Relative Singularity Categories

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Motivation

\[ X = \mathbb{C}^2 / \mathbb{Z}_2 \]
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\[ \pi \]

\[ Y \rightarrow X \]
Motivation

\[ \mathcal{D}^b(Y) \xleftarrow{\mathbb{L}\pi^*} \xrightarrow{\pi} \text{Perf}(X) \]
Motivation

\[ \mathcal{D}^b\left( \Pi( \circ \begin{array}{c} \rightsquigarrow \\ \end{array} \circ ) \right) \cong \mathcal{D}^b(Y) \]

Kapranov & Vasserot
Bridgeland, King & Reid

\[ \mathbb{L}_\pi \pi^* \]

Derived McKay Correspondence

\[ \mathcal{D}^b(Y) \]

\[ \pi \]

\[ \text{Perf}(X) \]
Idea (Van den Bergh)

Replace $D^b(Y)$ by $D^b(A)$ for a “nice” algebra $A$ (e.g. $\text{gl. dim}(A) < \infty$) and consider it as categorical resolution of $X$ if there is an embedding

$$\text{Perf}(X) \hookrightarrow D^b(A).$$
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Let $k$ be an algebraically closed field and $(R, \mathfrak{m})$ be a commutative complete Gorenstein $k$-algebra with isolated singularity and $R/\mathfrak{m} \cong k$.

Definition

Let $M \in \text{MCM}(R) := \{ N \in \text{mod} - R | \text{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \}$ be a maximal Cohen–Macaulay module and $A := \text{End}_R(R \oplus M)$. 

Martin Kalck (Bielefeld)

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Two triangulated categories

- $R$: Gorenstein Singularity
Two triangulated categories

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$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(\text{mod} - R)}{\mathcal{K}^b(\text{proj} - R)}$

Classical Singularity Category
Two triangulated categories

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‘measures complexity of singularities of Spec(\( R \))’
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Relative Singularity Category

- Buchweitz Orlov
- Chen Thanhoffer de Völcsy & Van den Bergh
- Burban & Kalck

Classical Singularity Category 'measures complexity of singularities of Spec($R$)'

Relative Singularity Category 'measures difference between categorical resolution $D^b(A)$ and smooth part $D^b(R) \supseteq K^b(\text{proj } - R) \subseteq D^b(A)$'
Let $A = \text{End}_R(R \oplus M)$ be a NCR of a Gorenstein $k$-algebra $R$ as above. The relative singularity category $\Delta_R(A) := D^b(A)/K^b(\text{proj } R)$ satisfies:

1. $\Delta_R(A)$ is Hom-finite, by [TV] or [KY].
2. There is an equivalence of triangulated categories $\Delta_R(A) \sim D_{sg}(R)$, where $S_1, \ldots, S_t$ denote the simple modules over the stable endomorphism algebra $A = \text{End}_R(M)$, by [TV] or [KY].
3. If $\text{add } M$ has almost split sequences, then $\text{thick } (S_1, \ldots, S_t)$ has a Serre functor $\nu$, whose action on the generators $S_i$ is given by $\nu^n(S_i) \sim = S_i[2^n]$ (fractionally CY), where $n(S_i)$ is given by the length of the $\tau$–orbit of $M_i$, by [KY].
Properties of the relative singularity category

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A natural question

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- $A$: non-commutative resolution of $R$

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Relation?
Main result

A first answer to this question was obtained in joint work with Dong Yang.
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**Theorem**

Let $R$ and $R'$ be MCM–representation finite complete Gorenstein $k$-algebras with *Auslander algebras* $A = \text{Aus}(R)$ respectively $A' = \text{Aus}(R')$. 

**Remark**

Kn"orrer's Periodicity yields a wealth of non-trivial examples for (i):

$$D_{sg}(S/(f)) \sim \rightarrow D_{sg}(S/J_{x,y}K/(f + xy)),$$

where $S = kJ_0, \ldots, z_d K$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$. 
A first answer to this question was obtained in joint work with Dong Yang.

**Theorem**

Let \( R \) and \( R' \) be MCM–representation finite complete Gorenstein \( \kappa \)-algebras with \textbf{Auslander algebras} \( A = \text{Aus}(R) \) respectively \( A' = \text{Aus}(R') \). Then the following statements are equivalent.

(i) There is a triangle equivalence \( \mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R') \).
Main result

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Theorem

Let $R$ and $R'$ be MCM–representation finite complete Gorenstein $k$-algebras with Auslander algebras $A = \text{Aus}(R)$ respectively $A' = \text{Aus}(R')$. Then the following statements are equivalent.

(i) There is a triangle equivalence $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$.

(ii) There is a triangle equivalence $\Delta_R(A) \cong \Delta_{R'}(A')$.
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The implication (ii) $\Rightarrow$ (i) holds more generally for arbitrary NCRs $A$ and $A'$ of arbitrary isolated Gorenstein singularities $R$ and $R'$.
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Kn"orrer’s Periodicity yields a wealth of non-trivial examples for (i):

$$\mathcal{D}_{sg}(S/(f)) \xrightarrow{\sim} \mathcal{D}_{sg}(S[[x, y]]/(f + xy)),$$

where $S = k[z_0, \ldots, z_d]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$. 
Let \( R = \mathbb{C}[x]/(x^2) \) and \( R' = \mathbb{C}[x, y, z]/(x^2 + yz) \). Knörrer’s equivalence and our theorem above yield a triangle equivalence

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\Delta_R(\text{Aus}(R)) \cong \Delta_{R'}(\text{Aus}(R')),
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where the right-hand side is the completion of the preprojective algebra of the Kronecker quiver \( \Pi \).
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which may be written explicitly as

$$\mathcal{D}^b\left(\begin{array}{c}
1 & \rightarrow & 2 \\
\rightarrow & i & \rightarrow & \rightarrow \\
K^b(\text{add } P_1)
\end{array}\right) / (pi) \sim \mathcal{D}^b\left(\begin{array}{c}
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\end{array}\right) / (xy - yx).$$
Let $R = \mathbb{C}[x]/(x^2)$ and $R' = \mathbb{C}[x, y, z]/(x^2 + yz)$. Knörrer’s equivalence and our theorem above yield a triangle equivalence

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$$D^b\left( \frac{1 \quad p \quad 2}{i \quad (pi)} \right) \cong K^b(\text{add } P_1) \quad \rightsquigarrow \quad D^b\left( \frac{1 \quad x \quad y \quad 2}{y \quad p \quad x} \right) \cong K^b(\text{add } P_1).$$

The quiver algebra on the right hand side is the **completion** of the preprojective algebra of the Kronecker quiver $\Pi(\circ \quad \circ \quad \circ)$. 

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**Example**
Idea of the proof

Key Statement (Kalck & Yang)

There exists a dg-algebra $\Lambda_{dg}(R)$ such that

- $\text{per } \Lambda_{dg}(R) \cong \Delta_R(\text{Aus}(R))$.
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### Key Statement (Kalck & Yang)

There exists a dg-algebra $\Lambda_{dg}(R)$ such that

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We call $\Lambda_{dg}(R)$ the dg-Auslander algebra of $\mathcal{D}_{sg}(R)$. 
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We call $\Lambda_{dg}(R)$ the dg-Auslander algebra of $\mathcal{D}_{sg}(R)$.

Corollary

$\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$ implies $\Delta_R(\text{Aus}(R)) \cong \Delta_{R'}(\text{Aus}(R'))$. 
Example: The dg-Auslander algebra of an odd-dimensional $\mathbb{E}_8$-singularity: $f = z_0^3 + z_1^5 + z_2^2 + \ldots + z_d^2$

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$$\begin{align*}
\text{degree } (\rightarrow) &= 0 \\
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$$d(\rho_4) = \alpha_2 \alpha_2^* + \alpha_3 \alpha_4^* = \text{mesh relation}$$
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**Proposition**

Let \( A \) be a Noetherian ring and let \( e \in A \) be an idempotent. The exact functor \( \text{Hom}_A(eA, -) \) induces an equivalence of triangulated categories

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\frac{\mathcal{D}^b(\text{mod} -A)/\text{thick}(eA)}{\text{thick}(\text{mod} -A/AeA)} \sim \mathcal{D}_{sg}(eAe).
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Let \( A = \text{End}_R(R \oplus M) \) be an NCR and \( e := \text{id}_R \in A \).
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A “purely commutative” application
Rational surface singularities

Setup

Let \((R, m)\) be a complete local \textbf{rational} normal surface singularity over an algebraically closed field of characteristic zero,
Rational surface singularities

Setup

Let \((R, \mathfrak{m})\) be a complete local rational normal surface singularity over an algebraically closed field of characteristic zero, i.e. \(H^1(X, \mathcal{O}_X) = 0\), where \(X \to \text{Spec}(R)\) is a resolution of singularities.

Example (Brieskorn) Quotient singularities \(\mathbb{C}^2/G\) are rational.

Definition A maximal Cohen–Macaulay \(R\)-module is special if \(\text{Ext}^1_R(M, R) = 0\).

Theorem (Wunram’s generalization of the McKay-Correspondence) There is a natural bijection between indecomposable special MCMs and irreducible components of the exceptional curve \(E = \pi^{-1}(\mathfrak{m})\), where \(\pi: Y \to \text{Spec}(R)\) is the minimal resolution of singularities.
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Let \((R, \mathfrak{m})\) be a complete local **rational** normal surface singularity over an algebraically closed field of characteristic zero, i.e. \(H^1(X, \mathcal{O}_X) = 0\), where \(X \to \text{Spec}(R)\) is a resolution of singularities.

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\textit{There is a \textbf{natural bijection} between indecomposable \textbf{special} MCMs}
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There is a natural bijection between indecomposable special MCMs and irreducible components of the exceptional curve $E = \pi^{-1}(\mathfrak{m})$, where $\pi : Y \to \text{Spec}(R)$ is the minimal resolution of singularities.
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- Wunram’s Theorem shows that $\text{SCM}(R)$ is representation-finite.
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The category of special MCMs ($\text{SCM}(R)$) has the following properties:
- Wunram’s Theorem shows that $\text{SCM}(R)$ is **representation-finite**. Let $M_0 = R, M_1, \ldots, M_t$ be the set of **indecomposable** special MCMs. The **reconstruction algebra** $\Lambda := \text{End}_R(\bigoplus_i M_i)$
In general, $R$ is not Gorenstein. In that case, $\text{MCM}(R)$ is not Frobenius. Moreover, the singularity category is not Krull-Schmidt.

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- Wunram’s Theorem shows that $\text{SCM}(R)$ is representation-finite. Let $M_0 = R, M_1, \ldots, M_t$ be the set of indecomposable special MCMs. The reconstruction algebra $\Lambda := \text{End}_R(\bigoplus_i M_i)$ has gl. dim $\leq 3$, by work of Wemyss.
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Answer

We may take the stable category $\text{SCM}(R)$. 
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We may take the **stable category** $\text{SCM}(R)$. If $R$ is Gorenstein, then $\text{SCM}(R) \cong \text{MCM}(R) \cong \mathcal{D}_{sg}(R)$. 

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Description of the stable category $\text{SCM}(R)$

This is joint work with Osamu Iyama, Michael Wemyss & Dong Yang.

**Theorem**

*Let $R$ be a rational surface singularity with minimal resolution $Y$.  

Then there are triangle equivalences  

$$\text{SCM}(R) \sim = D_{sg}(X) \sim = \bigoplus_{x \in \text{Sing}(X)} \text{MCM}(\hat{O}_x).$$

In particular, $\text{SCM}(R)$ is $1$-CY and there is a natural isomorphism $\sim = \text{id}$.  

The second equivalence follows from Orlov's Localization Theorem and Buchweitz' equivalence.*
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**Theorem**

Let $R$ be a rational surface singularity with minimal resolution $Y$. Let $X$ be obtained from $Y$ by **contracting the exceptional** $(-2)$-curves.

It is well-known that $\text{Sing}(X)$ consists of isolated singularities, which are rational double points. Then there are triangle equivalences $\text{SCM}(R) \sim D_{\text{sg}}(X) \sim \bigoplus_{x \in \text{Sing}(X)} MCM(\hat{\mathcal{O}}_x)$.

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Sketch of the proof of $\text{SCM}(R) \cong D_{sg}(X)$

- $D^b(X) \cong D^b(e\Lambda e)$,
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- $D^b(X) \cong D^b(e\Lambda e)$, where
  - $X$: rational double point resolution of $R$,
  - $\Lambda$: reconstruction algebra,
  - $e = 1_P \in \Lambda$: idempotent (where $\text{add}(P) = \text{proj SCM}(R)$).

Recall from Part I that we always have a triangle equivalence $D^b(\text{mod } -\Lambda) / \text{thick}(e\Lambda) \cong \text{SCM}(R)$.
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$\Lambda \xrightarrow{} \Lambda e \xleftarrow{} \bigoplus_i M_i$

- $\text{gl. dim}(\Lambda) < \infty \Rightarrow D_{sg}(e\Lambda e) \cong \text{SCM}(R)$
Example

Let $G \subseteq \text{GL}(2, \mathbb{C})$ be the cyclic group of order 27 generated by

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{19} \end{pmatrix} \in \text{GL}(2, \mathbb{C}),$$

where $\zeta$ is a primitive 27th root of unity.
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\[
\begin{array}{c}
A_2 \\
\bullet \\
E_3 \\
\bullet \\
A_3 \\
-5 \\
\end{array}
\]

\( \subseteq X \)

Our Theorem yields a description of the stable category of SCMs:

\[
\underline{\text{SCM}}(R_{27,19}) \cong \mathcal{D}_{sg}(X) \cong \text{MCM} \left( \frac{\mathbb{C}[x,y,z]}{(x^3 + yz)} \right) \oplus \text{MCM} \left( \frac{\mathbb{C}[x,y,z]}{(x^4 + yz)} \right)
\]
Thank you!