a torsionfree commutative ring, then $\alpha^2 = 0$.

5. Consider $X = X_1 \vee X_2$ and assume that X_1, X_2 are well-pointed. Then by Proposition 1.9.7
$$H^*(X) \cong H^*(X_1) \times H^*(X_2)$$
as rings. For $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ with $\alpha_i, \beta_i \in H^*(X_i)$ in positive degrees, the cup product can be shown to be
$$\alpha \cup \beta = (\alpha_1 + \alpha_2) \cup (\beta_1 + \beta_2) = \alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2.$$

6. If X can be covered like $X = X_1 \cup \ldots \cup X_r$ with $H^*(X_i) = 0$ for $* \geqslant 1$ and X_i path-connected, then in $H^*(X)$ all r-fold cup products of elements of positive degree vanish. We prove the case $r = 2$; the general claim then follows by induction. So assume $X = X_1 \cup X_2$ such that the X_i have vanishing cohomology groups in all positive degrees and let $i_j \colon X_j \hookrightarrow X$ be the inclusion of X_j into X ($j = 1, 2$). Then for all $\alpha \in H^*(X)$, $i_j^*(\alpha) = 0$. Consider the exact sequence

$$H^*(X, X_j) \longrightarrow H^*(X) \xrightarrow{i_j^*} H^*(X_j)\ .$$

Therefore, for any $\alpha \in H^*(X)$, there exists $\alpha' \in H^*(X, X_1)$ that is mapped isomorphically to α. Similarly, for $\beta \in H^*(X)$ there is an $\beta' \in H^*(X, X_2)$ that corresponds to β. The cup product $\alpha \cup \beta$ then corresponds to $\alpha' \cup \beta'$ but this is an element of $H^*(X, X_1 \cup X_2) = H^*(X, X) = 0$.

7. Consider a product of spheres, $X = \mathbb{S}^n \times \mathbb{S}^m$ with $n, m \geqslant 1$. The Künneth formula and the universal coefficient theorem imply that as an abelian group

$$H^*(\mathbb{S}^n \times \mathbb{S}^m) \cong H^*(\mathbb{S}^n) \otimes H^*(\mathbb{S}^m).$$

We have four additive generators

$$1 \times 1,\ \ \alpha_n := \nu_n \times 1,\ \ \beta_m := 1 \times \nu_m \quad \text{and} \quad \gamma_{n+m} := \nu_n \times \nu_m.$$

The square α_n^2 is trivial for degree reasons:

$$\alpha_n^2 = (\nu_n \times 1) \cup (\nu_n \times 1) = (\nu_n \cup \nu_n) \times (1 \cup 1) = 0.$$

Similarly, $\beta_m^2 = 0 = \gamma_{n+m}^2$. But the products

$$\alpha_n \cup \beta_m = \nu_n \times \nu_m = \gamma_{n+m} \quad \text{and} \quad \beta_m \cup \alpha_n = (-1)^{mn}\gamma_{n+m}$$

are non-trivial. This determines the ring structure of $H^*(\mathbb{S}^n \times \mathbb{S}^m)$.

8. Additively, as a graded abelian group, this is isomorphic to the cohomology ring $H^*(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$, which has generators $\tilde{\alpha}_n, \tilde{\beta}_m$ and $\tilde{\gamma}_{n+1}$ in degrees n, m and $n+m$. However, by 5.

$$\tilde{\alpha}_n \cup \tilde{\beta}_m = (\tilde{\alpha}_n + 0) \cup (0 + \tilde{\beta}_m) = 0 + 0 = 0$$

so that the cohomology ring $H^*(\mathbb{S}^n \times \mathbb{S}^m)$ is *not* isomorphic to the cohomology ring $H^*(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$ as a ring. Thus the graded cohomology ring is a finer invariant than the cohomology groups.

Note that the cohomology rings of the suspensions $\Sigma(\mathbb{S}^n \times \mathbb{S}^m)$ and $\Sigma(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$ *are* isomorphic (cf. exercise). But here, we actually have

$$\Sigma(\mathbb{S}^n \times \mathbb{S}^m) \simeq \Sigma(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m}).$$

2.6 Orientability of manifolds

We now consider topological spaces with more properties.

Definition 2.6.1
A topological space X is called locally euclidean of dimension m, if every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset $V \subset \mathbb{R}^m$.

- *A homeomorphism $\varphi\colon M \supset U \to V \subset \mathbb{R}^m$ is called a* <u>*chart*</u>*.*

- *A set of charts is called* <u>*atlas*</u>*, if the corresponding $U \subset X$ cover X.*

Example 2.6.2.
Consider the line with two origins, i.e.

$$X = \{(x,1)|x \in \mathbb{R}\} \cup \{(x,-1)|x \in \mathbb{R}\}/\sim, \quad (x,1) \sim (x,-1) \text{ for } x \neq 0.$$

Then X is locally euclidean, but X is not a particularly nice space. For instance, it is not Hausdorff: one cannot separate the two origins.

Definition 2.6.3
A topological space X is an m-dimensional (topological) <u>*manifold*</u> *(or m-manifold for short) if X is a locally euclidean space of dimension m that is Hausdorff and has a countable basis for its topology.*

With this definition, topological manifolds are paracompact, i.e. every open cover has a locally finite open refinement.

Examples 2.6.4.

1. Let $U \subset \mathbb{R}^m$ an open subset, then U is a topological manifold of dimension m.
2. The n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an n-manifold and $\mathbb{S}^n = (\mathbb{S}^n \setminus N) \cup (\mathbb{S}^n \setminus S)$ is an atlas of $\mathbb{S}^n$.
3. The 2-dimensional torus $T \cong \mathbb{S}^1 \times \mathbb{S}^1$ is a 2-manifold and more generally, the surfaces F_g are 2-manifolds. Charts can be easily given via the $4g$-gon whose quotient F_g is.
4. The open Möbius strip $[-1,1] \times (-1,1)/\sim$ with $(-1,t) \sim (1,-t)$ for $-1 < t < 1$ is a 2-manifold.

Let M be a connected manifold of dimension $m \geqslant 2$. We denote the open charts by $U_\alpha \subset M$. Without loss of generality we can assume that the coordinate patches are homeomorphic to open balls in $\mathbb{R}^m$:

$$\varphi\colon U_\alpha \xrightarrow{\sim} \mathring{\mathbb{D}}^m \subset \mathbb{R}^m .$$

For any $x \in M$, we can find a chart $\varphi : U_x \xrightarrow{\sim} \mathring{\mathbb{D}}^m$ with $\varphi(x) = 0$. Excision for $(M \setminus U_x) \subset (M \setminus \{x\}) \subset M$ tells us that for all $x \in M$

$$H_m(M, M \setminus \{x\}) \cong H_m(U_x, U_x \setminus \{x\}) \cong H_m(\mathring{\mathbb{D}}^m, \mathring{\mathbb{D}}^m \setminus \{0\}) \cong H_{m-1}(\mathring{\mathbb{D}}^m \setminus \{0\}) \cong \mathbb{Z}$$

for $m \geqslant 2$. Here the chart φ was used for the second isomorphism. Since we have fixed in Definition 1.8.4 a generator in $H_{m-1}(\mathring{\mathbb{D}}^m \setminus \{0\}) = H_{m-1}(\mathbb{S}^{m-1})$, any chart provides us with a generator in $H_m(M, M \setminus \{x\})$.

For a triple $B \subset A \subset M$, there are maps of pairs

$$\varrho_{B,A}\colon (M, M \setminus A) \longrightarrow (M, M \setminus B) .$$

In particular, for $x \in U \subset M$, we get a map of pairs

$$\varrho_{x,U}\colon (M, M \setminus U) \to (M, M \setminus \{x\}) .$$

Definition 2.6.5
Let M be an m-manifold.

1. *A choice of generators $o_x \in H_m(M, M \setminus \{x\})$ for all $x \in M$ is called coherent, if for all $x \in M$ there is an open neighbourhood U of x and a class $o_U \in H_m(M, M \setminus U)$ such that for all $y \in U$ we have that $(\varrho_{y,U})_* o_U = o_y$.*

2. *An m-manifold M is called orientable (with respect to homology with values in $\mathbb{Z}$), if there exists a coherent choice of generators $o_x \in H_m(M, M \setminus \{x\})$.*

3. *If such a choice is possible, then the family $(o_x | x \in M)$ is called an orientation of M.*

In the sequel, we will write $\varrho_{x,U}$ also for the map on homology, i.e. we drop the lower star in $(\varrho_{x_2,U})_*$.

Remarks 2.6.6.

1. Assume that U is a small ball in M so that $(\varrho_{x,U})_* : H_m(M, M \setminus U) \to H_m(M, M \setminus \{x\})$ is an isomorphism for each $x \in U$. For a coherent choice of generators, we have for all $x_1, x_2 \in U$ the compatibility condition

$$o_{x_2} = \varrho_{x_2,U} \circ (\varrho_{x_1,U})^{-1}(o_{x_1}).$$

$$\begin{array}{ccccc} & & H_m(M, M \setminus U) & & \\ & \swarrow^{\varrho_{x_1,U}} & & \searrow^{\varrho_{x_2,U}} & \\ o_{x_1} \in H_m(M, M \setminus \{x_1\}) & & & & H_m(M, M \setminus \{x_2\}) \ni o_{x_2} \end{array}$$

2. Given an orientation $(o_x | x \in M)$, the family $(-o_x | x \in M)$ is an orientation of M as well.

Example 2.6.7.

1. If M is the open Möbius strip and you pick a generator $o_x \in H_2(M, M \setminus \{x\})$ and you walk once around the Möbius strip, you end up at $-o_x$.

2. If we choose other coefficients, these problems can disappear. For instance for $G = \mathbb{Z}/2\mathbb{Z}$ there is no problem to choose coherent generators for $H_2(M, M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, so the Möbius strip is $\mathbb{Z}/2\mathbb{Z}$-orientable. In general, $\mathbb{Z}$-orientability implies $\mathbb{Z}_p$ orientability. The converse holds for p an odd prime.

   Orientability can also be considered for more general homology theories.

3. Now, we consider integral coefficients again. Suppose that the family $(o_x | x \in M)$ is an orientation of M. In this case, we want to obtain a global class $o_M \in H_m(M; \mathbb{Z}) = H_m(M)$, an orientation class, that determines the orientation in the sense that

$$\varrho_{x,M} =: \varrho_x : H_m(M) \to H_m(M, M \setminus \{x\}), \quad \varrho_x(o_M) = o_x \ .$$

   For example, for $\mathbb{R}P^2$, we have $H_2(\mathbb{R}P^2) = 0$ by Example 1.9.6 and cannot have such a class.

For questions of orientability, compact subsets play a particularly important role. We will derive a global characterization of orientability for M compact in Theorem 2.6.11.

Definition 2.6.8

Let $K \subset M$ be a compact subset of M. We call a class $o_K \in H_m(M, M \setminus K)$ an orientation of

M along K, if the collection of classes $o_x := (\varrho_{x,K})_*(o_K)$ for all $x \in K$ constitutes a coherent choice of generators for all $x \in K$.

Clearly, if we have a global class $o_M \in H_m(M)$, then we get coherent generators o_x for all $x \in M$ and also a class $o_K = (\varrho_{K,M})_*(o_M)$ as above for all compact $K \subset M$.

Lemma 2.6.9.
Let M be a connected topological manifold of dimension m and assume that M is orientable. Let $K \subset M$ be compact.

1. Then $H_q(M, M \setminus K) = 0$ for all $q > m$
2. Let $a \in H_m(M, M \setminus K)$. Then a is trivial, if and only if $(\varrho_{x,K})_*(a) = 0$ for all $x \in K$.

In particular, if M is compact, then $H_q(M, M \setminus M) = H_q(M) = 0$ for $q > m$.
The following method of proof is a standard method in the theory of manifolds.

Proof.

1. We first show that, if the two claims hold for compact subsets $A, B \subset M$ and for $A \cap B$, then they hold for the union $A \cup B$.

   Consider the following part of a relative Mayer-Vietoris sequence, cf. Theorem 1.8.6:

   $$\begin{aligned} 0 &= H_{n+1}(M, M \setminus (A \cap B)) \to H_n(M, M \setminus (A \cup B)) \stackrel{\Phi}{\to} \\ &\quad H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \stackrel{\Psi}{\to} H_n(M, M \setminus A \cap B) \end{aligned}$$

   For $n \geqslant m$, the leftmost zero comes from our assumption 1. on $A \cap B$. All terms $H_i(M \setminus (A \cup B))$ with $i > m$ appear between terms equalling zero, hence are zero. This shows the first claim for $A \cup B$. If a class $\alpha \in H_m(M \setminus A \cup B)$ has zero image in all $H_m(M, M \setminus \{x\})$, then its images under Φ, as restrictions, have the same property, hence are zero by assumption.

2. First, consider the special case when $M = \mathbb{R}^m$ and K is *convex* and compact in M. In this case we can assume without loss of generality that $K \subset \mathring{\mathbb{D}}^m$. We calculate

   $$H_q(M, M \setminus K) = H_q(\mathbb{R}^m, \mathbb{R}^m \setminus K) \cong H_q(\mathring{\mathbb{D}}^m, \mathring{\mathbb{D}}^m \setminus x) = 0, \text{ for } q > m.$$

   All identifications are isomorphisms also for $q = m$ and this gives the second claim as well.

3. Using the statement in 1. and induction shows that claim for the case when $M = \mathbb{R}^m$ and $K = K_1 \cup \ldots \cup K_r$ is a union with K_i convex and compact, as in 2.

4. Let $M = \mathbb{R}^m$ and let K be an arbitrary compact subset and let $a \in H_q(M, M \setminus K)$ with $q > m$. Choose a chain $\psi \in S_q(\mathbb{R}^m)$ representing the class a. The boundary of ψ, $\partial(\psi)$, has to be of the form

   $$\partial(\psi) = \sum_{j=1}^{\ell} \lambda_j \tau_j$$

   with finitely many $(q-1)$-simplices $\tau_j \colon \Delta^{q-1} \to \mathbb{R}^m \setminus K$ with values in $\mathbb{R}^m \setminus K$. As the standard simplex Δ^{q-1} is compact, the union of the images

   $$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \subset \mathbb{R}^m \setminus K$$

is compact as well.

Hence, there exists an open neighborhood U of the compact subset K in $\mathbb{R}^m$ that does not meet the simplices:

$$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \cap U = \varnothing.$$

Therefore, the specific q-chain ψ on $\mathbb{R}^m$ also defines a cycle in $S_*(\mathbb{R}^m, \mathbb{R}^m \setminus U)$; let $a' \in H_q(\mathbb{R}^m, \mathbb{R}^m \setminus U)$ be the corresponding class. Since the classes a and a' are defined by the same cycle, we have

$$(\varrho_{K,U})_*(a') = a.$$

To get compact convex subsets as in 3., choose finitely many closed balls $B_1, \ldots, B_r \subset \mathbb{R}^m$ with $B_i \subset U$ for all i and $K \cap B_i \neq \varnothing$ such that $K \subset \bigcup_{i=1}^r B_i$. Consider the chain of restriction maps

$$(\mathbb{R}^m, \mathbb{R}^m \setminus U) \xrightarrow{\varrho_{\bigcup B_i, U}} (\mathbb{R}^m, \mathbb{R}^m \setminus \textstyle\bigcup_{i=1}^r B_i) \xrightarrow{\varrho_{K, \bigcup B_i}} (\mathbb{R}^m, \mathbb{R}^m \setminus K).$$

Define a'' as $a'' := (\varrho_{\bigcup B_i, U})_*(a')$. Note that $(\varrho_{K, \bigcup B_i})_*(a'') = a$.

The closed balls B_i are convex and compact subsets of $\mathbb{R}^m$ and therefore by 3.

$$(\varrho_{\bigcup B_i, U})_*(a') = 0 = a'', \text{ for all } q > m$$

and hence also $a = 0$. This shows the first claim for all compact subsets of $\mathbb{R}^m$.

For the second claim, let $q = m$ and assume that $(\varrho_{x,K})_*(a) = 0$ for all $x \in K$. We have to show that a is trivial. We express $(\varrho_{x,K})_*(a)$ as above as

$$(\varrho_{x,K})_*(a) = (\varrho_{x,K})_* \circ (\varrho_{K, \bigcup B_i})_*(a'') = (\varrho_{x, \bigcup B_i})_*(a'') = 0$$

for all $x \in K$. For every $x \in B_j \cap K$ the above composition is equal to $(\varrho_{x,B_j})_* \circ (\varrho_{B_j, \bigcup B_i})_*(a'')$, but $(\varrho_{x,B_j})_*$ is an isomorphism and hence $(\varrho_{B_j, \bigcup B_i})_*(a'') = 0$. This implies $(\varrho_{y,B_j})_* \circ (\varrho_{B_j, \bigcup B_i})_*(a'') = 0$ for all $y \in B_j$ and in addition $(\varrho_{y, \bigcup B_i})_*(a'') = 0$ for all $y \in \bigcup B_i$. According to case 3., this implies that $a'' = 0$ and therefore $a = (\varrho_{K, \bigcup B_i})_*(a'')$ is trivial as well.

5. Now let M be an arbitrary manifold. Suppose that the compact subset K is contained in a domain U_α of a chart, i.e. $K \subset U_\alpha \cong \mathbb{R}^m$. Therefore, by excision for $M \setminus U_\alpha \subset M \setminus K \subset M$

$$H_q(M, M \setminus K) \cong H_q(U_\alpha, U_\alpha \setminus K) \cong H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \mathrm{im}(K)).$$

As the image of K is compact in $\mathbb{R}^m$, the claim follows from 4.

6. If both the manifold M and the compact subset K are arbitrary, then write $K = K_{\alpha_1} \cup \ldots \cup K_{\alpha_r}$ and each K_{α_i} contained in the domain U_{α_i} of a chart. Then 5. and 1. imply the claim.

□

Proposition 2.6.10.
Assume that M is oriented with $(o_x \in H_m(M, M \setminus \{x\} \mid x \in M)$. Let $K \subset M$ be any compact subset. Then there is a unique orientation of M along K, which is compatible with the orientation of M, i.e. there is a unique class $o_K \in H_m(M, M \setminus K)$ such that $(\varrho_{x,K})_*(o_K) = o_x$ for all $x \in K$.

Proof.

- First we show uniqueness. Let o_K and $\tilde{o}_K$ be two orientations of M along K. By assumption we have that

$$(\varrho_{x,K})_*(o_K) - (\varrho_{x,K})_*(\tilde{o}_K) = (\varrho_{x,K})_*(o_K - \tilde{o}_K) \;;$$

  on the other hand, this equals $o_x - o_x = 0$. According to Lemma 2.6.9.2 this is only the case if $o_K - \tilde{o}_K = 0$.

- In order to prove the existence of o_K we first consider the case where $K \subset U_\alpha \cong \mathring{\mathbb{D}}^m$ and hence $M \setminus U_\alpha \subset M \setminus K$. Let $x \in K$. We denote the isomorphism $H_m(M, M \setminus U_\alpha) \cong H_m(M, M \setminus \{x\})$ by ϕ_x.

  We define o_K as

$$o_K := (\varrho_{K,U_\alpha})_*((\phi_x^{-1})(o_x)).$$

- For $K = K_1 \cup K_2$, with K_i contained in the codomain of a chart, the previous argument ensures the existence of classes $o_{K_1} \in H_m(M, M \setminus K_1)$ and $o_{K_2} \in H_m(M, M \setminus K_2)$. Let $K_0 = K_1 \cap K_2$ and consider the Mayer-Vietoris sequence

$$0 \to H_m(M, M \setminus K) \xrightarrow{i} H_m(M, M \setminus K_1) \oplus H_m(M, M \setminus K_2) \xrightarrow{\kappa} H_m(M, M \setminus K_0) \to \dots$$

  The uniqueness of the orientation along the intersection K_0 implies that

$$\kappa(o_{K_1}, o_{K_2}) = (\varrho_{K_0,K_1})_*(o_{K_1}) - (\varrho_{K_0,K_2})_*(o_{K_2}) = 0.$$

  By exactness, there is a unique class $o_K \in H_m(M, M \setminus K)$ with $i(o_K) = (o_{K_1}, o_{K_2})$.

- For the general case we consider a compact subset K and we know that $K = K_1 \cup \ldots \cup K_r$ with $K_i \subset U_{\alpha_i}$. An induction then finishes the proof.

□

Theorem 2.6.11.
Let M be a connected and *compact* manifold of dimension m. The following statements are equivalent:

1. M is orientable,
2. There is an orientation class $o_M \in H_m(M; \mathbb{Z})$,
3. $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof.

- Proposition 2.6.10 yields that (1) implies (2).

- Now assume that (2) holds, thus there is a class $o_M \in H_m(M)$ restricting to the local orientation classes o_x. Then the class o_M cannot be trivial, because its restriction $(\varrho_{x,M})_* o_M = o_x$ is a generator and hence non-trivial. Furthermore, o_M cannot be of finite order: the relation $k o_M = 0$ implies $k o_x = 0$ for all $x \in M$, contradicting the property that o_x generates the free abelian group $H_m(M, M \setminus \{x\})$.

Let $a \in H_m(M)$ be an arbitrary element. Thus $(\varrho_{x,M})_*(a) = ko_x$ for some integer k. As the o_x are coherent in x, this k has to be constant on the connected manifold M. We let $b := ko_M - a$ and find $(\varrho_{x,M})_* b = 0$ for all x. Since M is compact, Lemma 2.6.9 implies that $b = 0$. Therefore $a = ko_M$, thus every element in $H_m(M)$ is a multiple of o_M and $H_m(M) \cong \mathbb{Z}$.

- Assuming (3), there are two possible generators in $H_m(M)$. Choose one of them and call it o_M. Then $((\varrho_{x,M})_* o_M | x \in M)$ is an orientation of M.

□

Definition 2.6.12
Let M be a compact, connected, orientable manifold. Given an orientation on M, the class o_M as in Theorem 2.6.11 is also called fundamental class of M and is often denoted by $[M] = o_M$.

Example 2.6.13.
For the m-sphere, $M = \mathbb{S}^m$ we can choose $\mu_m \in H_m(\mathbb{S}^m)$ as in Definition 1.8.4 as a generator. Thus

$$[\mathbb{S}^m] = o_{\mathbb{S}^m} = \mu_m.$$

In particular, spheres are orientable. It follows from Theorem 2.6.11 and Example 1.12.8 that $\mathbb{R}P^n$ is orientable, iff n is odd.

Remarks 2.6.14.
All results about orientations can be transferred to a setting with coefficients in a commutative ring R with unit 1_R.

1. Then M is called R-orientable if and only if there is a coherent choice of generators of the group $H_m(M, M \setminus x; R) \cong R$ for all $x \in M$.

2. Suppose M is a compact manifold. If M is not R-orientable, the map $H_n(M; R) \to H_n(M, M \setminus \{x\}; R) \cong R$ is injective for all $x \in M$ with image $\{r \in R \,|\, 2r = 0\}$, cf. Hatcher, Theorem 3.26 p. 236. In particular, for $R = \mathbb{Z}$, M is not orientable, if and only if $H_n(M; \mathbb{Z}) = 0$.

3. The results we obtained have formulations relative R: Lemma 2.6.9 goes through, and if M has an R-orientation $(o_x^R | x \in M)$, then for all compact $K \subset M$ there is an R-orientation of M along K, i.e. a class $o_K^R \in H_m(M, M \setminus K; R)$ that restricts to the local classes. The R-version of Theorem 2.6.11 yields for a compact manifold M a class $o_M^R \in H_m(M; R)$ restricting to the o_x^R. The class o_M^R is then called the fundamental class of M with respect to R and is denoted by $[M; R]$.

Returning to integral coefficients, we know from Theorem 2.6.11 that for compact connected orientable manifolds of the same dimension we get a copy of the integers in the homology of the highest degree, with the fundamental class as a generator. This motivates the following definition:

Definition 2.6.15
Let M and N be two oriented compact connected manifolds of the same dimension $m \geqslant 1$ and let $f\colon M \to N$ be continuous. Then the degree of f is the integer $\deg(f)$ that is given by

$$H_m(f)[M] = \deg(f)[N].$$

Of course, this definition extends the notion of the degree of a map we introduced in Definition 1.10.1 for self-maps of spheres.

Proposition 2.6.16.
Let M, N_1, N_2 be oriented compact connected manifolds and let $f\colon M \to N_1$ and $g\colon N_1 \to N_2$ be continuous maps.

1. The degree is multiplicative,

$$\deg(g \circ f) = \deg(g)\deg(f).$$

2. If $\bar{M}$ is the same manifold as M but with opposite orientation, then

$$\deg(f) = \deg(f\colon \bar{M} \to \bar{N}_1) = -\deg(f\colon \bar{M} \to N_1) = -\deg(f\colon M \to \bar{N}_1).$$

3. If the degree of f is not trivial, then f is surjective.

Proof.
The first claim follows directly from the definition of the degree. For the second claim, note that $[\bar{M}] = -[M]$, because we need to have

$$(\varrho_{x,M})_*[\bar{M}] = -o_x$$

if $(o_x|x \in M)$ is the given orientation of M.

For (3) assume that f is not surjective, thus there is a point $y \in N$, that is not contained in the image of M under f. Consider the composition

$$H_m(M) \xrightarrow{H_m(f)} H_m(N) \xrightarrow{(\varrho_{y,N})_*} H_m(N, N \setminus y).$$

This composition is trivial, since $y \notin \mathrm{im}(f)$. On the other hand $(\varrho_{y,N})_*$ is an isomorphism. Hence $H_m(f) = 0$ and f has trivial degree. □

2.7 Cohomology with compact support

So far, orientation theory works fine if we restrict our attention to compact manifolds. We are aiming at *Poincaré duality*: if M is a compact connected oriented manifold of dimension m, then taking the cap product with the orientation class $[M] = o_M$ gives a map

$$(-) \cap o_M\colon H^q(M) \to H_{m-q}(M).$$

Our aim is to show that this gives an isomorphism, but we also want to extend the result to non-compact manifolds M. To this end we start with the following:

Definition 2.7.1
Let X be an arbitrary topological space and let R be a commutative ring with unit 1_R.

1. *Then the singular n-cochains with compact support singular cochains with compact support are*

$$S_c^n(X;R) = \{\varphi\colon S_n(X) \to R | \exists K_\varphi \subset X \text{ compact}, \varphi(\sigma) = 0$$
$$\text{for all singular simplices } \sigma\colon \Delta^n \to X \text{ with } \sigma(\Delta^n) \cap K_\varphi = \varnothing.\}$$

2. *The nth cohomology group with compact support of X with coefficients in R is*

$$H_c^n(X;R) := H^n(S_c^*(X;R)) \ .$$

Remarks 2.7.2.

1. The condition of compact support is formulated in a weak sense. One could have imagined to restrict to cochains φ that are non-zero only on simplices contained in a given compact subset K_φ depending on φ. Then, however, a differential cannot be defined: if φ is a 0-cochain on $\mathbb{R}$ assigning non-zero value only to the 0-simplex contained in $x = 0$, then its differential assigns non-zero values to arbitrarily large 1-simplices, i.e. all those starting or ending in $x = 0$.

2. Note that $S_c^*(X;R) \subset S^*(X;R)$ is a sub-complex. This inclusion of complexes induces a map on cohomology
$$H_c^n(X;R) \longrightarrow H^n(X;R).$$
If X is compact, then obviously $H_c^n(X;R) \cong H^n(X;R)$ for all n.

   A map from singular cohomology to singular cohomology with compact support is much more subtle; indeed, we only get in Proposition 2.7.7 a map from a collection of relative singular cohomologies, involving all the compact subsets of a space.

Observation 2.7.3.

1. Let $K \subset X$ be compact. The map of pairs
$$\varrho_{K,X} \colon (X, X \setminus X) = (X, \varnothing) \longrightarrow (X, X \setminus K)$$
induces a map of cochain complexes
$$\varrho_{K,X}^n \colon S^n(X, X \setminus K; R) \longrightarrow S^n(X;R) \ .$$
We claim that the image of $\varrho_{K,X}^n$ is contained in $S_c^n(X;R)$. Indeed, for an n-cochain $\varphi \in S^n(X;R)$ in the image, there exists $\psi \in S^n(X, X \setminus K; R)$ with $\varrho_{K,X}^n(\psi) = \varphi$. The functional ψ is trivial on all simplices $\sigma \colon \Delta^n \to X$ with $\sigma(\Delta^n) \cap K = \varnothing$. Therefore, for such a simplex σ
$$\varphi(\sigma) = \varrho_{K,X}^n(\psi)(\sigma) = 0 \ .$$

2. For compact subsets $K \subset L \subset X$ we have maps of pairs
$$(X, X \setminus X) \stackrel{\varrho_{L,X}}{\rightarrow} (X, X \setminus L) \stackrel{\varrho_{K,L}}{\rightarrow} (X, X \setminus K)$$
such that $\varrho_{K,L} \circ \varrho_{L,X} = \varrho_{K,X}$.

We summarize:

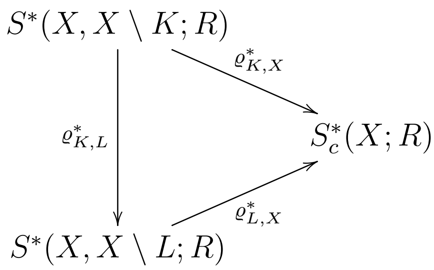
Lemma 2.7.4.
Let X be a topological space.

1. For any compact subset $K \subset X$, the map $\varrho_{K,X}^*$ gives a cochain map $S^*(X, X \setminus K; R) \longrightarrow S_c^*(X;R)$. In particular we get an induced map
$$H^*(\varrho_{K,X}) \colon H^*(X, X \setminus K; R) \longrightarrow H_c^*(X;R).$$

2. For compact subsets $K \subset L \subset X$ we have

$$\varrho_{K,L} \circ \varrho_{L,X} = \varrho_{K,X}$$

and therefore the diagram of cochain complexes:



commutes.

Remarks 2.7.5.

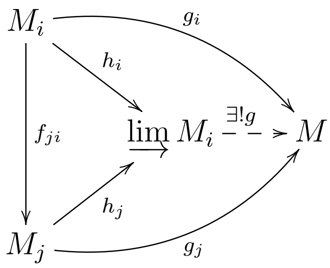
1. Recall that a poset I is called directed, if for all $i, j \in I$ there is a $k \in I$ with $i, j \leqslant k$
2. The compact subsets of a space X form a directed system: if $K \subset X$ and $L \subset X$ are compact, both are subsets of the compact subset $K \cup L \subset X$.
3. Given a poset I, we can consider diagrams (of modules, of abelian groups, of chain complexes) of the shape I: for each $i \in I$, there is an object M_i and for all $i \leqslant j$ there is a map $f_{ji}\colon M_i \to M_j$; with $f_{kj} \circ f_{ji} = f_{ki}$ for $i \leqslant j \leqslant k$ and $f_{ii} = \mathrm{id}_{M_i}$ for all i. If I is directed, then we call the system $(M_i)_{i \in I}$ a directed system.
4. Lemma 2.7.4 says that the system $K \mapsto S^*(X, X \setminus K; R)$ is a *direct system of cochain complexes*: For $K \subset L \subset L'$ we have

$$\varrho^*_{K,L'} = \varrho^*_{L,L'} \circ \varrho^*_{K,L} \ .$$

5. We recall some facts about the direct limit of a direct system $(M_i)_{i \in I}$ of R-modules and of (co)chain complexes of R-modules.

   The direct limit $\varinjlim M_i$ of a direct system (M_i) is an R-module $\varinjlim M_i$, together with a family of maps $(h_i\colon M_i \to \varinjlim M_i)_{i \in I}$ with the following universal property: for every family of R-module maps $g_i\colon M_i \to M$ that satisfy $g_j \circ f_{ji} = g_i$ for all $i \leqslant j$, there is a unique morphism of R-modules $g\colon \varinjlim M_i \to M$ such that $g \circ h_i = g_i$ for all $i \in I$.

   As a commuting diagram:



   This universal property determines the R-module $\varinjlim M_i$ up to unique isomorphism.

6. For a direct system $(M_i, i \in I)$ of R-modules we can construct the direct limit $\varinjlim M_i$ as

$$\varinjlim M_i = \left(\bigoplus_{i \in I} M_i \right) / U$$

where U is the submodule of $\bigoplus_{i\in I} M_i$ generated by all differences of the form $m_i - f_{ji}(m_i), i \leqslant j$. The map $g_j : M_j \to \varinjlim M_i$ is the composition of the injection for the direct sum, followed by the canonical projection to the quotient.

7. We need an explicit construction of the direct limit of a direct system $((C_i)_*)_{i\in I}$ of (co)chain complexes: we write $L := \varinjlim(C_i)$. In degree n, we set

$$L_n := \varinjlim((C_i)_n) \ .$$

All diagrams constructed from the boundary operators

$$\begin{array}{ccccc} (C_i)_n & \xrightarrow{d_i} & (C_i)_{n-1} & & \\ \downarrow{\scriptstyle f_{ji}} & & \downarrow{\scriptstyle f_{ji}} & \searrow & \\ & & & & L_{n-1} \\ & & & \nearrow & \\ (C_j)_n & \xrightarrow[d_j]{} & (C_j)_{n-1} & & \end{array}$$

commute and thus, by the universal property of L_n, induce a map

$$\begin{array}{ccccc} (C_i)_n & & \longrightarrow & & \\ & \searrow & & \searrow & \\ \downarrow{\scriptstyle f_{ji}} & & L_n & \overset{d}{\dashrightarrow} & L_{n-1} \\ & \nearrow & & \nearrow & \\ (C_j)_n & & \longrightarrow & & \end{array}$$

This gives a boundary map

$$d\colon L_n = (\varinjlim(C_i))_n \longrightarrow (\varinjlim(C_i))_{n-1} = L_{n-1} \ .$$

More generally, any morphism of a directed system induces a morphism between the direct limits.

8. Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ be three direct systems of R-modules. If

$$0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$$

is a short exact sequence for all $i \in I$ and if $f_{ji}^B \circ \phi_i = \phi_j \circ f_{ji}^A$, $f_{ji}^C \circ \psi_i = \psi_j \circ f_{ji}^B$ for all $i \leqslant j$, then we call

$$0 \to (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \to 0$$

a short exact sequence of direct systems.

The composition of the map $\phi_i\colon A_i \to B_i$ with $h_i\colon B_i \to \varinjlim B_i$ gives maps $A_i \to \varinjlim B_i$. These maps yield, by the universal property of $\varinjlim A_i$, a unique map

$$\phi\colon \varinjlim A_i \longrightarrow \varinjlim B_i \ .$$

Similarly, we get a map $\psi : \varinjlim B_i \to \varinjlim C_i$.

Lemma 2.7.6.

1. If
$$0 \to (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \to 0$$
is a short exact sequence of directed systems of R-modules, then the sequence of R-modules
$$0 \to \varinjlim A_i \xrightarrow{\phi} \varinjlim B_i \xrightarrow{\psi} \varinjlim C_i \to 0$$
is short exact.

2. If $(A_i)_{i \in I}$ is a directed system of chain complexes, then
$$\varinjlim H_m(A_i) \cong H_m(\varinjlim A_i).$$

Proof.
One has to show that i) ϕ is injective, ii) the kernel of ψ is the image of ϕ and iii) ψ is surjective. We show i) and leave ii) and iii) and the second assertion as an exercise.

Let $a \in \varinjlim A_i$ with $\phi(a) = 0 \in \varinjlim B_i$. Write $a = [\sum_{j=1}^n \lambda_j a_j]$ with $a_j \in A_{i_j}$. Choose $k \geqslant i_1, \ldots, i_n$, then $a = [a_k]$ for some $a_k \in A_k$. By assumption $\phi(a) = [\phi_k(a_k)] = 0$. Thus there is an $N \geqslant k$ with $f^B_{Nk}\phi_k(a_k) = 0$ and by the fact that the families ϕ_k are maps of directed systems, we have $0 = f^B_{Nk} \circ \phi_k(a_k) = \phi_N \circ f^A_{Nk}(a_k)$. But ϕ_N is a monomorphism and therefore $f^A_{Nk}(a_k) = 0 \in \varinjlim A_i$, hence $a = [a_k] = [f^A_{Nk}(a_k)] = 0$. □

We can use this algebraic result to approximate singular cohomology with compact support via relative singular cohomology groups.

Proposition 2.7.7.
For all spaces X we have isomorphisms
$$\varinjlim S^*(X, X \setminus K; R) \xrightarrow{\cong} S^*_c(X; R)$$
and hence
$$\varinjlim H^*(X, X \setminus K; R) \xrightarrow{\cong} H^*_c(X; R).$$
Here the directed system runs over the poset of compact subsets $K \subset X$.

Proof.
By the universal property of $\varinjlim S^*(X, X \setminus K; R)$, the chain maps
$$\varrho^*_{K,X} \colon S^*(X, X \setminus K; R) \longrightarrow S^*_c(X; R)$$
from Lemma 2.7.4 combine into a single chain map
$$\varinjlim S^*(X, X \setminus K; R) \longrightarrow S^*_c(X; R)$$

A cochain $\varphi \in S^n(X; R)$ is an element of $S^n_c(X; R)$, if and only if there is a compact $K = K_\varphi$ such that $\varphi(\sigma) = 0$ for all σ with $\sigma(\Delta^n) \cap K = \varnothing$ and this is the case if and only if $\varphi \in S^n(X, X \setminus K; R)$. This shows that the map is surjective. Injectivity is direct. Then apply Lemma 2.7.6.2. □

To the eyes of compact cohomology, $\mathbb{R}^m$ looks like a sphere:

Proposition 2.7.8.

$$H_c^*(\mathbb{R}^m; R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) \cong \begin{cases} R, & * = m, \\ 0, & * \neq m. \end{cases}$$

Proof.
If $K \subset \mathbb{R}^m$ is compact, then there is a closed ball $B_{r_K}(0)$ of radius r_K around the origin, with $K \subset B_{r_K}(0)$. Without loss of generality we can assume that r_K is a natural number. Thus we can take the direct limit over the subsystem of such balls:

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus K; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R)$$

where the direct system on the right runs over all natural numbers $r \in \mathbb{N}$. But

$$H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R)$$

for all r and the diagrams

$$\begin{array}{ccc} H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) & \longrightarrow & H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_{r+1}(0); R) \\ \downarrow & & \downarrow \\ H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) & \xrightarrow{\text{id}} & H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) \end{array}$$

commute. Therefore

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R)$$

is an isomorphism, but the system on the right is constant and therefore

$$H_c^*(\mathbb{R}^m; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R).$$

□

Note that $\mathbb{R}^m$ is homotopy equivalent to a one-point space which is compact and for which compactly supported cohomology and ordinary cohomology coincide, cf. Remark 2.7.3.2. Thus cohomology with compact support is not homotopy invariant; it cannot be characterized by axioms of Eilenberg-Steenrod type.

2.8 Poincaré duality

Observation 2.8.1.
Let R be a commutative unital ring.

- Let M be a connected m-dimensional manifold with an R-orientation $(o_x | x \in M)$. For a compact subset $L \subset M$, following Proposition 2.6.10, let $o_L \in H_m(M, M \setminus L)$ be the orientation of M along L. For an inclusion $K \subset L$ of compact subsets, we have that

  $$(\varrho_{K,L})_*(o_L) = o_K$$

  because $(\varrho_{x,K})_*(o_K) = o_x = (\varrho_{x,L})_*(o_L) = (\varrho_{x,K})_* \circ (\varrho_{K,L})_*(o_L)$ for all $x \in K$ and, by Lemma 2.6.9.2, the class o_K is uniquely characterized by this property.

- Consider for any compact subset $K \subset M$ the cap-product

$$\begin{array}{rcl}(-) \cap o_K \colon H^{m-p}(M, M \setminus K; R) & \longrightarrow & H_p(M; R) \\ \alpha & \mapsto & \alpha \cap o_K = F(o_K) \otimes \langle \alpha, R(o_K) \rangle \ .\end{array}$$

  For an inclusion $K \subset L$ of compact subsets, we have for $\alpha \in H^{m-p}(M, M \setminus K; R)$ that $(\varrho_{K,L})^*(\alpha) \in H^{m-p}(M, M \setminus L; R)$ and

$$(\varrho_{K,L})^*(\alpha) \cap o_L = \alpha \cap (\varrho_{K,L})_*(o_L) = \alpha \cap o_K \in H_p(M; R) \ .$$

  because by Proposition 2.4.4.2 the cap product is natural.

  By the universal property, the maps produced by the cap products combine into a map

$$\varinjlim(- \cap o_K) \colon \varinjlim H^{m-p}(M, M \setminus K; R) \overset{2.7.7}{=} H_c^{m-p}(M; R) \longrightarrow H_p(M; R).$$

Definition 2.8.2
Let M be a connected m-manifold with R-orientation $(o_x | x \in M)$. The map

$$\varinjlim(- \cap o_K) \colon H_c^{m-p}(M; R) \to H_p(M; R)$$

is called Poincaré duality map and is denoted by PD *or* PD_M.

We can now state the main result of this section:

Theorem 2.8.3 (Poincaré Duality).
Let M be a connected m-manifold with R-orientation $(o_x | x \in M)$. Then the Poincaré duality map

$$\mathrm{PD} \colon H_c^{m-p}(M; R) \longrightarrow H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Corollary 2.8.4 (Poincaré duality for compact manifolds).
Let M be a connected compact manifold of dimension m with an R-orientation $(o_x | x \in M)$ and let $[M] = o_M$ be the fundamental class of M, then

$$\mathrm{PD} = (-) \cap [M] \colon H^{m-p}(M; R) \longrightarrow H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$. In particular, we have for a compact connected R-oriented m-manifold $H^m(M, R) = H_0(M; R) = R$.

Example 2.8.5.
Any connected compact manifold of dimension m possesses a $\mathbb{Z}/2\mathbb{Z}$-orientation and thus a fundamental class $o_M^{\mathbb{Z}/2\mathbb{Z}} \in H_m(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and thus for all p

$$(-) \cap o_M^{\mathbb{Z}/2\mathbb{Z}} \colon H^{m-p}(M; \mathbb{Z}/2\mathbb{Z}) \cong H_p(M; \mathbb{Z}/2\mathbb{Z}).$$

For instance the cohomology of $\mathbb{R}P^n$ and its homology satisfy Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$-coefficients, regardless of the parity of n.

Proof. **of Theorem 2.8.3**

1. First we consider the case of $M = \mathbb{R}^m$. We know from Proposition 2.7.8 that

$$H_c^{m-p}(\mathbb{R}^m) \cong \begin{cases} R, & p = 0, \\ 0, & p \neq 0 \end{cases}$$

and this is isomorphic to $H_p(\mathbb{R}^m; R)$. Therefore, abstractly, both graded R-modules are isomorphic. Let B_r be the closed r-ball centered at the origin. We have to understand the map

$$(-) \cap o_{B_r} \colon H_c^m(\mathbb{R}^m) \to H_0(\mathbb{R}^m; R).$$

We know from Example 2.4.7.2 that $\langle 1, \alpha \cap o_{B_r} \rangle = \langle \alpha, o_{B_r} \rangle$ for all $\alpha \in H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R)$. But

$$\langle -, o_{B_r} \rangle \colon H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \longrightarrow R, \quad u \mapsto \langle u, o_{B_r} \rangle$$

is bijective because of the universal coefficient theorem 2.2.5:

$$H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \cong \mathrm{Hom}(H_m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r), R) \oplus \mathrm{Ext}(H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \setminus B_r), R)$$

The last summand is trivial because $H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \setminus B_r) = 0$. Thus we obtain that for all r the map $(-) \cap o_{B_r}$ is the map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\langle o_{B_r} \rangle, R) \to R \quad \text{with} \quad \varphi \mapsto \varphi(o_{B_r})$$

and thus bijective and therefore its direct limit

$$\varinjlim (-) \cap o_{B_r} \colon \varinjlim H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \longrightarrow H_0(\mathbb{R}^m; R)$$

is an isomorphism as well.

2. Now assume that $M = U \cup V$ such that the claim holds for the open subsets U, V and $U \cap V$ which are m-dimensional manifolds themselves, i.e. the maps $\mathrm{PD}_U, \mathrm{PD}_V$ and $\mathrm{PD}_{U \cap V}$ are isomorphisms and each of them uses the orientation that is induced from the orientation of M. Assume that $K \subset U$ and $L \subset V$ are compact and consider the relative version of the Mayer-Vietoris sequences in cohomology

$$\cdots \longrightarrow H^p(M, M \setminus (K \cap L); R) \longrightarrow H^p(M, M \setminus K; R) \oplus H^p(M, M \setminus L; R) \longrightarrow H^p(M, M \setminus (K \cup L); R) \longrightarrow H^{p+1}(M, M \setminus (K \cap L); R) \longrightarrow \cdots$$

Excision for $M \setminus U \subset M \setminus K$ tells us

$$H^p(M, M \setminus K; R) \cong H^p(U, U \setminus K; R) \ .$$

Similarly, we find for $M \setminus V \subset M \setminus L$ and $M \setminus (U \cap V) \subset M \setminus (K \cap L)$

$$H^p(M, M \setminus (K \cap L); R) \cong H^p((U \cap V), (U \cap V) \setminus (K \cap L); R)$$
$$H^p(M, M \setminus L; R) \cong H^p(V, V \setminus L; R).$$

We obtain a map of exact sequences

$$\begin{array}{ccc}
H_c^{m-p}(U\cap V;R) & \xrightarrow{\cap o_{U\cap V}} & H_p(U\cap V;R) \\
\downarrow & & \downarrow \\
H_c^{m-p}(U;R)\oplus H_c^{m-p}(V;R) & \xrightarrow{\cap o_U \oplus \cap o_V} & H_p(U;R)\oplus H_p(V;R) \\
\downarrow & & \downarrow \\
H_c^{m-p}(M;R) & \xrightarrow{\cap o_M} & H_p(M;R) \\
\downarrow & & \downarrow \\
H_c^{m-p+1}(U\cap V;R) & \xrightarrow{\cap o_{U\cap V}} & H_{p-1}(U\cap V;R) \\
\downarrow & & \downarrow \\
H_c^{m-p+1}(U;R)\oplus H_c^{m-p+1}(V;R) & \xrightarrow{\cap o_U \oplus \cap o_V} & H_{p-1}(U;R)\oplus H_{p-1}(V;R)
\end{array}$$

The right column is exact by the Mayer-Vietoris sequence 1.8.1 in homology; the exactness of the left column follows from the Mayer-Vietoris sequence in cohomology we just considered and the isomorphisms obtained by excision by taking the limit. By assumption, the top two and the two bottom horizontal arrows are isomorphisms. The five lemma 1.8.5 thus proves the case $M = U \cup V$.

3. Now assume $M = \bigcup_{i=1}^{\infty} U_i$ with open subsets U_i that exhaust M, i.e. such that $U_1 \subset U_2 \subset \ldots$. We will show that if the claim holds for all open subsets U_i with the orientation induced by the one of M, then the claim holds for M.

   To that end, let $U \subset M$ be an arbitrary open subset and let $K \subset U$ be compact. Excision for $(M\setminus U) \subset (M\setminus K) \subset M$ gives us an isomorphism

   $$H^p(M, M\setminus K; R) \cong H^p(U, U\setminus K; R)$$

   and we denote by φ_K the inverse of this map. The direct limit of these φ_K over all $K \subset U$ for fixed U induces a map

   $$\varphi_U^M := \varinjlim \varphi_K \colon H_c^p(U;R) \longrightarrow H_c^p(M;R).$$

   In general, this map is *not* an isomorphism (U is 'too small to see enough of M'), but now we vary the open set U. For $U \subset V \subset W$ we get

   $$\varphi_U^W = \varphi_V^W \circ \varphi_U^V, \quad \varphi_U^U = \mathrm{id}.$$

   As the excision isomorphism is induced by the inclusion $(U, U\setminus K) \hookrightarrow (M, M\setminus K)$, we get that the following diagram commutes:

   $$\begin{array}{ccc}
   H_c^{m-p}(U;R) & \xrightarrow{\varphi_U^M} & H_c^{m-p}(M;R) \\
   \downarrow{\scriptstyle \mathrm{PD}_U} & & \downarrow{\scriptstyle \mathrm{PD}_M} \\
   H_p(U;R) & \xrightarrow{(i_U^M)_*} & H_p(M;R)
   \end{array}$$

   and hence the corresponding diagram

   $$\begin{array}{ccc}
   \varinjlim H_c^{m-p}(U_i;R) & \xrightarrow{\varinjlim \varphi_{U_i}^M} & H_c^{m-p}(M;R) \\
   \downarrow{\scriptstyle \varinjlim \mathrm{PD}_{U_i}} & & \downarrow{\scriptstyle \mathrm{PD}_M} \\
   \varinjlim H_p(U_i;R) & \xrightarrow{\varinjlim (i_{U_i}^M)_*} & H_p(M;R)
   \end{array}$$

commutes as well. The map $\varinjlim \varphi^M_{U_i}$ is an isomorphism because every compact subset $K \subset M$ ends up in some open set U_i eventually. By assumption, each PD_{U_i} is an isomorphism and so is their limit. Similarly the limit of the $(i^M_{U_i})_*$ is an isomorphism and therefore PD_M is an isomorphism.

4. We show that the claim is valid for arbitrary open subsets $M \subset \mathbb{R}^m$. We express M as a countable union $M = \bigcup_{r=1}^{\infty} \mathring{B}_r$, where the B_r are m-balls. This is possible because the topology of $\mathbb{R}^m$ has a countable basis.

   Each open ball $\mathring{B}_r$ in $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^m$, thus by (1) the claim holds for these balls $\mathring{B}_r$.

   Set $U_i := \bigcup_{r=1}^{i} \mathring{B}_r$, then of course
   $$U_1 \subset U_2 \subset \dots$$
   The claim then holds by (2) for the U_i as finite unions, and because of (3) it then holds for M.

5. Finally, we assume that M is as in the theorem: a connected m-manifold with some fixed R-orientation. Every point in M has a neighborhood which is homeomorphic to some open subset of $\mathbb{R}^m$ and we can choose the homeomorphism in such a way that it preserves the orientation. We know that M has a countable basis for its topology and thus there are open subsets $V_1, V_2, \dots \subset M$ such that $V_i \cong W_i \subset \mathbb{R}^m$ and the V_i cover M. Define $U_i := \bigcup_{j=1}^{i} V_j$, thus $M = \bigcup_i U_i$. The claim holds for the V_j by (4) and therefore it holds for the finite unions U_i by (2) and thus by (3) for M.

□

2.9 Alexander-Lefschetz duality

We will derive a relative version of Poincaré duality $H^{m-q}(M;R) \cong H_q(M;R)$ and some geometric applications. First, we consider Čech cohomology.

Observation 2.9.1.
Let X be an arbitrary topological space and let $A \subset B \subset X$ a pair in X. We want to associate to the pair a cohomology group. The rough idea of Čech cohomology is to approximate $H^q(B, A)$ by $H^q(V, U)$ where the open neighborhoods come closer and closer to (B, A).

- We consider open neighborhoods (V, U) of (B, A), i.e. open subsets $U \subset V \subset X$ with $A \subset U$ and $B \subset V$.
- From the inclusion $(V, U) \subset (V', U')$ we get induced maps in relative cohomology
  $$H^q(V', U') \longrightarrow H^q(V, U).$$
- We use this property to construct for a fixed pair $A \subset B$ in X a directed system, so we set $(V', U') \leqslant (V, U)$ if and only if $V \subset V'$ and $U \subset U'$.

Definition 2.9.2
Čech cohomology of the pair (B, A) with $A \subset B \subset X$ is defined as the limit
$$\check{H}^p(B, A) = \varinjlim H^p(V, U) \ .$$

In this generality, Čech cohomology has very bad properties.

Remarks 2.9.3.

1. A space Y is called a euclidean neighborhood retract, if Y is homeomorphic to a subset $X \subset \mathbb{R}^n$ for some n such that X is a retract of a neighborhood $X \subset U \subset \mathbb{R}^n$.

2. If the space X is a euclidean neighborhood retract and $A \subset B \subset X$ are locally compact, then $\check{H}^p(B, A)$ only depends on B and A and *not* on X.

3. If in addition A and B are euclidean neighborhood retracts themselves, then $\check{H}^p(B, A)$ is actually isomorphic to $H^p(B, A)$. For more background on Čech cohomology see Dold's book *Lectures on Algebraic Topology*, reprint in: Classics in Mathematics. Springer-Verlag, Berlin, 1995, VIII §6.

Observation 2.9.4.

- Now let M be a connected m-dimensional manifold and let $K \subset L \subset M$ be compact subsets in M. We assume that there is an orientation class $o_L \in H_m(M, M \setminus L)$ of M along L (possibly with coefficients in R, but we suppress coefficients from the notation). We aim at a cap-pairing of Čech-cohomology $\check{H}^*(L, K)$ with relative homology $H_*(M, M \setminus L)$ in which the class o_L is.

- For $(L, K) \subset (V, U)$ we set up a map on the level of chains and cochains
$$S^p(V, U) \otimes \left(\frac{S_k(U) + S_k(V \setminus K)}{S_k(V \setminus L)} \right) \longrightarrow S_{k-p}(V \setminus K, V \setminus L) \quad (*).$$
For this note the trivial inclusion $V \setminus L \subset (U \cup (V \setminus K)) = V$. For $\alpha \in S^p(V, U)$ and $a + b \in \frac{S_k(U)+S_k(V\setminus K)}{S_k(V\setminus L)}$ we have
$$\alpha \cap (a + b) = \alpha \cap a + \alpha \cap b = 0 + \alpha \cap b$$
and this ends up in the correct chain group.

- The homology of $\frac{S_*(U)+S_*(V\setminus K)}{S_*(V\setminus L)}$ is isomorphic to $H_*(V, V \setminus L)$ and this in turn is isomorphic to $H_*(M, M \setminus L)$ via excision for $(M \setminus V) \subset (M \setminus L) \subset M$. This allows us to rewrite the second tensorand on the left hand side of $(*)$, as desired.

- Excision for $(M \setminus V) \subset (M \setminus L) \subset (M \setminus K)$ tells us as well that
$$H_*(V \setminus K, V \setminus L; R) \cong H_*(M \setminus K, M \setminus L; R) \ .$$
This allows us to rewrite the right hand side.

- As Čech cohomology is the direct limit $\varinjlim H^*(V, U)$ and as everything is compatible (which we did not really show), the above gives a well-defined map
$$\mathrm{PD}\colon \check{H}^q(L, K) \otimes H_m(M, M \setminus L) \longrightarrow H_{m-q}(M \setminus K, M \setminus L), \quad \alpha \otimes o_L \mapsto \alpha \cap o_L.$$

Proposition 2.9.5 (Alexander-Lefschetz duality)**.**
Let M be a connected m-dimensional manifold and let $K \subset L \subset M$ with K, L compact. Let M be oriented along L with respect to R. Then the map
$$\mathrm{PD} = (-) \cap o_L \colon \check{H}^q(L, K; R) \longrightarrow H_{m-q}(M \setminus K, M \setminus L; R)$$
is an isomorphism for all integers q.

Before we prove this result, we collect some properties of this form of the Poincaré duality map.

Remarks 2.9.6.

1. This PD map still satisfies that $\mathrm{PD}(1) = o_L$ for $K = \varnothing$ and $1 \in H^0(L;R)$.

2. The PD-map is natural in the following sense: for any map of pairs $i : (L, K) \hookrightarrow (L', K')$ in M, we have also a map $\tilde{i} : (M \setminus K', M \setminus L') \hookrightarrow (M \setminus K, M \setminus L)$, and the diagram

$$\begin{array}{ccc} \check{H}^q(L', K') & \xrightarrow{(-)\cap o_{L'}} & H_{m-q}(M \setminus K', M \setminus L') \\ \downarrow{\scriptstyle \check{H}^q(i)} & & \downarrow{\scriptstyle H_{m-q}(\tilde{i})} \\ \check{H}^q(L, K) & \xrightarrow{(-)\cap o_L} & H_{m-q}(M \setminus K, M \setminus L) \end{array}$$

commutes.

3. We will not prove the following fact (cf. Bredon Lemma VI.8.1). The diagram

$$\begin{array}{ccccccccc} \ldots \to & \check{H}^q(L,K) & \to & \check{H}^q(L) & \to & \check{H}^q(K) & \to & \check{H}^{q+1}(L,K) & \to \ldots \\ & \downarrow{\scriptstyle \cap o_L} & & \downarrow{\scriptstyle \cap o_L} & & \downarrow{\scriptstyle \cap o_K} & & \downarrow{\scriptstyle \cap o_L} & \\ \ldots \to & H_{m-q}(M \setminus K, M \setminus L) & \to & H_{m-q}(M, M \setminus L) & \to & H_{m-q}(M, M \setminus K) & \to & H_{m-q-1}(M \setminus K, M \setminus L) & \to \ldots \end{array}$$

commutes, and therefore (using the five lemma) it suffices to show the absolute version of Alexander-Lefschetz duality,

$$\check{H}^q(L) \xrightarrow{\cap o_L} H_{m-q}(M, M \setminus L) \ .$$

Lemma 2.9.7.
Let K and L are compact subsets of M with an orientation class $o_{K\cup L}$ along $K \cup L$ and induced orientation classes o_K and o_L. Then the diagram

$$\begin{array}{ccccccc} \ldots \xrightarrow{\partial} & \check{H}^q(K\cup L) & \to & \check{H}^q(K)\oplus\check{H}^q(L) & \to & \check{H}^q(K\cap L) & \xrightarrow{\partial} \ldots \\ & \downarrow{\scriptstyle \cap o_{K\cup L}} & & \downarrow{\scriptstyle \cap o_K \oplus \cap o_L} & & \downarrow{\scriptstyle o_{K\cap L}} & \\ \ldots \xrightarrow{\delta} & H_{m-q}(M, M\setminus(K\cup L)) & \to & H_{m-q}(M,M\setminus K)\oplus H_{m-q}(M,M\setminus L) & \to & H_{m-q}(M,M\setminus(K\cap L)) & \xrightarrow{\delta} \ldots \end{array}$$

commutes and has exact rows.

Proof.

- The only critical squares are the ones that are slightly out of the focus of the above diagram, the ones with the connecting homomorphisms. The $\check{H}^*$-sequence in the upper line comes from taking the direct limit of

$$0 \to \mathrm{Hom}(S_*(U) + S_*(V), R) \longrightarrow \mathrm{Hom}(S_*(U), R) \oplus \mathrm{Hom}(S_*(V), R) \longrightarrow \mathrm{Hom}(S_*(U \cap V), R) \to 0$$

over all open subsets U, V with $K \subset U$ and $L \subset V$. (Note that by Lemma 2.7.6 taking the direct limit is exact.)

- Let $\alpha \in \check{H}^q(K \cap L; R)$. Choose a representing cocycle f with $\alpha = [f]$, i.e. $\delta f = 0$ on $U \cap V$ and let ∂ be the connecting homomorphism for ordinary singular cohomology. What is $\partial(\alpha)$? A preimage for f in the direct sum is a pair $(f, 0)$ and its coboundary is $(\delta f, 0)$, so if we define $h \in \mathrm{Hom}(S_*U + S_*V, R)$ by the property $h(u + v) = \delta f(u)$ for $u \in S_*(U)$, $v \in S_*(V)$, then

$$\partial(\alpha) = [h].$$

We can extend h to a cochain on M (for instance by defining it to be trivial on the chains that are supported on the complement).

- We want to compare $\partial(\alpha) \cap o_{K\cup L}$ and $\delta(\alpha \cap o_{K\cap L})$. For the first term we express the orientation class $o_{K\cup L} = [a]$ as a sum

$$a = b + c + d + e \in S_*(U \cap V) + S_*(U \setminus L) + S_*(V \setminus K) + S_*(M \setminus (K \cup L)).$$

This is possible, since the subsets $U \cap V, U \setminus L, V \setminus K$ and $M \setminus (K \cup L)$ are open and therefore we can work with small chains for this open cover. With the notation as above we get

$$\partial(\alpha) \cap o_{K\cup L} = [h \cap (b + c + d + e)] = [h \cap c] \ .$$

As h is only non-trivial on chains in U, only the terms involving b and c can contribute. Since $\delta(f)$ is trivial on $U \cap V$, h is only non-trivial on the complement of V in U.

- For $\alpha \cap o_{K\cap L}$ we write $[f \cap a]$ and as the lower exact row comes from the short exact sequence of complexes

$$0 \to \frac{S_*(M)}{S_*(M \setminus K \cup L)} \longrightarrow \frac{S_*(M)}{S_*(M \setminus K)} \oplus \frac{S_*(M)}{S_*(M \setminus L)} \longrightarrow \frac{S_*(M)}{S_*(M \setminus K) + S_*(M \setminus L)} \to 0$$

we view $f \cap a$ as an element modulo $S_*(M \setminus K) + S_*(M \setminus L)$. The connecting homomorphism picks $(f \cap a, 0)$ as a pre-image of $f \cap a$, then takes its boundary $(\partial(f \cap a), 0)$. But the latter is up to sign by the Leibniz rule

$$(\partial(f \cap a), 0) = (\delta(f) \cap a), 0) \pm (f \cap \partial a, 0).$$

Writing a as $a = b + c + d + e$ as above and using that f ignores b and e we obtain that the above is $(\delta f \cap c + \delta f \cap d \pm f \cap \partial a, 0)$. But $\delta f \cap d$ and $f \cap \partial a$ are elements in $S_*(M \setminus K)$ and hence all that remains when we pick a preimage is $(\delta f \cap c, 0)$, thus

$$\delta(\alpha \cap o_{K\cap L}) = [\delta f \cap c] = [h \cap c].$$

□

Now we can prove Alexander-Lefschetz duality.

Proof. of Proposition 2.9.5

Remark 2.9.6.3 implies that it suffices to prove the absolute case, i.e. to show that for any compact subset $K \subset M$

$$(-) \cap o_K \colon \check{H}^q(K) \longrightarrow H_{m-q}(M, M \setminus K)$$

is an isomorphism for all q.

1. If K is empty, then we get the true statement that $\check{H}^q(\emptyset) = 0 = H_{m-q}(M, M)$. For K a point we only get something non-trivial for degree $q = 0$ and here $1 \in R = \check{H}^0(K)$ is sent to $o_K = o_x$ via Poincaré duality. Similarly, if $M = \mathbb{R}^m$ and K is convex and compact we can proceed as in the case of a point.

2. If $K = K_1 \cup \ldots \cup K_r$ with K_i compact and convex and M is still $\mathbb{R}^m$ an induction over r using Lemma 2.9.7 and (1) proves the claim.

3. For $M = \mathbb{R}^m$ and K arbitrary we can find a neighborhood U of K of the form $U = \bigcup_{i=1}^N U_i$ with the U_i being convex. Such U suffice to calculate the direct limit $\varinjlim H^q(U)$ for the Čech cohomology of K. For such U we have

$$H_{m-q}(\mathbb{R}^m, \mathbb{R}^m \setminus K) \cong \varinjlim H_{m-q}(\mathbb{R}^m, \mathbb{R}^m \setminus U)$$

because $\mathbb{R}^m \setminus K = \bigcup_U \mathbb{R}^m \setminus U$. The U satisfy Alexander-Lefschetz duality by (2) and hence K does.

4. Finally let M and K be arbitrary, but satisfying the conditions of Proposition 2.9.5. Express $K = K_1 \cup \ldots \cup K_r$ such that the K_i are contained in a chart that is homeomorphic to $\mathbb{R}^m$ and proceed as in the case before.

□

2.10 Application of duality

We specialize to the case when the manifold M is $\mathbb{R}^m$ with the standard orientation.

Proposition 2.10.1 (Classical Alexander duality)**.**
Let $K \subset \mathbb{R}^m$ be compact. Then

$$\check{H}^q(K) \cong H_{m-q}(\mathbb{R}^m, \mathbb{R}^m \setminus K) \cong \tilde{H}_{m-q-1}(\mathbb{R}^m \setminus K).$$

Proof.
Here the first isomorphism is the absolute version of Alexander-Lefschetz duality 2.9.2 for $M = \mathbb{R}^m$. The second one is a result of the long exact sequence of pairs in homology. □

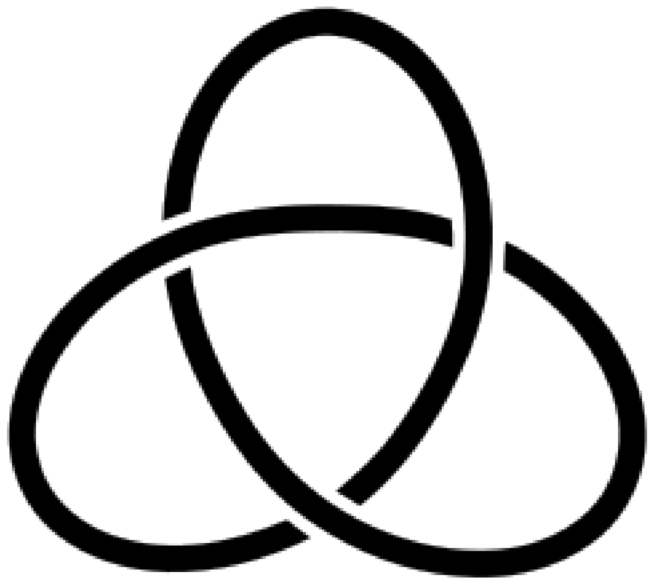
Remark 2.10.2.

- This is bad news for knot complements. A knot K is the homeomorphic image of $\mathbb{S}^1$ in $\mathbb{R}^3$. Proposition 2.10.1 implies that

$$H_1(\mathbb{R}^3 \setminus K) \cong \check{H}^1(K)$$

but the circle is a euclidean neighborhood retract and therefore Čech cohomology concides with ordinary singular cohomology. Since $H^1(K) \cong \mathbb{Z}$, the first homology group of any knot complement is isomorphic to the integers, thus it does not help to distinguish knots.

- The *fundamental group* of the knot complement does a better job. Here the un-knot gives the integers, but for instance the complement of the trefoil knot



has a fundamental group that is *not* isomorphic to the integers, but is isomorphic to the group $\langle a, b | a^2 = b^3 \rangle$. This group is actually isomorphic to the braid group on three strands. (This can be computed using the Wirtinger presentation derived from the link diagram of a knot, see Section 4.2.3 and 4.2.4 of J. Stillwell. Classical Topology and Combinatorial Group Theory. Springer Graduate Text in Mathematics 72, 1993.

Proposition 2.10.3.
Let M be a compact oriented connected m-manifold and let $\varnothing \neq K \subset M$ be compact. If the first homology $H_1(M)$ of the ambient manifold is trivial, then $\check{H}^{m-1}(K)$ is a free abelian group. The complement $M \setminus K$ then has $\mathrm{rank}\check{H}^{m-1}(K) + 1$ connected components.

Proof.
Let $k = |\pi_0(M \setminus K)|$ be the number of components of the complement of K in M. By Corollary 1.3.3,

$$k = \mathrm{rank} H_0(M \setminus K) = 1 + \mathrm{rank}\tilde{H}_0(M \setminus K).$$

By assumption $H_1(M) = 0 = \tilde{H}_0(M)$ and therefore we know from the long exact sequence and duality that

$$\tilde{H}_0(M \setminus K) \cong H_1(M, M \setminus K) \cong \check{H}^{m-1}(K).$$

Since the group $\check{H}^{m-1}(K)$ is isomorphic to a zeroth homology group, it is free abelian. The statement about k is now the combination of the two equations. □

Proposition 2.10.4.
If M is a compact connected orientable m-manifold and if the first homology group of M with integral coefficients vanishes, then all compact submanifolds of M without boundary of dimension $(m-1)$ are orientable.

Compact manifolds without boundary are often called closed.

Proof.
A submanifold $N \subset M$ is a euclidean neighborhood retract and therefore

$$H^{m-1}(N) \cong \check{H}^{m-1}(N) \cong H_1(M, M \setminus N) \cong \tilde{H}_0(M \setminus N) \ .$$

Thus $H^{m-1}(N)$ is free abelian. Theorem 2.6.11 implies that the components of N are orientable. □

Corollary 2.10.5.
It is not possible to embed real projective space $\mathbb{R}P^2$ into $\mathbb{R}^3$.

Proof.
If one could, then one could embed $\mathbb{R}P^2$ into $\mathbb{S}^3$ as the one-point compactification of $\mathbb{R}^3$. Due to $H_1(\mathbb{S}^3) = 0$, the 2-manifold $\mathbb{R}P^2$ would be orientable, but we know from Example 2.6.13 that this is not true. □

At Oberwolfach Research Institute for Mathematics there is a model of the Boy surface. This is a model of an immersion of $\mathbb{R}P^2$ into three-space. `http://www.mfo.de/general/boy/`

Proposition 2.10.6.
Let M be a compact connected and orientable m-manifold and let

$$\beta_i := \dim_{\mathbb{Q}} H_i(M;\mathbb{Q})$$

be the ith Betti number of M. Then $\beta_i = \beta_{m-i}$.

Proof.
Note that in this case Čech cohomology $\check{H}^*(M) = \check{H}^*(M,\emptyset)$ is isomorphic to $H^*(M)$ because a limit is to be taken over the relative cohomology groups $H^*(M,U)$ for the directed system of pairs (M,U) with U any open set which has $(M,\emptyset)$ as a maximal element. Duality 2.9.5 then implies that

$$\beta_{m-i} = \dim_{\mathbb{Q}} H_{m-i}(M;\mathbb{Q}) \overset{2.9.5}{=} \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$$

As the group $\mathbb{Q}$ is divisible, Remark 2.2.2.7 implies that there is no Ext-term arising in the universal coefficient theorem 2.2.4 and thus

$$\dim_{\mathbb{Q}} H^i(M;\mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{Hom}(H_i(M),\mathbb{Q})) \ .$$

The right hand side is equal to the dimension of the vector space of the homomorphisms from the free part of the homology group $H_i(M)$ to $\mathbb{Q}$ which is equal to the rank of $H_i(M)$. Since tensoring with $\mathbb{Q}$ is exact, there is no Tor-term and thus $H_i(M;\mathbb{Q}) = H_i(M)\otimes_{\mathbb{Z}}\mathbb{Q}$; thus the rank of $H_i(M)$ is equal to $\beta_i = \dim_{\mathbb{Q}} H_i(M;\mathbb{Q})$. □

Corollary 2.10.7.
Let M be a compact connected and orientable m-manifold of *odd* dimension. Then the Euler characteristic $\chi(M) = \sum_{i=0}^{m}(-1)^i\beta_i$ vanishes.

Proof.
We compute

$$\chi(M) = \sum_{i=0}^{m}(-1)^i\beta_i \overset{2.10.6}{=} \sum_{i=0}^{m}(-1)^i\beta_{m-i} = (-1)^m\chi(M) \ .$$

□

Proposition 2.10.8.
For M a compact connected orientable m-manifold with boundary the duality holds

$$\check{H}^q(M,\partial M) \cong H_{m-q}(M) \ .$$

Proof.
Glue a collar to M, i.e., consider the auxiliary manifold

$$W := M \cup (\partial M \times [0,1)) =: M \cup W'.$$

Then W is an m-manifold without boundary; thus duality 2.9.5 applies to the pair of compact subsets $\partial M \subset M$:

$$\check{H}^q(M,\partial M) \cong H_{m-q}(W\setminus\partial M, W\setminus M) \ .$$

Now note that

$$W \setminus \partial M \simeq M \setminus \partial M \sqcup W' \setminus \partial M \quad \text{and } W \setminus M = W' \setminus \partial M$$

so that the right hand side becomes

$$H_{m-q}(M \setminus \partial M) \cong H_{m-q}(M) \ .$$

For the last isomorphism, we used that taking the complement of the boundary ∂M in M gives a space that is homotopy equivalent to M. □

Corollary 2.10.9.
If M is a compact connected orientable m-manifold, then the Euler characteristic of the boundary ∂M is always even.

Proof.
With W as above, the homotopy equivalence $W \simeq M$ implies $\chi(M) = \chi(W)$. The long exact sequence of the pair $W \setminus M \subset W$ gives

$$\chi(W) = \chi(W \setminus M) + \chi(W, W \setminus M) \ .$$

The homotopy equivalence $W \setminus M \simeq \partial M$ yields $\chi(W \setminus M) = \chi(\partial M)$ and duality 2.9.5 guarantees that $\chi(W, W \setminus M) = (-1)^m \chi(M)$. Therefore

$$\chi(\partial M) = (1 + (-1)^{m-1})\chi(M)$$

and this is always an even number. □

Remark 2.10.10.

1. Recall from from Example 1.12.8 that the real projective space has the structure of a CW complex with one cell in each dimension. Thus $\mathbb{R}P^{2m}$ has Euler characteristic 1 and by Corollary 2.10.9 cannot be a boundary.

2. For the calculations of of the Euler characteristic of complex and quaternionic projective spaces, recall from Example 1.12.8 that for complex projective space of dimension $2m$ we have cells in dimension up to $4m$, but only in even dimensions. Similarly, for quaternion projective space of dimension $2m$ cells occur up to dimension $8m$, but only in degrees divisible by 4.

   Thus

   $$\chi(\mathbb{C}P^{2m}) = \sum_{i=0}^{2m} (-1)^{2i} = 2m + 1$$

   and

   $$\chi(\mathbb{H}P^{2m}) = \sum_{i=0}^{2m} (-1)^{4i} = 2m + 1 \ .$$

   By Corollary 2.10.9, all these projective spaces do not occur as boundaries of connected compact orientable manifolds.

3. These facts are important in *bordism theory*: one can introduce an equivalence relation on manifolds by saying that two m-manifolds M and N are bordant, if there is an $(m+1)$-manifold W whose boundary is the disjoint union of M and N, $\partial W = M \sqcup N$. Thus the projective spaces give non-trivial equivalence classes under the bordism relation.

2.11 Duality and cup products

Let M be a connected closed m-manifold with an R-orientation o_M^R for some commutative ring R. We consider the composition

$$\begin{array}{ccc} H^k(M;R) \otimes_R H^{m-k}(M;R) & \xrightarrow{\cup} & H^m(M;R) \\ & & \big\downarrow {\scriptstyle (-)\cap o_M^R} \\ & & H_0(M;R) \cong R \end{array}$$

Definition 2.11.1
Let M be a connected closed m-manifold with an R-orientation for some commutative ring R. For $\alpha \in H^k(M;R)$, $\beta \in H^{m-k}(M;R)$ the map

$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, o_M^R \rangle$$

with values in R is called the cup product pairing *of M.*

Proposition 2.11.2.
If R is a field or if $R = \mathbb{Z}$ and all homology groups of M are torsion-free, the cup product pairing is non-singular in the sense that the two induced maps

$$\begin{array}{rcl} H^k(M;R) & \to & \mathrm{Hom}_R(H^{m-k}(M;R), R) \\ \alpha & \mapsto & \big(\beta \mapsto \langle \alpha \cup \beta, o_M^R \rangle \in R\big) \end{array} \quad \text{and} \quad \begin{array}{rcl} H^{m-k}(M;R) & \to & \mathrm{Hom}_R(H^k(M;R), R) \\ \beta & \mapsto & \big(\alpha \mapsto \langle \alpha \cup \beta, o_M^R \rangle \in R\big) \end{array}$$

are both isomorphisms.

Proposition 2.11.2 holds as long as one restricts attention to the free part of the cohomology groups: let $FH^k(M;R)$ denote the free part of $H^k(M;R)$ then there is a non-singular pairing

$$FH^k(M;R) \otimes_R FH^{m-k}(M;R) \to R.$$

In geometric applications the ground ring is often $R = \mathbb{R}$.
Proof.
The Kronecker pairing, cf. Lemma 2.1.6, yields a map

$$\kappa \colon H^k(M;R) \to \mathrm{Hom}_R(H_k(M;R), R)$$

and Poincaré duality 2.8.3 tells us that capping with o_M^R is an isomorphism between $H_k(M;R)$ and $H^{m-k}(M;R)$. The composite is

$$\begin{array}{rcl} H^k(M;R) & \stackrel{\kappa}{\to} & \mathrm{Hom}_R(H_k(M;R), R) \cong \mathrm{Hom}_R(H^{m-k}(M;R), R), \\ \alpha & \mapsto & \langle \alpha, (-) \cap o_M^R \rangle \ . \end{array}$$

Over a field, κ and hence the composite is an isomorphism. We finally use the duality relation

$$\langle \alpha \cup \beta, o_M^R \rangle \stackrel{2.4.7.2}{=} \langle 1, (\alpha \cup \beta) \cap o_M^R \rangle \stackrel{2.5.11.2}{=} \langle 1, \alpha \cap (\beta \cap o_M^R) \rangle \stackrel{2.4.7.2}{=} \langle \alpha, \beta \cap o_M^R \rangle$$

In the torsion-free setting, we obtain an isomorphism as well. □

Definition 2.11.3

Let M be a connected closed m-manifold with an R-orientation for some commutative ring R. Dual to the cup product pairing, we define the intersection form:

$$H_p(M;R) \otimes H_{m-p}(M;R) \to R$$

with $a \otimes b \mapsto \langle \mathrm{PD}^{-1}(a) \cup \mathrm{PD}^{-1}(b), o_M^R \rangle$.

For even-dimensional manifolds, the signature of this form is a particularly important invariant in differential topology. For instance one can show that for a compact oriented manifold W such that $\partial W = M$ with a $4n$-dimensional manifold M, the signature of the intersection form on M is trivial.

For explicit computations of cohomology rings, the following Lemma is useful:

Lemma 2.11.4.
Let M be a connected closed m-manifold with a $\mathbb{Z}$-orientation and with torsion-free homology groups. If $H^p(M) \cong \mathbb{Z} \cong H^{m-p}(M)$ and if $\alpha \in H^p(M)$, $\beta \in H^{m-p}(M)$ are generators, then $\alpha \cup \beta$ is a generator of the group $H^m(M) \cong \mathbb{Z}$.

Proof.
Since by Proposition 2.11.2 the cup product pairing is non-degenerate for torsion free cohomology, there exists for any generator $\alpha \in H^p(M)$ an element $\beta' \in H^{m-p}(M)$ with

$$\langle \alpha \cup \beta', o_M \rangle = 1 \ .$$

Note that this implies that $\alpha \cup \beta'$ is a generator of $H^m(M)$, as a dual of the generator o_M.

As β is a generator of $H^{m-p}(M)$, we know that $\beta' = k\beta$ for some integer k and hence

$$1 = \langle \alpha \cup \beta', o_M \rangle = \langle \alpha \cup k\beta, o_M \rangle = k \langle \alpha \cup \beta, o_M \rangle.$$

But $\langle \alpha \cup \beta, o_M \rangle$ is an integer as well, so k has to be ± 1 and therefore $\alpha \cup \beta$ generates the group $H^m(M)$ as well. □

We will use this result to calculate the cohomology *rings* of projective spaces.

Lemma 2.11.5.
If $\alpha \in H^2(\mathbb{C}P^m)$ is an additive generator, then $\alpha^q = \alpha^{\cup q} \in H^{2q}(\mathbb{C}P^m)$ is an additive generator as well for all $q \leqslant m$.

Proof.
We have to show by induction on the complex dimension m that α^q is an additive generator of $H^{2q}(\mathbb{C}P^m)$.

- For $m = 1$ there is nothing to prove because $\mathbb{C}P^1 \cong \mathbb{S}^2$ and there $\alpha^2 = 0$.
- Consider the inclusion $i \colon \mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m$. The CW structure of $\mathbb{C}P^m$ explained in Example 2.12.9 implies $\mathbb{C}P^m \cong \mathbb{C}P^{m-1} \cup_f \mathbb{D}^{2m}$ for attaching the $2m$-cell. For $m > 1$

  $$i^* \colon H^{2i}(\mathbb{C}P^m) \to H^{2i}(\mathbb{C}P^{m-1})$$

  is an isomorphism for $1 \leqslant i \leqslant m-1$. In particular, $i^*(\alpha)$ additively generates $H^2(\mathbb{C}P^{m-1})$. Induction over m then shows that $(i^*(\alpha))^q$ generates $H^{2q}(\mathbb{C}P^{m-1})$ for all $1 \leqslant q \leqslant m-1$. But $(i^*(\alpha))^q = i^*(\alpha^q)$, by Proposition 2.5.9.4, and i^* is an isomorphism, so α^q additively generates $H^{2q}(\mathbb{C}P^m)$ for $1 \leqslant q \leqslant m-1$. Lemma 2.11.4 then shows that $\alpha \cup \alpha^{m-1} = \alpha^m$ generates $H^{2m}(\mathbb{C}P^m)$.

□

Corollary 2.11.6.
As a graded ring, we have

$$H^*(\mathbb{C}P^m) \cong \mathbb{Z}[\alpha]/\alpha^{m+1} \text{ with } |\alpha| = 2.$$

Similarly,

$$H^*(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/\alpha^{m+1} \text{ with } |\alpha| = 1.$$

(Taking coefficients in $\mathbb{Z}/2\mathbb{Z}$ leads to a complex with vanishing differentials, cf. Example 1.13.3.)

There are two geometric consequences that follow from this calculation.

Proposition 2.11.7.
For $0 < m < n$ the inclusion $j\colon \mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ is not a weak retract.

Proof.
Let us assume that there exists $r\colon \mathbb{C}P^n \to \mathbb{C}P^m$ with $r \circ j \simeq \mathrm{id}$. On the second cohomology groups, the map j induces an isomorphism

$$j^*\colon H^2(\mathbb{C}P^n) \to H^2(\mathbb{C}P^m)\ .$$

Let $\alpha \in H^2(\mathbb{C}P^m)$ be an additive generator. Because of $j^* \circ r^* = \mathrm{id}$, the element $\beta := r^*(\alpha) \in H^2(\mathbb{C}P^n)$ is an additive generator as well. As $\alpha^{m+1} = 0$ we get

$$\beta^{m+1} = r^*(\alpha)^{m+1} = r^*(\alpha^{m+1}) = r^*(0) = 0.$$

But by Corollary 2.11.6 $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\beta]/\beta^{n+1}$ and hence $\beta^{m+1} \neq 0$. □

Proposition 2.11.8.
The attaching map of the $2n$-cell in $\mathbb{C}P^n$ is not null-homotopic.

Proof.
Let $\varphi\colon \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ be the attaching map, thus

$$\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_\varphi \mathbb{D}^{2n}.$$

If φ were null-homotopic, then there is a homotopy $H : \mathbb{S}^{2n-1} \times [0,1] \to \mathbb{C}P^{n-1}$ with $H_1 = \varphi$ and H_0 constant. Since H_0 is constant, H factorizes to a map $\tilde{H}$

$$\begin{array}{ccc} \mathbb{S}^{2n-1} \times [0,1] & \xrightarrow{H} & \mathbb{C}P^{n-1} \\ \downarrow & \nearrow_{\tilde{H}} & \\ \mathbb{D}^{2n} & & \end{array}$$

with the vertical map being $(x,t) \mapsto tx$. Then

$$\tilde{H} \sqcup \mathrm{id}_{\mathbb{C}P^{n-1}} : \mathbb{D}^{2n} \sqcup \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$$

factorizes to a retract $r : \mathbb{C}P^n \to \mathbb{C}P^{n-1}$,

$$\begin{array}{ccc} \mathbb{D}^{2n} \sqcup \mathbb{C}P^{n-1} & \xrightarrow{\tilde{H} \sqcup \mathrm{id}} & \mathbb{C}P^{n-1} \\ \downarrow & \nearrow_{r} & \\ \mathbb{C}P^n = \mathbb{D}^{2n} \cup_\varphi \mathbb{C}P^{n-1} & & \end{array}$$

in contradiction to Proposition 2.11.7. □

Remark 2.11.9.
A famous example of this phenomenon is the Hopf fibration

$$h\colon \mathbb{S}^3 \to \mathbb{C}P^1 = \mathbb{S}^2 = \mathbb{C} \cup \infty \ .$$

Consider $\mathbb{S}^3 \subset \mathbb{C}^2$ and send $\mathbb{S}^3 \ni (u, v)$ to

$$h(u,v) := \begin{cases} \frac{u}{v}, & v \neq 0, \\ \infty, & v = 0. \end{cases}$$

Up to a homeomorphism of $\mathbb{S}^2$, this is the attaching map for the 4-dimensional cell of $\mathbb{C}P^2$ and thus by Proposition 2.11.8 not null-homotopic. In fact, the map h generates the homotopy group $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

2.12 The Milnor sequence

The aim is to calculate the cohomology rings of infinite dimensional projective spaces and more generally to understand cohomology groups for infinite dimensional CW complexes.

We start with some algebraic structures: let $(M_i)_{i\in\mathbb{N}_0}$ be a family of R-modules together with a sequence of maps

$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \ldots$$

We call such a family $(M_i, f_i)_{i\in\mathbb{N}_0}$ an <u>inverse system</u> (over the poset $(\mathbb{N}_0, \leqslant)$).

Definition 2.12.1
The <u>inverse limit</u> of the inverse system $(M_i)_{i\in\mathbb{N}_0}$ is the R-module

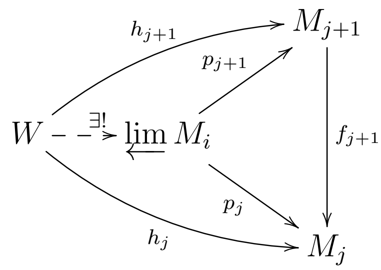
$$\varprojlim M_i = \{(x_0, x_1, \ldots) \in \prod_{i\in\mathbb{N}_0} M_i | f_{i+1}(x_{i+1}) = x_i, i \geqslant 0\}.$$

Remarks 2.12.2.

1. The restrictions of the projections of the product endow the the inverse limit with a system of maps such that the diagrams

$$\begin{array}{ccc} \varprojlim M_i & \xrightarrow{p_{j+1}} & M_{j+1} \\ & \searrow^{p_j} & \downarrow f_{j+1} \\ & & M_j \end{array}$$

commute for all $j \in \mathbb{N}_0$. With their use, we can characterize the inverse limit by the following universal property:



2. If ξ denotes the map that sends the element $(x_0, x_1, \ldots) \in \prod_{i\in\mathbb{N}_0} M_i$ to $(x_0 - f_1(x_1), x_1 - f_2(x_2), \ldots)$ then we can express the inverse limit as the kernel of ξ,

$$0 \to \varprojlim M_i \longrightarrow \prod_{i\in\mathbb{N}_0} M_i \xrightarrow{\xi} \prod_{i\in\mathbb{N}_0} M_i \ .$$

Definition 2.12.3
Let $\varprojlim{}^1 M_i$ *be the R-module* $\operatorname{coker}(\xi)$.

By definition, we have an exact sequence

$$0 \to \varprojlim M_i \longrightarrow \prod_{i\in\mathbb{N}_0} M_i \xrightarrow{\xi} \prod_{i\in\mathbb{N}_0} M_i \longrightarrow \varprojlim{}^1 M_i \to 0 \ .$$

Lemma 2.12.4.
If

$$0 \to (M_i, f_i) \longrightarrow (N_i, g_i) \longrightarrow (Q_i, h_i) \to 0$$

is a short exact sequence of inverse systems (cf. Remark 2.7.6.8 for exact sequences of direct systems), then the sequence

$$0 \to \varprojlim M_i \longrightarrow \varprojlim N_i \longrightarrow \varprojlim Q_i \longrightarrow \varprojlim{}^1 M_i \longrightarrow \varprojlim{}^1 N_i \longrightarrow \varprojlim{}^1 Q_i \to 0$$

is exact.

Proof.
Consider the map $\xi\colon \prod_i M_i \to \prod_i M_i$ as a chain complex C_* that is non-trivial only in two degrees 0 and 1. Then the first homology group is the inverse limit and the zeroth homology group is the lim-one term

$$H_1 C_* = \ker\xi \cong \varprojlim M_i \quad \text{and} \quad H_0 C_* = \operatorname{coker}\xi \cong \varprojlim{}^1 M_i \ .$$

We can translate the short exact sequence of inverse systems into a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_i M_i & \longrightarrow & \prod_i N_i & \longrightarrow & \prod_i Q_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \\ 0 & \longrightarrow & \prod_i M_i & \longrightarrow & \prod_i N_i & \longrightarrow & \prod_i Q_i & \longrightarrow & 0 \end{array}$$

and the associated long exact sequence (cf. Proposition 1.5.6) gives precisely our claim. □

Therefore the lim-one terms measure how non-exact inverse limits are. We present a criterion which ensures that we have exactness.

Lemma 2.12.5 (Mittag-Leffler condition).
Let (M_i, f_i) be an inverse system. Assume that for every $n \geqslant 0$ there exists $N = N(n)$ such that for all $m \geqslant N$ that the image of $f_{n+1} \circ \ldots \circ f_m \colon M_m \to M_n$ is equal to the image of $f_{n+1} \circ \ldots \circ f_N \colon M_N \to M_n$. Then

$$\varprojlim{}^1 M_i = 0.$$

Proof.
Without loss of generality, we can assume that the sequence $N(n)$ is monotonously increasing in n. We have to show that the cokernel of ξ is trivial. This means that we have to show that any sequence $(a_i)_i \in \prod_i M_i$ is in the image of ξ, if the Mittag-Leffler condition holds.

- As a first case, we deal with sequences $(a_i)_i$ such that every a_i is in the image of $f_{i+1} \circ \ldots \circ f_{N(i)} \colon M_{N(i)} \to M_i$.

  By induction on k, we construct elements $b_0, \ldots, b_k$ with

  $$b_i \in \operatorname{im}(f_{i+1} \circ \ldots \circ f_{N(i)}) \subset M_i$$

  such that $a_i = b_i - f_{i+1} b_{i+1}$ for all $i < k$. Then we have $(a_i) = \xi(b_i)$.

  We start with $a_0 = b_0 \in M_0$. The condition is empty for $k = 0$.

  Assume that elements $b_0, b_1, \ldots, b_k$ have been found. Because both a_k and b_k are in $\operatorname{im}(f_{k+1} \circ \ldots \circ f_{N(k)})$ and because by the assumption that the image of $f_{k+1} \circ \ldots \circ f_{N(k+1)}$ is equal to the image of $f_{k+1} \circ \ldots \circ f_{N(k)}$, we can find $y \in M_{N(k+1)}$ with

  $$a_k - b_k = f_{k+1} \circ \ldots \circ f_{N(k+1)}(y).$$

  Define

  $$b_{k+1} := -f_{k+2} \circ \ldots \circ f_{N(k+1)}(y).$$

  Then

  $$b_k - f_{k+1} b_{k+1} = b_k + a_k - b_k = a_k \; .$$

  Thus $(a_k) \in \operatorname{im} \xi$.

- If for some i the element a_i is not in the image $f_{i+1} \circ \ldots \circ f_{N(i)} \colon M_{N(i)} \to M_i$, then we consider the sum

  $$a'_i := a_i + f_{i+1} a_{i+1} + \ldots + f_{i+1} \circ \ldots \circ f_{N(i)}(a_{N(i)}).$$

  We check that

  $$\begin{aligned} a_i - (a'_i - f_{i+1}(a'_{i+1})) &= a_i - a_i - f_{i+1}(a_{i+1}) - \ldots - f_{i+1} \circ \ldots \circ f_{N(i)}(a_{N(i)}) \\ &\quad + f_{i+1}(a_{i+1}) + f_{i+1} \circ f_{i+2}(a_{i+2}) + \ldots + f_{i+1} \circ \ldots \circ f_{N(i+1)}(a_{N(i+1)}) \\ &= f_{i+1} \circ \ldots \circ f_{N(i)+1}(a_{N(i)+1}) + \ldots + f_{i+1} \circ \ldots \circ f_{N(i+1)}(a_{N(i+1)}) \end{aligned}$$

  and therefore $a_i - (a'_i - f_{i+1}(a'_{i+1}))$ is in the image of $f_{i+1} \circ \ldots \circ f_{N(i+1)}$. As in the preceeding case, we write $a_i - (a'_i - f_{i+1}(a'_{i+1}))$ as $b_i - f_{i+1} b_{i+1}$. Thus

  $$a_i = c_i - f_{i+1}(c_{i+1})$$

  with $c_i := b_i + a'_i$.

□

Examples 2.12.6.

1. If every map f_i is surjective, then the inverse system (M_i, f_i) satisfies the Mittag-Leffler criterion with $N(n) = n + 1$. For instance, the inverse system of rings

$$\mathbb{Z}/p\mathbb{Z} \longleftarrow \mathbb{Z}/p^2\mathbb{Z} \longleftarrow \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \dots$$

satisfies this condition. The inverse limit of this system is a ring, called the <u>p-adic integers</u>. These are denoted by $\hat{\mathbb{Z}}_p$ and they are the p-adic completion of the ring of integers.

2. We want to apply Lemma 2.12.5 to inverse systems of cochain complexes.

Assume that X is a CW complex and that $(X_n)_n$ is a sequence of subcomplexes with $X_n \subset X_{n+1}$ and $X = \bigcup_n X_n$. For instance, we could take $X_n = X^n$, the n-skeleton of X. Consider for each n the cochain complex

$$S_n^*(X) := S^*(X_n).$$

The inclusion maps $X_n \subset X_{n+1}$ induce maps of cochain complexes

$$f_{n+1} \colon \quad S_{n+1}^*(X) \longrightarrow S_n^*(X)\ .$$

We therefore have an inverse system

$$S_0^*(X) \xleftarrow{f_1} S_1^*(X) \xleftarrow{f_2} \dots$$

which are maps of cochain complexes, i.e. commute with the coboundary maps

$$\begin{array}{ccc} S^i_{n+1}(X) & \xrightarrow{f_{n+1}} & S^i_n(X) \\ \downarrow \delta & & \downarrow \delta \\ S^{i+1}_{n+1}(X) & \xrightarrow{f_{n+1}} & S^{i+1}_n(X). \end{array}$$

Lemma 2.12.7.
If (C_n^*, f_n) is an inverse system of cochain complexes, such that for every cochain degree m the inverse system (C_n^m, f_n) satisfies the Mittag-Leffler condition, then the sequence

$$0 \to \varprojlim{}^1 H^{m-1}(C_n^*) \longrightarrow H^m(\varprojlim C_n^*) \longrightarrow \varprojlim H^m(C_n^*) \to 0$$

is exact.

Proof.
We consider for fixed degree m the two obvious exact sequences

$$0 \to B_n^m \longrightarrow Z_n^m \longrightarrow H^m(C_n^*) \to 0 \tag{2}$$

and

$$0 \to Z_n^m \longrightarrow C_n^m \xrightarrow{\delta_n} B_n^{m+1} \to 0. \tag{3}$$

1. As the C_n^m are supposed to satisfy the Mittag-Leffler condition, Lemma 2.12.5 implies that
$$\varprojlim{}^1 C_n^m = 0, \qquad \text{for all } m. \tag{4}$$
Lemma 2.12.4 applied to the short exact sequence (3) implies that the sequence
$$\varprojlim{}^1 C_n^m \longrightarrow \varprojlim{}^1 B_n^{m+1} \to 0$$
is exact and thus $\varprojlim{}^1 B_n^{m+1} = 0$. Therefore the sequence (2) yields that
$$\varprojlim{}^1 Z_n^m \cong \varprojlim{}^1 H^m(C_n^*)\ .$$

2. In addition we know, again from Lemma 2.12.4 applied to (3), that the sequence
$$0 \to \varprojlim Z_n^m \longrightarrow \varprojlim C_n^m \xrightarrow{\varprojlim \delta_n} \varprojlim B_n^{m+1}$$
is exact and hence the inverse limit of the m-cocycles is equal to the module of m-cocycles in the inverse limit complex, i.e.
$$\varprojlim Z_n^m \cong Z^m(\varprojlim C_n^*)\ .$$

3. As the lim-one term on the inverse system of coboundaries is trivial by 1., we obtain from (2) that the sequence
$$0 \to \varprojlim B_n^m \longrightarrow \varprojlim Z_n^m \longrightarrow \varprojlim H^m(C_n^*) \to 0$$
is exact as well. Lemma 2.12.4 applied to (3) tells us that the kernel of the connecting homomorphism
$$\partial\colon \varprojlim B_n^m \longrightarrow \varprojlim{}^1 Z_n^{m-1} \to 0 \qquad (*)$$
is isomorphic to the image of the map
$$\varprojlim C_n^{m-1} \xrightarrow{\varprojlim \delta_n} \varprojlim B_n^m$$
and thus to the coboundaries, i.e.
$$B^m(\varprojlim C_n^*) \cong \ker \partial\ .$$
Thus, we get an inclusion $B^m(\varprojlim C_n^*) \subset \varprojlim B_n^m$. Therefore we get the following sequence of inclusions
$$B^m(\varprojlim C_n^*) \subset \varprojlim B_n^m \subset \varprojlim Z_n^m = Z^m(\varprojlim C_n^*) \qquad (**)\ ,$$
where the last identity is 2.

4. Recall that for any inclusion $A \subset B \subset C$ of submodules, the diagram
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\text{id}} & A & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & C/B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & B/A & \longrightarrow & C/A & \longrightarrow & C/B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

has exact rows and the first two lines are exact so that by the nine-lemma the last row is exact.

5. Applied to the inclusion $(**)$ of submodules this yields the short exact sequence
$$0 \to \frac{\varprojlim B_n^m}{B^m(\varprojlim C_n^*)} \longrightarrow \frac{Z^m(\varprojlim C_n^*)}{B^m(\varprojlim C_n^*)} \longrightarrow \frac{\varprojlim Z_n^m}{\varprojlim B_n^m} \to 0$$
is exact. The middle term is the cohomology $H^m(\varprojlim C_n^*)$ of the inverse limit complex. The right term is isomorphic to $\varprojlim H^m(C_n^*)$ and the left term is isomorphic to the lim-one term $\varprojlim{}^1 H^{m-1}(C_n^*)$ because the kernel of ∂ is $B^m(\varprojlim C_n^*)$ and thus by $(*)$ the quotient is $\varprojlim{}^1 Z_n^{m-1} \cong \varprojlim{}^1 H^{m-1}(C_n^*)$ by 2.

□

Theorem 2.12.8 (Milnor sequence)**.**
If X is a CW complex with a filtration $X_0 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ of subcomplexes with $X = \bigcup_n X_n$, then the sequence
$$0 \to \varprojlim{}^1 H^{m-1}(X_n; G) \longrightarrow H^m(X; G) \longrightarrow \varprojlim H^m(X_n; G) \to 0$$
is exact for all abelian groups G.

Proof.

- We define $C_n^* = \mathrm{Hom}(S_*(X_n), G)$. This system of cochain complexes satisfies the Mittag-Leffler condition because the inclusions of chains
$$S_m(X_n) \hookrightarrow S_m(X_{n+1})$$
dualize to epimorphisms
$$\mathrm{Hom}(S_m(X_{n+1}), G) \longrightarrow \mathrm{Hom}(S_m(X_n), G) \ .$$
- The only thing we have to show to apply Lemma 2.12.7 is that for the term in the middle
$$H^m(X; G) \cong H^m(\varprojlim \mathrm{Hom}(S_*(X_n), G)) \ .$$
By Remark 2.12.3.1, the inverse limit has a universal property dual to the one of the direct limit and the maps induced from the inclusions $X_n \hookrightarrow X$
$$\mathrm{Hom}(S_*(X), G) \longrightarrow \mathrm{Hom}(S_*(X_n), G)$$
induce a homomorphism
$$\mathrm{Hom}(S_*(X), G) \to \varprojlim \mathrm{Hom}(S_*(X_n), G) \ .$$
- To see that this is an isomorphism, first note that if a space X is the union of a directed system of subspaces X_α with the property that each compact subset of X is contained in some X_α, then for homology the map
$$\varinjlim S_i(X_\alpha; G) \to S_i(X; G)$$
is an isomorphism for all abelian groups G. (Note that by Corollary 1.11.16.1, this applies to a filtration by subcomplexes.)

  Indeed, for surjectivity, represent a cycle on X by a sum of finitely many simplices. The union of their images is compact in X and thus contained in some X_α, which ensures surjectivity. For injectivity, if a cycle in some X_α is a boundary in X, by compactness, it is a boundary in some $X_\beta \supset X_\alpha$, hence represents zero in $\varinjlim H_i(X_\alpha; G)$.

- The dual of this argument then shows the claim.

□

Example 2.12.9.
We consider the infinite complex projective space $\mathbb{C}P^\infty$. It is defined as a limit $\varinjlim \mathbb{C}P^n$. This space has a natural structure of a CW complex with a cell in every even dimension. To apply Theorem 2.12.8, we consider the skeleton filtration, i.e.

$$X_0 = \mathrm{pt} \subset X_1 = \mathbb{C}P^1 \subset X_2 = \mathbb{C}P^2 \subset \dots$$

Thus X_n is the $2n$-skeleton of $\mathbb{C}P^\infty$. The Milnor sequence 2.12.8 in this case is for each m

$$0 \to \varprojlim{}^1 H^{m-1}(\mathbb{C}P^n) \longrightarrow H^m(\mathbb{C}P^\infty) \longrightarrow \varprojlim H^m(\mathbb{C}P^n) \to 0 \quad (*) \ .$$

However, the maps $H^{m-1}(\mathbb{C}P^{n+1}) \to H^{m-1}(\mathbb{C}P^n)$ are surjective. By Remark 2.12.6.1, this inverse system satisfies the Mittag-Leffler condition and thus by Lemma 2.12.5

$$\varprojlim{}^1 H^{m-1}(\mathbb{C}P^n) = 0$$

and therefore $(*)$ gives isomorphisms

$$H^m(\mathbb{C}P^\infty) \cong \varprojlim H^m(\mathbb{C}P^n).$$

The inverse limit of truncated polynomial rings $\mathbb{Z}[\alpha]/\alpha^{n+1}$ is isomorphic to the ring of formal power series. Recall that for a commutative ring R, the ring $R[[z]]$ of formal power series is the set $R^{\mathbb{N}}$ of sequences with value in R with addition $(a_n) + (b_n) = (a_n + b_n)$ and multiplication given by a Cauchy product $(a_n) \cdot (b_n) = (\sum_{k=1}^n a_k b_{n-k})$.

Corollary 2.12.10.

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[\alpha]], \quad \text{with } |\alpha| = 2 \ ,$$

where $\mathbb{Z}[[\alpha]]$ denotes the ring of formal power series in α.

The arguments are analogous for the infinite real and quaternionic projective spaces, $\mathbb{R}P^\infty$ and $\mathbb{H}P^\infty$.

Corollary 2.12.11.

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[[\alpha]], \quad \text{with } |\alpha| = 1$$

and

$$H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[[\alpha]], \quad \text{with } |\alpha| = 4.$$

Remark 2.12.12.

1. At times, the cohomology of a space is considered as a direct sum

$$H^*(X;G) = \bigoplus_{n \geqslant 0} H^n(X;G) \ .$$

From that point of view, we only have finite sums in $H^*(X;G)$ so that this interpretation yields the identification of $H^m(\mathbb{C}P^\infty)$ and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ as a polynomial ring: the formulae

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\alpha] \quad \text{with } |\alpha| = 2$$

and

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha] \text{ with } |\alpha| = 1.$$

can be found in the literature as well.

2. However, viewing homology $H_*(X)$ as a direct sum $\bigoplus_n H_n(X)$ and for free $H_*(X)$ the cohomology as a dual, then the description of $H^*(X)$ as a product $\prod_n H^n(X)$ is more natural.

2.13 Lens spaces

Observation 2.13.1.

1. Let $m \in \mathbb{N}$ and let $\ell_1, \dots, \ell_n$ be natural numbers with $\gcd(m, \ell_i) = 1$ for all i and assume $n \geqslant 2$. Choose a primitive n-root of unity $\zeta := e^{\frac{2\pi i}{m}}$ and define an action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{S}^{2n-1}$ by
$$\begin{array}{rcl} \varrho\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{S}^{2n-1} & \to & \mathbb{S}^{2n-1}, \\ (\zeta; z_1, \dots, z_n) & \mapsto & (\zeta^{\ell_1} z_1, \dots, \zeta^{\ell_n} z_n) \ , \end{array}$$
where we view $\mathbb{S}^{2n-1}$ as a subspace of $\mathbb{C}^n$.

2. This action is free: if $\varrho(\zeta^r; z_1, \dots, z_n) = (z_1, \dots, z_n)$ for some $(z_1, \dots, z_n)$, then we have $\zeta^{r\ell_i} z_i = z_i$ for all i. Since there exists i such that $z_i \neq 0$, we find $\zeta^{r\ell_i} = 1$ and thus $r\ell_i = 0 \bmod m$. Since ℓ_i is invertible modulo m, we find $r = 0 \bmod m$.

Example 2.13.2.
If $m = 2$, then the all integers ℓ_i must be odd and therefore the action
$$\varrho\colon \mathbb{Z}/2\mathbb{Z} \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$$
is the antipodal action.

We consider the quotient spaces $\mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$.

Definition 2.13.3
The space $L = L(m; \ell_1, \dots, \ell_n) = \mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$ with the action as described in Observation 2.13.1 as above is called lens space with parameters $(m; \ell_1, \dots, \ell_n)$.

Remarks 2.13.4.

1. For $m = 2$ we get the real projective spaces $L(2; \ell_1, \dots, \ell_n) = \mathbb{R}P^{2n-1}$ as lens spaces.

2. The classical case is the three manifold case: For integers p, q with $\gcd(p, q) = 1$ one considers $L(p, q) := L(p; 1, q) = \mathbb{S}^3/\mathbb{Z}_p$ with $(\zeta; z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_2)$

3. Note that the projection map $\pi\colon \mathbb{S}^{2n-1} \longrightarrow L(m; \ell_1, \dots, \ell_n)$ is a covering map, because the $\mathbb{Z}/m\mathbb{Z}$-action is free.

We now want to consider CW structures on lens spaces that generalize the CW structures on projective spaces.

Observation 2.13.5.

1. We start with a CW structure on $\mathbb{S}^1$ that has m zero cells $\{e^{\frac{2\pi ij}{m}}, 1 \leqslant j \leqslant m\}$ and m one cells.

2. Let B_j^{2n-2} be the subset of $\mathbb{C}^n$
$$\begin{array}{rcl} B_j^{2n-2} & := & \{\cos\theta(0, \dots, 0, e^{\frac{2\pi ij}{m}}) + \sin\theta(z_1, \dots, z_{n-1}, 0) | \\ & & 0 \leqslant \theta \leqslant \pi/2, (z_1, \dots, z_{n-1}) \in \mathbb{S}^{2n-3}\}, \end{array}$$
i.e., we connect the point $(0, \dots, 0, e^{\frac{2\pi ij}{m}})$ with all the points $(z_1, \dots, z_{n-1}) \in \mathbb{S}^{2n-3}$ via quarters of a circle. Thus we obtain a space homeomorphic to a $(2n-2)$-dimensional disc, $B_j^{2n-2} \cong \mathbb{D}^{2n-2}$. A calculation shows that $B_j^{2n-2} \subset \mathbb{S}^{2n-1}$.

3. If we connect all points on the circular arc in $\mathbb{S}^1$ between $e^{\frac{2\pi ij}{m}}$ and $e^{\frac{2\pi i(j+1)}{m}}$ with $\mathbb{S}^{2n-3}$, again via quarters of a circle, we get a $(2n-1)$-dimensional ball B_j^{2n-1} contained in $\mathbb{S}^{2n-1}$ with boundary
$$\partial B_j^{2n-1} = B_j^{2n-2} \cup B_{j+1}^{2n-2} \ . \qquad (*)$$
The two boundary discs B_j^{2n-2} and B_{j+1}^{2n-2} are attached to each other via their common boundary $\mathbb{S}^{2n-3}$. Thus B_j^{2n-1} looks like a $(2n-1)$-dimensional lens. The union of all B_j^{2n-1} is $\mathbb{S}^{2n-1}$.

4. We have to understand the $\mathbb{Z}/m\mathbb{Z}$-action ϱ on these cells. It restricts to the subspace $\mathbb{S}^{2n-3}$, i.e. $\varrho(\mathbb{S}^{2n-3}) \subset \mathbb{S}^{2n-3}$. The arcs between the points $e^{\frac{2\pi ij}{m}}$ and $e^{\frac{2\pi i(j+1)}{m}}$ on $\mathbb{S}^1$ are permuted by ϱ and therefore ϱ permutes the $(2n-2)$-dimensional balls B_j^{2n-2} and the $(2n-1)$-dimensional balls B_j^{2n-1}.

   For any $r \in \mathbb{N}$ with $r\ell_n = 1 \bmod m$, the map ϱ^r has order m as well and
$$\varrho^r|_{B_j^{2n-2}} \colon B_j^{2n-2} \longrightarrow B_{j+1}^{2n-2},$$
   because
$$\zeta^{r\ell_n} e^{\frac{2\pi ij}{m}} = e^{\frac{2\pi i r\ell_n}{m}} e^{\frac{2\pi ij}{m}} = e^{\frac{2\pi i(j+1)}{m}} .$$
   Thus, ϱ^r identifies the two faces of B_j^{2n-1}, cf. $(*)$. Each of the balls B_j^{2n-1} is a fundamental domain of the ϱ^r-action. Thus
$$L \cong B_j^{2n-1}/\varrho^r$$
   for any $j = 1, \dots, m$.

5. There is a natural inclusion
$$L(m; \ell_1, \dots, \ell_{n-1}) \subset L(m; \ell_1, \dots, \ell_n)$$
   which is given by mapping the class $[(z_1, \dots, z_{n-1})]$ to $[(z_1, \dots, z_{n-1}, 0)]$. Representing the $(2n-3)$-dimensional lens space $L(m; \ell_1, \dots, \ell_{n-1})$ as $B_j^{2n-3}/\sim$, we see that we can build $L(m; \ell_1, \dots, \ell_n)$ out of $L(m; \ell_1, \dots, \ell_{n-1})$ by attaching the $(2n-1)$-cell B_j^{2n-1} and a $(2n-2)$-cell B_j^{2n-2}. Note that we really just have to take one of the latter, because B_j^{2n-2} is identified with its neighbour B_{j-1}^{2n-2} in the quotient.

   Inductively we get a cell structure of L with one cell in each dimension up to $2n-1$.

Example 2.13.6.
For $n = 2$, the lens spaces are quotients of $\mathbb{S}^3$. Let $m = 5$ and $\ell_1 = 1$ and $\ell_2 = 2$, so $\zeta = e^{\frac{2\pi i}{5}}$.

Then the B_j^3 are 3-balls with boundary B_j^2 and B_{j+1}^2. The 2-balls B_j^2 consist of elements $\cos\theta(0, e^{\frac{2\pi ij}{5}}) + \sin\theta(z, 0)$ for $z \in \mathbb{S}^1$ and $0 \leqslant \theta \leqslant \frac{\pi}{2}$; these are the two-dimensional discs
$$(\sin\theta z, \cos\theta e^{\frac{2\pi ij}{5}}) \in \mathbb{S}^3 \subset \mathbb{C}^2 \ .$$

Observation 2.13.7.

1. Let us consider the cellular chain complex of the lens spaces. In Observation 2.13.5.5, we constructed a CW structure such that
$$C_*(L) = \mathbb{Z}, \quad * = 0, \dots, 2n-1 \ .$$
   Let σ^k be the cell corresponding to the ball B_j^k. We need to compute the boundary maps.

2. The top cell has trivial boundary,

$$d(\sigma^{2n-1}) = \sigma^{2n-2} - \sigma^{2n-2} = 0$$

because the topological boundary of B_j^{2n-1} is the union of two balls one dimension lower which are identified in the quotient.

3. The boundary of the cell σ^{2n-2} is $\mathbb{S}^{2n-3}$ and the attaching map is the quotient map

$$\mathbb{S}^{2n-3} \longrightarrow L(m; \ell_1, \ldots, \ell_{n-1}) \ .$$

The action ϱ permutes the cells cyclically, and we get degree m:

$$d(\sigma^{2n-2}) = m\sigma^{2n-3} \ .$$

By induction we see that the boundary maps are given by multiplication by zero respectively m. Thus the homology of the lens space is the homology of the cellular complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \ldots \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

and thus

$$H_*(L(m; \ell_1, \ldots, \ell_n)) = \begin{cases} \mathbb{Z}, & * = 0, 2n-1, \\ \mathbb{Z}/m\mathbb{Z}, & * \text{ odd and } < 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we also get $H_1(L) = \pi_1(L) = \mathbb{Z}/m\mathbb{Z}$ from covering theory because $\pi_1(\mathbb{S}^{2n-1}) = 0$ for $n \geqslant 2$ and thus $\mathbb{S}^{2n-1}$ is a universal cover of L.

4. As the top homology group is $\mathbb{Z}$, Theorem 2.6.11 implies that lens spaces are compact connected orientable manifolds of dimension $2n-1$.

The universal coefficient theorem 2.2.5 immediately gives for cohomology with coefficients in $\mathbb{Z}_m$:

Lemma 2.13.8.
The additive cohomology groups are

$$H^*(L; \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m\mathbb{Z}, & \text{for all degrees } 0 \leqslant * \leqslant 2n-1 \\ 0, & * > 2n-1. \end{cases}$$

Note that the homology groups of L with coefficients in $\mathbb{Z}/m\mathbb{Z}$ are isomorphic to the cohomology groups just by using the universal coefficient theorem 1.14.17

$$H_k(L; \mathbb{Z}/m\mathbb{Z}) \cong H_k(L; \mathbb{Z}) \otimes \mathbb{Z}/m\mathbb{Z} \oplus \mathrm{Tor}(H_{k-1}(L), \mathbb{Z}/m\mathbb{Z})$$

or by applying Poincaré duality 2.8.3, since L is compact and orientable.

We now focus on the case when $m = p$ is a prime.

Proposition 2.13.9.
Let $L = L(p; \ell_1, \ldots, \ell_{n+1})$ be a lens space. Denote by $\alpha \in H^1(L; \mathbb{Z}/p\mathbb{Z})$ and $\beta \in H^2(L; \mathbb{Z}/p\mathbb{Z})$ (additive) generators. The cohomology group $H^j(L; \mathbb{Z}/p\mathbb{Z})$ is then generated by

$$\begin{cases} \beta^i, & \text{for } j = 2i \\ \alpha\beta^i, & \text{for } j = 2i+1. \end{cases}$$

Proof.
We prove the claim by induction on n.

- For $n = 1$, we have a three-dimensional lens space $L = L(p; \ell_1, \ell_2)$. If $\alpha \in H^1(L; \mathbb{Z}/p\mathbb{Z})$ and $\beta \in H^2(L; \mathbb{Z}/p\mathbb{Z})$ are generators, then a cup product pairing argument from Lemma 2.11.4 shows that $\alpha \cup \beta$ is a generator in degree three. We have to understand what α^2 is: if p is odd, then by the graded symmetry of the cup product, we have $\alpha^2 = 0$. For $p = 2$ we know that the lens space is $\mathbb{R}P^3$. In this case, by Corollary 2.11.6, α^2 is a generator of $H^2(L, \mathbb{Z}/2\mathbb{Z})$. Thus, it is equal to β. In all other degrees, the cohomology groups are trivial.

- Assume now that the claim is true up to degree n. We consider the inclusion

$$L(p; \ell_1, \ldots, \ell_n) \hookrightarrow L(p; \ell_1, \ldots, \ell_{n+1}) =: L^{2n+1}.$$

  Up to degree $2n-1$ this inclusion gives rise to an isomorphism on cohomology groups. We know that β^i generates the cohomology groups up in even degrees $j = 2i < 2n-1$ and $\alpha\beta^i$ generates the cohomology groups in odd degrees $j = 2i+1 \leqslant 2n-1$. An argument as for projective spaces, cf. Lemma 2.11.5, then shows that $\beta \cup \beta^{n-1}$ generates $H^{2n}(L^{2n+1}; \mathbb{Z}/p\mathbb{Z})$ and $\beta \cup \alpha\beta^{n-1} = \alpha\beta^n$ generates $H^{2n+1}(L^{2n+1}; \mathbb{Z}/p\mathbb{Z})$.

□

Corollary 2.13.10.

1. As graded rings

$$H^*(L(p; \ell_1, \ldots, \ell_{n+1}); \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \Lambda(\alpha) \otimes \mathbb{Z}/p\mathbb{Z}[\beta]/\beta^{n+1}, & p > 2, \\ \mathbb{Z}/p\mathbb{Z}[\alpha]/\alpha^{2n+2}, & p = 2. \end{cases}$$

2. Fix a prime p and a sequence $(\ell_1, \ell_2, \ldots)$ of integers coprime to p. Let L denote the direct limit of any system of the form

$$L(p; \ell_1, \ldots, \ell_{n+1}) \subset L(p; \ell_1, \ldots, \ell_{n+2}) \subset \ldots$$

   then

$$H^*(L; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \Lambda(\alpha) \otimes \mathbb{Z}/p\mathbb{Z}[[\beta]], & p > 2, \\ \mathbb{Z}/p\mathbb{Z}[[\alpha]], & p = 2. \end{cases}$$

Proof.
The second claim follows with the help of the Milnor sequence 2.12.8 as in Example 2.12.9. □

Remark 2.13.11.
Note that these cohomology groups do not dependent on the ℓ_i's.

Lens spaces of dimension three give rise to important examples of orientable connected and compact 3-manifolds that have the same fundamental group and homology groups, but that are not homotopy equivalent. For instance the lens spaces $L(5; 1, 1)$ and $L(5; 1, 2)$ are not homotopy equivalent (cf. Hatcher, exercise 3.E.2), but have the same fundamental groups and the same homology groups.

Observation 2.13.12.
We can interpret the generator β in terms of the so-called *Bockstein-homomorphism.*

The two short exact sequences of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}/p\mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

give rise to short exact sequences of cochain complexes

$$\begin{array}{lccc} 0 \to & S^*(X;\mathbb{Z}) \to & S^*(X;\mathbb{Z}) \to & S^*(X;\mathbb{Z}/p\mathbb{Z}) \to \;\; 0 \\ 0 \to & S^*(X;\mathbb{Z}/p\mathbb{Z}) \to & S^*(X;\mathbb{Z}/p^2\mathbb{Z}) \to & S^*(X;\mathbb{Z}/p\mathbb{Z}) \to \;\; 0 \end{array}$$

and we get by Lemma 1.5.6 a corresponding long exact sequences of cohomology groups. Let

$$\tilde{\beta}\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$$

be the connecting homomorphism for the first sequence, let

$$\beta\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$$

be the connecting homomorphism for the second sequence and let

$$\rho_*\colon H^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$$

be induced by the reduction of the coefficients mod p. Then β is called the Bockstein homomorphism.

Lemma 2.13.13.
For all n, the diagram

$$\begin{array}{ccc} H^n(X;\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X;\mathbb{Z}) \\ & \searrow^{\beta} & \downarrow{\rho_*} \\ & & H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \end{array}$$

commutes.

Proof.
For the proof just note that the diagram relating the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \xrightarrow{\rho} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\rho} & & \downarrow{\rho_2} & & \downarrow{\mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \end{array}$$

commutes and therefore we obtain the commutativity of the connecting homomorphisms, the naturality statement of Proposition 1.5.5, implies

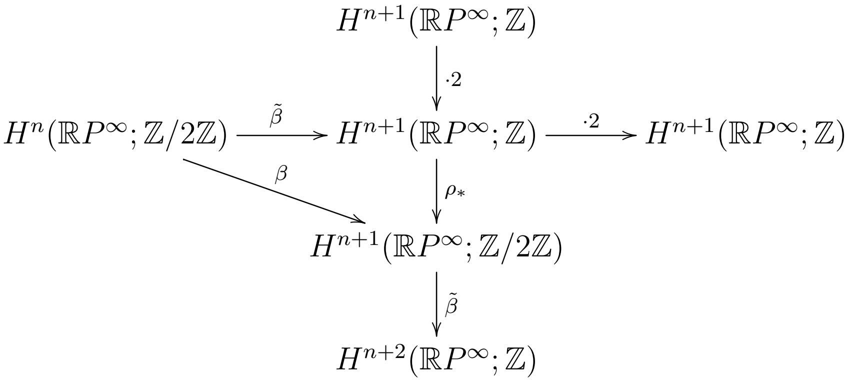
$$\begin{array}{ccc} H^n(X;\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X;\mathbb{Z}) \\ \downarrow{\mathrm{id}} & & \downarrow{\rho_*} \\ H^n(X;\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\beta} & H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}). \end{array}$$

□

With the help of this auxiliary result we will show that the class $\beta \in H^2(L(p;\ell_1,\dots,\ell_{n+1});\mathbb{Z}/p\mathbb{Z})$ in Proposition 2.13.9 is the image of the Bockstein homomorphism applied to α, i.e. $\beta = \beta(\alpha)$. We discuss the example $p = 2$, i.e. the cases of real projective spaces of odd dimension in detail; the cases for odd prime are similar.

Proposition 2.13.14.
The Bockstein homomorphism $\beta\colon H^n(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z})$ is an isomorphism for odd n and is trivial for even n. In particular, $\beta(\alpha) = \alpha^2$.

Proof.
Consider the diagram



- If n is odd, then $n+1 = 2k$ for some k and then $H^{2k}(\mathbb{R}P^\infty;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ so that the multiplication by 2 is trivial. The horizontal exact sequence then implies that $\tilde\beta$ is surjective. But both adjacent groups are $\mathbb{Z}/2\mathbb{Z}$, thus $\tilde\beta$ is an isomorphism, since any surjective endomorphism of $\mathbb{Z}/2\mathbb{Z}$ is an isomorphism.
- For even n, the groups $H^{n+1}(\mathbb{R}P^\infty;\mathbb{Z})$ are trivial, hence in these degrees $\tilde\beta = 0$, and also the Bockstein homomorphism $\beta = \rho_* \circ \tilde\beta$ vanishes for even n.
- The same fact implies that for odd n, the lowest arrow in the exact column is zero. Thus $\rho_*\colon H^{n+1}(\mathbb{R}P^\infty;\mathbb{Z}) \to H^{n+1}(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z})$ is surjective and, by the same arguments, an isomorphism and therefore β is an isomorphism.

□

Remark 2.13.15.
Using that β is a connecting homomorphism and thus defined using a coboundary, one can use the Leibniz rule 2.5.10.3 to show that it is a derivation with respect to the cup product:

$$\beta(\alpha \cup \gamma) = \beta(\alpha) \cup \gamma + (-1)^{|\alpha|}\alpha \cup \beta(\gamma).$$

The Bockstein homomorphism is one example of a cohomology operation.

2.14 A first quick glance at homotopy theory

The definition of a fundamental group has an obvious generalization:

Definition 2.14.1
Denote by $I := [0,1]$ the standard interval and by $I^n \subset \mathbb{R}^n$ the n-dimensional unit cube.

1. *For a space X with base point $x_0 \in X$, we denote by $\pi_n(X, x_0)$ set of homotopy classes of maps*
$$F : (I^n, \partial I^n) \to (X, x_0)\ ,$$

where homotopies are required to satisfy $F_t(\partial I^n) = x_0$ for all $t \in [0,1]$. (For $n = 0$, take I^0 to be a point and $\partial I^0 = \emptyset$ so that $\pi_0(X)$ is the set of path-connected components of X.)

2. *For a subset $A \subset X$ and a base point $x_0 \in A$, take for $n \geqslant 1$ the subset $I^{n-1} \subset \partial I^n$ the points with last coordinate $s_n = 0$; they are homeomorphic to an $(n-1)$-dimensional cube. Finally let J^{n-1} be the closure of the complement $\partial I^n \setminus I^{n-1}$. Then $\pi_n(X, A, x_0)$ is the set of homotopy classes of maps*

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \quad .$$

Observation 2.14.2.

1. Generalizing the case of the fundamental group $n = 1$, we turn $\pi_n(X, x_0)$ into a group with composition defined by operations involving the first coordinate,

$$(f+g)(s_1, s_2, \ldots, s_n) := \begin{cases} f(2s_1, \ldots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, \ldots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

   This is well-defined on homotopy classes; since only the first coordinate is involved, the same arguments as for the fundamental group show that we obtain a group structure. The group $\pi_n(X, x_0)$ is called the n-th homotopy group of X.

2. A sequence of homotopies shows that for $n \geqslant 2$, the group $\pi_n(X, x_0)$ is abelian. Note that for this process, we only need the two coordinates s_1, s_2.

   Similar arguments as for the fundamental group show that the different base points yield isomorphic homotopy groups. Indeed, the fundamental group $\pi_1(X, x_0)$ acts on all groups $\pi_n(X, x_0)$; for $n = 1$ this is the inner action of $\pi_1(X, x_0)$ on itself.

3. In the relative case, the last coordinate s_n plays a special role. For this reason, $\pi_n(X, A, x_0)$ has the structure of a group only for $n \geqslant 2$. It is abelian for $n \geqslant 3$. We call it the relative homotopy group. (The set $\pi_1(X, A, x_0)$ is the set of homotopy classes of paths in X from a varying point in A to the base point x_0.)

Familiar features of the fundamental group can be extended:

Observation 2.14.3.

1. A covering $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces an isomorphism $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ for $n \geqslant 2$.

   Indeed, injectivity follows from [Topologie, 2.7.13] and surjectivity from the fact that the maps $I^n \to X$ factorize to maps $\mathbb{S}^n \to X$ and that $\mathbb{S}^n$ is simply connected for $n \geqslant 2$.

2. As a consequence, $\pi_n(X, x_0)$ with $n \geqslant 2$ vanishes for all spaces X with a contractible cover, e.g. the sphere $\mathbb{S}^1$ with cover $\mathbb{R}$ or the torus T^n with cover $\mathbb{R}^n$. Such spaces are called aspherical.

3. Base-point preserving maps of pairs $\varphi : (X, A, x_0) \to (Y, B, y_0)$ give rise to maps $\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$. They obey the familiar relations $\mathrm{id}_* = \mathrm{id}$ and $(f \circ g)_* = f_* \circ g_*$. Homotopic maps $\varphi, \varphi' : (X, A, x_0) \to (Y, B, y_0)$ give the same maps in relative homotopy, $\varphi_* = \varphi'_*$.

4. We have for arbitrary products $\pi_n(\prod_\alpha X_\alpha) = \prod_\alpha \pi_n(X_\alpha)$.

5. Given an imbedding $i : A \to X$ and a base point $x_0 \in A$, one gets a long exact sequence

$$\dots \pi_n(A, x_0) \xrightarrow{\iota_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \dots \to \pi_0(X, x_0)$$

of homotopy groups. Here $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ and and ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to X$ to I^{n-1}.

6. There is, however, no general analogue of the excision property for homology or the Seifert-van Kampen theorem for the fundamental group, see however Theorem 2.14.8. This makes homotopy groups difficult to compute, even for spheres.

However, homotopy is theoretically important, because it gives strong invariants, in particular for CW complexes:

Theorem 2.14.4 (Whitehead).
If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n \geqslant 0$, then f is a homotopy equivalence.

In the case when f is the inclusion of a subcomplex with the same property, then X is even a deformation retract of Y.

The proof is based on the following

Lemma 2.14.5 (Compression lemma).
Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. Assume that for each n such that $X \setminus A$ has cells of degree n, we have $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative A to a map $X \to B$, i.e. we can homotop the map such that the image is in $B \subset Y$.

Proof.
Assume inductively that f has been homotoped to take the skeleton X^{k-1} to the subspace B. Let ϕ be the characteristic map of a k-cell e^k of $X \setminus A$. The composition

$$f \circ \phi : \quad (\mathbb{D}^k, \partial \mathbb{D}^k) \to (Y, B)$$

can be homotoped relative $\partial \mathbb{D}^k$ into B, since we assumed $\pi_k(Y, B, y_0) = 0$ for all $y_0 \in B$.

This homotopy induces on the quotient space

$$X^{k-1} \sqcup \mathbb{D}^k \to X^{k-1} \cup_\phi e^k$$

a homotopy relative X^{k-1}. We do this for all k-cells of $X \setminus A$ at once, take the constant homotopy on A and get a homotopy of the restriction $f|_{X^k \cup A}$ to a map into B.

Inductively, we settle the case when the dimension of the cells of $X \setminus A$ is bounded. In general, deal with X^k during the t-interval $[1 - 2^{-k}, 1 - 2^{-(k+1)}]$. □

Proof.
of Whitehead's Theorem 2.14.4.

- First suppose that f is the inclusion of a subcomplex. Consider the long exact sequence 2.14.3.5 for the pair (Y, X). Since f induces isomorphisms on the homotopy groups, the relative homotopy groups $\pi_n(Y, X)$ vanish. Applying Lemma 2.14.5 to the identity $(Y, X) \to (Y, X)$ yields a deformation retraction of Y onto X, as claimed.

- Now consider the mapping cylinder M_f of $f : X \to Y$: this is the quotient

$$X \times I \sqcup Y \to M_f$$

under the identification $(x, 1) \sim f(x)$. Thus M_f contains $X = X \times \{0\}$ and Y as subspaces. M_f deformation retracts to Y. Thus f is a composition

$$X \hookrightarrow M_f \to Y$$

of an inclusion and a retraction. A retraction is a homotopy equivalence; thus it suffices to show that M_f retracts onto X, if f induces isomorphisms on the homotopy groups (or, equivalently, if all relative groups $\pi_n(M_f, X)$ vanish).

- If f happens to be cellular, then (M_f, X) is a CW pair and we are done by the first part of the proof. In the general case, one can invoke a theorem that f is homotopic to a cellular map.

□

Remarks 2.14.6.

1. We do not claim that any two CW complexes with isomorphic homotopy group are homotopy equivalent; rather the existence of a map f inducing the isomorphisms in homotopy is required.

   As a counterexample, consider $X = \mathbb{R}P^2$ and $Y = \mathbb{S}^2 \times \mathbb{R}P^\infty$. Both have fundamental group $\mathbb{Z}_2$; their universal covers are $\tilde{X} = \mathbb{S}^2$ and $\tilde{Y} = \mathbb{S}^2 \times \mathbb{S}^\infty$ which are homotopy equivalent, since $\mathbb{S}^\infty$ is contractible. Thus Observation 2.14.3.1 implies that the homotopy groups are all isomorphic.

   But the two spaces cannot be homotopy equivalent, since their homology differs: since $Y = \mathbb{S}^2 \times \mathbb{R}P^\infty$ retracts to $\mathbb{R}P^\infty$, it has non-vanishing homology in infinitely many components, in contrast to X.

2. There is a CW complex, unique up to homotopy, which has the property that it has a single non-vanishing homotopy group G in degree n. Such a space $K(G, n)$ is called an Eilenberg-Mac Lane space. (The group G has to be abelian for $n > 1$.)

   Cohomology classes for a CW complex X correspond bijectively to homotopy classes of maps $X \to K(G, n)$: for any abelian group G and for all CW complexes X, there is for $n > 0$ a natural bijection

$$T : [X, K(G, n)] \to H^n(X; G) \; .$$

   More precisely, there is a distinguished class $\alpha \in H^n(K(G, n); G)$ such that $T(f) = f^*(\alpha)$. This is a strong link between homotopy theory and cohomology theory.

3. For CW complexes, we can replace maps within the same homotopy class by cellular maps: every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then the homotopy may be taken to be stationary on the subcomplex A.

   As a consequence, $\pi_n(\mathbb{S}^k) = 0$ for $n < k$. Indeed, with with usual CW structure on spheres consisting of a 0-cell and a top-dimensional cell, cf. Example 1.11.4.3, and the 0-cells as base points, any map $\mathbb{S}^n \to \mathbb{S}^k$ can be homotoped fixing the based point to a cellular map which is thus constant.

We finally state a generalization of Proposition 1.3.8 which relates homotopy groups to homology groups:

Theorem 2.14.7 (Hurewicz)**.**
If a space X is $(n-1)$-connected with $n \geqslant 2$, i.e. if $\pi_i(X) = 0$ for all $i \leqslant n-1$, then $\tilde{H}^i(X) = 0$ for $i \leqslant n-1$ and $\pi_n(X) = H_n(X)$.

Thus the first nonzero homotopy and homology group of a simply-connected space occur in the same degree and are isomorphic.

We finally explain the best available analogue of excision:

Theorem 2.14.8.
Let X be a CW complex that is decomposed as the union of subcomplexes A and B with non-empty connected intersection $C := A \cap B$. Suppose that (A, C) is n-connected and (B, C) is m-connected, with $m, n \geqslant 0$. Then the map

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.

This yields

Corollary 2.14.9 (Freudenthal suspension theorem)**.**
The suspension map

$$\pi_i(\mathbb{S}^n) \to \pi_{i+1}(\mathbb{S}^{n+1})$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$. More generally, this holds for the suspension $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ of any $(n-1)$-connected CW complex X. (Note that by Remark 2.14.6.3 $X = \mathbb{S}^n$ is $(n-1)$-connected.)

Proof.
Decompose the suspension ΣX as the union of two cones $C_\pm X$ intersecting in a copy of X. Inclusion gives us by Theorem 2.14.8 a morphism

$$\pi_{i+1}(C_+X, X) \to \pi_{i+1}(\Sigma X, C_-X) \qquad (*)\ .$$

The long exact sequence for (C_+X, X)

$$\ldots \to \pi_i(X) \to \pi_i(C_+X) \to \pi_i(C_+X, X) \to \pi_{i-1}(X) \to \pi_{i-1}(C_+X) \to \ldots$$

together with the fact that the cone C_+X is contractible shows for the left hand side of $(*)$ the isomorphism $\pi_{i+1}(C_+X, X) \cong \pi_i(X)$. The long exact sequence for $(\Sigma X, C_-X)$

$$\ldots \to 0 = \pi_i(C_-X) \to \pi_i(\Sigma X) \to \pi_i(\Sigma X; C_-X) \to \pi_{i-1}(C_-X) = 0 \to \ldots$$

shows that $\pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X)$.

Now suppose that X is $(n-1)$-connected. Then the first long exact sequence implies that the pairs $(C_\pm X, X)$ are n-connected. By Theorem 2.14.8 the inclusion map is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$.

□

Corollary 2.14.10.
We have $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ for all $n \geqslant 1$, with the identity map as a generator. In particular, the degree provides an isomorphism $\pi_n(\mathbb{S}^n) \to \mathbb{Z}$.

Proof.
From Corollary 2.14.9, we know that in the suspension sequence

$$\pi_1(\mathbb{S}^1) \to \pi_2(\mathbb{S}^2) \to \pi_3(\mathbb{S}^3) \to \dots$$

the first map is surjective and all other maps are isomorphisms. Since $\pi_1(\mathbb{S}^1)$ is infinite cyclic, generated by the identity map, it follows that all other groups $\pi_n(\mathbb{S}^n)$ are finite or infinite cyclic groups generated by the identity map.

The group cannot be finite: there exist base-point preserving maps $\mathbb{S}^n \to \mathbb{S}^n$ of arbitrary degree, cf. Lemma 1.10.3 for a weaker statement, and the degree is a homotopy invariant.

The degree map is an isomorphism, since the map $z \mapsto z$ of $\mathbb{S}^1$ has degree 1 and so do by Lemma 1.10.3 its iterated suspensions. □

A English - German glossary

English	German
boundary	Rand
chain complex	Kettenkomplex
chart	Karte
cone	Kegel
connecting homomorphism	Verbindungshomomorphismus
cycle	Zykel
excision	Ausschneidung
hairy ball theorem	Satz vom gekämmten Igel
lens space	Linsenraum
manifold	Mannigfaltigkeit
skeleton	Gerüst
support	Träger
suspension	Einhängung

Index