On classification of nonabelian bundle gerbes with connections

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Abstract
We discuss nonabelian bundle gerbes and their differential geometry using simplicial methods. Associated to any crossed module \((H \to D)\) there is a simplicial group \(NC_{(H \to D)}\), the nerve of the 1-category defined by the crossed module and its geometric realization \(|NC_{(H \to D)}|\). Equivalence classes of principal bundles with structure group \(|NC_{(H \to D)}|\) are shown to be one-to-one with stable equivalence classes of what we call crossed module gerbes bundle gerbes. We can also associate to a crossed module a 2-category \(\tilde{C}_{(H \to D)}\). Then there are to equivalent ways how to view the classifying space of \(NC_{(H \to D)}\)-bundles and hence of \(|NC_{(H \to D)}|\)-bundles. We can either to apply the \(W\)-construction to \(NC_{(H \to D)}\) or take the nerve od the 2-category \(\tilde{C}_{(H \to D)}\). We shortly discuss string structures from this point of view. Also simplicial principal bundle can be equipped with a simplicial connection and a \(B\)-field. It is shown how in the case of a simplicial principal \(NC_{(H \to D)}\)-bundle these simplicial objects give the bundle gerbe connection and the bundle gerbe \(B\)-field.

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Nonabelian gebres arose in the realms of nonabelian cohomolgy [1] and higher category [2]. Their differential geometry was described thoroughly by Breen ans Messing [3] from the algebraic geometry point of view (see [4] for the combinatorial description). In [5] nonabelian bundle gerbes, generalizing the nice concept of an abelian bundle gebre [6], were introduced. These have to be shown (along with their connections and curvings) very natural objects in classical fibre bundle theory. There a is hope that in this form gebres can be useful in physics (see e.g. examples of higher Yang-Mills theories [7] and anomaly cancelation of M5-branes [8]). Closely related to crossed modules bundle gerbes are 2-bundles introduced in [9] and discussed together with their connections and curvings in [10].

In this paper we discuss classification of bundle gerbes associated with crossed modules. These are bundle gerbes equipped with modules in the terminology of [5]. This is done using some well-known simplicial constructions. Then the relation between simplicial principal bundles and crossed module bundle gerbes is used to describe bundle gerbe connections and curvings in simplicial language.

The first section is devoted to simplicial principal bundles. We describe them as twisted Cartesian products following [11] and recall the construction of a universal bundle.

Connections on simplicial bundles are introduced in Section 2. This is done in a straightforward way which we believe is the relevant one for our purposes. Next we shortly discuss the corresponding notion of a curvature.

Our task in Section 3 is to define the next in an infinite sequence of relevant differential geometric objects associated with simplicial principal bundles, the $B$-field.

In Section 4 we describe some simplicial constructions related to a crossed module $(H \to D)$. We can view a crossed module as a 1-category (actually 1-groupoid) $\mathcal{C}(H \to D)$ or as a 2-category (actually a 2-groupoid) $\hat{\mathcal{C}}(H \to D)$. We can form the corresponding nerves $N\mathcal{C}(H \to D)$ and $N\hat{\mathcal{C}}(H \to D)$ respectively. The geometric realization $|N\mathcal{C}(H \to D)|$ is the classifying space of $H$-principal bundles with a chosen trivialization when we change the structure group from $H$ to $D$. If $H$ and $D$ are Lie groups $\mathcal{C}(H \to D)$ is a simplicial Lie group and its geometric realization $|N\mathcal{C}(H \to D)|$ is a topological group. String group of [12], [13] is an example.

Crossed module bundle gerbes are introduced in Section 5. The geometric realization $|N\hat{\mathcal{C}}(H \to D)|$ of the nerve of the 2-category related to our crossed module gives the classifying space of such bundle gerbes. These are shown to be the same as principal bundles with structure group $|N\mathcal{C}(H \to D)|$. In particular string structures [14] can be described equivalently in terms of nonabelian bundle gerbes. Locally crossed module bundle gerbes can be described using simplicial maps between the nerve of the 1-category coming from an open covering of the manifold and a nerve of the 2-category associated with a crossed module (which is equal to the classifying space $WNC(H \to D)$ of principal $N\mathcal{C}(H \to D)$-bundles). Here we have to mention closely related work of D. Stevenson [15]. In the last section we describe how the connection and $B$-field on simplicial principal $N\mathcal{C}(H \to D)$-bundle give rise to a connection and an $B$-field on the corresponding crossed module bundle gerbe. Here as well in sections 2 and 3 we work in the category of manifolds. However as J. Baez pointed out it might be more appropriate to work in the category of “smooth spaces” studied in the appendix of [10].
1. Simplicial principal bundles

We start by recalling some relevant properties of simplicial principal bundles following mainly [11]. Let \( \pi : P \to X \) to be a simplicial (left) principal \( G \)-bundle, with \( P \) and \( X \) simplicial sets and \( G \) a simplicial group. As usually we will use \( \partial_i \) and \( s_i \) for the corresponding face and degeneracy maps. In the rest of the paper we always assume, without spelling it out explicitly, \( P \to X \) to possess a pseudo-cross section \( \sigma : X \to P \) such that \( \pi \sigma = id_X \) and \( \partial_i \sigma = \sigma \partial_i \) if \( i > 0 \) and \( s_i \sigma = \sigma s_i \) if \( i \geq 0 \). Associated with a pseudo-cross section \( \sigma \) we have the twisting function \( \tau : X_n \to G_{n-1} \) defined as

\[
\partial_0 \sigma(x) = \tau(x).\sigma(\partial_0 x).
\]

We will use the following description of \( G \)-bundles which we alternatively can use as a definition.

1.1. Twistoings. To make this section self-contained we have to describe the twisting first. For a function \( \tau : X_n \to G_{n-1} \) to be a twisting the following conditions should be fulfilled:

\[
\begin{align*}
\partial_0 \tau(x) &= \tau(\partial_1 x)(\tau(\partial_0 x))^{-1}, \\
\partial_i \tau(x) &= \tau(\partial_{i+1} x) \quad \text{for} \quad i > 0, \\
s_i \tau(x) &= \tau(s_{i+1} x) \quad \text{for} \quad i \geq 0, \\
\tau(s_0 x) &= e_n \quad \text{for} \quad x \in X_n.
\end{align*}
\]

1.2. Principal bundles as twisted Cartesian products. A principal \( G \)-bundle \( p : P \to X \) with a pseudo-cross section can be identified with the simplicial set \( P(\tau) = G \times_{\tau} X \), which satisfies

\[
P(\tau)_n = G_n \times X_n
\]

and has the following face and degeneracy maps

\[
\begin{align*}
(i) \quad \partial_i(g, x) &= (\partial_i g, \partial_i x) \quad \text{for} \quad i > 0, \\
(ii) \quad \partial_0(g, x) &= (\partial_0 g, \tau(x), \partial_0 x), \\
(iii) \quad s_i(g, x) &= (s_i g, s_i x) \quad \text{for} \quad i \geq 0.
\end{align*}
\]

Moreover there is a canonical choice for the pseudo-cross section \( \sigma(x) = (e_n, x), x \in X_n \) and \( e_n \) the identity in \( G_n \).

Equivalence of two \( G \)-bundles \( P(\tau) \) and \( P(\tau') \) over the same \( X \) is described in terms of twisting as follows.

1.3. Equivalence of principal bundles. We call two twistings \( \tau' \) and \( \tau \) equivalent if there exists a simplicial map \( \psi : X \to G \) such that

\[
\begin{align*}
\partial_0 \psi(x).\tau'(x) &= \tau(x).\psi(\partial_0 x), \\
\partial_i \psi(x) &= \psi(\partial_i x) \quad \text{if} \quad i > 0, \\
s_i \psi(x) &= \psi(s_i x) \quad \text{if} \quad i \geq 0.
\end{align*}
\]

It will be convenient to introduce the equivariant map \( \bar{\sigma} : P \to G, \bar{\sigma}(gp) = g.\bar{\sigma}(p) \) by the equation \( p = \bar{\sigma}(p)\sigma(x) \). In the rest we will always assume the canonical choice of the pseudo-cross section is made in which case \( \bar{\sigma}(g_n, g_{n-1}, \ldots, g_0) = (g_n, e_{n-1}, \ldots, e_0) \). We have

\[
\partial_0 \bar{\sigma}(p) = \bar{\sigma}(\partial_0 p)\tau(x).
\]
As with ordinary bundles simplicial bundles can be pulled back and their structure groups can be changed using simplicial group homomorphisms. Pseudo-cross sections and twistings transform under these operations in the usual way.

1.4. **Universal $G$-bundle.** There is a canonical choice of the classifying space of $G$-bundles denoted as $\mathcal{W}G$ and constructed as follows. $\mathcal{W}G_0$ has one element $*$ and $\mathcal{W}G_n = G_{n-1} \times G_{n-2} \times \ldots \times G_0$ for $n > 0$. Face and degeneracy maps are

$$s_0(*) = (e_0), \quad \partial_i(g_0) = * \quad \text{for} \quad i = 0 \text{ or } 1$$

and

$$\partial_0(g_n, \ldots, g_0) = (g_{n-1}, \ldots, g_0),$$

$$\partial_{i+1}(g_n, \ldots, g_0) = (\partial_1 g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i}, g_{n-i-1}, g_{n-i-2}, \ldots, g_0),$$

$$s_0(g_{n-1}, \ldots, g_0) = (e_n, g_{n-1}, \ldots, g_0),$$

$$s_{i+1}(g_{n-1}, \ldots, g_0) = (s_i g_n, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0),$$

if $n > 0$. With the choice of a twisting given by

$$\tau(g_{n-1}, \ldots, g_0) = g_{n-1}$$

we have the universal $G$-principal bundle

$$\mathcal{W}G = G \times_\tau \mathcal{W}G.$$

As with ordinary bundles we have that $\mathcal{W}G$ is contractible and is universal in the following sense.

1.5. **Theorem.** Let us assign to any simplicial map

$$f : X \to \mathcal{W}G$$

the induced bundle $f^*(\mathcal{W}G) \to X$. This defines a one-to-one correspondence between homotopy classes of maps $[X, \mathcal{W}G]$ and the equivalence classes of principal $G$-bundles over the base $X$.

2. **Simplicial connection, curvature and $B$-field**

Here we introduce the notion of a connection on a simplicial bundle. Of course now we assume that $G$ is a simplicial Lie group and $P$ and $X$ are simplicial manifolds. Also all maps and actions are smooth. We use the shorthand notation $\Omega^k(Y) \otimes \text{Lie}(G)$ for the collection of all $\text{Lie}(G_n)$-valued $k$-forms on $Y_n$ for all $n$ and any simplicial manifold $Y$. Here of course $\text{Lie}(G)$ is the corresponding simplicial Lie algebra $\text{Lie}(G)_n = \text{Lie}(G_n)$ with the induced face and degeneracy maps. For purposes of this paper the following definition of a simplicial connection seems to be adequate.
2.1. Definition. Let \( A \in \Omega^1(P) \otimes \text{Lie}(G) \) be a collection of one forms \( A_n \in \Omega^1(P_n) \otimes \text{Lie}(G_n) \). We call \( A \) a connection on the simplicial principal \( G \)-bundle \( P \to X \) if it fulfils the following conditions:

\[
(i) \quad \partial^*_i A = \partial_i A \quad \text{and} \quad s^*_i A = s_i A
\]

where \( \partial^*_i \) on the left is the pullback of the face map acting on the one-form part of \( A \) and \( \partial_i A \) on the right is the simplicial Lie algebra face map acting on the simplicial Lie algebra part of \( A \) and similarly for degeneracies.

\[
(ii) \quad A \text{ is equivariant with respect to the left } G \text{-action on } P \quad g^* A = gA g^{-1}
\]

and

\[
(iii) \quad \text{its pullback to the fibre under } \bar{\sigma} : P \to G \text{ is the Cartan-Maurer form } gdg^{-1}, \text{ i.e. the collection of elements } g_n dg_n^{-1} \in \Omega^1(G_n) \otimes \text{Lie}(G_n).
\]

2.2. Local connection forms. Let us consider a collection of one forms \( A \in \Omega^1(X) \otimes \text{Lie}(G) \) with the property

\[
\partial_0 A = \tau^{-1} \partial^*_0 A \tau + \tau^{-1} d\tau,
\]

\[
\partial^*_i A = \partial_i A \quad \text{for} \quad i > 0
\]

and

\[
s^*_i A = s_i A \quad \text{for} \quad i \geq 0.
\]

We call such an \( A \) a local connection.

The following proposition is obvious.

2.3. Proposition. Any connection \( A \) is of the form

\[
A = \bar{\sigma} A \bar{\sigma}^{-1} + \bar{\sigma} d \bar{\sigma}^{-1}
\]

with

\[
A = \sigma^* A.
\]

Pullbacks and change of the structure group work as usually.

2.4. Curvature. Curvature is defined exactly the same way as in the case of ordinary bundles. It is a collection of two forms \( F \in \Omega^2(P) \otimes \text{Lie}(G) \) defined as \( F = dA + A \wedge A \) and it has the following properties:

\[
(i) \quad \partial^*_i F = \partial_i F \quad \text{and} \quad s^*_i F = s_i F
\]

\[
(ii) \quad F \text{ is equivariant with respect to the left } G \text{-action on } P \quad g^* F = gF g^{-1}
\]

and

\[
(iii) \quad F \text{ is of the form } F = \bar{\sigma} F \bar{\sigma}^{-1} \text{ with } F \in \Omega^2(X) \otimes \text{Lie}(G), \text{ i.e. it is horizontal.}
\]

Of course \( F = dA + A \wedge A \).

Let us note that

\[
\partial_0 F = \tau^{-1} \partial^*_0 F \tau \quad \text{and} \quad \partial_i F = \partial^*_i F \quad \text{for} \quad i > 0
\]
and
\[ s_i F = s_i^* F \quad \text{for} \quad i \geq 0 \]

3. \( \bar{B} \)-field

Let
\[ \bar{G}_0 = 0, \]
\[ \bar{G}_n = \ker \partial_1 \ldots \partial_n \subset G_n. \]

Let us note that \( \partial_0 \bar{G}_{n+1} \) is a normal subgroup in \( G_n \). Also \( \partial_i \bar{G}_{n+1} \subset \bar{G}_n \) for \( i > 0 \). From now on we will assume that there exists an action of \( G_n \) on \( \bar{G}_{n+1} \);
\[ g_n \times \bar{g}_{n+1} \mapsto g_n \bar{g}_{n+1} \]
such that
\[ \partial_0 (g_n \bar{g}_{n+1}) = g_n \partial_0 (\bar{g}_{n+1}) g_n^{-1} \]
and
\[ \partial_0 g_{n+1} \bar{g}_n' = g_{n+1} \bar{g}_n' g_n^{-1}. \]

These conditions will be automatically satisfied in the next sections when we consider simplicial groups with simplicial homotopy groups \( \pi_i(G) = 0, \) for \( i \geq 2. \)

3.1. Definition. \( \bar{B} \)-field is a collection of two-forms \( \bar{B}_{n+1} \in \Omega^2(X_n) \otimes (\bar{G}_{n+1}) \) such that
\[ \partial_0^* \bar{B} = \tau \partial_1^* \bar{B} \]
and
\[ \partial_i^* \bar{B} = \partial_{i+1}^* \bar{B} \quad \text{for} \quad i > 0 \]
and
\[ s_i^* \bar{B} = s_{i+1}^* \bar{B} \quad \text{for} \quad i \geq 0. \]

Finally we introduce collection of two forms \( \nu \in \Omega^2(X) \otimes \text{Lie}(G) \) as
\[ \nu_n = F_n + \partial_0 \bar{B}_{n+1}. \]

Obviously \( \nu \) has the same properties with respect to face and degeneration maps as \( F. \)

3.2. Remark. Of course there is no reason to stop with connection \( A \) and \( B \)-field here. One can introduce \( C \)-field etc. ad infinitum. We will however not to do so here as we are really interested only in simplicial groups which are algebraic models of homotopy 2-type (crossed modules). Also it seems that it would be more proper to treat all this fields together, see remark (6.5).

4. Crossed modules

4.1. Definition. Let \( H \) and \( D \) be two Lie groups. We say that \( H \) is a crossed \( D \)-module if there is a group homomorphism \( \alpha : H \to D \) and an action of \( D \) on \( H \) denoted as \( (d, h) \mapsto d^h \) such that
\[ \alpha(h') h' = hh'^{-1} \quad \text{for} \quad h, h' \in H \]
and
\[ \alpha(dh) = da \alpha(h) d^{-1} \quad \text{for} \quad h \in H, d \in D. \]

holds true.

We will use the the following notation \((H \to D)\) for a crossed module. If the groups are infinite dimensional we have assume that these are Frechét Lie Groups.

There are two canonical categorial construction associated with any crossed module.
4.2. **Crossed module as a 1-category.** Let us denote $\mathcal{C}_{(H \rightarrow D)}$ the (topological) category with objects being group elements $d \in D$ and morphisms (1-arrows) group elements $(h, d)$ of the semidirect product $H \rtimes D$.

As with any category we can now form the simplicial space, the nerve $NC_{(H \rightarrow D)}$ of $\mathcal{C}_{(H \rightarrow D)}$ and its geometric realization $|NC_{(H \rightarrow D)}|$. The nerve is naturally a simplicial Lie group and its geometric realization becomes naturally a topological group [12]. We will use the following pictorial representation for the simplicial group $NC_{(H \rightarrow D)}$:

The (opposite) group multiplication is given by horizontal composition

$$d_0 \rightarrow \cdot \quad \rightarrow \quad d_0 = d_0d_1$$

etc.

Simplicial homotopy groups of $NC_{(H \rightarrow D)}$ are trivial except $\pi_0(NC_{(H \rightarrow D)}) = \text{coker } \alpha$ and $\pi_1(NC_{(H \rightarrow D)}) = \text{ker } \alpha$.

4.3. **Proposition.** $|NC_{(H \rightarrow D)}|$ is the classification space of principal $H$-bundles equipped with a chosen trivialization when the structure group is changed to $D$ using the homomorphism $\alpha$.

**Proof.** In other words $|NC_{(H \rightarrow D)}|$ is the homotopy fiber of $BH \rightarrow BD$. This is the pullback under $B\alpha : BH \rightarrow BD$ of the based path bundle $P_0BD \rightarrow BD$ and as a principal $\Omega BD \sim D$-bundle it can be identified with the the homotopy quotient $D//H = EH \rtimes_\alpha D$ of $D$ by $H$. Let us recall the $EH$ is the geometric realization of the following simplicial space

$$h_0 \rightarrow \equiv \quad \equiv \quad h_0 \quad \equiv \quad \cdots$$
From here we get $EH \times_{\alpha} D$ as the geometric realization of the simplicial space

\[
\begin{array}{ccc}
  d_0 & \leq & d_1 \\
  & & d_0 \quad \cdots \\
  h_{01} & \leq & \\
\end{array}
\]

and we see that this really identical to the simplicial group $NC_{(H \to D)}$.

4.4. **Remark.** Bundles described in (4.3) are automatically left and right $H$-principal bundles with the two principal $H$-actions commuting. Moreover the multiplication in (4.2) gives naturally a multiplication of such bundles. This follows from proposition 4 in [5]. We will refer to such bundles as *crossed module bundles*.

If $P$ and $P'$ are two crossed module bundles and $f$ and $f'$ the corresponding classifying maps, then the pointwise product map $f.f'$ is a classifying map for a bundle equivalent to the product bundle $P.P'$.

If $P \to X$ is a crossed module bundle then the corresponding trivial $D$-bundle $P \times_{\alpha} D$ is an example of what is called a $(D - H)$-bundle according to definition 5 in [5].

4.5. **String group.** Together with a crossed module $(H \to D)$ we can consider also crossed modules $(H \to \text{Im} \alpha)$ and $(1 \to \text{coker} \alpha)$. This gives an exact sequence of (topological) groups

\[
1 \longrightarrow |NC_{(H \to \text{Im} \alpha)}| \longrightarrow |NC_{(H \to D)}| \longrightarrow |NC_{(1 \to \text{coker} \alpha)}| = \text{coker} \alpha \longrightarrow 1
\]

String group *String* is a nice example of the above construction. Let $G$ be a simply connected compact simple Lie group. The crossed module in question is $H = \widehat{\Omega}G$ the centrally extended group of based loops and $D = P_0G$ is the group of based paths [12], [13] or some modification of these [14].

Of course we can as well consider crossed modules $(\ker \alpha \to e)$ and $(\text{Im} \alpha \to D)$ in which case we obtain the exact sequence

\[
1 \longrightarrow |NC_{(\ker \alpha \to e)}| = B \ker \alpha \longrightarrow |NC_{(H \to D)}| \longrightarrow |NC_{(\text{Im} \alpha \to D)}| \longrightarrow 1.
\]

We can view homotopy quotient $D/H = EH \times_{\alpha} D$ also as a bundle with the base space $\text{coker} \alpha$. Then the fibre is $B \ker \alpha$ and we see that the two exact sequences in (4.4) identical to

\[
1 \longrightarrow B \ker \alpha \longrightarrow |NC_{(H \to D)}| \longrightarrow \text{coker} \alpha \longrightarrow 1.
\]

4.6. **Crossed module as a 2-category.** Similarly we denote $\tilde{C}_{(H \to D)}$ the (topological) 2-category with just one object, 1-arrows group elements $d \in D$ and 2-arrows group elements $(h, d)$ of $H \rtimes D$. Again we can form the corresponding nerve $N\tilde{C}_{(H \to D)}$ [16]. This simplicial manifold can be pictorially represented as:

\[
\begin{array}{ccc}
  * & \leq & d_{01} \\
  & & d_{01} \quad \cdots \\
  d_{12} & \leq & h_{012} \\
\end{array}
\]
Simplicial homotopy groups of \( N \tilde{C}_{(H \to D)} \) are trivial except \( \pi_1(N \tilde{C}_{(H \to D)}) = \text{coker} \alpha \) and \( \pi_2(N \tilde{C}_{(H \to D)}) = \text{ker} \alpha \).

5. Bundle gerbes

Let us recall the definition of a nonabelian bundle gerbe in the form given in [5]. Consider a submersion \( \varphi : Y \to X \) (i.e. a map onto with differential onto). It follows we can always find a covering \( \{O_\alpha\} \) of \( M \) with local sections \( \sigma_\alpha : O_\alpha \to Y \), i.e. \( \varphi \circ \sigma_\alpha = id \).

The manifold \( Y \) will always be equipped with the submersion \( \varphi : Y \to X \). We also consider \( Y^{[n]} = Y \times_X Y \times_X \ldots \times_X Y \) the \( n \)-fold fiber product of \( Y \), i.e. \( Y^{[n]} \equiv \{(y_1, \ldots, y_n) \in Y^n | \varphi(y_1) = \varphi(y_2) = \ldots = \varphi(y_n)\} \).

Given a \((H \to D)\)-crossed module bundle \( \mathcal{E} \) over \( Y^{[2]} \) we denote by \( \mathcal{E}_{12} = p_{12}^* \mathcal{E} \) the crossed module bundle on \( Y^{[3]} \) obtained as pull-back of \( p_{12} : Y^{[3]} \to Y^{[2]} \) (\( p_{12} \) is the identity on its first two arguments); similarly for \( \mathcal{E}_{13} \) and \( \mathcal{E}_{23} \).

Consider the quadruple \((\mathcal{E}, Y, X, h)\) where \( \mathcal{E} \) is a crossed module bundle, \( Y \to M \) a submersion and \( h \) an isomorphism of crossed module bundles \( h : \mathcal{E}_{12} \mathcal{E}_{23} \to \mathcal{E}_{13} \) (let us recall that two crossed module bundles can be multiplied to obtain again a crossed module bundle). We now consider \( Y^{[4]} \) and the bundles \( \mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{13}, \mathcal{E}_{24}, \mathcal{E}_{34}, \mathcal{E}_{14} \) on \( Y^{[4]} \) relative to the projections \( p_{12} : Y^{[4]} \to Y^{[2]} \) etc. and also the crossed module isomorphisms \( h_{123}, h_{124}, h_{132}, h_{234} \) induced by projections \( p_{123} : Y^{[4]} \to Y^{[3]} \) etc.

5.1. Definition. The quadruple \((\mathcal{E}, Y, X, h)\) is called a crossed module bundle gerbe if \( h \) satisfies the cocycle condition (associativity) on \( Y^{[4]} \)

\[
\begin{array}{ccc}
\mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{34} & \xrightarrow{h_{123} h_{234}} & \mathcal{E}_{12}\mathcal{E}_{24} \\
\downarrow & & \downarrow \\
\mathcal{E}_{13}\mathcal{E}_{34} & \xrightarrow{h_{134}} & \mathcal{E}_{14}.
\end{array}
\]

5.2. Definition. Two crossed module bundle gerbes \((\mathcal{E}, Y, X, h)\) and \((\mathcal{E}', Y', X, h')\) are stably isomorphic if there exist a crossed module bundle \( N \to Z = Y \times_X Y' \) such that over \( Z^{[2]} \) the crossed module bundles \( q^* \mathcal{E} N_2 \) and \( N_1 q^* \mathcal{E}' \) are isomorphic. The corresponding isomorphism \( \ell : q^* \mathcal{E} N_2 \to N_1 q^* \mathcal{E}' \) should satisfy on \( Y^{[3]} \) the condition

\[ \ell_{13} \circ h = h' \circ \ell_{12} \circ \ell_{23}. \]

Here \( q \) and \( q' \) are projections onto first and second factor of \( Z = Y \times_X Y' \). \( N_1 \) and \( N_2 \) are the pullbacks of \( N \to Z \) to \( Z^{[2]} \) under respective projections form \( Z^{[2]} \) to \( Z \) etc.

5.3. Remark. Locally bundle gerbes can be described in terms of cocycles as follows. We can consider the trivializing cover \( \{O_\alpha\} \) of the submersion \( Y \to X \) be a good one.

Then a crossed module bundle gerbe can be described by a cocycle \( \{d_{\alpha\beta}, h_{\alpha\beta\gamma}\} \) where the maps \( d_{\alpha\beta} : O_\alpha \cap O_\beta \to D \) and \( h_{\alpha\beta\gamma} : O_\alpha \cap O_\beta \cap O_\gamma \to H \) fulfill the following cocycle condition

\[ d_{\alpha\beta} d_{\beta\gamma} = \alpha(h_{\alpha\beta\gamma}) d_{\alpha\gamma} \quad \text{on} \quad O_\alpha \cap O_\beta \cap O_\gamma \]

and

\[ h_{\alpha\beta\gamma} h_{\alpha\gamma\delta} = d_{\alpha\beta} h_{\beta\gamma\delta} h_{\alpha\beta\delta} \quad \text{on} \quad O_\alpha \cap O_\beta \cap O_\gamma \cap O_\delta. \]
Two crossed module bundle gerbes are stably equivalent if their respective cocycles \( \{d_{\alpha\beta}, h_{\alpha\beta}\} \) and \( \{d'_{\alpha\beta}, h'_{\alpha\beta}\} \) are related as

\[
d'_{\alpha\beta} = d_{\alpha\beta}(h_{\alpha\beta})d_{\alpha\beta}d_{\beta}^{-1}
\]

and

\[
h'_{\alpha'\beta'} = d_{\alpha\beta}h_{\alpha\beta}d_{\alpha\beta}d_{\alpha\beta'}h_{\alpha\beta'}^{-1}d_{\alpha\beta}^{-1}
\]

with \( d_{\alpha} : O_{\alpha} \to D \) and \( h_{\alpha\beta} : O_{\alpha} \cap O_{\beta} \to H \).

Pullback of a bundle gerbe is obtained pulling back the corresponding cocycle.

5.4. **Universal \( \mathcal{NC}(H \to D) \) bundle.** In (4.2) we have described the simplicial group \( \mathcal{NC}(H \to D) \). Now we can construct the corresponding universal bundle. As a result we get simplicial manifolds \( W\mathcal{NC}_H \) and \( W\mathcal{NC}_H \) which are pictorially represented as

\[
\begin{array}{ccc}
* & \overset{d_{01}}{\longrightarrow} & \ast \\
\downarrow & & \downarrow \\
& h_{012} & h_{012} \\
\uparrow & & \uparrow \\
& d_{12} & d_{12} \\
\end{array}
\]

\[
\begin{array}{ccc}
\ast & \overset{d_{0}}{\longrightarrow} & \ast \\
\downarrow & & \downarrow \\
& h_{01} & h_{01} \\
\uparrow & & \uparrow \\
& d_{01} & d_{01} \\
\end{array}
\]

Comparing to (4.6) gives

5.5. **Proposition.** \( W\mathcal{NC}(H \to D) = \mathcal{NC}(H \to D) \).

Now we can touch upon the question of the classification of crossed module bundle gerbes.

5.6. **Theorem.** Equivalence classes of principal \( |\mathcal{NC}(H \to D)| \) bundles are one to one with stable equivalence classes of \( (H \to D) \) crossed module bundle gerbes. The geometric realization \( |W\mathcal{NC}(H \to D)| = E|\mathcal{NC}(H \to D)| \to |W\mathcal{NC}(H \to D)| = B|\mathcal{NC}(H \to D)| \) gives the universal \( |\mathcal{NC}(H \to D)| \) bundle as well as the universal crossed module bundle gerbe.

**Proof.** The proof is just a slight generalization of section 5 in [5], where the lifting bundle gerbes (crossed module bundle gerbes with ker \( \alpha = 0 \)) are discussed in detail. Let \( f : X \to B|\mathcal{NC}(H \to D)| \) be the classification map for an \( |\mathcal{NC}(H \to D)| \) principal bundle \( P \). Associated with \( P \) there is a map \( P^{[2]} \to |\mathcal{NC}(H \to D)| \) which sends \( (p, p') \in P \) in the same fibre into unique group element \( g \in |\mathcal{NC}(H \to D)| \) which relates \( p \) and \( p' \). As \( |\mathcal{NC}(H \to D)| \) is the classification space for crossed module bundles, we obtain that way a crossed module bundle \( \mathcal{E} \to P^{[2]} \). As it follow from (4.4) \( \mathcal{E}_{12}\mathcal{E}_{23} \) is isomorphic to \( \mathcal{E}_{13} \) and it is easy to check that this isomorphism fulfills the cocycle condition of (5.1). So we obtain a bundle gerbe with \( Y = P \). If we start with an equivalent bundle \( P' \) we obtain an stably equivalent gerbe.

Conversely if we start with a crossed module bundle gerbe, the classification map of crossed module bundle \( \mathcal{E} \to Y^{[2]} \) is a map from \( f : Y^{[2]} \to |\mathcal{NC}(H \to D)| \) fulfilling the on \( Y^{[3]} \) the cocycle condition \( f(y_1, y_2)f(y_2, y_3) = f(y_1, y_3) \). Using local sections \( O_{\alpha} \to Y \) we get...
a cocycle \( f_{\alpha\beta} : O_{\alpha} \cap O_{\beta} \to \vert NC_{(H \to D)} \vert \): \( f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma} \). Thus we have transition functions for an \( \vert NC_{(H \to D)} \vert \)-bundle. Starting from a stably equivalent gerbe we get an equivalent bundle.

5.7. **Remark.** Let us recall that by definition under a nonabelian \( H \)-bundle gerbe we understand an \( (H \to \text{Aut}(H)) \)-bundle gerbe [5]. So the universal \( H \)-bundle gerbe is the same as the universal \( \vert NC_{(H \to \text{Aut}(H))} \vert \)-bundle.

5.8. **String structures.** Now can apply the classifying space functor \( B \) to the exact sequence of (4.5), or which is the same the \( W \) to the corresponding simplicial groups. Hence have the following exact sequence (\( \ker \alpha \) is abelian)

\[
1 \to B \ker \alpha \to \vert NC_{(H \to D)} \vert \to \text{coker} \alpha \to B^2 \ker \alpha \to B \vert NC_{(H \to D)} \vert \to B \text{coker} \alpha \to B^3 \ker \alpha.
\]

It follows that a lift of a principal coker \( \alpha \)-bundle to a principal \( \vert NC_{(H \to D)} \vert \)-bundle is the same as a lift of an \( (\alpha(H) \to D) \)-bundle gerbe to an \( (H \to D) \)-bundle gerbe.

In the case of \( \text{String} \) we do have

\[
1 \to K(\mathbb{Z}, 2) \to \text{String} \to \text{Spin} \to K(\mathbb{Z}, 3) \to B\text{String} \to B\text{Spin} \to K(\mathbb{Z}, 4).
\]

String structure is a lift of the structure group of a principal \( \text{Spin} \)-bundle to the string group \( \text{String} \) [14]. So the string structure is also lift of an \( (\Omega \text{Spin} \to P_0 \text{Spin}) \)-bundle gerbe to an \( (\Omega \text{Spin} \to P_0 \text{Spin}) \)-bundle gerbe.

5.9. **Remark.** A crossed module bundle gerbe is canonically equipped with a module (see section 6 of [5] for the definition of a bundle gerbe module). The trivial \( D \)-principal bundle \( D \times Y \to Y \) fullfills all the axioms of a module. This is shown in [5] in the case \( D = \text{Aut}(H) \) and applies word by word to the more general situation as well.

5.10. **Remark.** Let us consider the (topological) category described as follows. Objects are pairs \( (x, O_{\alpha}) \) with \( x \in O_{\alpha} \) and there is unique morphism \( (x, O_{\alpha}) \to (y, O_{\beta}) \) iff \( x = y \in O_{\alpha} \cap O_{\beta} \). Let \( N\{O_{\alpha}\} \) denote the nerve of this category. Consider a map of simplicial sets \( N\{O_{\alpha}\} \to W NC_{(H \to D)} \). Then the maps between 1- 2- and 3-simplexes give us the cocycle for gerbe transition functions in (5.1). We also see that the simplicial \( \tau_1 \) is identified with \( d_{\alpha\beta}, \tau_2 \) identifies with \( d_{\alpha\gamma}d_{\beta\gamma}^{-1}h_{\alpha\beta\gamma}d_{\alpha\beta} \) etc. So from (1.3) and (5.3) we can conclude that locally the stable equivalence classes of crossed module gerbes are described by homotopy classes of simplicial maps \( N\{O_{\alpha}\} \to W NC_{(H \to D)} = \tilde{NC}_{(H \to D)} \).

6. **Connection and B-field on a bundle gerbe**

In the previous section we have established a correspondence between \( \vert NC_{(H \to D)} \vert \)-principal bundles and \( (H \to D) \)-crossed module bundle gerbes. Now we would like to extend this relationship to connections, and also discuss the \( B \)-field from this point of view. However let us recall that \( \vert NC_{(H \to D)} \vert \) is only a topological group so in general there is no differential geometric connection on a principal \( \vert NC_{(H \to D)} \vert \)-bundle over a manifold \( X \). But we have the simplicial connection as described in section 2 on any simplicial \( NC_{(H \to D)} \)-bundle \( P \to X \).

The notion of a bundle gerbe connection (and that of a bundle gebre \( B \)-field as well) are quite subtle and we are not going to repeat them here in their global formulations (see [5], [3] for that). Instead we will give their local description using cocycles. This description is perfectly well suited for our purposes as we will relate the bundle gerbe connection and \( B \)-field to the simplicial connection and simplicial \( \tilde{B} \)-field as they were
introduced in sections 2 and 3 in the case of a simplicial $NC(H\to D)$-bundle over $N\{O\}_\alpha$
described by a classifying map $N\{O\}_\alpha \to \hat{W}NC(H\to D) = NC(H\to D)$ (see remark 5.10).

Let us now recall the cocycle description of what a connection on an crossed module bundle gerbe is. Again let $\{O\}_\alpha$ be a covering of a manifold $X$.

6.1. Bundle gerbe connection. A collection $\{A_\alpha, a_{\alpha\beta}\}$, with $A_\alpha \in \Omega^1(O_\alpha) \otimes \text{Lie}(D)$ and $a_{\alpha\beta} \in \Omega^1(O_\alpha \cap O_\beta) \otimes \text{Lie}(H)$ is called a connection on crossed module bundle gerbe (characterized by a nonabelian cocycle $\{a_{\alpha\beta}, h_{\alpha\beta}\}$) if it fulfils the following conditions

$$A_\alpha = d_{\alpha\beta}A_\beta d_{\alpha\beta}^{-1} + d_{\alpha\beta}dd_{\alpha\beta}^{-1} + \alpha(a_{\alpha\beta}) \text{ on } O_\alpha \cap O_\beta$$

and

$$a_{\alpha\beta} + d_{\alpha\beta}a_{\beta\gamma} = h_{\alpha\beta\gamma}a_{\alpha\gamma}h_{\alpha\beta\gamma}^{-1} + h_{\alpha\beta\gamma}dh_{\alpha\beta\gamma}^{-1} + A_\alpha - \alpha(h_{\alpha\beta\gamma})A_\alpha h_{\alpha\beta\gamma}^{-1} \text{ on } O_\alpha \cap O_\beta \cap O_\gamma.$$

The curvature $F$ is given by a collection of local two-forms $F_\alpha \in \Omega^2(O_\alpha) \otimes \text{Lie}(D)$ defined as $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$; the corresponding cocycle conditions follow from the definition. We will not repeat the explicit formulas here, interested reader can find them in e.g. [5], [3]. Now we can compare the above definition with the definition of a simplicial connection. We will not repeat the explicit formulas here, interested reader can find them in e.g. [5], [3]. Now we can compare the above definition with the definition of a simplicial connection on a $NC(H\to D)$-principal bundle $P \to N\{O\}_\alpha$. Realizing that $\tau_1$ corresponds $d_{\alpha\beta}, \tau_2$ corresponds to $d_{\alpha\beta}d_{\alpha\beta}^{-1} a_{\alpha\beta}, A_0$ corresponds to $A_\alpha, (\partial_0 A_1 \to \partial_1 A_1)$ corresponds to $d_{\alpha\beta}a_{\alpha\beta}$ etc., we easily obtain

6.2. Proposition. A connection on a crossed module bundle gerbe defines a simplicial connection on the corresponding $NC(H\to D)$-principal bundle over $N\{O\}_\alpha$ and vice versa.

Similar discussion applies to $B$-field as well.

6.3. Bundle gerbe $B$-field. $B$-field on a crossed module bundle gerbe equipped with a connection is a collection $\{B_\alpha, \delta_{\alpha\beta}\}$ of local two-forms $B_\alpha \in H^2(O_\alpha) \otimes \text{Lie}(H)$ and $\delta_{\alpha\beta} \in H^2(O_{\alpha\beta}) \otimes \text{Lie}(H)$ such that

$$B_\alpha = d_{\alpha\beta}B_\beta + \delta_{\alpha\beta} \text{ on } O_\alpha \cap O_\beta$$

and

$$\delta_{\alpha\beta} + d_{\alpha\beta}\delta_{\beta\gamma} = h_{\alpha\beta\gamma}\delta_{\alpha\gamma}h_{\alpha\beta\gamma}^{-1} + B_\alpha - h_{\alpha\beta\gamma}B_\alpha h_{\alpha\beta\gamma}^{-1} \text{ on } O_\alpha \cap O_\beta \cap O_\gamma.$$

Given a simplicial $\tilde{B}$ in the present case then the bundle gerbe $B$-field is identified as the morphism $B$ in the $\tilde{B}_1 = (\partial_0 \tilde{B}_1 - B \to 0)$ part of the simplicial $\tilde{B}$ and the simplicial $\tau_1(\partial_1 \tilde{B}_2 - \partial_2 \tilde{B}_2)$ is identified with the bundle gerbe $\delta$, we obtain the following proposition.

6.4. Proposition. A simplicial $\tilde{B}$-field on a $NC(H\to D)$ principal bundle over $N\{O\}_\alpha$ gives a $B$-field on the corresponding bundle gerbe and vice versa.

The bundle gerbe $\nu$-field is defined as $\nu = F + \alpha(B)$. This definition guaranties that it is the same as the simplicial one in the present case.

6.5. Remark. It is generally true only in the case of abelian $H$ that connection $A$ and the $B$-field can be chosen such that $\nu_\alpha = d_{\alpha\beta}\nu_\beta d_{\alpha\beta}^{-1}$. We are are not sure what kind of condition should replace this in the case of nonabelian $H$.

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