Relative Singularity Categories

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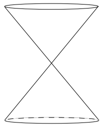
Motivation



$X = \mathbb{C}^2/\mathbb{Z}_2$

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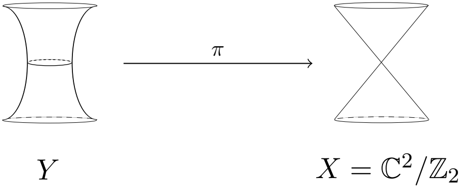

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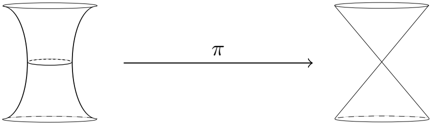

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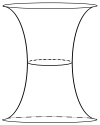
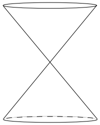


$$\xrightarrow{\pi}$$



$$\mathcal{D}^b\Big(\Pi\big(\circ \rightrightarrows \circ\big)\Big) \xleftarrow[\substack{\text{Kapranov \& Vasserot} \\ \text{Bridgeland, King \& Reid}}]{\cong} \mathcal{D}^b(Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

Derived McKay Correspondence

Introduction

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ *by* $\mathcal{D}^b(A)$ *for a "nice" algebra* A *(e.g.* $\operatorname{gl.dim}(A) < \infty$*) and consider it as* **categorical resolution** *of* X *if there is an embedding*

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Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod} - R \,\middle|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a **maximal Cohen–Macaulay module** and $A := \mathrm{End}_R(R \oplus M)$.

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  where $n = n(S_i)$ is given by the length of the τ–orbit of M_i, by [KY].

A natural question

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- A: non-commutative resolution of R

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Remark

Knörrer's Periodicity yields a wealth of non-trivial examples for (i):

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x, y]\!]/(f + xy)\big),$$

where $S = k[\![z_0, \ldots, z_d]\!]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$.

Example

Let $R = \mathbb{C}[\![x]\!]/(x^2)$ and $R' = \mathbb{C}[\![x, y, z]\!]/(x^2 + yz)$. **Knörrer's equivalence** and our theorem above yield a triangle equivalence

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which may be written explicitly as

$$\frac{\mathcal{D}^b\left(\begin{tikzcd} 1 \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "i"] \end{tikzcd} \Big/ (pi) \right)}{K^b(\mathsf{add}\, P_1)} \xrightarrow{\sim} \frac{\mathcal{D}^b\left(\begin{tikzcd} 1 \arrow[r, bend left=40, "x"] \arrow[r, bend left=20, "y"'] & 2 \arrow[l, bend left=20, "y"'] \arrow[l, bend left=40, "x"] \end{tikzcd} \Big/ (xy-yx) \right)}{K^b(\mathsf{add}\, P_1)}.$$

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The quiver algebra on the right hand side is the **completion** of the preprojective algebra of the Kronecker quiver $\Pi\left(\begin{tikzcd} \circ \arrow[r, bend left] \arrow[r, bend right] & \circ \end{tikzcd}\right)$.

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Key Statement (Kalck & Yang)

There exists a dg-algebra $\Lambda_{dg}(R)$ *such that*

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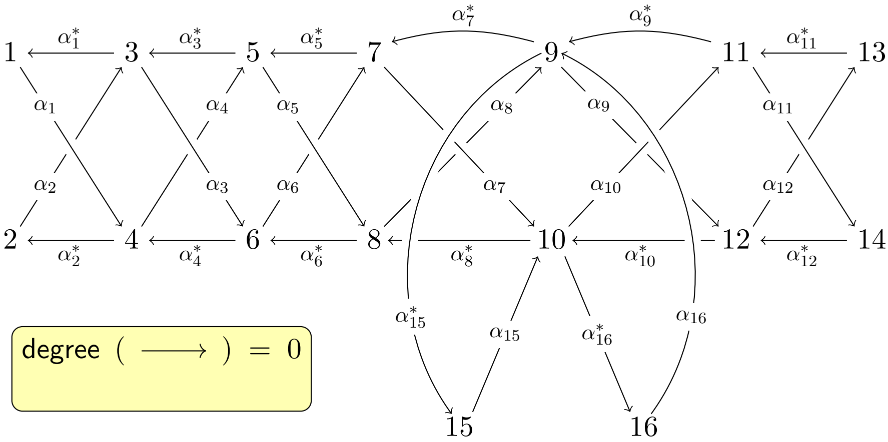
Corollary

$\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$ *implies* $\Delta_R(\mathsf{Aus}(R)) \cong \Delta_{R'}(\mathsf{Aus}(R'))$.

Example: The dg-Auslander algebra of an odd-dimensional $\mathbb{E}_8$-singularity: $f = z_0^3 + z_1^5 + z_2^2 + \ldots + z_d^2$



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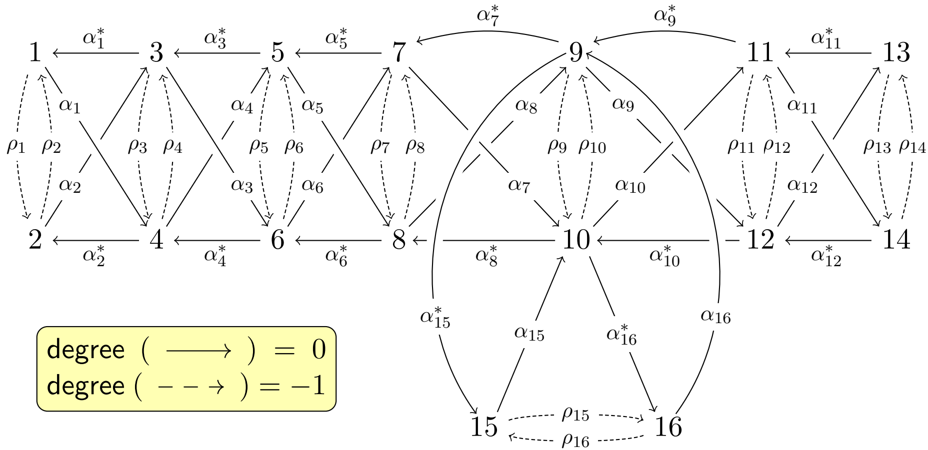

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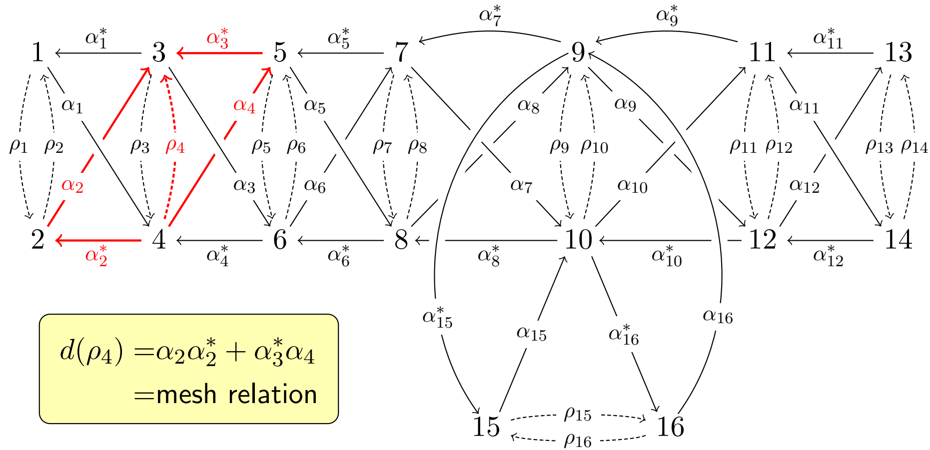

$$\begin{aligned} d(\rho_4) &= \alpha_2\alpha_2^* + \alpha_3^*\alpha_4 \\ &= \text{mesh relation} \end{aligned}$$

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A "purely commutative" application

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The Frobenius category of Special MCM-modules

In general, R is **not Gorenstein**. In that case, $\mathrm{MCM}(R)$ is **not Frobenius**. Moreover, the singularity category is **not Krull-Schmidt**.

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We may take the **stable category** $\underline{\underline{\mathrm{SCM}}}(R)$. If R is Gorenstein, then $\underline{\underline{\mathrm{SCM}}}(R) \cong \underline{\mathrm{MCM}}(R) \cong \mathcal{D}_{sg}(R)$.

Description of the stable category $\underline{\underline{\mathrm{SCM}}}(R)$

This is joint work with Osamu Iyama, Michael Wemyss & Dong Yang.

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$$\frac{\mathcal{D}^b(\mathrm{mod} - \Lambda)/\mathsf{thick}(e\Lambda)}{\mathsf{thick}(\mathrm{mod} - \Lambda/\Lambda e \Lambda)} \xrightarrow{\;\sim\;} \mathcal{D}_{sg}(e\Lambda e)$$

$$\Lambda \longmapsto \Lambda e$$

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$$\begin{tikzcd}
\dfrac{\mathcal{D}^b(\mathrm{mod}-\Lambda)/\mathsf{thick}(e\Lambda)}{\mathsf{thick}(\mathrm{mod}-\Lambda/\Lambda e\Lambda)} \arrow[r, "\sim"] & \mathcal{D}_{sg}(e\Lambda e) & \underline{\underline{\mathsf{SCM}}}(R) \arrow[l, hook', "\text{Keller \&}"', "\text{Vossieck}"] \\
\Lambda \arrow[r, mapsto] & \Lambda e & \bigoplus_i M_i \arrow[l, mapsto]
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- $\mathsf{gl.dim}(\Lambda) < \infty \qquad \Longrightarrow \qquad \mathcal{D}_{sg}(e\Lambda e) \cong \underline{\underline{\mathsf{SCM}}}(R)$

Example

Let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be the cyclic group of order 27 generated by

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{19} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

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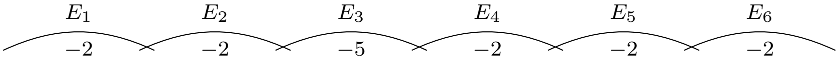

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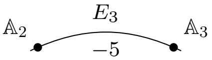

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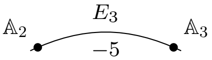

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Our Theorem yields a description of the stable category of SCMs:

$$\underline{\underline{\mathrm{SCM}}}(R_{27,19}) \cong \mathcal{D}_{sg}(X) \cong \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x,y,z]\!]}{(x^3+yz)}\right) \oplus \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x,y,z]\!]}{(x^4+yz)}\right)$$

Thank you!