Algebra and Geometry of LG Orbifolds for Invertible Polynomials in Three Variables

Atsushi TAKAHASHI

OSAKA UNIVERSITY

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Invertible polynomials

Let $f = f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial. This means that there are *positive* integers w_1, w_2, w_3, d such that

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \lambda^{w_3} x_3) = \lambda^d f(x_1, x_2, x_3), \ \lambda \in \mathbb{C}^*$$

Definition 1

A weighted homogeneous polynomial f is called *invertible* if

1. The number of variables = the number of monomials

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} a_i \prod_{j=1}^{3} x_j^{E_{ij}}, \ a_i \in \mathbb{C}^*, \ E_{ij} \in \mathbb{Z}_{\geq 0}. \qquad (0.1)$$

2. The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.
3. f defines an isolated singularity at the origin.

Definition 2

Let f be an invertible polynomial. The *canonical system of weights* is the system of weights $(w_1, w_2, w_3; d)$ given by the unique solution of the equation

$$E \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \det(E) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ d := \det(E). \tag{0.2}$$

Maximal grading L_f

Definition 3

The *maximal grading* L_f of f is the abelian group defined by

$$L_f := \left(\bigoplus_{i=1}^{3} \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{f} \right) \Big/ M_f , \tag{0.3}$$

where M_f is the subgroup generated by the elements

$$\vec{f} - \sum_{j=1}^{3} E_{ij}\vec{x_j}, \quad i = 1, 2, 3. \tag{0.4}$$

L_f is an abelian group of rank one which is not necessarily free.

Definition 4

The *degree map* is the surjective homomorphism given by

$$\deg : L_f \longrightarrow \mathbb{Z}, \quad \vec{x}_i \mapsto \frac{w_i}{c_f}, \tag{0.5}$$

where $c_f := \gcd(w_1, w_2, w_3, d)$.

Definition 5

The element $\vec{\varepsilon}_f \in L_f$ defined by

$$\vec{\varepsilon}_f := \left(\sum_{i=1}^{n} \vec{x}_i \right) - \vec{f} \tag{0.6}$$

is called the *inverse dualizing element*. Set

$$\varepsilon_f := \deg(\vec{\varepsilon}_f). \tag{0.7}$$

Examples

Type	$f(x,y,z)$	Singularity
I	$x^2+y^2+z^{2k+1},\ k\geq 1$	A_{2k}
	$x^2+y^2+z^{2k},\ k\geq 2$	A_{2k-1}
	$x^2+y^3+z^3$	D_4
	$x^2+y^3+z^4$	E_6
	$x^2+y^3+z^5$	E_8
II	$x^2+y^2+yz^k,\ k\geq 2$	A_{2k-1}
	$x^2+y^{k-1}+yz^2,\ k\geq 4$	D_k
	$x^3+y^2+yz^2$	E_6
	$x^2+y^3+yz^3$	E_7
III	$x^2+zy^2+yz^{k+1},\ k\geq 1$	D_{2k+2}
IV	$x^l+xy+yz^k,\ k,l\geq 2$	A_{kl-1}
	$x^2+xy^k+yz^2,\ k\geq 2$	D_{2k+1}
V	$xy+y^kz+z^lx,\ k,l\geq 1$	A_{kl}

Table : Invertible polynomials with $\varepsilon_f > 0$

Type	$f(x,y,z)$	Singularity
I	$x^2+y^3+z^6$	$\widetilde{E}_8$
	$x^2+y^4+z^4$	$\widetilde{E}_7$
	$x^3+y^3+z^3$	$\widetilde{E}_6$
II	$x^2+y^3+yz^4$	$\widetilde{E}_8$
	$x^2+y^4+yz^3$	$\widetilde{E}_7$
	$x^4+y^2+yz^2$	$\widetilde{E}_7$
	$x^3+y^2+yz^3$	$\widetilde{E}_8$
	$x^3+y^3+yz^2$	$\widetilde{E}_6$
III	$x^2+zy^3+yz^3$	$\widetilde{E}_7$
	$x^3+zy^2+yz^2$	$\widetilde{E}_6$
IV	$x^2+xy^2+yz^3$	$\widetilde{E}_7$
	$x^3+xy^2+yz^2$	$\widetilde{E}_6$
V	$x^2y+y^2z+z^2x$	$\widetilde{E}_6$

Table : Invertible polynomials with $\varepsilon_f = 0$

Definition 6

Define a group $\widetilde{G}_f$ by

$$\widetilde{G}_f := \mathrm{Spec}(\mathbb{C}L_f)(\mathbb{C}), \tag{0.8}$$

where $\mathbb{C}L_f$ denotes the group ring of L_f. Note that

$$\widetilde{G}_f = \left\{ (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 \,\middle|\, \prod_{j=1}^{3} \lambda_j^{E_{1j}} = \prod_{j=1}^{3} \lambda_j^{E_{2j}} = \prod_{j=1}^{3} \lambda_j^{E_{3j}} \right\}. \tag{0.9}$$

We have

$$f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = \lambda f(x_1, x_2, x_3) \tag{0.10}$$

for $(\lambda_1, \lambda_2, \lambda_3) \in \widetilde{G}_f$ where $\lambda := \prod_{j=1}^{3} \lambda_j^{E_{1j}}$.

We have the following short exact sequence

$$1 \longrightarrow G_f \longrightarrow \widetilde{G}_f \longrightarrow \mathbb{C}^* \longrightarrow 1. \tag{0.11}$$

where

$$G_f = \left\{ (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 \;\middle|\; \prod_{j=1}^{3} \lambda_j^{E_{1j}} = \prod_{j=1}^{3} \lambda_j^{E_{2j}} = \prod_{j=1}^{3} \lambda_j^{E_{3j}} = 1 \right\}. \tag{0.12}$$

Note that G_f always contains the exponential grading operator

$$g_0 := \left(\mathbf{e}[\frac{w_1}{d}], \mathbf{e}[\frac{w_2}{d}], \mathbf{e}[\frac{w_3}{d}] \right), \ \mathbf{e}[*] := e^{2\pi\sqrt{-1}*}.$$

Denote by G_0 the subgroup of G_f generated by g_0.

Duality

Definition 7 (Berglund–Hübsch)

$$f^T(x_1, x_2, x_3) := \sum_{i=1}^{3} \prod_{j=1}^{3} x_j^{E_{ij}^T} = \sum_{i=1}^{3} \prod_{j=1}^{3} x_j^{E_{ji}}. \tag{0.13}$$

Definition 8 (Berglund–Henningson)

For a subgroup G of G_f, define the *dual group* G^T by

$$G^T := \mathrm{Hom}(G_f/G, \mathbb{C}^*). \tag{0.14}$$

Proposition 9 (Ebeling–T '12)

1. *There is a natural group isomorphism*

$$G_{f^T} \simeq \mathrm{Hom}(G_f, \mathbb{C}^*). \tag{0.15}$$

2. *For a subgroup* G *of* G_f, *one has* $(G^T)^T = G$.
3. *There is a natural group isomorphism*

$$(G_0)^T \simeq \mathrm{SL}_3(\mathbb{C}) \cap G_{f^T}. \tag{0.16}$$

$$\left(\text{In particular, } G \supset G_0 \implies G^T \subset \mathrm{SL}_3(\mathbb{C}) \cap G_{f^T}.\right)$$

4. $|G_f| = |G_{f^T}| = d = \det E$.
5. $|(G_0)^T| = |\mathrm{SL}_3(\mathbb{C}) \cap G_{f^T}| = c_f$.

Landau–Ginzburg Orbifolds

Let G be a subgroup of G_f.

The pair (f, G) is called a *Landau-Ginzburg orbifold*.

From now on, assume the condition

$$G \supset G_0 \left(\Longleftrightarrow G^T \subset \mathrm{SL}_3(\mathbb{C}) \cap G_{f^T} \right).$$

Mirror Symmetry

Problem 10

1. Compare *algebraic* objects associated to (f, G) with *geometric* objects associated to (f^T, G^T).
2. Describe these objects combinatorially in terms of E and G.

(objects $\ni$ matrix factorization, vanishing cycle, ...)

Algebraic Aspects

Under the assumption $G \supset G_0$, we have an abelian group L such that

- the degree map $\deg : L_f \longrightarrow \mathbb{Z}$ factors through L,
- we have a short exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow \mathbb{C}^* \longrightarrow 1, \tag{0.17}$$

  where $\widetilde{G} := \mathrm{Spec}(\mathbb{C}L)(\mathbb{C})$.

Set $S := \mathbb{C}[x_1, x_2, x_3]$ and $R_f := S/(f)$.
S and R_f are naturally L-graded.

Orbifold Curves

Define a stack $\mathcal{C}_{(f,G)}$ by

$$\mathcal{C}_{(f,G)} := [(\mathrm{Spec}(R_f)\backslash\{0\}) / \mathrm{Spec}(\mathbb{C}L)] . \tag{0.18}$$

Since f defines an isolated singularity at the origin, the curve

$$C_{(f,G)} := (\mathrm{Spec}(R_f)\backslash\{0\}) / \mathrm{Spec}(\mathbb{C}L) \tag{0.19}$$

is smooth.

Definition 11

The genus of $C_{(f,G)}$ is denoted by $g_{(f,G)}$. The orders $a_1, \ldots, a_r$ of the isotropy groups at the isotropic points of $\mathcal{C}_{(f,G)}$ is called the *Dolgachev numbers* of the pair (f, G) and denoted by $A_{(f,G)}$.

Weighted Projective Lines

Let $A = (a_1, \ldots, a_r)$ be an r-tuple of positive integers ($r \geq 3$) and $\Lambda = (\lambda_1, \ldots, \lambda_r)$ be an r-tuple of pairwise distinct elements of $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\}$.

- Define a ring $R_{A,\Lambda}$ by

$$R_{A,\Lambda} := \mathbb{C}[X_1, \ldots, X_r] \,/\, I_\Lambda \,, \tag{0.20a}$$

where I_Λ is an ideal generated by

$$X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}, \quad i = 3, \ldots, r. \tag{0.20b}$$

- Denote by L_A an abelian group defined as the quotient

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \,/\, M_A \,, \tag{0.21a}$$

where M_A is the subgroup generated by the elements

$$a_i \vec{X}_i - a_j \vec{X}_j, \quad 1 \leq i < j \leq r. \tag{0.21b}$$

Definition 12

Let r, A and Λ be as above. Define a stack $\mathbb{P}^1_{A,\Lambda}$ by

$$\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(R_{A,\Lambda}) \backslash \{0\} \right) / \mathrm{Spec}(\mathbb{C} L_A) \right], \qquad (0.22)$$

which is called the *weighted projective line* of type (A, Λ).

Theorem 13 (Geigle–Lenzing)

The category $D^b \mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ admits a full strongly exceptional collection.

Theorem 14 (Ebeling–T '11)

There exists a triple $A' = (a'_1, a'_2, a'_3)$ *of positive integers and an isomorphism of stacks*

$$\mathcal{C}_{(f,G_f)} \cong \mathbb{P}^1_{A'}. \qquad (0.23)$$

In particular, we have $A_{(f,G_f)} = A' = (a'_1, a'_2, a'_3)$.

Theorem 15 (Ebeling–T '12)

Let $H_i \subset G_f$ *be the minimal subgroup containing* G *and the isotropy group of the point* p_i, $i = 1, 2, 3$ *on* $\mathcal{C}_{(f,G_f)}$. *Then we have the following formula for the Dolgachev numbers*:

$$A_{(f,G)} = \left(\frac{a'_i}{|H_i/G|} * |G_f/H_i|, i = 1, 2, 3 \right). \qquad (0.24)$$

Matrix Factorizations

Denote by $\mathrm{HMF}^{L}_{S}(f)$ the triangulated category of L-graded matrix factorizations.

Theorem 16 (Hirano–T)

$\mathrm{HMF}^{L_f}_{S}(f)$ *admits a full strongly exceptional collection.*

Theorem 17 (Kajiura–Saito–T, Ueda, Futaki–Ueda, ...)

If $\varepsilon_f > 0$, *then* $\mathrm{HMF}^{L}_{S}(f)$ *admits a full strongly exceptional collection.*

Type	$f^T(x,y,z)$	G^T	$\mathrm{HMF}^L_S(f)$	Remark
I	$x^2+y^2+z^{2k+1},\ k\geq 1$	$\{1\}$	$D^b(\mathbb{C}\vec{A}_{2k})$	
	$x^2+y^2+z^{2k+1},\ k\geq 1$	$\mathbb{Z}/2\mathbb{Z}$	$D^b(\mathbb{C}\vec{A}_{2k+1})\times D^b(\mathbb{C}\vec{A}_{2k+1})$	
	$x^2+y^2+z^{2k},\ k\geq 2$	$\{1\}$	$D^b(\mathbb{C}\vec{A}_{2k-1})$	
	$x^2+y^2+z^{2k},\ k\geq 2$	$\left\langle(\mathbf{e}[\frac{1}{2}],\mathbf{e}[\frac{1}{2}],0)\right\rangle$	$D^b(\mathbb{C}\vec{A}_{2k})\times D^b(\mathbb{C}\vec{A}_{2k})$	
	$x^2+y^2+z^{2k},\ k\geq 2$	$\left\langle(\mathbf{e}[\frac{1}{2}],0,\mathbf{e}[\frac{1}{2}])\right\rangle$	$D^b(\mathbb{C}\vec{D}_{k+1})$	
	$x^2+y^2+z^{2k},\ k\geq 2$	$\left\langle(0,\mathbf{e}[\frac{1}{2}],\mathbf{e}[\frac{1}{2}])\right\rangle$	$D^b(\mathbb{C}\vec{D}_{k+1})$	
	$x^2+y^2+z^{2k},\ k\geq 2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$D^b(\mathbb{C}\vec{A}_{2k-1})$	KST
	$x^2+y^3+z^3$	$\{1\}$	$D^b(\mathbb{C}\vec{D}_4)$	
	$x^2+y^3+z^3$	$\mathbb{Z}/3\mathbb{Z}$	$D^b(\mathbb{C}\vec{D}_4)$	KST
	$x^2+y^3+z^4$	$\{1\}$	$D^b(\mathbb{C}\vec{E}_6)$	
	$x^2+y^3+z^4$	$\mathbb{Z}/2\mathbb{Z}$	$D^b(\mathbb{C}\vec{E}_6)$	KST
	$x^2+y^3+z^5$	$\{1\}$	$D^b(\mathbb{C}\vec{E}_8)$	
II	$x^2+y^2+yz^k,\ k\geq 2$	$\{1\}$	$D^b(\mathbb{C}\vec{A}_{2k-1})$	
	$x^2+y^2+yz^k,\ k\geq 2$	$\mathbb{Z}/2\mathbb{Z}$	$D^b(\mathbb{C}\vec{D}_{k+1})$	KST
	$x^2+y^{k-1}+yz^2,\ k\geq 4$	$\{1\}$	$D^b(\mathbb{C}\vec{D}_k)$	
	$x^2+y^{k-1}+yz^2,\ k\geq 4$	$\mathbb{Z}/2\mathbb{Z}$	$D^b(\mathbb{C}\vec{A}_{2k-3})$	KST
	$x^3+y^2+yz^2$	$\{1\}$	$D^b(\mathbb{C}\vec{E}_6)$	KST
	$x^2+y^3+yz^3$	$\{1\}$	$D^b(\mathbb{C}\vec{E}_7)$	KST
III	$x^2+zy^2+yz^{k+1},\ k\geq 1$	$\{1\}$	$D^b(\mathbb{C}\vec{D}_{2k+2})$	KST
IV	$x^l+xy+yz^k,\ k,l\geq 2$	$\{1\}$	$D^b(\mathbb{C}\vec{A}_{kl-1})$	KST
	$x^2+xy^k+yz^2,\ k\geq 2$	$\{1\}$	$D^b(\mathbb{C}\vec{D}_{2k+1})$	KST
V	$xy+y^kz+z^lx,\ k,l\geq 1$	$\{1\}$	$D^b(\mathbb{C}\vec{A}_{kl})$	KST

Table : $\mathrm{HMF}^L_S(f)$ for $\varepsilon_f > 0$

Theorem 18 (Orlov's theorem, L-graded version)

Suppose that $\epsilon_f = 0$. Then we have

$$\mathrm{HMF}^L_S(f) \cong D^b\mathrm{coh}(\mathcal{C}_{(f,G)}). \tag{0.25}$$

If $\vec{\varepsilon}_f = \vec{0}$, then $G = G_0$ and $\mathcal{C}_{(f,G)}$ is an elliptic curve.

Theorem 19 (Geigle–Lenzing, T)

Suppose that $\varepsilon_f = 0$ and $\vec{\varepsilon}_f \neq 0$. For some A and Λ,

$$\mathcal{C}_{(f,G)} \cong \mathbb{P}^1_{A,\Lambda}. \tag{0.26}$$

Corollary 20

Suppose that $\varepsilon_f = 0$ and $\vec{\varepsilon}_f \neq 0$. Then $\mathrm{HMF}^L_S(f)$ admits a full strongly exceptional collection.

(also by Oda–Saito–T without using Orlov's theorem)

$\mathrm{HMF}^L_S(f)$ for $\varepsilon_f = 0$

Type	$f^T(x,y,z)$	G^T	$\mathrm{HMF}^L_S(f)$	Remark
I	$x^2+y^3+z^6$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,3,6)})$	Ueda
	$x^2+y^3+z^6$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^2+y^3+z^6$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^2+y^3+z^6$	$\mathbb{Z}/6\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^2+y^4+z^4$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	Ueda
	$x^2+y^4+z^4$	$\left\langle(\mathbf{e}[\frac{1}{2}],\mathbf{e}[\frac{1}{2}],0)\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^2+y^4+z^4$	$\left\langle(\mathbf{e}[\frac{1}{2}],0,\mathbf{e}[\frac{1}{2}])\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^2+y^4+z^4$	$\left\langle(0,\mathbf{e}[\frac{1}{2}],\mathbf{e}[\frac{1}{2}])\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^2+y^4+z^4$	$\left\langle(\mathbf{e}[\frac{1}{2}],\mathbf{e}[\frac{1}{4}],\mathbf{e}[\frac{1}{4}])\right\rangle$	$D^b\mathrm{coh}(E)$	Orlov
	$x^2+y^4+z^4$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+y^3+z^3$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	Ueda
	$x^3+y^3+z^3$	$\left\langle(0,\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{2}{3}])\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+y^3+z^3$	$\left\langle(\mathbf{e}[\frac{2}{3}],0,\mathbf{e}[\frac{1}{3}])\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+y^3+z^3$	$\left\langle(\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{2}{3}],0)\right\rangle$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+y^3+z^3$	$\left\langle(\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{1}{3}])\right\rangle$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+y^3+z^3$	$(\mathbb{Z}/3\mathbb{Z})^2$	$D^b\mathrm{coh}(E)$	Orlov

Type	$f^T(x,y,z)$	G^T	$\mathrm{HMF}^L_S(f)$	Remark
II	$x^2+y^3+yz^4$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,3,6)})$	
	$x^2+y^3+yz^4$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^2+y^3+yz^4$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^2+y^3+yz^4$	$\mathbb{Z}/6\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^2+y^4+yz^3$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^2+y^4+yz^3$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^2+y^4+yz^3$	$\mathbb{Z}/4\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^4+y^2+yz^2$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^4+y^2+yz^2$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^4+y^2+yz^2$	$\mathbb{Z}/4\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+y^2+yz^3$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,3,6)})$	
	$x^3+y^2+yz^3$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+y^2+yz^3$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^3+y^2+yz^3$	$\mathbb{Z}/6\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+y^3+yz^2$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+y^3+yz^2$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov

Type	$f^T(x,y,z)$	G^T	$\mathrm{HMF}^L_S(f)$	Remark
III	$x^2+zy^3+yz^3$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^2+zy^3+yz^3$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	
	$x^2+zy^3+yz^3$	$\mathbb{Z}/4\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+zy^2+yz^2$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+zy^2+yz^2$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
IV	$x^2+xy^2+yz^3$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,4,4)})$	
	$x^2+xy^2+yz^3$	$\mathbb{Z}/2\mathbb{Z}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(2,2,2,2)})$	Geigle–Lenzing (Thm 16)
	$x^2+xy^2+yz^3$	$\mathbb{Z}/4\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
	$x^3+xy^2+yz^2$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^3+xy^2+yz^2$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov
V	$x^2y+y^2z+z^2x$	$\{1\}$	$D^b\mathrm{coh}(\mathbb{P}^1_{(3,3,3)})$	
	$x^2y+y^2z+z^2x$	$\mathbb{Z}/3\mathbb{Z}$	$D^b\mathrm{coh}(E)$	Orlov

Geometric Aspects

Consider f^T as a holomorphic map:

$$f^T : \mathbb{C}^3 \longrightarrow \mathbb{C}. \tag{0.27}$$

Since G^T is a subgroup of $SL_3(\mathbb{C})$, we obtain the following holomorphic map

$$\widetilde{f}^T : \widetilde{\mathbb{C}^3/G} \longrightarrow \mathbb{C}. \tag{0.28}$$

where $\widetilde{\mathbb{C}^3/G}$ is a crepant resolution of $\mathbb{C}^3/G$ (e.g. $G\text{-}\mathrm{Hilb}(\mathbb{C}^3)$).

Want to study the relative homology

$$H_i(\widetilde{\mathbb{C}^3/G}, (\widetilde{f}^T)^{-1}(1); \mathbb{Q}), \;\; i = 2, 3, 4.$$

However, it is so difficult in general.
Instead, we shall consider the holomorphic map

$$g := \widetilde{f}^T - cx_1x_2x_3 : \widetilde{\mathbb{C}^3/G} \longrightarrow \mathbb{C}, \;\; c \gg 0, \tag{0.29}$$

and

$$H_i(\widetilde{\mathbb{C}^3/G}, g^{-1}(1); \mathbb{Q}), \;\; i = 2, 3, 4.$$

Theorem 21 (Ebeling-T '11)

By a "suitable" change of coordinates, $g = f^T - cx_1x_2x_3$ becomes a polynomial of the form

$$g = z_1^{\gamma_1'} + z_2^{\gamma_2'} + z_2^{\gamma_3'} - c' z_1 z_2 z_3, \;\; c' \gg 0, \qquad (0.30)$$

for positive integers $\gamma_1', \gamma_2', \gamma_3'$ given explicitly in terms of E^T.

Remark 22

The group G^T acts diagonally on the new coordinates z_1, z_2, z_3.

Each element $g \in G^T$ has a unique expression of the form

$$g = \mathrm{diag}(e^{\frac{2\pi b_1}{r}}, e^{\frac{2\pi b_2}{r}}, e^{\frac{2\pi b_3}{r}}) \quad \text{with } 0 \leq b_i < r, \tag{0.31}$$

where r is the order of g. The age of g is defined as

$$\mathrm{age}(g) := \frac{1}{r} \sum_{i=1}^{3} b_i. \tag{0.32}$$

Since we assume that $G^T \subset \mathrm{SL}_3(\mathbb{C})$, then $\mathrm{age}(g) = 0, 1, 2$.

Definition 23

The number of elements $g \in G^T$ such that $\mathrm{age}(g) = 1$ and $\mathrm{Fix}(g) = \{0\}$ is denoted by j_{G^T}.

Using the McKay correspondence, we have the following

Theorem 24 (Ebeling-T '13)

$$\dim_{\mathbb{Q}} H_2(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q}) = \dim_{\mathbb{Q}} H_4(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q}) = j_{G^T}$$

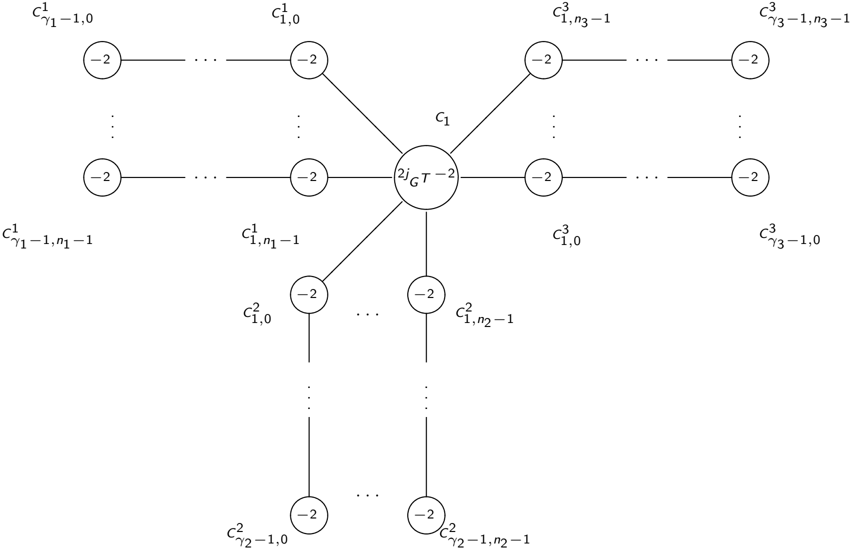
Since the map $H_3(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q}) \longrightarrow H_2(g^{-1}(1); \mathbb{Q})$ is injective, $H_3(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q})$ has an intersection form.

Theorem 25 (Ebeling-T '13)

There is a subset $\mathcal{B}$ of $H_3(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q})$ consisting of vanishing classes which represents a $\mathbb{Q}$-basis of $H_3(\widetilde{\mathbb{C}^3/G^T}, g^{-1}(1); \mathbb{Q})/\langle\delta_0\rangle$ whose intersection numbers are given by the Coxeter-Dynkin diagram on the next slide.

δ_0 : a cycle in the radical ($\Longleftarrow \{|z_1| = |z_2| = |z_3| = 1\} \subset \mathbb{C}^3$)

For $i = 1, 2, 3$, K_i denotes the maximal subgroup of G^T fixing the i-th coordinate z_i, whose order $|K_i|$ is denoted by n_i.
Set $\gamma_i := \frac{\gamma_i'}{|G^T/K_i|}$.


$C^1_{\gamma_1-1,0}$
$C^1_{1,0}$
C^3_{1,n_3-1}
$C^3_{\gamma_3-1,n_3-1}$
C_1
$^{2j}{}_GT^{-2}$
-2
$C^1_{\gamma_1-1,n_1-1}$
C^1_{1,n_1-1}
$C^3_{1,0}$
$C^3_{\gamma_3-1,0}$
$C^2_{1,0}$
C^2_{1,n_2-1}
$C^2_{\gamma_2-1,0}$
$C^2_{\gamma_2-1,n_2-1}$

Definition 26

The lengths of arms of the Coxeter-Dynkin diagram

$$\Gamma_{(f^T,G^T)} := (\gamma_i * n_i, i = 1,2,3) = \left(\frac{\gamma_i'}{|G^T/K_i|} * |K_i|, i = 1,2,3 \right)$$

is called the *Gabrielov numbers* for (f^T, G^T).

We have a generalization of Arnold's strange duality:

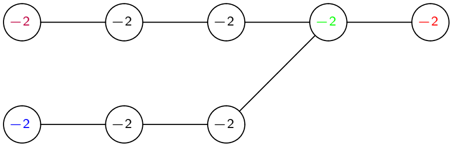
Theorem 27 (Ebeling-T '12)

$$g_{(f,G)} = j_{G^T}. \tag{0.33}$$

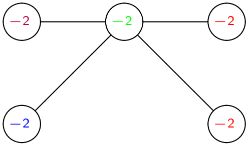
$$A_{(f,G)} = \Gamma_{(f^T,G^T)}. \tag{0.34}$$

Example: $f = x^4 - y^4 + z^2 = f^T$

$(f^T - cxyz, \{1\}) \Longleftrightarrow \mathbb{P}^1_{(4,4,2)}$, $(f^T, \{1\}) \Longleftrightarrow HMF^{L_f}_S(f)$
$A = \Gamma = (4, 4, 2)$



$(f^T - cxyz, \mathbb{Z}/2\mathbb{Z}) \Longleftrightarrow \mathbb{P}^1_{(2,2,2,2),(-1)}$, $(f^T, \mathbb{Z}/2\mathbb{Z}) \Longleftrightarrow HMF^{L}_S(f)$
$A = \Gamma = (4/2 * 1, 4/2 * 1, 2/1 * 2) = (2, 2, 2, 2)$



Thank you very much!