Half-flat structures on Lie groups

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Introduction

Let M be a smooth real six-dimensional manifold. An almost special Hermitian structure on M is defined as a tuple (g, J, ω, Ψ) consisting of a Riemannian metric g, an orthogonal almost complex structure J, the fundamental two-form $\omega = g(\,.\,, J\,.\,)$ and a complex-valued $(3,0)$-form Ψ of non-zero constant length. The simultaneous stabiliser in $\mathrm{GL}(T_pM)$ of these tensors evaluated in a point $p \in M$ is isomorphic to $\mathrm{SU}(3)$. Thus, an almost special Hermitian structure is essentially the same as an SU(3)-structure, i.e. a reduction of the frame bundle of M from $\mathrm{GL}(6,\mathbb{R})$ to $\mathrm{SU}(3)$. As thoroughly explained in chapter 1, an SU(3)-structure (g, J, ω, Ψ) is completely determined by the pair $(\omega, \mathrm{Re}\Psi)$. Alternatively, an SU(3)-structure can be reconstructed from the Riemannian metric g and a non-trivial spinor field, see [**LM**, ch. IV, Proposition 9.13, p. 341].

The most important class of SU(3)-structures is the class of torsionfree SU(3)-structures characterised by the existence of a torsionfree SU(3)-connection. Such a connection exists if and only if the holonomy of the metric is contained in SU(3), which implies in particular that (M, J) is a complex manifold, g is a Ricci-flat Kähler metric and Ψ is a holomorphic section of the canonical bundle. In other words, a compact torsionfree almost special Hermitian manifold M is a Calabi-Yau manifold. Ever since the existence of non-flat examples was proved by Yau in the end of the 70's, Calabi-Yau manifolds have attracted great attention in both mathematics and physics.

The deviation of an SU(3)-structure from being torsionfree is measured by the so-called intrinsic torsion or structure function which is a section of a vector bundle of rank 42 with fibre isomorphic to $T_p^*M \otimes \mathfrak{su}^\perp(3)$ where $\mathfrak{su}^\perp(3)$ denotes the orthogonal complement of $\mathfrak{su}(3)$ in $\mathfrak{so}(6,\mathbb{R})$. In fact, the complete information on the intrinsic torsion is contained in the covariant derivatives $\nabla^g\omega$ and $\nabla^g\Psi$ with respect to the Levi-Civita connection ∇^g, or, equivalently according to [**ChSa**], in the exterior derivatives $d\omega$ and $d\Psi$. For instance, a torsionfree SU(3)-structure is also characterised by the exterior system

$$d\omega = 0\,, \qquad d\Psi = 0.$$

In this thesis, we like to draw the attention to the following class of SU(3)-structures. A *half-flat structure* is defined as an SU(3)-structure (g, J, ω, Ψ) satisfying the exterior differential system

$$(\star) \qquad d(\omega \wedge \omega) = 0\,, \qquad d(\mathrm{Re}\Psi) = 0.$$

The name has been chosen in [**ChSa**] referring to the fact that the exterior system is satisfied if and only if the intrinsic torsion of the structure is contained in a certain subbundle of rank 21 in the rank 42 bundle $T^*M \otimes \mathfrak{su}^\perp(3)$. Obviously, torsionfree SU(3)-structures are in particular half-flat.

In the following, we roughly sketch the role of SU(3)-structures in string compactifications.

SU(3)**-structures in string theory.** The spacetime background for the five standard superstring theories is a real ten-dimensional manifold Y endowed, amongst other structure, with a pseudo-Riemannian metric, a spin structure and a number of spinor fields encoding the supersymmetry of the theory. A process called (Kaluza-Klein) compactification splits the pseudo-Riemannian background manifold in a six dimensional compact Riemannian manifold M, which is called internal space, and a four-dimensional Lorentz spacetime X, which is usually assumed to be flat:

$$Y^{10} = X^4 \times M^6.$$

Analogous splittings are also studied in ten-dimensional supergravity theories, which arise as the low energy limit of the superstring theories. In both cases, the geometry and topology of the six-manifold M is directly related to the four-dimensional "effective" theory. For instance, a standard requirement is the preservation of (a part of) the supersymmetry implying that there is a non-trivial global spinor field, i.e. an SU(3)-structure, on the six-manifold M. Traditionally, the spinor field on the six-manifold is moreover assumed to be parallel such that the SU(3)-structure on M is in fact Calabi-Yau, see for instance [**CHSW**].

Moreover, we like to mention briefly that there are important dualities between different string theories leading in particular to deep mathematical conjectures relating the geometry of the corresponding internal spaces. For instance, the famous mirror symmetry mysteriously interchanges the symplectic and complex geometry of two even topologically different Calabi-Yau mirror partner manifolds. However, for the motivation of the structures studied in this thesis, it is more important that the string theory literature also suggests non-torsionfree SU(3)-structures on the internal six-manifold, starting already in the 80's with [**Str**], see also [**CCDLMZ**] and references [2]–[26] therein. Non-torsionfree SU(3)-structures turn out to be advantageous if geometrical background fluxes are to be considered which can be encoded directly in the defining exterior forms ω and Ψ of the SU(3)-structure. Non-torsionfree structures also appear when the four-dimensional Lorentz spacetimes are not assumed to be flat.

For the first time in 2002, half-flat structures have been proposed in [**GLMW**] as natural candidates for internal spaces of type IIA string theory with background flux. In fact, it is argued that the half-flat structures arise as mirror partners of certain Calabi-Yau internal spaces of type IIB string theory. Different string theory scenarios with half-flat internal spaces are studied for instance in [**GM**], [**GLM1**], [**GLM2**], [**To**].

Evolution of half-flat structures. From the mathematical point of view, the main motivation for studying half-flat structures is the possibility to construct metrics with holonomy contained in G_2 out of half-flat structures. In this context, half-flat structures appeared first in the literature in [**Hi1**], although the term half-flat is not used. From the physical point of view, this possibility relates string theories with half-flat internal spaces to eleven-dimensional $\mathcal{M}$-theory with seven-dimensional internal spaces. In fact, similar to the case of ten-dimensional string theories, compactifications of $\mathcal{M}$-theory are demanded to preserve supersymmetry such that the internal spaces are usually endowed with a metric with holonomy contained in G_2.

A Riemannian metric with holonomy contained in G_2 on a seven-manifold M is in fact equivalent to a torsionfree G_2-structure on M, which is also called a parallel G_2-structure. Now, on the one hand, a parallel G_2-structure on a seven-manifold induces a half-flat SU(3)-structure on every oriented hypersurface. On the other hand, a six-manifold M with half-flat SU(3)-structure can be embedded in a seven-manifold with parallel G_2-structure by the following evolution process.

In Hitchin [**Hi1**], a k-form φ on a differentiable manifold M is called *stable* if the orbit of φ_p under $\mathrm{GL}(T_pM)$ is open in $\Lambda^k T_p^* M$ for all $p \in M$. Stability is a rare phenomenom occurring only in small dimension. Given an SU(3)-structure $(g, J, \omega, \Psi = \rho + \mathrm{i}J_\rho^*\rho)$ on a six-manifold M, the two-form ω and the three-forms ρ and $J_\rho^*\rho$ are stable. Moreover, given a family of SU(3)-structures depending on a time-parameter $t \in I$ defined by $(\omega(t), \rho(t))$, the three-form

$$\varphi = \omega(t) \wedge dt + \rho(t)$$

is stable on $M \times I$ with stabiliser G_2, i.e. φ defines a G_2-structure. Since a G_2-structure is parallel if and only if

$$d\varphi = 0\ , \qquad d *_\varphi \varphi = 0,$$

it is straightforward to check that $\varphi = \omega \wedge dt + \rho$ is parallel if and only if $(\omega(t), \rho(t))$ is half-flat for all t and if it satisfies the evolution equations

$$\frac{\partial}{\partial t}\rho = d\omega, \quad \frac{\partial}{\partial t}(\omega^2) = d(J_\rho^*\rho).$$

These evolution equations are often referred to as the Hitchin flow equations for the following reason. For compact manifolds M, Hitchin proved in [**Hi1**] that a family of stable forms $(\omega(t), \rho(t))$ which satisfies the evolution equations on an interval I and which is a half-flat SU(3)-structure at an initial value t_0 is automatically a half-flat SU(3)-structure for all t. Thus, such a family defines in particular a metric with holonomy contained in G_2.

In order to prove his result, Hitchin considers the product of the infinite-dimensional cohomology classes of the closed three-form $\rho(t_0)$ and the closed four-form $\omega^2(t_0)$ which is endowed with a natural symplectic structure defined using integration over the compact manifold. Now, it is possible to translate the evolution equations into a Hamiltonian system on this symplectic manifold and to prove the result using in particular the theorem of Stokes several times.

A main result of [**CLSS**], a collaboration of the author with Vicente Cortés, Thomas Leistner and Lars Schäfer, is a new, direct proof of Hitchin's result which avoids integration and which also holds for non-compact six-manifolds. As the author contributed to this proof and as this theorem is the most important motivation for studying half-flat structures, the theorem is also contained in this thesis in chapter 6, Theorem 1.2. At the same time, the proof is extended to indefinite metrics, i.e. $\mathrm{SU}(p,q)$-structures, $p+q = 3$, and $\mathrm{SL}(3,\mathbb{R})$-structures, where $\mathrm{SL}(3,\mathbb{R})$ is embedded in $\mathrm{SO}(3,3)$ and stabilises a so-called special para-Hermitian structure which is explained in detail in section 1.4 of chapter 1. When replacing SU(3) by a noncompact group, the stabiliser of the three-form $\varphi = \omega \wedge dt + \rho$ on $M \times I$ is the noncompact form G_2^* of $G_2^{\mathbb{C}}$ and we obtain a pseudo-Riemannian metric of signature $(3,4)$ with holonomy group in G_2^*. As an application, we prove that any six-manifold endowed with a real analytic half-flat G-structure,

$G = \mathrm{SU}(p,q)$ or $G = \mathrm{SL}(3,\mathbb{R})$, can be extended to a Ricci-flat seven-manifold with holonomy group in G_2 or G_2^*, depending on whether G is compact or noncompact.

More generally, an SU(3)-structure (ω, ρ) is called *nearly half-flat* if

$$d\rho = \omega^2$$

and a G_2-structure defined by a three-form φ is called *nearly parallel* if

$$d\varphi = *_\varphi \varphi.$$

For compact manifolds M, it was proven by Stock [**St**], generalising the proof of Hitchin, that any solution $I \ni t \mapsto \Big(\omega(t) = 2\widehat{d\rho}(t), \rho(t)\Big)$ of the evolution equation

$$\dot{\rho} = d\omega - \varepsilon J_\rho^* \rho$$

evolving from a nearly half-flat SU(3)-structure $(\omega(0), \rho(0))$ on M defines a nearly parallel G_2-structure on $M \times I$. Here, the two-form $\widehat{\sigma}$ is the stable two-form uniquely associated to a stable four-form σ and an orientation by demanding $\widehat{\sigma}^2 = \sigma$ and $\widehat{\sigma}^3 > 0$. In fact, our new method of proving Hitchin's result generalises to this situation as well, such that the assumptions that M is compact and that the considered metrics are Riemannian can be dropped resulting in Theorem 2.2 in chapter 6.

There is another type of Hitchin flow linking dimensions seven and eight. A G_2- or G_2^*-structure defined by a three-form φ is called *cocalibrated* if

$$d *_\varphi \varphi = 0.$$

Hitchin proposed the following equation for the evolution of a cocalibrated G_2-structure $\varphi(0)$:

$$\frac{\partial}{\partial t}(*_\varphi \varphi) = d\varphi.$$

He proved that any solution $I \ni t \mapsto \varphi(t)$ on a compact manifold M defines a Riemannian metric on $M \times I$ with holonomy group in Spin(7). We also generalise this theorem to noncompact manifolds and show that any solution of the evolution equation starting from a cocalibrated G_2^*-structure defines a pseudo-Riemannian metric of signature $(4,4)$ and holonomy group in $\mathrm{Spin}_0(3,4)$, see chapter 6, Theorem 3.1.

Existence of half-flat structures on Lie groups. We return to half-flat structures in dimension six which shall be the main objects to be studied in this thesis. Considering left-invariant half-flat structures on a Lie group G, the defining partial differential equations $(\star)$ reduce to a system of algebraic equations on the Lie algebra $\mathfrak{g}$ of G. In the literature, half-flat structures are mainly studied on nilmanifolds, assuming that the structures are left-invariant. For instance, a classification of nilmanifolds admitting left-invariant half-flat SU(3)-structures with different additional premises is obtained in [**CF**], [**ChSw**] and [**CT**]. Very recently, the classification of nilmanifolds admitting left-invariant half-flat SU(3)-structures without any further restrictions has been obtained in [**Con**]. Apart from the nilpotent case, examples and constructions of half-flat SU(3)-structures can be found in [**TV**] and [**AFFU**]. The Ricci curvature of a half-flat SU(3)-structure is computed in [**BV**] and [**AC1**].

The main goal of this thesis was to obtain new examples and classification results concerning half-flat structures. In order to reasonably confine the class of considered structures and produce new examples which are not nilmanifolds, we focus the attention on direct products of two three-dimensional Lie groups. Concerning existence, we ask the question which of these products admit a left-invariant half-flat SU(3)-structure. There are 12 isomorphism classes of three-dimensional Lie algebras including three classes depending on a parameter (see tables 1 and 2 in chapter 4). Although Bianchi counted only two classes depending on a parameter in his original classification, we had to split one of the Bianchi classes into two classes which turned out to have different properties. Thus, we have to consider $78 = \binom{13}{2}$ classes of direct sums in total.

Initially, we tried to find a classification by a direct proof which avoids the verification of the existence or non-existence case by case. However, this was only successful when we asked for the existence of a half-flat SU(3)-structure (g, J, ω, Ψ) such that the two factors are orthogonal with respect to the metric g. The resulting classification result is presented in section 2 of chapter 4. We prove that exactly 15 classes admit such an SU(3)-structure, 11 of which are unimodular and comply with a regular pattern, whereas the remaining four do not seem to share many properties. Given, that the additional assumption is rather strong and the proof is already quite technical, an answer to the initial question with this method cannot be expected. However, an advantage of the assumption of a Riemannian product is the fact that the curvature is completely determined by the Ricci tensors of the three-dimensional factors and that the possible Ricci tensors of left-invariant metrics on three-dimensional Lie groups are classified in [**Mi**].

A completely different method is used in [**Con**] for classifying the nilmanifolds admitting an arbitrary half-flat SU(3)-structure. The result is that 24 out of the 34 isomorphism classes of nilpotent six-dimensional Lie algebras admit a half-flat structure which is proved by giving an explicit example. The non-existence on 8 out of the remaining 10 classes is proved by introducing an obstruction to the existence of a half-flat SU(3)-structure in terms of the cohomology of a certain double complex. However, two classes resist the obstruction and the non-existence is proved by a different method.

In our situation, such a double complex can be constructed if and only if both Lie groups are solvable. However, as the methods of homological algebra turn out not to be advantageous for our problem, we prove a simplified version of the obstruction condition in section 3.1 of chapter 4. This obstruction can be applied directly to 41 isomorphism classes of direct sums in section 3.2 of chapter 4. Two classes resist the obstruction, similar as in [**Con**], although they do not admit a half-flat structure. The non-existence in these cases can be shown by an individual refinement of our standard obstruction. The remaining 35 direct sums, including all unimodular direct sums and all non-solvable direct sums, admit a half-flat SU(3)-structure which is proved by giving an explicit example in each case. We point out that the products of unimodular three-dimensional Lie groups are particularly interesting since they admit co-compact lattices, see [**RV**].

In fact, the most time-consuming part of the classification was the construction of examples of half-flat structures for the 20=35-15 classes which do not admit an "orthogonal" half-flat SU(3)-structure. The construction essentially relies on the fact that a left-invariant half-flat SU(3)-structure is defined by a pair $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ of stable forms which satisfy

$$(\star\star) \qquad \omega \wedge \rho = 0, \quad d\omega^2 = 0, \quad d\rho = 0$$

and which induce a Riemannian metric. Identifying $\mathfrak{g}^*$ with the left-invariant one-forms, the Lie bracket of $\mathfrak{g}$ contains the same information as the exterior derivative restricted to $\mathfrak{g}^*$. Thus, fixing the structure constants of the Lie bracket, two of the equations are quadratic and one is linear in the coefficients of ω and ρ. For each case separately, it is thus straightforward to construct solutions of the system of equations ($\star\star$) with the help of a computer algebra system, for instance Maple. However, even after Maple was taught to compute the induced metric, finding a solution inducing a positive definite metric required a certain persistence, in particular for the non-unimodular direct sums. We remark that in each case, all solutions of ($\star\star$) in a small neighbourhood of the constructed example give rise to a family of half-flat SU(3)-structures since the definiteness of the metric is an open condition.

Considering more generally $\mathrm{SU}(p,q)$-structures, $p+q=3$, of arbitrary signature, we give an obstruction to the existence of half-flat structures in section 4 of chapter 4 which is stronger than the obstruction established for the definite case and applies to 15 classes. Apart from giving an example of a Lie algebra admitting a half-flat $\mathrm{SU}(1,2)$-structure, but no half-flat SU(3)-structure, we abstain from completing the classification in the indefinite case since it would involve constructing approximately 62=78-15-1 explicit examples of half-flat $\mathrm{SU}(1,2)$-structures.

In section 5 of chapter 4, we turn to the para-complex case of $\mathrm{SL}(3,\mathbb{R})$-structures. Again, we give an example of a Lie algebra admitting a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure, but no half-flat $\mathrm{SU}(p,q)$-structure for any signature. Then, we consider half-flat $\mathrm{SL}(3,\mathbb{R})$-structures on direct sums such that the summands are mutually orthogonal, as before, and with the additional assumption, that the metric restricted to each summand is definite. It turns out that the proof of the classification of "orthogonal" half-flat SU(3)-structures generalises with some sign modifications and we end up with the same list of 15 Lie algebras. Finally, we consider half-flat $\mathrm{SL}(3,\mathbb{R})$-structures such that both summands are isotropic. The straightforward result is that such a structure is admitted on a direct sum of three-dimensional Lie algebras if and only if both summands are unimodular.

All results of chapter 4 concerning the existence problem of half-flat structures on direct sums of three-dimensional Lie algebras are already submitted in [**SH**].

Further results on half-flat structures. In chapter 5, the problem of the uniqueness of half-flat structures is studied. More precisely, we ask the question how many half-flat structures exist on a fixed Lie algebra modulo Lie algebra automorphisms. We focus on the probably most interesting direct sums: The only compact direct sum $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and the only nilpotent direct sum $\mathfrak{h}_3\oplus\mathfrak{h}_3$ where $\mathfrak{h}_3$ denotes the three-dimensional real Heisenberg algebra. In both cases, most of the results apply to SU(3)-structures, $\mathrm{SU}(1,2)$-structures and $\mathrm{SL}(3,\mathbb{R})$-structures simultaneously.

The case of $\mathfrak{g}=\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ is discussed in section 1 and we obtain a complete and explicit description of the space of all half-flat structures modulo $\mathrm{Aut}(\mathfrak{g})$. The method is to fix a standard basis for the Lie bracket and then to carefully classify the normal forms of the defining stable forms ω and ρ under the action of $\mathrm{Aut}(\mathfrak{g})$. In fact, we are also able to explicitly describe all nearly half-flat structures modulo $\mathrm{Aut}(\mathfrak{g})$ and those which are half-flat such that the opposite structure $(\omega, J^*\rho)$ is nearly half-flat at the same time.

As another application of this method, we prove that there is no left-invariant complex structure on $S^3 \times S^3$ admitting a holomorphic section of the canonical bundle.

The half-flat structures modulo $\mathrm{Aut}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ are studied in section 2 of chapter 5, which is also contained in [**CLSS**]. To begin with, we explicitly determine the orbits of the $\mathrm{Aut}(\mathfrak{h}_3 \oplus \mathfrak{h}_3)$-action on the space of non-degenerate two-forms ω satisfying $d\omega^2 = 0$. It turns out that there are exactly five orbits and we give a simple standard representative in each case. Based on this result, we are able to describe all left-invariant half-flat structures (ω, ρ) on $H_3 \times H_3$. In particular, it is shown that half-flat $\mathrm{SU}(3)$-structures (ω, ρ) inducing a Riemannian metric are only possible if ω belongs to the unique orbit characterised by $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$ where $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$. A surprising rigidity phenomenon occurs in indefinite signature. Under the assumption $\omega(\mathfrak{z}, \mathfrak{z}) = 0$, which corresponds to the vanishing of the projection of ω on a one-dimensional space, the metric induced by a half-flat structure (ω, ρ) is always isometric to the product of a flat $\mathbb{R}^2$ and the unique four-dimensional para-hyper-Kähler symmetric space with abelian holonomy.

Due to the explicit characterisation of all half-flat structures on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, we are in fact able to explicitly solve the Hitchin flow in all possible cases which is carried out in section 5 of chapter 6, also being part of [**CLSS**]. First of all, we remark that the evolution equations reduce from a PDE to an ODE due to the assumption of left-invariance. Moreover, Lemma 5.1, ch. 6, shows how to simplify effectively the solution ansatz for a number of nilpotent Lie algebras including $\mathfrak{h}_3 \oplus \mathfrak{h}_3$. Using this ansatz for $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ and assuming that $\omega(\mathfrak{z}, \mathfrak{z}) = 0$, which is only possible for indefinite metrics, the evolution turns out to be affine linear (Proposition 5.4). However, this evolution produces only metrics that are decomposable and have one-dimensional holonomy group. The opposite case $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$ is solved in Proposition 5.6: We are able to give an explicit formula in Proposition 5.6 for the parallel three-form φ resulting from the evolution of an *arbitrary* half-flat structure (ω, ρ) with $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$. Even more surprising, the formula we obtain is completely algebraic such that the integration of the differential equation is circumvented.

In particular, we give a number of explicit examples of half-flat structures of the second kind on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ which evolve to new metrics with holonomy group *equal* to G_2 and G_2^*. Moreover, we construct an eight-parameter family of half-flat deformations of the half-flat examples which lift to an eight-parameter family of deformations of the corresponding parallel stable three-forms in dimension seven. For obvious reasons, those examples of $\mathrm{G}_2^{(*)}$-metrics on $M \times (a, b)$ for which $(a, b) \neq \mathbb{R}$ are geodesically incomplete. However, for $\mathrm{SU}(3)$-structures on compact M, a conformal transformation produces complete Riemannian metrics on $M \times \mathbb{R}$ that are conformally parallel G_2.

Results on nearly Kähler six-manifolds. Finally, this thesis contains some results obtained by the author in collaboration with Lars Schäfer on nearly pseudo-Kähler and nearly para-Kähler six-manifolds which form a subclass of the class of half-flat structures. These results are already submitted in [**SSH**].

A nearly Kähler manifold is defined as an almost Hermitian manifold (M, g, J, ω) such that $(\nabla^g_X J)X = 0$. These manifolds were first studied by A. Gray in a series of papers summarised in [**G3**]. Since nearly Kähler manifolds are Einstein, admit Killing spinors and admit a Hermitian connection with skew-symmetric torsion, these structures are very appealing to differential geometers, but also for physicists working in the context of string compactifications. It is remarkable that, with the exception of dimension six, nearly Kähler manifolds are classified in [**Na1**] and [**Na2**]. In dimension six, there are only four known examples, S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}^3$, all of which are homogeneous three-symmetric spaces. Another fact which is special to dimension six is the following. Due to a result of [**RC**], a nearly Kähler six-manifold with $\nabla^g J \neq 0$ is equivalently given by an SU(3)-structure (ω, ρ) satisfying the exterior system

$$d\omega = 3\rho, \tag{NK1}$$

$$d(J_\rho^* \rho) = 2\,\omega^2. \tag{NK2}$$

Hence, nearly Kähler manifolds induce a natural half-flat SU(3)-structure.

Almost all of the literature on nearly Kähler manifolds considers only Riemannian metrics, one of the few exceptions is [**G2**]. When considering indefinite metrics and even when replacing the almost complex structure J by an almost para-complex structure, many properties are very similar to the definite case. However, some completely new phenomena occur. For instance, Levi-Civita flat strict nearly pseudo-Kähler manifolds only exist for split signature, see [**CS1**] and also [**CS2**] for the analogous para-complex case.

The main question asked in [**SSH**] is the following. There is a left-invariant nearly Kähler structure on $S^3 \times S^3$ which arises from a classical construction of three-symmetric spaces by Ledger and Obata [**LO**]. It is shown in [**Bu1**] (compare also [**Bu2**]) that this nearly Kähler structure is the only one on $S^3 \times S^3$ up to homothety. In fact, the proof of this uniqueness result has been the most difficult step in the classification of homogeneous nearly Kähler structures in dimension six. It is easy to see that the construction of Ledger and Obata also yields a nearly pseudo-Kähler structure on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with indefinite signature and the question arises whether the uniqueness result of [**Bu1**] also holds for this structure.

In section 3 of chapter 5, we include the proof of the main theorem of [**SSH**] stating that the left-invariant nearly pseudo-Kähler structure on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is also unique up to homothety. It fits well into chapter 5 since the method of the proof is very similar to the description of the "moduli space" of left-invariant half-flat structures on $S^3 \times S^3$. In fact, the idea is essentially the same as the one used in the Riemannian case by [**Bu1**], however, the technical problems increase substantially in the indefinite case. As a by-product, we prove that neither $S^3 \times S^3$ nor $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ admit a left-invariant nearly para-Kähler structure.

Since the starting point of the proof is the characterisation of a nearly Kähler manifold by the exterior system (NK1), (NK2), first of all, this characterisation has to be extended to nearly pseudo-Kähler manifolds with indefinite metrics. Similar as in [**SSH**], we spend some effort in chapter 3 in order to give a self-contained proof of this characterisation which also applies to the para-complex case. Unlike the original proof of [**RC**] in the Riemannian case, we clarify the structure of the proof by elaborating the role of the Nijenhuis tensor. Along the way, we obtain some useful characterisations for the skew-symmetry of the Nijenhuis tensor for $\mathrm{U}(p,q)$-, $\mathrm{GL}(m,\mathbb{R})$-, $\mathrm{SU}(p,q)$- and $SL(m,\mathbb{R})$-structures in terms of the intrinsic torsion. As another application of the characterisation of nearly pseudo- and para-Kähler manifolds by an exterior system, we discuss the evolution of these structures under the Hitchin flow in section 4 of chapter 6. In fact, this section is also contained in [**CLSS**].

The structure of the thesis. The previous sections contain a detailed description of all results obtained in this thesis including references where to find these results in this thesis. Nevertheless, we briefly summarise the structure of the thesis.

Chapter 1 collects all necessary algebraic preliminaries. We have to be rather explicit since we introduce a unified language which allows us to treat special pseudo- and special para-Hermitian structures simultaneously, compare [**AC2**], [**SSH**]. In particular, we introduce the algebraic models for all G-structures appearing in this thesis and extend many important algebraic identities to all possible signatures. Moreover, an introduction to the formalism of stable forms is contained and the algebraic constructions linking dimensions six, seven and eight are presented and extended to arbitrary signature. The most important algebraic concept for this thesis is probably the characterisation of the groups $\mathrm{SU}(p,q)$, $p+q=3$, by a certain pair of a two-form and a three-form. To the author's best knowledge, our generalisation of this formalism to the groups $\mathrm{SU}(p,q)$, $p+q=2l-1$, $l \geq 2$, in sections 1.2 and 1.4, ch. 1, is not contained in the literature so far.

Chapter 2 is a brief introduction on G-structures in general focusing on the concept of intrinsic torsion. We review different methods for characterising the intrinsic torsion, in particular for structures with $G \subset \mathrm{O}(p,q)$, and the relation to the concept of holonomy.

In chapter 3, we deal with G-structures for the groups $\mathrm{U}(p,q)$-, $\mathrm{GL}(m,\mathbb{R})$-, $\mathrm{SU}(p,q)$- and $SL(m,\mathbb{R})$ and discuss the characterisation of these structures in terms of the intrinsic torsion. As mentioned already before, we focus on properties of the Nijenhuis tensor, half-flat structures and nearly Kähler manifolds.

In fact, the first three introductory chapters contain extensions and unifications of many well-known properties from the Hermitian to the pseudo-Hermitian and para-Hermitian context. Although usually proved similarly or identically as in the Riemannian case, many of these extensions are hard to find or not contained at all in the literature.

In chapter 4, we present our classification results concerning the existence of left-invariant half-flat structures on direct products of three-dimensional Lie groups. In chapter 5, we examine the half-flat and nearly pseudo- and para-Kähler structures on the groups $S^3 \times S^3$, $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $H_3 \times H_3$, where H_3 denotes the real three-dimensional Heisenberg group. The final chapter 6 deals with the Hitchin flow and we present proofs of its main properties. Examples for solutions of the Hitchin flow are provided by discussing the evolution of nearly pseudo- and para-Kähler manifolds and the evolution of an arbitrary left-invariant half-flat structure on $H_3 \times H_3$.

Computer support. Some results of this thesis would have been hard to obtain without computer support and we like to comment on the applied software which is in fact self-programmed in substantial parts. Before doing so, we add the remark that almost all of the proofs in this thesis are written in such a way that they can be completely verified by the reader without using a computer.

The main tool has been the package "difforms" which is contained by default in Maple, at least in Maple 10, 11 and 12. Although there are packages dealing more profoundly with differential forms, this package has the following advantage. It is possible to explicitly set the exterior differential d of a frame of left-invariant one-forms on a Lie group G. Thus, the Lie bracket of the corresponding Lie algebra $\mathfrak{g}$ can be implemented with respect to a basis of $\mathfrak{g}^*$ in a way which is well adapted to left-invariant exterior systems. With respect to this basis, the wedge product of k-forms and the exterior differential of k-forms can be computed for all k. Except for some minor additional procedures, the described functions already exhaust the scope of operation of this package which is not sufficient for our purposes.

In order to compute explicitly the almost complex structure and the metric induced by a compatible pair of stable forms, we needed to teach Maple to compute the contraction of a (multi-)vector and a k-form. In the course of time, we were able to implement a large extension of the package difforms, written in the programming language contained in Maple, which contains the following functionality. For instance, the new package explicitly computes not only contractions, but also directly the tensors induced by stable forms as well as the Hodge dual, pullbacks or infinitesimal pullbacks of forms, always working with respect to a basis. Moreover it contains many useful routines for comparing differential forms, for instance it is possible to translate an exterior system into a system of coefficient equations with respect to a basis in such a way that the output is directly accessible for the solve procedure of Maple. After applying the solve procedure to the coefficient equations, there is another function translating the solution into assignments which eliminate some of the coefficients.

Given any system of algebraic equations, the solve procedure of Maple uses recent computer algebra methods, for instance Gröbner bases, in order to obtain solutions. However, when a certain degree of complexity is exceeded, the return values of this procedure are not useful or do not seem to be reliable. Even if the result seems reasonable, a manual verification is needed, proving that indeed *all* solutions are found, in order to obtain classification results. Nevertheless, we admit that Maple's solve function often turned out to be an excellent source of inspiration. At times, an elegant basis-independent proof is much easier to find after having verified the assertion by brute computational force. Apart from that, the quick forming and solving of equations involving left-invariant forms turned out to be useful when constructing examples. On the other hand, the current computer algebra systems are not very helpful when trying to find a solution of a system of inequalities. Despite this disadvantage, the construction of the half-flat structures given in tables 3, 4 and 5 of chapter 4 inducing positive definite metrics would have been hardly possible without computer support.

Although there appear more and more packages for Maple and Mathematica or independent software containing implementations of almost all differential geometric constructions, there is currently no other package, to the author's best knowledge, which is as well adapted to solving left-invariant exterior systems on Lie groups (or homogeneous spaces) as our package.

Finally, we mention that the package "tensor", also contained in Maple by default, has been used in the version contained in Maple 10 in section 2, ch. 5, and section 5.2, ch. 6, to compute the curvature of a number of explicit left-invariant metrics on $H_3 \times H_3$.

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Hamburg
March 2010 — Fabian Schulte-Hengesbach

CHAPTER 1

Algebraic preliminaries

Notational conventions. When $e_1, \ldots, e_n$ denotes the basis of a vector space or a local frame of a manifold, we denote by $e^1, \ldots, e^n$ the dual basis or dual frame. At least in the context of indefinite metrics, this differs from the convention of some authors who refer to the metric dual one-forms of an orthonormal basis by this notation. The wedge product of basis vectors e_i and e_j is abbreviated by e_{ij} and the same abbreviation applies to forms, i.e. $e^{ij} = e^i \wedge e^j$. Given a k-form α, the power α^n denotes the n-fold wedge product of α with itself.

We denote the contraction or interior product of a vector and a l-form by $\lrcorner$ and extend this operator to the contraction

$$\lrcorner : \Lambda^k V \times \Lambda^l V^* \to \Lambda^{l-k} V^*$$

of k-vectors and l-forms, $k \leq l$, by setting

$$(x_1 \wedge \ldots \wedge x_k) \lrcorner \alpha = x_k \lrcorner (x_{k-1} \lrcorner \ldots \lrcorner (x_1 \lrcorner \alpha))$$

for decomposable k-vectors. With this convention, the contraction coincides with the adjoint operator of the wedge product satisfying

$$(v \lrcorner \alpha)(w) = \alpha(v \wedge w)$$

for all $v \in \Lambda^k V$, $\alpha \in \Lambda^l V^*$ and $w \in \Lambda^{l-k} V$. If a contraction and a wedge product appear in an identity, we assume that the contraction is evaluated first.

Moreover, we will always follow the convention $v \wedge w = v \otimes w - w \otimes v$ for the wedge product, but $v \cdot w = \frac{1}{2}(v \otimes w + w \otimes v)$ for the symmetric product. The first convention seems to reduce the number of fractions in calculations in a basis and is not applied to the symmetric powers in order to have $v^2 = v \cdot v = v \otimes v$.

For an endomorphism A of a vector space V, we denote by A^* the endomorphism of $(V^*)^{\otimes k}$ given by

$$A^* \alpha(X_1, \ldots, X_k) \quad = \quad \alpha(AX_1, \ldots, AX_k)$$

for $\alpha \in (V^*)^{\otimes k}$ and $X_i \in V$.

1. Structures on vector spaces defined by linear Lie groups

In this section, we discuss the algebraic models for the G-structures which appear in this thesis. Although we are mainly interested in dimension six, we establish the basic material for arbitrary dimension.

In the following, let V denote an n-dimensional real vector space and $\mathrm{GL}(V)$ the real general linear group of this vector space. If V is oriented, we denote by $\mathrm{GL}^+(V)$ the group of orientation preserving automorphisms.

1.1. The complex and the para-complex linear group. A *complex structure* J on a $2m$-dimensional vector space V is an anti-involution of V. The stabiliser in $\mathrm{GL}(V)$ of a complex structure J is the *complex general linear group* $\mathrm{GL}(V, J) \cong \mathrm{GL}(m, \mathbb{C})$.

A *product structure* P on an n-dimensional vector space is an involution of V. Equivalently, $V = V^+ \oplus V^-$ splits as a direct sum where we denote by $V^\pm$ the ± 1-eigenspace of P. A *para-complex structure* J on a $2m$-dimensional vector space V is a product structure with $\dim V^+ = \dim V^- = m$. The stabiliser in $\mathrm{GL}(V)$ of a para-complex structure J is the *para-complex general linear group* $\mathrm{GL}(V, J) \cong \mathrm{GL}(V^+) \oplus \mathrm{GL}(V^-) \cong \mathrm{GL}(m, \mathbb{R}) \oplus \mathrm{GL}(m, \mathbb{R})$.

It turns out that many well-known notions for complex structures can be extended to para-complex structures. In order to benefit from the similarities of complex and para-complex structures, we unify the notation as follows, see also [**AC2**], [**SSH**].

DEFINITION 1.1. Let $\varepsilon = 1$ or $\varepsilon = -1$ and let i_ε be a symbol satisfying ${\mathrm{i}_\varepsilon}^2 = \varepsilon$.

(i) Let the *ε-complex numbers* be defined as $\mathbb{C}_\varepsilon = \mathbb{R}[\mathrm{i}_\varepsilon]$. We will use the name *para-complex numbers*[1] for the real algebra $C := \mathbb{C}_1 \cong \mathbb{R} \oplus \mathbb{R}$ following [**CMMS**].
(ii) An *ε-complex structure* J on V is defined as an endomorphism which satisfies $J^2 = \varepsilon \mathrm{Id}$ and $\dim V^+ = \dim V^- = m$ for $\varepsilon = 1$. The pair (V, J) is called an *ε-complex vector space.*
(iii) The stabiliser of an ε-complex structure J in $\mathrm{GL}(V)$ is called the *ε-complex general linear group* $\mathrm{GL}(V, J)$.
(iv) The *ε-complexification* of a real vector space V is defined as the $\mathbb{C}_\varepsilon$-module $V_{\mathbb{C}_\varepsilon} = V \otimes_{\mathbb{R}} \mathbb{C}_\varepsilon \cong V \oplus \mathrm{i}_\varepsilon V$.

Obviously, the unified language is chosen such that ε can be replaced by the word "para" if $\varepsilon = 1$ whereas it can be omitted for $\varepsilon = -1$. The familiar notions of real part, imaginary part and complex conjugation of a complex number are extended literally to ε-complex numbers. We remark that the standard real inner product $\langle z_1, z_2 \rangle = \mathrm{Re}(z_1 \bar{z}_2)$ on $\mathbb{C}_\varepsilon$ has signature $(1, 1)$ for $\varepsilon = 1$ such that the norm square $|z|^2 = z\bar{z}$ may vanish in the para-complex world.

To summarise, the ε-complex general linear group satisfies

$$\mathrm{GL}(V, J) \cong \mathrm{GL}(m, \mathbb{C}_\varepsilon) := \begin{cases} \mathrm{GL}(m, \mathbb{C}) & \text{if } \varepsilon = -1, \\ \mathrm{GL}(m, \mathbb{R}) \oplus \mathrm{GL}(m, \mathbb{R}) & \text{if } \varepsilon = 1. \end{cases}$$

In order to fix the notation, we discuss explicitly a number of properties of ε-complex structures which are standard for complex structures.

Let (V, J) be an ε-complex vector space. By $V^{1,0}$ and $V^{0,1}$, we denote the $+\mathrm{i}_\varepsilon$- and $-\mathrm{i}_\varepsilon$-eigenspace of J acting on the ε-complexification $V_{\mathbb{C}_\varepsilon}$ which therefore splits into

$$V_{\mathbb{C}_\varepsilon} = V^{1,0} \oplus V^{0,1} = V^{1,0} \oplus \overline{V^{1,0}}.$$

Any $v \in V_{\mathbb{C}_\varepsilon}$ can be projected onto $V^{1,0}$ and $V^{0,1}$ by

$$v^{1,0} = \frac{1}{2}(v + \varepsilon \mathrm{i}_\varepsilon J v) \in V^{1,0}\,, \quad v^{0,1} = \frac{1}{2}(v - \varepsilon \mathrm{i}_\varepsilon J v) \in V^{0,1},$$

[1] In fact, there are many different names for the algebra C, for instance "split-complex numbers" or "double numbers".

and the corresponding restrictions $V \to V^{1,0}$ and $V \to V^{0,1}$ are isomorphisms of real vector spaces.

The dual ε-complex vector space (V^*, J^*) satisfies $(V^{1,0})^* = (V^*)^{1,0}$. We define the vector space of *forms of type (r,s)* or *(r,s)-forms* as

$$\Lambda^{r,s}V^* := \Lambda^r(V^*)^{1,0} \otimes \Lambda^s(V^*)^{0,1}$$

such that the space of ε-complex k-forms decomposes into

$$\Lambda^k V^*_{\mathbb{C}_\varepsilon} = \bigoplus_{r+s=k} \Lambda^{r,s}V^*$$

for all k.

As we will be mainly interested in real representations, we define for $r \neq s$ the space of *(real) forms of type* $(r,s)+(s,r)$, denoted by $[\![\Lambda^{r,s}V^*]\!]$, and for $r = s$ the space of *(real) forms of type* (r,r), denoted by $[\Lambda^{r,r}V^*]$, by the properties

$$\begin{aligned} [\![\Lambda^{r,s}V^*]\!] \otimes \mathbb{C}_\varepsilon = & \quad [\![\Lambda^{r,s}V^*]\!] \oplus \mathrm{i}_\varepsilon [\![\Lambda^{r,s}V^*]\!] & = \Lambda^{r,s}V^* \oplus \Lambda^{s,r}V^*, \\ [\Lambda^{r,r}V^*] \otimes \mathbb{C}_\varepsilon = & \quad [\Lambda^{r,r}V^*] \oplus \mathrm{i}_\varepsilon [\Lambda^{r,r}V^*] & = \Lambda^{r,r}V^*. \end{aligned}$$

In consequence, we have the decompositions

$$\begin{aligned} \Lambda^{2l}V^* &= \bigoplus_{s=0}^{l-1} [\![\Lambda^{2l-s,s}V^*]\!] \oplus [\Lambda^{l,l}V^*], \\ \Lambda^{2l+1}V^* &= \bigoplus_{s=0}^{l} [\![\Lambda^{2l+1-s,s}V^*]\!] \end{aligned} \tag{1.1}$$

as real $\mathrm{GL}(V,J)$-modules for all l.

We collect a number of useful characterisations for some (r,s)-forms using the operator $J_{(i)}$ which is defined by

$$J_{(i)}\alpha(X_1, \dots, X_k) = \alpha(X_1, \dots, JX_i, \dots, X_k)$$

for $\alpha \in \Lambda^k V^*$. The real forms of type $(r,0)+(0,r)$ are given by

$$\begin{aligned} [\![\Lambda^{r,0}V^*]\!] &= \{\alpha \in \Lambda^r V^* \mid J_{(1)}J_{(2)}\alpha = \varepsilon\alpha\} \\ &= \{\alpha \in \Lambda^r V^* \mid J_{(i)}J_{(j)}\alpha = \varepsilon\alpha \text{ for all } i \neq j\}. \end{aligned} \tag{1.2}$$

The first identity can easily be seen when expanding the expression $\alpha(X^{1,0}, Y^{0,1}, \dots) = 0$ holding for any $X, Y \in V$ and the second identity is obvious. Using a similar argument, we find

$$[\Lambda^{1,1}V^*] = \{\alpha \in \Lambda^2 V^* \mid J^*\alpha = -\varepsilon\alpha\}\,, \tag{1.3}$$

$$[\![\Lambda^{2,1}V^*]\!] = \{\alpha \in \Lambda^3 V^* \mid J_{(i)}J_{(j)}\alpha = -\varepsilon\alpha \text{ for all } i \neq j\}, \tag{1.4}$$

$$[\Lambda^{2,2}V^*] = \{\alpha \in \Lambda^4 V^* \mid J_{(i)}J_{(j)}\alpha = -\varepsilon\alpha \text{ for all } i \neq j\}, \tag{1.5}$$

$$[\![\Lambda^{3,1}V^*]\!] = \{\alpha \in \Lambda^4 V^* \mid J^*\alpha = -\alpha\}. \tag{1.6}$$

In the para-complex world, there is of course a more natural decomposition of real k-forms induced by the real eigendecomposition $V = V^+ \oplus V^-$. Indeed, the decomposition

$$\Lambda^k V^* = \bigoplus_{r+s=k} \Lambda^r(V^+)^* \otimes \Lambda^s(V^-)^*$$

is $\mathrm{GL}(V,J)$-invariant as it consists of J^*-eigenspaces. However, it is useful to introduce also in the para-complex context the usual type decomposition, although it is more complicated and not irreducible. The main reason is that it turns out to be possible to generalise well-known results for complex structures to para-complex structures in a straightforward way. In particular, it is often convenient to give unified proofs of identities involving ε-complex structures without separating the two cases.

In fact, the two decompositions are related as follows:

$$\begin{aligned}
[\![\Lambda^{r,s}V^*]\!] &= \Lambda^r(V^+)^*\otimes\Lambda^s(V^-)^*\oplus\Lambda^s(V^+)^*\otimes\Lambda^r(V^-)^*,\\
[\Lambda^{r,r}V^*] &= \Lambda^r(V^+)^*\otimes\Lambda^r(V^-)^*.
\end{aligned}\tag{1.7}$$

In the following, it will be advantageous to remember that it holds

$$[\![\Lambda^{r,0}V^*]\!] = \Lambda^r(V^+)^*\oplus\Lambda^r(V^-)^*.\tag{1.8}$$

1.2. The complex and the para-complex special linear group. A *complex volume form* on a complex $2m$-dimensional vector space (V,J) is a non-zero complex-valued $(m,0)$-form Ψ. The stabiliser in $\mathrm{GL}(V)$ of a complex volume form Ψ is the *complex special linear group* $\mathrm{SL}(V,\Psi)\cong\mathrm{SL}(m,\mathbb{C})$. Note that the (m,m)-form $\Psi\wedge\bar\Psi$ is non-trivial for a complex volume form. This motivates the following definition.

Definition 1.2. Let (V,J) be an $2m$-dimensional ε-complex vector space.

(i) An ε-complex m-form $\Psi=\psi_+ + i_\varepsilon\psi_-\in\Lambda^m V^*_{\mathbb{C}_\varepsilon}$ is called non-degenerate if $\Psi\wedge\bar\Psi\neq 0$.
(ii) An *ε-complex volume form* is defined as a non-degenerate ε-complex $(m,0)$-form Ψ.
(iii) The stabiliser in $\mathrm{GL}(V)$ of an ε-complex volume form Ψ is the *ε-complex special linear group* $\mathrm{SL}(V,\Psi)$.

For $\varepsilon=1$, the additional condition $\Psi\wedge\bar\Psi\neq 0$ guarantees that Ψ is not contained in the para-complexification of one of the real summands of (1.8).

We collect a number of identities for ε-complex volume forms.

Lemma 1.3. *Let $\Psi=\psi_+ + i_\varepsilon\psi_-$ be an ε-complex volume form on an ε-complex vector space (V^{2m},J), $m\geq 2$, and let $X\in V$ and $\alpha\in V^*$ be arbitrary. The real part and the imaginary part of Ψ are related by the formulae*

$$X\lrcorner\psi_- = \varepsilon JX\lrcorner\psi_+,\qquad X\lrcorner\psi_+ = JX\lrcorner\psi_-,\tag{1.9}$$

$$\alpha\wedge\psi_- = -\varepsilon J^*\alpha\wedge\psi_+,\qquad \alpha\wedge\psi_+ = -J^*\alpha\wedge\psi_-,\tag{1.10}$$

$$X\lrcorner\psi_+\wedge\psi_- = -X\lrcorner\psi_-\wedge\psi_+,\qquad X\lrcorner\psi_+\wedge\psi_+ = -\varepsilon\, X\lrcorner\psi_-\wedge\psi_-,\tag{1.11}$$

and

$$J^*\psi_+ = \varepsilon^l\,\psi_-,\qquad J^*\psi_- = \varepsilon^{l-1}\,\psi_+,\qquad \textit{if } m=2l-1 \textit{ is odd,}\tag{1.12}$$

$$J^*\psi_+ = \varepsilon^l\,\psi_+,\qquad J^*\psi_- = \varepsilon^{l}\,\psi_-,\qquad \textit{if } m=2l \textit{ is even.}\tag{1.13}$$

Moreover, the $2m$-form $\Psi\wedge\bar\Psi$ satisfies

$$\Psi\wedge\bar\Psi = 2i_\varepsilon\psi_-\wedge\psi_+,\qquad (\psi_+)^2=(\psi_-)^2=0,\qquad \textit{if } m \textit{ is odd,}\tag{1.14}$$

$$\Psi\wedge\bar\Psi = 2(\psi_+)^2 = -2\varepsilon(\psi_-)^2,\qquad \psi_+\wedge\psi_-=0,\qquad \textit{if } m \textit{ is even,}\tag{1.15}$$

and both ψ_+ and ψ_- are non-degenerate in the sense that

$$X\lrcorner\psi_+ = 0\Rightarrow X=0,\qquad \alpha\wedge\psi_+=0\Rightarrow\alpha=0\tag{1.16}$$

$$X \lrcorner \psi_- = 0 \Rightarrow X = 0, \qquad \alpha \wedge \psi_- = 0 \Rightarrow \alpha = 0$$

for all $X \in V$ and $\alpha \in V^*$. Finally, we have the useful characterisation

$$[\![\Lambda^{m,1}V^*]\!] = \psi^+ \wedge V^* = \psi^- \wedge V^*. \tag{1.17}$$

PROOF. Since $(V^{1,0})^* = (V^*)^{1,0}$, the $(m-1)$-form $X^{0,1} \lrcorner \Psi$ is zero and the identities (1.9) are immediate when expanding this expression. In analogy, the equations (1.10) follow since the wedge product of the $(m, 0)$-form Ψ and the $(1, 0)$-form $\alpha+\varepsilon \mathrm{i}_\varepsilon J^*\alpha$ vanishes. The two formulae (1.11) correspond to the vanishing of the real and imaginary part of the $(2m-1, 0)$-form $X \lrcorner \Psi \wedge \Psi$.

Taking the relation (1.2) into account, the identity (1.13) is obvious and (1.12) is equivalent to (1.9). In the odd case, the properties (1.14) of the (m, m)-form $\Psi \wedge \bar{\Psi}$ are just a consequence of the skew-symmetry of the wedge product. In the even case, the properties (1.15) can be seen for instance by expanding the vanishing $(2m, 0)$-form Ψ^2 into real and imaginary part.

In order to show the non-degeneracy of ψ_+ and ψ_-, we assume that $X \lrcorner \psi_+ = 0$ or $X \lrcorner \psi_- = 0$. In the even case, it follows $X \lrcorner (\Psi \wedge \bar{\Psi}) = 0$ by (1.15). In the odd case, the same identity holds taking (1.14) and (1.11) into account. Therefore, X has to be zero in both cases since $\Psi \wedge \bar{\Psi}$ is a real volume form by the definition of an ε-complex volume form. If $\alpha \wedge \psi_+ = 0$ or $\alpha \wedge \psi_- = 0$, the same arguments show that $\alpha(X)\Psi \wedge \bar{\Psi} = \alpha \wedge (X \lrcorner (\Psi \wedge \bar{\Psi})) = 0$ for all $X \in V$, which clearly yields $\alpha = 0$.

The identification (1.17) can be seen as follows. Every real form of type $(m, 1)+(1, m)$ is the real part of an ε-complex $(m, 1)$-form $\Psi \wedge \theta^{0,1}$ for some real one-form $\theta \in V^*$. Because of (1.10), the real part of $\Psi \wedge \theta^{0,1}$ is exactly $\psi^+ \wedge \theta = -\psi^- \wedge J^*\theta$ and the proof is finished. □

Notice that an ε-complex $(m, 0)$-form is always decomposable since the rank of the $\mathbb{C}_\varepsilon$-module $\Lambda^{m,0}V^*$ is one.

In the following, we show that an ε-complex volume form Ψ completely determines the ε-complex structure J, thus proving $\mathrm{SL}(V, \Psi) \subset \mathrm{GL}(V, J)$ and

$$\mathrm{SL}(V, \Psi) = \{A \in \mathrm{GL}(V) \,|\, A^*\Psi = \Psi\} \cong \mathrm{SL}(m, \mathbb{C}_\varepsilon) := \begin{cases} \mathrm{SL}(m, \mathbb{C}) & \text{if } \varepsilon = -1, \\ \mathrm{SL}(m, \mathbb{R}) \oplus \mathrm{SL}(m, \mathbb{R}) & \text{if } \varepsilon = 1. \end{cases}$$

More precisely, we give an explicit formula for the ε-complex structure J associated to an ε-complex non-degenerate decomposable m-form Ψ such that Ψ is of type $(m, 0)$ with respect to J.

Let κ denote the natural isomorphism

$$\kappa : \Lambda^k V^* \to \Lambda^{m-k} V^* \otimes \Lambda^{2m} V^*, \; \xi \mapsto X \otimes \phi \quad \text{with } X \lrcorner \phi = \xi.$$

PROPOSITION 1.4. *Let $\Psi = \psi_+ + \mathrm{i}_\varepsilon \psi_-$ be an ε-complex non-degenerate decomposable m-form on a real vector space V^{2m}, $m \geq 2$. Then, the $2m$-form*

$$\phi(\Psi) := \begin{cases} \frac{1}{4}\Psi \wedge \bar{\Psi} & \textit{if } m \textit{ is even,} \\ \frac{1}{4\mathrm{i}_\varepsilon}\Psi \wedge \bar{\Psi} & \textit{if } m \textit{ is odd,} \end{cases} \tag{1.18}$$

is a real volume form and there is a unique ε-complex structure J such that Ψ is of type $(m,0)$. The ε-complex structure J is characterised by the identities

$$JX = \begin{cases} \frac{1}{\phi(\Psi)}\,\kappa(X\lrcorner\psi_+\wedge\psi_+), & \text{if } m \text{ is odd,} \\ \frac{1}{\phi(\Psi)}\kappa(X\lrcorner\psi_+\wedge\psi_-), & \text{if } m \text{ is even.} \end{cases} \tag{1.19}$$

for $X\in V$ or, equivalently, by

$$(JX\lrcorner\alpha)\,\phi(\Psi) = \begin{cases} \alpha\wedge(X\lrcorner\psi_+)\wedge\psi_+, & \text{if } m \text{ is odd,} \\ \alpha\wedge(X\lrcorner\psi_+)\wedge\psi_-, & \text{if } m \text{ is even,} \end{cases} \tag{1.20}$$

for $X\in V$, $\alpha\in V^*$.

Proof. First of all, we observe that, given an ε-complex decomposable m-form

$$\Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_- = (e^1+\varepsilon\mathrm{i}_\varepsilon e^2)\wedge\ldots\wedge(e^{2m-1}+\varepsilon\mathrm{i}_\varepsilon e^{2m})$$

for some $e^i\in V^*$, the $2m$-form (1.18) associated to Ψ is

$$\begin{aligned} \phi(\Psi) &= (-1)^{\frac{m(m-1)}{2}}(-1)^m\varepsilon^m\,2^{m-2}e^{1..2m}\cdot\begin{cases} \mathrm{i}_\varepsilon{}^{m-1} & \text{if } m=2l-1 \text{ is odd} \\ \mathrm{i}_\varepsilon{}^{m} & \text{if } m=2l \text{ is even,} \end{cases} \\ &= (-\varepsilon)^l 2^{m-2}e^{1..2m} \end{aligned} \tag{1.21}$$

for both $m=2l$ and $m=2l-1$. In particular, $e_1,\ldots,e_{2m}$ is a basis since Ψ is non-degenerate. With respect to this basis, the unique ε-complex structure such that Ψ is of type $(m,0)$ is given by

$$J^*e^{2k-1}=e^{2k},\qquad J^*e^{2k}=\varepsilon e^{2k-1},\qquad 1\le k\le m, \tag{1.22}$$

on V^* since each factor $e^{2k-1}+\varepsilon\mathrm{i}_\varepsilon e^{2k}$ has to be a $(1,0)$-form.

For odd m, the explicit formula (1.19) follows from the computation

$$JX\lrcorner\phi(\Psi) \overset{(1.14)}{=} \frac{1}{2}JX\lrcorner(\psi_-\wedge\psi_+) \overset{(1.11)}{=} JX\lrcorner\psi_-\wedge\psi_+ \overset{(1.9)}{=} X\lrcorner\psi_+\wedge\psi_+$$

and the definition of κ. The identity (1.20) is equivalent since

$$\alpha\wedge(X\lrcorner\psi_+)\wedge\psi_+ = \alpha\wedge(JX\lrcorner\phi(\Psi)) = (JX\lrcorner\alpha)\,\phi(\Psi).$$

For even m, the proof is completely analogous using the corresponding formulas of Lemma 1.3. $\square$

Notice that the real volume form associated to $\Psi=\psi_++\mathrm{i}_\varepsilon\psi_-$ satisfies

$$\phi(\Psi) = \begin{cases} \frac{1}{4}\,\Psi\wedge\bar\Psi = \frac{1}{2}(\psi_+)^2 & \text{if } m \text{ is even,} \\ \frac{1}{4\mathrm{i}_\varepsilon}\Psi\wedge\bar\Psi = \frac{1}{2}\psi_-\wedge\psi_+ & \text{if } m \text{ is odd,} \end{cases} \tag{1.23}$$

due to (1.14) and (1.15).

If m is odd, the equality (1.19) suggests that it already suffices to know the real part or equivalently the imaginary part, compare (1.11). However, it remains to construct the volume form $\phi(\Psi)=\frac{1}{2}\psi_-\wedge\psi_+$ without knowing ψ_-. This can be achieved as follows by generalising the construction established in [**Hi2**] for dimension six.

Let V^{2m} be oriented and $m = 2l - 1 \geq 3$ *odd*. For a real m-form ρ, we define the map $K_\rho : V \to V \otimes \Lambda^{2m} V^*$ by

$$K_\rho X = \kappa((X \lrcorner \rho) \wedge \rho) \tag{1.24}$$

and the quartic invariant

$$\lambda(\rho) = \frac{1}{2m} \mathrm{tr}(K_\rho^2) \ \in (\Lambda^{2m} V^*)^{\otimes 2}. \tag{1.25}$$

Recall that, for any one-dimensional vector space L, an element $u \in L^{\otimes 2r}$ is defined to be positive, $u > 0$, if $u = s^{\otimes 2r}$ for some $s \in L$ and negative if $-u > 0$. Therefore, the norm of an element $u \in L^{\otimes 2r}$ is well-defined and we set

$$\phi(\rho) = \sqrt{|\lambda(\rho)|} \tag{1.26}$$

for the positively oriented square root. If $\phi(\rho) \neq 0$, we furthermore define

$$J_\rho = \frac{1}{\phi(\rho)} K_\rho. \tag{1.27}$$

PROPOSITION 1.5. *Let V^{2m} be oriented, $m = 2l - 1 \geq 3$ odd, and let ρ be a real m-form lying in the $\mathrm{GL}^+(V)$-orbit of the real part of an ε-complex non-degenerate decomposable m-form Ψ with $\phi(\Psi) > 0$. Then, the $2m$-form $\phi(\rho)$ is a real volume form and the endomorphism J_ρ is an ε-complex structure such that $\Psi = \Psi_\rho = \rho + \varepsilon^l\, \mathrm{i}_\varepsilon J_\rho^* \rho$ is of type $(m, 0)$ and $\phi(\Psi) = \phi(\rho)$.*

PROOF. Considering the $\mathrm{GL}^+(V)$-equivariance of all involved expressions, Proposition 1.4 and formula (1.12), it suffices to show that $\phi(\Psi) = \phi(\psi_+)$ for an ε-complex non-degenerate decomposable m-form $\Psi = \psi_+ + \mathrm{i}_\varepsilon \psi_-$ with $\phi(\Psi) > 0$. To see this, we observe that $\frac{K_{\psi_+}}{\phi(\Psi)}$ is an ε-complex structure due to identity (1.19). Thus, $\mathrm{tr}(\frac{K_{\psi_+}}{\phi(\Psi)})^2 = 2\varepsilon m$ and, by definition, $\lambda(\psi_+) = \varepsilon(\phi(\Psi))^{\otimes 2}$. Now, the assertion $\phi(\Psi) = \phi(\psi_+)$ follows directly from the definition of $\phi(\psi_+)$ since $\phi(\Psi) > 0$ by assumption. $\square$

In fact, dimension six, i.e. $m = 3$, is distinguished by the property that the orbit of such a three-form ρ is open. Indeed, the orbit under $G = \mathrm{GL}(6, \mathbb{R})$ of a real three-form with stabiliser $H = \mathrm{SL}(3, \mathbb{C}_\varepsilon)$ has maximal dimension since

$$\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} H = 36 - 16 = 20 = \dim_{\mathbb{R}} \Lambda^3 (\mathbb{R}^6)^*.$$

This phenomenon only occurs in few special cases and is discussed at length in section 3. In particular, Proposition 3.5 collects a number of convenient characterisations of the two open $\mathrm{GL}^+(V)$-orbits on $\Lambda^3(\mathbb{R}^6)^*$.

1.3. The unitary and the para-unitary group. A *pseudo-Euclidean structure* on V is a real non-degenerate symmetric bilinear form g and the pair (V, g) is called a *pseudo-Euclidean vector space*.

A *pseudo-Hermitian structure* (g, J) on V consists of a pseudo-Euclidean structure g and an orthogonal complex structure J, i.e. $J^* g = g$. The non-degenerate two-form $\omega = g(\,.\,, J\,.\,)$ is called the *fundamental two-form* and the tuple (V, g, J) is called a *pseudo-Hermitian vector space*. The stabiliser in $\mathrm{GL}(V)$ of a pseudo-Hermitian structure (J, g)

is the *unitary group* $\mathrm{U}(V, g, J) \cong \mathrm{U}(p, q)$. Here, the pair (p, q) denotes the Hermitian signature of g such that $(2p, 2q)$ is the (real) signature[2] of g.

REMARK 1.6. We warn the reader that many authors define the fundamental two-form with the opposite sign.

Similarly, a *para-Hermitian structure* (g, J) on V consists of a pseudo-Euclidean structure g and a product structure J which is antiorthogonal in the sense that $J^*g = -g$. The two-form $\omega = g(\,.\,, J\,.\,)$ is also called the fundamental two-form and the tuple (V, g, J, ω) is called a *para-Hermitian vector space*. The stabiliser in $\mathrm{GL}(V)$ of a para-Hermitian structure (J, g) is the *para-unitary group* $\mathrm{U}(V, g, J)$.

Notice that J is automatically a para-complex structure and the signature of g is neutral. Indeed, due to the antiorthogonality, all eigenvectors $X^{\pm} = X \pm JX$, $X \in V$, of the product structure J are isotropic. Using moreover the non-degeneracy of g, it is easy to construct a basis ${e_i}^{\pm} = e_i \pm Je_i$ of isotropic vectors such that the eigenspaces $V^{\pm}$ of J are spanned by the $\{e_i^{\pm}\}$. Thus the para-unitary group is isomorphic to $\mathrm{GL}(m, \mathbb{R}) \subset \mathrm{SO}(m, m)$ acting by the standard representation on V^+ and by the dual representation on V^-.

As before for ε-complex structures, the analogy suggests to unify the language. In the Hermitian context, the notation is chosen in such a way that the prefix "ε" can be replaced by "para" for $\varepsilon = 1$ and by "pseudo" for $\varepsilon = -1$.

DEFINITION 1.7. An *ε-Hermitian structure* on V is defined as a pair (g, J) of a pseudo-Euclidean structure g and an endomorphism J which satisfies $J^2 = \varepsilon \mathrm{Id}$ and $J^*g = -\varepsilon g$. The tuple (V, g, J) is called an *ε-Hermitian vector space* and the stabiliser of an ε-Hermitian structure is the *ε-unitary group* $\mathrm{U}(V, g, J)$.

Due to the observation (1.3), the fundamental two-form is in both cases of real type $(1, 1)$. Moreover, we remark that an ε-Hermitian structure can equivalently be characterised by a pair (ω, J) of a non-degenerate two-form and an endomorphism J with $J^2 = \varepsilon \mathrm{Id}$ and $J^*\omega = -\varepsilon\omega$. Indeed, the induced pseudo-Euclidean structure defined by

$$g = \varepsilon\omega(\,.\,, J\,.\,) = -\varepsilon\omega(J\,.\,, \,.\,) \tag{1.28}$$

turns the tuple (V, g, J, ω) in an ε-Hermitian vector space. Similarly, the ε-complex structure can be reconstructed from a pair (g, ω) satisfying an adequate compatibility. On group level, these properties are reflected in the relation

$$\mathrm{U}(V, g, J, \omega) = \mathrm{GL}(V, J) \cap \mathrm{O}(V, g) = \mathrm{GL}(V, J) \cap \mathrm{Sp}(V, \omega) = \mathrm{O}(V, g) \cap \mathrm{Sp}(V, \omega),$$

where $\mathrm{Sp}(V, \omega)$ denotes the real symplectic group which is defined as the stabiliser of a non-degenerate two-form ω.

There are several ways to define an adapted basis. For instance, we can always choose a g-orthonormal basis $\{e_1, \ldots, e_{2m}\}$ such that

$$Je_{2k-1} = \varepsilon e_{2k},\ Je_{2k} = e_{2k-1}, \qquad 1 \leq k \leq m, \qquad \omega = \sum_{i=1}^{m} \sigma_{2i-1}\, e^{(2i-1)(2i)}. \tag{1.29}$$

Here, the sign $\sigma_j := g(e_j, e_j) \in \{\pm 1\}$ depends on the signature of g for $\varepsilon = -1$, and in this case we order the basis vectors such that the first $2p$ basis vectors are spacelike. However,

[2]We will follow the convention that 2p is the number of spacelike directions.

as the signature of g is always (m, m) for $\varepsilon = 1$, it is more convenient in this case to order the basis vectors such that $\sigma_{2i-1} = -\sigma_{2i} = 1$ for all i. A basis satisfying (1.29) and the sign conventions just explained will be denoted as an *ε-unitary basis*. Note that the dual ε-complex structure J^* in this basis is exactly (1.22).

The matrix group representing the ε-unitary group with respect to an ε-unitary basis shall be denoted by

$$\mathrm{U}^\varepsilon(p,q) \cong \begin{cases} \mathrm{U}(p,q) \subset \mathrm{SO}(2p,2q) & \text{for } \varepsilon = -1, \\ \mathrm{GL}(m,\mathbb{R}) \subset \mathrm{SO}(m,m) & \text{for } \varepsilon = 1. \end{cases}$$

An ε-Hermitian structure is naturally oriented by the fundamental two-form ω when defining the *Liouville volume form*

$$\phi(\omega) = \frac{1}{m!}\omega^m$$

as positively oriented. Since

$$\phi(\omega) = \begin{cases} (-1)^q e^{1\ldots 2m} & \text{for } \varepsilon = -1, \\ e^{1\ldots 2m} & \text{for } \varepsilon = 1. \end{cases} \tag{1.30}$$

in an ε-unitary basis, the Liouville volume form $\phi(\omega)$ is a metric volume form for both values of ε.

Later, we will need the following interesting formula which is easily proved in an ε-unitary basis.

LEMMA 1.8. *On an ε-Hermitian vector space (V, g, J, ω), the identity*

$$\alpha \wedge J^*\beta \wedge \frac{1}{(m-1)!}\omega^{m-1} = g(\alpha,\beta)\phi(\omega)$$

holds for all α, $\beta \in V^*$.

Finally, we briefly discuss the decomposition of the space of k-forms as a $\mathrm{U}^\varepsilon(p,q)$-module. Let (V, g, J, ω) be an ε-Hermitian vector space. Recall that there is an induced pseudo-Euclidean structure on $\Lambda^k V^*$ for all k defined by

$$g(x^1 \wedge \ldots \wedge x^k, y^1 \wedge \ldots \wedge y^k) = \det(a_{ij})\,, \qquad a_{ij} = g(x^i, y^j)$$

for decomposable k-forms and extending linearly.

DEFINITION 1.9. (i) The *Lefschetz operator* on the Grassmann algebra $\Lambda^* V^*$ is defined by wedging with the fundamental $(1,1)$-form ω.

(ii) A k-form α is called *primitive* or *effective* if α lies in the kernel of the adjoint operator of the Lefschetz operator. We denote the space of all primitive k-forms by $\Lambda^k_0 V^*$.

In fact, it is well-known, see for instance [**Huy**], that for $k > m$, every primitive k-form is trivial, and for $k \leq m$, the primitive k-forms are given by

$$\Lambda^k_0 V^* = \{\alpha \in \Lambda^k V^* \mid \alpha \wedge \omega^{m-k+1} = 0\}. \tag{1.31}$$

Since $J^*\omega = -\varepsilon\omega$, the Lefschetz operator preserves the type of a form such that the decompositions

$$[\![\Lambda^{r,s}V^*]\!] = \bigoplus_{i=0}^{\min\{r,s\}} \omega^i \wedge [\![\Lambda_0^{r-i,s-i}V^*]\!]\,, \tag{1.32}$$

$$[\Lambda^{r,r}V^*] = \bigoplus_{i=0}^{r} \omega^i \wedge [\Lambda_0^{r-i,r-i}V^*] \tag{1.33}$$

are $\mathrm{U}(V,g,J)$-invariant, in particular orthogonal with respect to g. For the compact form $\mathrm{U}(m)$, i.e. $\varepsilon = -1$ and g positive definite, these decompositions are in fact well-known to be irreducible. For the non-compact forms $\mathrm{U}(p,q)$, the irreducibility can be deduced from the irreducibility in the compact case which is explained thoroughly in section 2.

Combining (1.32) with the type decomposition (1.1), we have thus the decomposition of $\Lambda^k V^*$ into irreducible components as $\mathrm{U}(p,q)$-module.

In the para-complex context, $\varepsilon = 1$, the decomposition can be further refined using (1.7).

1.4. The special unitary and the special para-unitary group. When intersecting the groups defined in the previous two sections, we arrive at the structures we are most interested in.

Definition 1.10. A *special ε-Hermitian structure* (g, J, ω, Ψ) on V is an ε-Hermitian structure (g, J, ω) together with an ε-complex volume form $\Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_-$. The stabiliser in $\mathrm{GL}(V)$ of a special ε-Hermitian structure (J, g, ω, Ψ) is the *special ε-unitary group* $\mathrm{SU}(V, g, J, \omega, \Psi)$.

First of all, we note that the real forms ψ_+ and ψ_- are primitive since ω is of type $(1,1)$:

$$\omega \wedge \psi_+ = 0, \qquad \omega \wedge \psi_- = 0. \tag{1.34}$$

Another interesting identity is the following, expressing the metric in terms of ω and Ψ.

Lemma 1.11. *On a special ε-Hermitian vector space $(V^{2m}, g, J, \omega, \Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_-)$ with $m \geq 2$, we have*

$$g(X,Y)\phi(\Psi) = \begin{cases} X\lrcorner\omega \wedge Y\lrcorner\psi_+ \wedge \psi_+, & \text{if } m \text{ is odd,} \\ X\lrcorner\omega \wedge Y\lrcorner\psi_+ \wedge \psi_-, & \text{if } m \text{ is even} \end{cases} \tag{1.35}$$

for all $X, Y \in V$.

Proof. Since $g(X,Y) = JY\lrcorner X\lrcorner\omega$, the identity follows immediately from (1.20) when replacing α by $X\lrcorner\omega$. □

In fact, a special ε-Hermitian structure (g, J, ω, Ψ) can always be reconstructed from the forms ω and Ψ under the following assumptions.

Proposition 1.12. *Let V be a real $2m$-dimensional vector space, $m \geq 2$. Moreover let $\omega \in \Lambda^2 V^*$ be non-degenerate and let $\Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_- \in \Lambda^m V^*_{\mathbb{C}_\varepsilon}$ be non-degenerate, decomposable and compatible with ω in the sense that*

$$\omega \wedge \Psi = 0.$$

Then, the pair (ω, Ψ) can be extended to a unique special ε-complex structure (g, J, ω, Ψ) where J is characterised by (1.19) *and g by* (1.28) *or* (1.35).

Proof. By Proposition 1.4, the m-form Ψ is of type $(m, 0)$ with respect to the ε-complex structure J uniquely defined by (1.19). Due to the compatibility $\omega \wedge \Psi = 0$, the two-form ω is of type $(1,1)$ or equivalently, $J_\rho^*\omega = -\varepsilon\omega$ by (1.3). Hence, with the pseudo-Euclidian structure g uniquely defined by (1.28) or, equivalently, by (1.35), the tuple (g, J, ω, Ψ) is a special ε-Hermitian structure. □

In the case of odd ε-complex dimension, already the pair (ω, ψ_+) suffices under the following assumptions.

Proposition 1.13. *Let V be a $2m$-dimensional vector space with $m = 2l-1 \geq 3$ odd, let $\omega \in \Lambda^2 V^*$ be non-degenerate and let $\rho \in \Lambda^m V^*$ be a real m-form lying in the* $\mathrm{GL}(V)$*-orbit of the real part of an ε-complex non-degenerate decomposable m-form. Furthermore, assume that ω and ρ are compatible in the sense that*

$$\omega \wedge \rho = 0.$$

Then, there is a unique special ε-Hermitian structure (g, J, ω, Ψ) with fundamental two-form ω and $\mathrm{Re}(\Psi) = \rho$ such that $\phi(\Psi)$ and $\phi(\omega)$ induce the same orientation.

Proof. Let V be oriented by the Liouville volume form $\phi(\omega)$. By Proposition 1.5, there is a unique ε-complex structure J_ρ such that $\Psi_\rho = \rho + \varepsilon^l \mathrm{i}_\varepsilon J_\rho^*\rho$ is an $(m,0)$-form with $\phi(\Psi) > 0$. The explicit formula for J_ρ is given by (1.27). We claim that the vanishing of $\omega \wedge \rho$ also implies $\omega \wedge J_\rho^*\rho = 0$. Indeed, for all one-forms α, we have

$$0 = J_\rho^*\alpha \wedge \rho \wedge \omega \overset{(1.10)}{=} -\varepsilon^{l+1}\alpha \wedge J_\rho^*\rho \wedge \omega$$

and the claim follows. Since therefore $\omega \wedge \Psi = 0$, the two-form ω is of type $(1,1)$ or equivalently, $J_\rho^*\omega = -\varepsilon\omega$ by (1.3). Hence, with the pseudo-Euclidian structure g defined by (1.28) or, equivalently, by (1.35), the tuple $(g, J_\rho, \omega, \Psi)$ is a special ε-Hermitian structure. □

A standard basis for a special ε-Hermitian vector space (V, g, J, ω, Ψ) can be defined as follows. Since the rank of the $\mathbb{C}_\varepsilon$-module $\Lambda^{m,0}V$ is one, the complex volume form Ψ with respect to an ε-unitary basis $\{e_i\}$ is

$$\Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_- = z(e^1 + \varepsilon\mathrm{i}_\varepsilon e^2) \wedge \ldots \wedge (e^{2m-1} + \varepsilon\mathrm{i}_\varepsilon e^{2m}) \tag{1.36}$$

for an ε-complex number z with $z\bar{z} \neq 0$. Obviously, we can always choose an ε-unitary basis such that z is real. Such a basis will be called an ε-unitary basis adapted to the special ε-Hermitian structure. In an adapted basis, the special ε-unitary group is represented by the matrix group

$$\mathrm{SU}^\varepsilon(p,q) := \mathrm{U}^\varepsilon(p,q) \cap \mathrm{SL}(m, \mathbb{C}_\varepsilon) \cong \begin{cases} \mathrm{SU}(p,q) \subset \mathrm{SO}(2p, 2q) & \text{for } \varepsilon = -1, \\ \mathrm{SL}(m, \mathbb{R}) \subset \mathrm{SO}(m,m) & \text{for } \varepsilon = 1. \end{cases}$$

Obviously, the ε-complex volume form Ψ can always be multiplied by a constant such that $z = 1$ in an adapted basis. This normalisation turns out to be important when considering the corresponding G-structures on manifolds and we reformulate it as follows.

Definition 1.14. A special ε-Hermitian structure (g, J, ω, Ψ) is called *normalised* if

$$g_{\mathbb{C}_\varepsilon}(\Psi, \Psi) = \begin{cases} (-1)^q \, 2^m & \text{for } \varepsilon = -1, \\ \pm \, 2^m & \text{for } \varepsilon = 1. \end{cases}$$

Here, $g_{\mathbb{C}_\varepsilon}$ denotes the extension of g to a Hermitian form on $V_{\mathbb{C}_\varepsilon} = V \otimes \mathbb{C}_\varepsilon$ characterised by

$$g_{\mathbb{C}_\varepsilon}(v \otimes z_1, w \otimes z_2) = z_1 \bar{z}_2 \, g(v, w),$$

for $v, w \in V_{\mathbb{C}_\varepsilon}$, $z_1, z_2 \in \mathbb{C}_\varepsilon$ or, equivalently, by

$$g_{\mathbb{C}_\varepsilon}(v_+ + \mathrm{i}_\varepsilon v_-, w_+ + \mathrm{i}_\varepsilon w_-) = g(v_+, w_+) - \varepsilon g(v_-, w_-) + \mathrm{i}_\varepsilon (g(v_-, w_+) - g(v_+, w_-))$$

for v_+, v_-, w_+, $w_- \in V$. In analogy to the real case, there is an induced Hermitian form on $\Lambda^* V_{\mathbb{C}_\varepsilon}$, also denoted by $g_{\mathbb{C}_\varepsilon}$.

Lemma 1.15. *The following assertions are equivalent for a special ε-Hermitian structure (g, J, ω, Ψ).*

(i) The structure is normalised.

(ii) For every ε-unitary basis, the constant $z \in \mathbb{C}_\varepsilon$ appearing in (1.36) *satisfies $|z|^2 = \pm 1$.*

(iii) The volume form $\phi(\Psi)$ associated to Ψ by (1.23) *and the Liouville volume form $\phi(\omega)$ are related by the formula*

$$\phi(\Psi) = \begin{cases} (-1)^q \, 2^{m-2} \, \phi(\omega) & \text{for } \varepsilon = -1, \\ \pm \, 2^{m-2} \, \phi(\omega) & \text{for } \varepsilon = 1. \end{cases} \tag{1.37}$$

Proof. The equivalence of (i) and (ii) is easily seen when considering that $e^{2k} = J^* e^{2k-1}$ for all $k = 1, \ldots, m$ in an ε-unitary basis and thus

$$g_{\mathbb{C}_\varepsilon}(e^{2k-1} + \varepsilon \mathrm{i}_\varepsilon e^{2k}, e^{2k-1} + \varepsilon \mathrm{i}_\varepsilon e^{2k}) = g(e^{2k-1}, e^{2k-1}) - \varepsilon g(J^* e^{2k-1}, J^* e^{2k-1}) = 2\sigma_{2k-1}.$$

The equivalence of (ii) and (iii) follows when comparing the volume forms in an ε-unitary basis which we already computed in (1.21) and (1.30). ☐

Remark 1.16. Note that the third characterisation is very useful when reconstructing a special ε-Hermitian structure out of the pair (ω, ψ_+), or the tuple (ω, ψ_+, ψ_-), respectively.

For $\varepsilon = -1$, the following proposition yields the irreducible decomposition of the $\mathrm{SU}(p, q)$-module of k-forms. Again, this is a well-known fact for the compact form $\mathrm{SU}(m)$ and the result for the non-compact form follows with the arguments given in section 2.

Proposition 1.17. *Let (g, J, ω, Ψ) be a special pseudo-Hermitian structure. The decomposition* (1.32) *is also irreducible under* $\mathrm{SU}(p, q)$ *except for the forms of type $(m, 0) + (0, m)$ which can be decomposed in the one-dimensional summands*

$$[\![\Lambda^{m,0} V^*]\!] = \mathbb{R}\psi_+ \oplus \mathbb{R}\psi_-.$$

2. Representations of compact and non-compact forms of complex Lie groups

We shall apply several times the following facts on the irreducibility of complexified representations.

Let $\mathfrak{g}$ be a real Lie algebra and let V be a real $\mathfrak{g}$-module. The complexification $V_{\mathbb{C}} = V \otimes \mathbb{C} \cong V \oplus \mathrm{i}V$ is a complex $\mathfrak{g}$-module in the obvious way. The other way round, the realification $W_{\mathbb{R}}$ of a complex $\mathfrak{g}$-module W is simply the underlying real vector space regarded as a real $\mathfrak{g}$-module. Moreover, recall that a real structure on a complex vector space W is an antilinear involution. The real and complex representations are related as follows, for proofs see for instance [**On**], §8.

PROPOSITION 2.1. *Let $\mathfrak{g}$ be a real Lie algebra.*

(a) A real $\mathfrak{g}$-module V is irreducible if and only if it satisfies one of the following two conditions:
   (i) The complexification $V_{\mathbb{C}}$ is an irreducible complex $\mathfrak{g}$-module.
   (ii) V is the realification of an irreducible complex $\mathfrak{g}$-module W which does not admit an equivariant real structure and it holds $V_{\mathbb{C}} \cong W \oplus \bar{W}$.

(b) Moreover, two irreducible real $\mathfrak{g}$-modules V_1 and V_2 satisfying (i) are equivalent if and only if $(V_1)_{\mathbb{C}}$ and $(V_2)_{\mathbb{C}}$ are equivalent. Two irreducible real $\mathfrak{g}$-modules $V_1 = (W_1)_{\mathbb{R}}$ and $V_2 = (W_2)_{\mathbb{R}}$ satisfying (ii) are equivalent if and only if $W_1 \cong W_2$ or $W_1 \cong \bar{W}_2$ as complex $\mathfrak{g}$-modules. An irreducible real $\mathfrak{g}$-module satisfying (i) cannot be equivalent to an irreducible real $\mathfrak{g}$-module satisfying (ii).

In a second step, we can also complexify the Lie algebra $\mathfrak{g}$. Identifying $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, the extension of a complex $\mathfrak{g}$-module W to a $\mathfrak{g}_{\mathbb{C}}$-module is obvious.

LEMMA 2.2. *A complex $\mathfrak{g}$-module W of $\mathfrak{g}$ is irreducible if and only if W is irreducible as a $\mathfrak{g}_{\mathbb{C}}$-module. Two complex $\mathfrak{g}$-modules are equivalent if and only if the corresponding $g_{\mathbb{C}}$-modules are equivalent.*

More specifically, we turn to the real forms $\mathfrak{u}(p,q)$, $p+q=m$, and $\mathfrak{gl}(m,\mathbb{R})$ of $\mathfrak{gl}(m,\mathbb{C})$. If ω is a two-form on a real vector space V, we denote by $\omega_{\mathbb{C}}$ the complex linear extension of ω to $V_{\mathbb{C}}$.

LEMMA 2.3. *Let (J,ω) and (J',ω') be ε-Hermitian structures on a $2m$-dimensional real vector space V. Let the Lie algebras of the stabilisers be denoted by $\mathfrak{u}(J,\omega)$ and $\mathfrak{u}(J',\omega')$ such that $\mathfrak{gl}(m,\mathbb{C}) \cong \mathfrak{u}_{\mathbb{C}}(J,\omega) \cong \mathfrak{u}_{\mathbb{C}}(J',\omega')$.*

Then, there is an equivariant isomorphism of the $\mathfrak{u}_{\mathbb{C}}(J,\omega)$-module $(V_{\mathbb{C}}, J, \omega_{\mathbb{C}})$ and the $\mathfrak{u}_{\mathbb{C}}(J',\omega')$-module $(V_{\mathbb{C}}, J', \omega'_{\mathbb{C}})$.

PROOF. If $\varepsilon = -1$, an ε-complex structure J is completely determined by the decomposition $V_{\mathbb{C}} = W \oplus \bar{W}$ where $W = V^{1,0}$ and $\dim_{\mathbb{C}} W = m$. If $\varepsilon = 1$, an ε-complex structure corresponds to the real decomposition $V = V^+ \oplus V^-$ which can simply be tensored by $\mathbb{C}$ such that $V_{\mathbb{C}} = V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^-$ where the summands are complex m-dimensional as well.

Now, the complexified compatible two-form $\omega_{\mathbb{C}}$ is of type $(1,1)$, i.e. an element of $W^* \otimes \bar{W}^*$ for $\varepsilon = -1$ by definition and an element of $(V_{\mathbb{C}}^+)^* \otimes (V_{\mathbb{C}}^-)^*$ for $\varepsilon = 1$ by (1.7). Thus, independently of the sign of ε and the signature of the induced metric, there is a $\mathbb{C}$-linear isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$ mapping the decomposition corresponding to J to that corresponding to J' such that $\omega_{\mathbb{C}}$ is the pullback of $\omega'_{\mathbb{C}}$. This implies the assertion since $\mathfrak{u}_{\mathbb{C}}(J,\omega)$ and $\mathfrak{u}_{\mathbb{C}}(J',\omega')$ are exactly the subalgebras of $\mathfrak{gl}(V_{\mathbb{C}})$ preserving the decomposition of $V_{\mathbb{C}}$ corresponding to J or J' and annihilating $\omega_{\mathbb{C}}$ or $\omega'_{\mathbb{C}}$, respectively. □

REMARK 2.4. Notice that the main difference of the complex and the para-complex case is the fact that the defining representation is irreducible for $\varepsilon = -1$, whereas it splits into the irreducible representations $V^{\pm}$ for $\varepsilon = 1$.

Corollary 2.5. *Let* (J, ω, Ψ) *and* (J', ω', Ψ') *be two special pseudo-Hermitian structures on a* $2m$*-dimensional real vector space* V*. Let the Lie algebras of the stabilisers be denoted by* $\mathfrak{su}(J, \omega)$ *and* $\mathfrak{su}(J', \omega')$ *such that* $\mathfrak{sl}(m, \mathbb{C}) \cong \mathfrak{su}_{\mathbb{C}}(J, \omega) \cong \mathfrak{su}_{\mathbb{C}}(J', \omega')$*.*

Then, there is an equivariant isomorphism of the $\mathfrak{su}_{\mathbb{C}}(J, \omega)$*-module* $(V_{\mathbb{C}}, J, \omega_{\mathbb{C}}, \Psi)$ *and the* $\mathfrak{su}_{\mathbb{C}}(J', \omega')$*-module* $(V_{\mathbb{C}}, J', \omega'_{\mathbb{C}}, \Psi')$*.*

Proof. Without restriction, we can assume that $J = J'$. Let ϕ be the equivariant isomorphism of $V_{\mathbb{C}} = W \oplus \bar{W}$ which was constructed in the proof of the previous lemma. Since $\Psi, \Psi' \in \Lambda^m W$ and $\dim_{\mathbb{C}} \Lambda^m W = 1$, there is a $z \in \mathbb{C}$ such that $\Psi = z\Psi'$ and it is easy to modify ϕ such that (ω, Ψ) is the pullback of (ω', Ψ'). The rest of the argument is analogous to the proof of the previous lemma. $\square$

In combination, the lemmas can be applied as follows. Assume that we have a real $\mathrm{U}(m)$-module W contained in a tensor power $V^{\otimes r} \otimes (V^*)^{\otimes s}$ of the defining representation (V, g, J, ω). Moreover, assume that the decomposition into irreducible components is known and that the components can be written in terms of the defining tensors. One of the examples we have in mind is the decomposition of $\Lambda^k V$ as $\mathrm{U}(m)$-module, see (1.32). Now, when replacing the Euclidean structure g by a pseudo-Euclidean structure of signature (p, q), $p+q = m$, we obtain a corresponding $\mathrm{U}(p, q)$-module $\tilde{W}$ and an analogous decomposition into invariant components. The question is whether the components of the $\mathrm{U}(p, q)$-module $\tilde{W}$ are also irreducible for indefinite signature.

First of all, the discussion can be reduced to the corresponding modules of the Lie algebras $\mathfrak{u}(p, q)$ by the standard Lie theory arguments. Secondly, by Proposition 2.1, it suffices to compare the components of the complexified module $\tilde{W}_{\mathbb{C}}$ with the irreducible components of the corresponding module $W_{\mathbb{C}}$. Thirdly, due to Lemma 2.2, it suffices to show that all of the components of $W_{\mathbb{C}}$ and $\tilde{W}_{\mathbb{C}}$ are isomorphic as $\mathfrak{gl}(m, \mathbb{C})$-modules. However, an equivariant isomorphism of $\mathfrak{gl}(m, \mathbb{C})$-modules $W \to \tilde{W}$ is given by extending the isomorphism of the defining representations constructed in Lemma 2.3. The restriction to each component is an isomorphism since the components are characterised by the defining tensors.

Obviously, the same arguments can be applied to the groups $\mathrm{SU}(m)$ and $\mathrm{SU}(p, q)$, $p + q = m$. Similarly, one can also show the irreducibility of those representations of the non-compact form G_2^* of $G_2^{\mathbb{C}}$ which are defined completely analogous to well-known irreducible representations of the compact form G_2.

3. Stable forms

The following two sections are based on the first section of [**CLSS**].

The aim is to collect the basic facts about stable forms. Let V always denote an n-dimensional real vector space.

Definition 3.1. A k-form $\rho \in \Lambda^k V^*$ is called *stable* if its orbit under $\mathrm{GL}(V)$ is open.

It is easy to verify that a k-form ρ with stabiliser H in $\mathrm{GL}(V)$ is stable if and only if the dimension of the orbit $\mathrm{GL}(V)/H$ is maximal, i.e. if the dimension coincides with the dimension of $\Lambda^k V^*$. In fact, stability occurs only in the following special cases.

Proposition 3.2. *The group* $\mathrm{GL}(V)$ *has an open orbit in* $\Lambda^k V^*$*,* $0 \leq k \leq \left[\frac{n}{2}\right]$*, if and only if* $k \leq 2$ *or if* $k = 3$ *and* $n = 6, 7$ *or* 8*.*

PROOF. A real k-form ρ is stable if and only if the complex linear extension $\rho_{\mathbb{C}}$ of ρ to $\Lambda^k V_{\mathbb{C}}^*$ has an open orbit under $\mathrm{GL}(V_{\mathbb{C}})$. However, the representation of $\mathrm{GL}(V_{\mathbb{C}}) \cong \mathrm{GL}(n,\mathbb{C})$ on $\Lambda^k V_{\mathbb{C}}^* \cong \Lambda^k(\mathbb{C}^n)^*$ is irreducible and the result thus follows, for instance, from the classification of irreducible complex prehomogeneous vector spaces, [**KiSa**]. $\square$

REMARK 3.3. An open orbit is unique in the complex case, since an orbit which is open in the usual topology is also Zariski-open and Zariski-dense (Prop. 2.2, [**Ki**]). Over the reals, the number of open orbits is finite by a well-known theorem of Whitney.

PROPOSITION 3.4. *Let V be oriented and assume that $k \in \{2, n-2\}$ and n even, or $k \in \{3, n-3\}$ and $n = 6, 7$ or 8. Then, there is a $\mathrm{GL}^+(V)$-equivariant mapping*

$$\phi : \Lambda^k V^* \to \Lambda^n V^*,$$

homogeneous of degree $\frac{n}{k}$, which assigns a volume form to a stable k-form and which vanishes on non-stable forms. Given a stable k-form ρ, the derivative of ϕ in ρ defines a dual $(n-k)$-form $\hat{\rho} \in \Lambda^{n-k}V^*$ by the property

$$d\phi_\rho(\alpha) = \hat{\rho} \wedge \alpha \quad \text{for all } \alpha \in \Lambda^k V^*. \tag{3.1}$$

The dual form $\hat{\rho}$ is also stable and satisfies

$$(\mathrm{Stab}_{\mathrm{GL}(V)}(\rho))_0 = (\mathrm{Stab}_{\mathrm{GL}(V)}(\hat{\rho}))_0.$$

A stable form, its volume form and its dual are related by the formula

$$\hat{\rho} \wedge \rho = \frac{n}{k}\phi(\rho). \tag{3.2}$$

PROOF. This result can be viewed as a consequence of the theory of prehomogeneous vector spaces, [**Ki**], as follows. Replacing V and $\mathrm{GL}(V)$ by the complexifications $V_{\mathbb{C}}$ and $\mathrm{GL}(V_{\mathbb{C}})$, the situations we are considering correspond to examples 2.3, 2.5, 2.6 and 2.7 of §2, [**Ki**]. In all cases, the complement of the open orbit under $\mathrm{GL}(V_{\mathbb{C}})$ is a hypersurface in $\Lambda^k(V_{\mathbb{C}})^*$ defined by a complex irreducible non-degenerate homogeneous polynomial f which is invariant under $\mathrm{GL}(V_{\mathbb{C}})$ up to a non-trivial character $\chi : \mathrm{GL}(V_{\mathbb{C}}) \to \mathbb{C}^\times$.

Due to Proposition 4.1, [**Ki**], the polynomial f restricted to $\Lambda^k V^*$ is real-valued and the character χ restricts to $\chi : \mathrm{GL}(V) \to \mathbb{R}^\times$. Moreover, by Proposition 4.5, [**Ki**], the complement of the zero set of f in $\Lambda^k V^*$ has a finite number of connected components which are open $\mathrm{GL}(V)$-orbits. Since the only characters of $\mathrm{GL}(V)$ are the powers of the determinant, there is an equivariant mapping from $\Lambda^k V^*$ to $(\Lambda^n V^*)^{\otimes s}$ for some positive integer s. Taking the s-th root, which depends on the choice of an orientation if s is even, we obtain the $\mathrm{GL}^+(V)$-equivariant map ϕ. By construction, a k-form ρ is stable if and only if $\phi(\rho) \neq 0$. The equivariance under scalar matrices implies that the map ϕ is homogeneous of degree $\frac{n}{k}$.

The derivative

$$\Lambda^k V^* \to (\Lambda^k V^*)^* \otimes \Lambda^n V^* \overset{=}{\to} \Lambda^{n-k} V^*\,,\ \rho \mapsto d_\rho\phi \mapsto \hat{\rho}$$

inherits the $\mathrm{GL}^+(V)$-equivariance from ϕ and is an immersion since f is non-degenerate, compare Theorem 2.16, [**Ki**]. Therefore, it maps stable forms to stable forms such that the connected components of the stabilisers are identical. Finally, formula (3.2) is just Euler's formula for the homogeneous mapping ϕ. $\square$

In the following, we discuss the basic properties of the stable forms which are relevant in this thesis.

$\mathbf{k = 2, n = 2m}$. The orbit of a non-degenerate two-form is open since

$$\dim \mathrm{GL}(2m, \mathbb{R}) - \dim \mathrm{Sp}(2m, \mathbb{R}) = 4m^2 - m(2m+1) = m(2m-1) = \dim \Lambda^2 V^*$$

and there is only one open orbit in $\Lambda^2 V^*$. Thus, a two-form ω is stable if and only if it is non-degenerate and its stabiliser is isomorphic to $\mathrm{Sp}(2m, \mathbb{R})$. The polynomial invariant is the Pfaffian determinant. We normalise the associated equivariant volume form such that is corresponds to the Liouville volume form

$$\phi(\omega) = \frac{1}{m!}\omega^m.$$

Differentiation of the homogeneous polynomial map $\omega \mapsto \phi(\omega)$ yields

$$\hat{\omega} = \frac{1}{(m-1)!}\omega^{m-1}.$$

$\mathbf{k = (n-2), n = 2m, m}$ **even**. Let $\sigma \in \Lambda^{n-2} V^*$ be stable. Since $\Lambda^{n-2} V^* = \Lambda^2 V \otimes \Lambda^n V^*$, the power $\sigma^m \in (\Lambda^n V^*)^{\otimes (m-1)}$ is well-defined and the associated volume form can be defined as

$$\phi(\sigma) = (\frac{1}{m!}\sigma^m)^{\frac{1}{m-1}}.$$

The normalisation is chosen such that $\phi(\sigma) = \phi(\omega)$ for the unique two-form ω with $\sigma = \hat{\omega}$. Thus, the evaluation of (3.2) yields

$$\hat{\sigma} = \frac{1}{m-1}\omega.$$

The stabiliser of σ in $\mathrm{GL}(V)$ is again the real symplectic group and there is a unique open orbit in $\Lambda^{n-2} V^*$.

$\mathbf{k = (n-2), n = 2m, m}$ **odd**. Let V be oriented. Similar to the previous case, we define the volume form associated to a stable $(n-2)$-form σ by

$$\phi(\sigma) = |\frac{1}{m!}\sigma^m|^{\frac{1}{m-1}},$$

choosing the positively oriented root. In fact, there are two open orbits in $\Lambda^{n-2} V^* = \Lambda^2 V \otimes \Lambda^n V^*$: The first one consists of the forms ω^{m-1}, where ω is a stable two-form, the second one of the forms $-\omega^{m-1}$. According to (3.2), it holds

$$\hat{\sigma} = \frac{1}{m-1}\omega \tag{3.3}$$

for the unique two-form ω with $\hat{\omega} = \sigma$ and $\phi(\sigma) = \phi(\omega)$ if σ belongs to the first orbit and for the unique two-form ω with $\hat{\omega} = -\sigma$ and $\phi(\sigma) = -\phi(\omega)$ if σ belongs to the second orbit. The stabiliser of σ in $\mathrm{GL}(V)$ is the group of symplectic and anti-symplectic transformations.

$\mathbf{k = 3, n = 6}$. As we have already seen in section 1.2, the real part of an ε-complex volume form in dimension six is stable. It is shown in [**Hi2**] that the converse is also true. For the convenience of the reader, we summarise the properties of stable three-forms in dimension six repeating some notation introduced in section 1.2.

Let V be a six-dimensional oriented vector space. Recall that κ denoted the canonical isomorphism

$$\kappa : \Lambda^5 V^* \to V \otimes \Lambda^6 V^*, \; \xi \mapsto X \otimes \nu \quad \text{with } X \lrcorner \nu = \xi,$$

and that we defined for a three-form $\rho \in \Lambda^3 V^*$

$$K_\rho(v) = \kappa((v \lrcorner \rho) \wedge \rho) \quad \in V \otimes \Lambda^6 V^*, \tag{3.4}$$

$$\lambda(\rho) = \tfrac{1}{6} \operatorname{tr} K_\rho^2 \quad \in (\Lambda^6 V^*)^{\otimes 2}, \tag{3.5}$$

$$\phi(\rho) = \sqrt{\lambda(\rho)} \quad \in \Lambda^6 V^*, \tag{3.6}$$

where the positively oriented square root is chosen. If ρ satisfies moreover $\lambda(\rho) \neq 0$, we defined

$$J_\rho = \tfrac{1}{\phi(\rho)} K_\rho \quad \in \operatorname{End}(V). \tag{3.7}$$

PROPOSITION 3.5. *A three-form ρ on an oriented six-dimensional vector space V is stable if and only $\lambda(\rho) \neq 0$. There are two open orbits.*

One orbit consists of all three-forms ρ satisfying one of the following equivalent properties:

(a) The quartic invariant satisfies $\lambda(\rho) > 0$.
(b) There are uniquely defined real decomposable three-forms α and β such that $\rho = \alpha + \beta$ and $\alpha \wedge \beta > 0$.
(c) The stabiliser of ρ in $\mathrm{GL}^+(V)$ is $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$.
(d) It holds $\lambda(\rho) \neq 0$ and the endomorphism J_ρ is a para-complex structure on V.
(e) There is a basis $\{e_1, \ldots, e_6\}$ of V such that $\nu = e^{123456} > 0$ and

$$\rho = e^{123} + e^{456}. \tag{3.8}$$

In this basis, it holds $\lambda(\rho) = \nu^{\otimes 2}$, $J_\rho e_i = e_i$ for $i \in \{1,2,3\}$ and $J_\rho e_i = -e_i$ for $i \in \{4,5,6\}$.
(f) There is a unique para-complex decomposable three-form α such that $\rho = \operatorname{Re} \alpha$ and $\mathrm{i}_1(\bar{\alpha} \wedge \alpha) > 0$.
(g) There is a basis $\{e_1, \ldots, e_6\}$ of V such that $\nu = e^{123456} > 0$ and

$$\rho = e^{135} + e^{146} + e^{236} + e^{245}.$$

In this basis, it holds $\lambda(\rho) = 4\nu^{\otimes 2}$, $J_\rho e_i = e_{i+1}$ and $J_\rho e_{i+1} = e_i$ for $i \in \{1,3,5\}$.

The other orbit consists of all three-forms ρ satisfying one of the following equivalent properties:

(a) The quartic invariant satisfies $\lambda(\rho) < 0$.
(b) There is a uniquely defined complex decomposable three-form α such that $\rho = \operatorname{Re} \alpha$ and $\mathrm{i}(\bar{\alpha} \wedge \alpha) > 0$.
(c) The stabiliser of ρ in $\mathrm{GL}^+(V)$ is $\mathrm{SL}(3,\mathbb{C})$.
(d) It holds $\lambda(\rho) \neq 0$ and the endomorphism J_ρ is a complex structure on V.
(e) There is a basis $\{e_1, \ldots, e_6\}$ of V such that $\nu = e^{123456} > 0$ and

$$\rho = e^{135} - e^{146} - e^{236} - e^{245}.$$

In this basis, it holds $\lambda(\rho) = -4\nu^{\otimes 2}$, $J_\rho e_i = -e_{i+1}$ and $J_\rho e_{i+1} = e_i$ for $i \in \{1,3,5\}$.

PROOF. All properties are proved in section 2 of [**Hi2**] except for the characterisation (d), (f) and (g) of the first orbit. However, these are obvious considering the discussion of ε-complex structures and volume forms in section 1.2. $\square$

Using the unified language introduced in section 1.2, we can also describe both orbits simultaneously. Indeed, given a generic stable three-form ρ_ε and an orientation, there is an oriented basis $\{e_1, \ldots, e_6\}$ of V such that

$$\rho_\varepsilon = e^{135} + \varepsilon(e^{146} + e^{236} + e^{245}) \tag{3.9}$$

with $\lambda(\rho) = 4\varepsilon(e^{123456})^{\otimes 2}$. The induced ε-complex structure J_ρ is given by $J_\rho e_i = \varepsilon e_{i+1}$, $J_\rho e_{i+1} = e_i$ for $i \in \{1, 3, 5\}$ and it holds

$$J^*_{\rho_\varepsilon}\rho_\varepsilon = e^{246} + \varepsilon(e^{235} + e^{145} + e^{136}). \tag{3.10}$$

Moreover, we like to emphasise the following properties of a stable three-form.

LEMMA 3.6. *Let ρ be a stable three-form on a six-dimensional oriented vector space.*

(i) The dual stable form is given by

$$\hat\rho = J^*_\rho \rho. \tag{3.11}$$

(ii) For both orbits, the ε-complex three-form $\Psi_\rho = \rho + i_\varepsilon\hat\rho$ is a non-degenerate $(3,0)$-form with respect to the induced ε-complex structure J_ρ.

PROOF. (i) We already observed that the connected components of the stabilisers of ρ and $\hat\rho$ have to be identical. Therefore, since the space of real three-forms invariant under $\mathrm{SL}(3,\mathbb{C})$ respectively $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ is two-dimensional, we can make the ansatz

$$\hat\rho = c_1\rho + c_2 J^*_\rho\rho$$

with real constants c_1 and c_2. Computing

$$\frac{6}{3}\phi(\rho) \overset{(3.2)}{=} \hat\rho \wedge \rho = c_2\, J^*_\rho\rho \wedge \rho \overset{(3.9),(3.10)}{=} 2c_2\,\phi(\rho),$$

we find $c_2 = 1$.

In order to determine c_1, we compute $\lambda(\rho + tJ^*_\rho\rho) = 4\varepsilon(-\varepsilon + t^2)^2\,(e^{123456})^{\otimes 2}$ for the normal form (3.9). Thus, the derivative of λ in ρ in direction of $J^*_\rho\rho$ vanishes. However, by definition of ϕ and since

$$d_\rho\phi(J^*_\rho\rho) = \hat\rho \wedge J^*_\rho\rho = c_1\rho \wedge J^*_\rho\rho = -2c_1\phi(\rho),$$

the constant c_1 has to be zero.

(ii) The second part is now a special case of Proposition 1.5.

$\square$

It is a remarkable consequence of the lemma just proven that we can apply all identities for ε-complex volume forms to $\rho + i_\varepsilon\hat\rho$, see section 1.2. For instance, the lemma yields a convenient way to compute the dual of ρ without determining J_ρ.

COROLLARY 3.7. *(i) If $\lambda(\rho) > 0$ and $\rho = \alpha + \beta$ in terms of decomposables ordered such that $\alpha \wedge \beta > 0$, the dual three-form is $\hat\rho = \alpha - \beta$.*

(ii) If ρ is given as the real part of an ε-complex decomposable three-form α such that $i_\varepsilon(\bar\alpha \wedge \alpha) > 0$, the dual three-form $\hat\rho$ is the imaginary part of α.

Proof. The first assertion is obvious when evaluating (3.11) in a basis such that $\rho = e^{123} + e^{456}$ since $J_\rho e_i = e_i$ for $i \in \{1,2,3\}$ and $J_\rho e_i = -e_i$ for $i \in \{4,5,6\}$ in this basis. The second part follows from Lemma 3.6 and (1.12). $\square$

In fact, the corollary explicitly shows the equivalence of the two different definitions of $\rho \mapsto \hat{\rho}$ given in [**Hi1**] and [**Hi2**].

Finally, we note that for a fixed orientation, it holds

$$\hat{\hat{\rho}} = -\rho \quad \text{and} \quad J_{\hat{\rho}} = -\varepsilon J_\rho. \tag{3.12}$$

$\mathbf{k = 3, n = 7}$. Given any three-form φ, we define a symmetric bilinear form with values in $\Lambda^7 V^*$ by

$$b_\varphi(v,w) = \frac{1}{6}(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi. \tag{3.13}$$

Since the determinant of a scalar-valued bilinear form is an element of $(\Lambda^7 V^*)^{\otimes 2}$, we have $\det b \in (\Lambda^7 V^*)^{\otimes 9}$. If and only if φ is stable, the seven-form

$$\phi(\varphi) = (\det b_\varphi)^{\frac{1}{9}}$$

defines a volume form, independent of an orientation on V, and the scalar-valued symmetric bilinear form

$$g_\varphi = \frac{1}{\phi(\varphi)} b_\varphi$$

is non-degenerate. Notice that $\phi(\varphi) = \sqrt{\det g_\varphi}$ is the metric volume form.

It is known ([**Br1**], [**Har**]) that a stable three-form defines a multiplication "$\cdot$" and a vector cross product "$\times$" on V by the formula

$$\varphi(x,y,z) = g_\varphi(x, y \cdot z) = g_\varphi(x, y \times z), \tag{3.14}$$

such that $(V, \times)$ is isomorphic either to the imaginary octonions $\operatorname{Im}\mathbb{O}$ or to the imaginary split-octonions $\operatorname{Im}\tilde{\mathbb{O}}$. Thus, there are exactly two open orbits of stable three-forms having isotropy groups

$$\operatorname{Stab}_{\mathrm{GL}(V)}(\varphi) \cong \begin{cases} \mathrm{G}_2 \subset \mathrm{SO}(7), & \text{if } g_\varphi \text{ is positive definite,} \\ \mathrm{G}_2^* \subset \mathrm{SO}(3,4), & \text{if } g_\varphi \text{ is of signature (3,4).} \end{cases} \tag{3.15}$$

There is always a basis $\{e_1, \ldots, e_7\}$ of V such that

$$\varphi = \tau e^{124} + \sum_{i=2}^{7} e^{i\,(i+1)\,(i+3)} \tag{3.16}$$

with $\tau \in \{\pm 1\}$ and indices modulo 7. For $\tau = 1$, the induced metric g_φ is positive definite and the basis is orthonormal such that this basis corresponds to the Cayley basis of $\operatorname{Im}\mathbb{O}$. For $\tau = -1$, the metric is of signature (3,4) and the basis is pseudo-orthonormal with e_1, e_2 and e_4 being the three spacelike basis vectors.

The only four-forms having the same stabiliser as φ are the multiples of the Hodge dual $*_{g_\varphi}\varphi$, [**Br1**, Propositions 2.1, 2.2]. Since the normal form satisfies $g_\varphi(\varphi,\varphi) = 7$, we have by definition of the Hodge dual $\varphi \wedge *_{g_\varphi}\varphi = 7\,\phi(\varphi)$ and therefore

$$\hat{\varphi} = \frac{1}{3} *_{g_\varphi} \varphi, \tag{3.17}$$

by comparing with (3.2).

LEMMA 3.8. *Let φ be a stable three-form in a seven-dimensional vector space V. Let β be a one-form or a two-form. Then $\beta \wedge \varphi = 0$ if and only if $\beta = 0$.*

PROOF. For the compact case, see also [**Bo**]. If β is a one-form, the proof is very easy. If β is a two-form, we choose a basis such that φ is in the normal form (3.16) and $\beta = \sum_{i<j} b_{i,j}\, e^{ij}$ and compute

$$\begin{aligned} \beta \wedge \varphi &= (b_{2,3} - b_{1,6})\, e^{12356} + (b_{2,3} - b_{4,7})\, e^{23457} + (b_{1,6} + b_{4,7})\, e^{14567} \\ &+ (b_{5,7}\tau + b_{1,2})\, e^{12457} + (b_{3,6} - b_{5,7})\, e^{34567} + (b_{1,2} - b_{3,6}\tau)\, e^{12346} \\ &- (b_{3,7}\tau + b_{2,4})\, e^{12347} + (b_{5,6}\tau + b_{2,4})\, e^{12456} + (b_{3,7} + b_{5,6})\, e^{13567} \\ &+ (b_{2,5} - b_{4,6})\, e^{23456} + (b_{4,6} - b_{1,7})\, e^{13467} - (b_{2,5} + b_{1,7})\, e^{12357} \\ &+ (b_{4,5} + b_{2,6})\, e^{24567} - (b_{1,3} + b_{2,6})\, e^{12367} + (b_{4,5} + b_{1,3})\, e^{13457} \\ &+ (b_{3,5} + b_{6,7})\, e^{23567} + (b_{1,4} - b_{3,5}\tau)\, e^{12345} + (b_{6,7}\tau - b_{1,4})\, e^{12467} \\ &+ (b_{3,4} + b_{1,5})\, e^{13456} + (b_{2,7} - b_{1,5})\, e^{12567} + (b_{3,4} - b_{2,7})\, e^{23467}. \end{aligned}$$

The five-form is written as a linear combination of linearly independent forms and each line contains exactly three different coefficients of β. Inspecting the coefficient equations line by line, it is easy to see that all coefficients of β vanish if and only if $\beta \wedge \varphi = 0$. $\square$

$\mathbf{k = 3, n = 8}$. The stabiliser of a stable three-form in dimension eight is a real form of $PSL(3, \mathbb{C})$. However, as these groups are not considered in this thesis, we omit the discussion and refer to [**KiSa**], [**Hi1**] and [**Wi**] for more information.

4. Relation between stable forms in dimensions six and seven

There is a natural relation between stable forms in dimension six, stable forms in dimension seven and certain non-stable four-forms in dimensions eight which is explained in this section. This relation is in fact the algebraic construction underlying the Hitchin flow which is discussed in chapter 6.

4.1. Real forms of $\mathrm{SL}(3,\mathbb{C})$**.** Any real form of $\mathrm{SL}(3,\mathbb{C})$ can be written as a simultaneous stabiliser of a stable two-form and a stable three-form as follows.

Let V be a six-dimensional real vector space.

DEFINITION 4.1. A pair $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ of a stable two-form and a stable three-form is called *compatible* if

$$\omega \wedge \rho = 0$$

and *normalised* if

$$\phi(\rho) = 2\,\phi(\omega) \qquad \Longleftrightarrow \qquad J_\rho^*\rho \wedge \rho = \frac{2}{3}\,\omega^3.$$

By Proposition 1.13, a compatible pair (ω, ρ) of stable forms induces a unique special ε-Hermitian structure $(g_{(\omega,\rho)}, J_\rho, \omega, \rho + \mathrm{i}_\varepsilon J_\rho^*\rho)$ where the induced ε-complex structure J_ρ is given by (3.7), and the induced metric by

$$g_{(\omega,\rho)} = \varepsilon\omega(\,.\,, J_\rho\,.\,). \tag{4.1}$$

Thus, the stabiliser in $\mathrm{GL}(V)$ of a compatible pair is

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\rho,\omega) \cong \begin{cases} \mathrm{SU}(p,q) \subset \mathrm{SO}(2p,2q)\,, \quad p+q=3\,, & \text{if } \lambda(\rho)<0, \\ \mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SO}(3,3)\,, & \text{if } \lambda(\rho)>0, \end{cases}$$

where $\mathrm{SL}(3,\mathbb{R})$ is embedded in $\mathrm{SO}(3,3)$ such that it acts by the standard representation and its dual, respectively, on the maximally isotropic ± 1-eigenspaces of the para-complex structure J_ρ induced by ρ.

A special ε-Hermitian structure induced by a compatible and normalised pair is indeed normalised due to Lemma 1.15. In the case $\varepsilon = -1$, the metric $g_{(\omega,\rho)}$ induced by a normalised, compatible pair is either positive definite or of signature $(2,4)$ due to the sign $(-1)^q$ appearing in (1.37). This is no restriction of generality since any $\mathrm{SU}(p,q)$-structure, $p+q=3$, can be turned into a $\mathrm{SU}(3)$- or $\mathrm{SU}(1,2)$-structure by replacing g by $-g$ or, equivalently, ω by $-\omega$.

Following the conventions introduced in section 1.4, an adapted ε-unitary basis for a compatible and normalised pair (ω,ρ) is a pseudo-orthonormal basis $\{e_1,\ldots,e_6\}$ of V with dual basis $\{e^1,\ldots,e^6\}$ such that $\rho=\rho_\varepsilon$ is in the normal form (3.9) and

$$\omega \;=\; \tau(e^{12}+e^{34})+e^{56} \tag{4.2}$$

for $(\varepsilon,\tau) \in \{(-1,1),(-1,-1),(1,1)\}$. The signature of the induced metric with respect to this basis is

$$(\tau,-\varepsilon\tau,\tau,-\varepsilon\tau,1,-\varepsilon) = \begin{cases} (+,+,+,+,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=1, \\ (-,-,-,-,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=-1, \\ (+,-,+,-,+,-) & \text{for } \varepsilon=1 \text{ and } \tau=1, \end{cases} \tag{4.3}$$

and we have

$$\mathrm{Stab}_{\mathrm{GL}(6,\mathbb{R})}(\omega,\rho) \cong \begin{cases} \mathrm{SU}(3) \subset \mathrm{SO}(6) & \text{for } \varepsilon=-1 \text{ and } \tau=1, \\ \mathrm{SU}(1,2) \subset \mathrm{SO}(2,4) & \text{for } \varepsilon=-1 \text{ and } \tau=-1, \\ \mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SO}(3,3) & \text{for } \varepsilon=1. \end{cases}$$

For instance, the following observation is easily verified using the unified basis.

LEMMA 4.2. *Let (ω,ρ) be a compatible and normalised pair of stable forms on a six-dimensional vector space. Then, the volume form $\phi(\omega)$ is in fact a metric volume form with respect to the induced metric $g=g_{(\omega,\rho)}$ and the corresponding Hodge dual of ω and ρ is*

$$*_g\omega = -\varepsilon\hat{\omega}\,, \qquad *_g\rho = -\hat{\rho} \tag{4.4}$$

4.2. Relation between real forms of $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{G}_2^{\mathbb{C}}$. The relation between stable forms in dimension six and seven corresponding to the embedding $\mathrm{SU}(3) \subset \mathrm{G}_2$ is well-known. We extend this relation by including also the embeddings $\mathrm{SU}(1,2) \subset \mathrm{G}_2^*$ and $\mathrm{SL}(3,\mathbb{R}) \subset \mathrm{G}_2^*$ as follows.

PROPOSITION 4.3. *Let $V = W \oplus L$ be a seven-dimensional vector space decomposed as a direct sum of a six-dimensional subspace W and a line L. Let α be a non-trivial one-form in the annihilator W^0 of W and $(\omega,\rho) \in \Lambda^2 L^0 \times \Lambda^3 L^0$ a compatible and normalised pair of stable forms inducing the scalar product $h = h_{(\omega,\rho)}$ given in (4.1). Then, the* *three-form $\varphi \in \Lambda^3 V^*$ defined by*

$$\varphi = \omega \wedge \alpha + \rho \tag{4.5}$$

is stable and induces the scalar product

$$g_\varphi = h - \varepsilon \alpha \cdot \alpha \tag{4.6}$$

where ε denotes the sign of $\lambda(\rho)$ such that $J_\rho^2 = \varepsilon \mathrm{id}$. The stabiliser of φ in $\mathrm{GL}(V)$ is

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\varphi) \cong \begin{cases} \mathrm{G}_2 & \text{for } \varepsilon = -1 \text{ and positive definite } h, \\ \mathrm{G}_2^* & \text{otherwise.} \end{cases}$$

Proof. We choose a basis $\{e_1, \dots, e_6\}$ of L^0 such that ω and ρ are in the generic normal forms (3.9) and (4.2). With $e^7 = \alpha$, we have

$$\varphi = \tau(e^{127} + e^{347}) + e^{567} + e^{135} + \varepsilon(e^{146} + e^{236} + e^{245}). \tag{4.7}$$

The induced bilinear form (3.13) turns out to be

$$b_\varphi(v, w) = (-\varepsilon\tau v^1 w^1 + \tau v^2 w^2 - \varepsilon\tau v^3 w^3 + \tau v^4 w^4 - \varepsilon v^5 w^5 + v^6 w^6 + v^7 w^7) e^{1234567}$$

for $v = \sum v^i e_i$ and $w = \sum w^i e_i$. Hence, the three-form φ is stable for all signs of ε and τ and its associated volume form is

$$\phi(\varphi) = (\det b_\varphi)^{\frac{1}{9}} = -\varepsilon\, e^{1234567}.$$

The formula (4.6) for the metric g_φ induced by φ follows, since the basis $\{e_1, \dots, e_7\}$ of V is pseudo-orthonormal with respect to this metric of signature

$$(\tau, -\varepsilon\tau, \tau, -\varepsilon\tau, 1, -\varepsilon, -\varepsilon) = \begin{cases} (+,+,+,+,+,+,+) & \text{for } \varepsilon = -1 \text{ and } \tau = 1, \\ (-,-,-,-,+,+,+) & \text{for } \varepsilon = -1 \text{ and } \tau = -1, \\ (+,-,+,-,+,-,-) & \text{for } \varepsilon = 1 \text{ and } \tau = 1. \end{cases} \tag{4.8}$$

The assertion on the stabilisers now follows from (3.15). □

Lemma 4.4. *Under the assumptions of the previous proposition, the dual four-form of the stable three-form φ is*

$$3\hat\varphi = *_\varphi \varphi = -\,\varepsilon\,(\alpha \wedge \hat\rho + \hat\omega) = \varepsilon \alpha \wedge *_h \rho + *_h \omega, \tag{4.9}$$

where $*_\varphi$ denotes the Hodge dual with respect to the metric g_φ and the orientation induced by $\phi(\varphi)$.

Proof. In the basis of the previous proof, the Hodge dual of φ is

$$*_\varphi \varphi = -\varepsilon\tau(e^{3456} + e^{1256}) - \varepsilon e^{1234} + \varepsilon\, e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

The second equality follows when comparing this expression with $\varepsilon(e^7 \wedge \hat\rho + \frac{1}{2}\omega^2)$ in this basis using (3.10) and (4.2). The first and the third equality are just the formulas (3.17) and (4.4), respectively. □

The inverse process is given by the following construction.

Proposition 4.5. *Let V be a seven-dimensional real vector space and $\varphi \in \Lambda^3 V^*$ a stable three-form which induces the metric g_φ on V. Moreover, let $n \in V$ be a unit vector with $g_\varphi(n, n) = -\varepsilon \in \{\pm 1\}$ and let $W = n^\perp$ denote the orthogonal complement of $\mathbb{R}\cdot n$.*

Then, the pair $(\omega,\rho)\in\Lambda^2W^*\times\Lambda^3W^*$ *defined by*

$$\omega = (\,n\lrcorner\,\varphi)_{|W}\,,\qquad \rho=\varphi_{|W}\,, \tag{4.10}$$

is a pair of compatible normalised stable forms. The metric $h=h_{(\omega,\rho)}$ *induced by this pair on* W *satisfies* $h=(g_\varphi)_{|W}$ *and the stabiliser is*

$$\mathrm{Stab}_{\mathrm{GL}(W)}(\omega,\rho)\cong\begin{cases}\mathrm{SU}(3), & \textit{if } g_\varphi \textit{ is positive definite,}\\ \mathrm{SU}(1,2), & \textit{if } g_\varphi \textit{ is indefinite and } \varepsilon=-1,\\ \mathrm{SL}(3,\mathbb{R}), & \textit{if } \varepsilon=1.\end{cases}$$

When (V,φ) *is identified with the imaginary octonions, respectively, the imaginary split-octonions, by* (3.14), *the* ε*-complex structure induced by* ρ *is given by*

$$J_\rho v = -n\cdot v = -n\times v \qquad \textit{for } v\in V. \tag{4.11}$$

Proof. Due to the stability of φ, we can always choose a basis $\{e_1,\dots,e_7\}$ of V with $n=e_7$ such that φ is given by (4.7) where $\varepsilon=-g_\varphi(n,n)$ and $\tau\in\{\pm1\}$ depends on the signature of g_φ. As this basis is pseudo-orthonormal with signature given by (4.8), the vector n has indeed the right scalar square and $\{e_1,\dots,e_6\}$ is a pseudo-orthonormal basis of the complement $W=n^\perp$. Since the pair (ω,ρ) defined by (4.10) is now exactly in the generic normal form given by (3.9) and (4.2), it is stable, compatible and normalised and the induced endomorphism J_ρ is an ε-complex structure. The identity $h=(g_\varphi)_{|W}$ for the induced metric $h_{(\omega,\rho)}$ follows from comparing the signatures (4.8) and (4.3) and the assertion for the stabilisers is an immediate consequence. Finally, the formula for the induced ε-complex structure J_ρ is another consequence of $g=(g_\varphi)_{|W}$ since we have

$$g_\varphi(x,n\times y)\overset{(3.14)}{=}\varphi(x,n,y)=-\omega(x,y)=-h(x,J_\rho y)$$

for all $x,y\in W$. □

Notice that, for a fixed metric h of signature $(2,4)$ or $(3,3)$, the compatible and normalised pairs (ω,ρ) of stable forms inducing this metric are parametrised by the homogeneous spaces $\mathrm{SO}(2,4)/\,\mathrm{SU}(1,2)$ and $\mathrm{SO}(3,3)/\,\mathrm{SL}(3,\mathbb{R})$, respectively. Thus, the mapping $(\omega,\rho)\mapsto\varphi$ defined by formula (4.5) yields isomorphisms

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SU}(1,2)}\cong\frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*}\,,\qquad \frac{\mathrm{SO}(3,3)}{\mathrm{SL}(3,\mathbb{R})}\cong\frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*}\,,$$

since the metric h completely determines the metric g_φ by the formula (4.6).

4.3. Relation between real forms of $\mathrm{G}_2^{\mathbb{C}}$ and $\mathrm{Spin}(7,\mathbb{C})$. It is possible to extend this construction to dimension eight as follows. Starting with a stable three-form φ on a seven-dimensional space V, we can consider the four-form

$$\Phi = e^8\wedge\varphi + *_\varphi\varphi. \tag{4.12}$$

on the eight-dimensional space $V\oplus\mathbb{R}e_8$. Although the four-form Φ is not stable, it is shown in [**Br1**] that it induces the metric

$$g_\Phi = g_\varphi + (e^8)^2 \tag{4.13}$$

on $V \oplus \mathbb{R}e_8$ and that its stabiliser is

$$\mathrm{Stab}_{\mathrm{GL}(V\oplus\mathbb{R}e_8)}(\Phi) \cong \begin{cases} \mathrm{Spin}(7) \subset \mathrm{SO}(8), & \text{if } g_\varphi \text{ is positive definite,} \\ \mathrm{Spin}_0(3,4) \subset \mathrm{SO}(4,4), & \text{if } g_\varphi \text{ is indefinite.} \end{cases}$$

The index "0" denotes, as usual, the connected component. Starting conversely with a four-form Φ on $V \oplus \mathbb{R}e_8$ such that its stabiliser in $\mathrm{GL}(V \oplus \mathbb{R}e_8)$ is isomorphic to $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$, the process can be reversed by setting $\varphi = e_8 \lrcorner \Phi$. As before, the metric induced by Φ on $V \oplus \mathbb{R}e_8$ is determined by the metric g_φ induced by φ on V. Thus, the indefinite analogue of the well-known isomorphisms

$$\mathbb{RP}^7 \cong \frac{\mathrm{SO}(6)}{\mathrm{SU}(3)} \cong \frac{\mathrm{SO}(7)}{\mathrm{G}_2} \cong \frac{\mathrm{SO}(8)}{\mathrm{Spin}(7)}$$

is given by

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SU}(1,2)} \cong \frac{\mathrm{SO}(3,3)}{\mathrm{SL}(3,\mathbb{R})} \cong \frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*} \cong \frac{\mathrm{SO}(4,4)}{\mathrm{Spin}_0(3,4)}. \tag{4.14}$$

CHAPTER 2

Geometric structures defined by linear Lie groups

In this chapter, we review a number of general results on G-structures and holonomy groups and explain how to classify G-structures for a given group G in terms of the intrinsic torsion. Standard references for this material are for instance [**Sa1**] and [**Joy2**] or [**Joy3**].

1. G-structures and holonomy

If not otherwise stated, the manifolds, bundles and mappings in consideration will always assumed to be *smooth*.

Let M be a real n-dimensional manifold and V a real n-dimensional "model" vector space. The fibre of the *frame bundle* $\mathrm{GL}(M)$ over a point $p \in M$ consists of the isomorphisms $u : T_pM \to V$ and $\mathrm{GL}(M)$ is a principal bundle with fibre $\mathrm{GL}(V)$ where the free right action of $\mathrm{GL}(V)$ is given by $R_g(u) = g^{-1} \circ u$ for all $g \in \mathrm{GL}(V)$. A section from an open set $U \subset M$ into $\mathrm{GL}(M)$ is a *frame* which is equivalently given by an n-tuple $(X_i)_{1 \le i \le n}$ consisting of local vector fields $X_i \in \Gamma(U, TM)$ such that $\{(X_i)_p\}$ is a basis of T_pM for all $p \in U$. In particular, a *coordinate frame* is a local frame $\{\frac{\partial}{\partial x^i}\}$ defined by a coordinate chart $x = (x^1, \ldots, x^n)$ on an open set. Given a frame (X_i), there are coordinates (x^i) such that $X_i = \frac{\partial}{\partial x^i}$ for all i if and only the frame (X_i) is *integrable*, i.e. $[X_i, X_j] = 0$ for all i, j.

In general, the *reduction* of a principal bundle $P \to M$ with fibre G to a subgroup $H \subset G$ is a submanifold $Q \subset P$ which is invariant under the right action of H such that $Q \to M$ is a principal bundle with fibre H. The reductions of a principal bundle $P \to M$ with fibre G to a *closed* subgroup $H \subset G$ are parametrised by the global sections of the quotient bundle $P/H \to M$ with typical fibre G/H, [**KN**, Prop. I.5.6]. We note that $P \to P/H$ is a principal bundle with fibre H and that $P/H \to M$ can be identified with the bundle associated to P and the left action of G on G/H.

Definition 1.1. A *G-structure* on a manifold M is a reduction of the frame bundle $\mathrm{GL}(M)$ to a linear Lie group $G \subset \mathrm{GL}(V)$.

The existence of a G-structure on a manifold M for a given $G \subset \mathrm{GL}(V)$ is a purely topological question. For instance, reductions always exist if G/H is diffeomorphic with some $\mathbb{R}^N$, [**KN**, Prop. I.5.7]. Thus, a reduction to a maximal compact subgroup is always possible due to the existence of an Iwasawa decomposition for $\mathrm{GL}(V)$.

Many interesting G-structures arise from closed groups $G \subset \mathrm{GL}(V)$ which can be written as the stabiliser in $\mathrm{GL}(V)$ of one or several tensors on V. We will use the abbreviation $V^{r,s}$ for the $\mathrm{GL}(V)$-module $V^{\otimes r} \otimes (V^*)^{\otimes s}$ and similarly, we set $TM^{r,s} = TM^{\otimes r} \otimes (TM^*)^{\otimes s}$.

Proposition 1.2. *Let P be a G-structure on a manifold M and let H be a subgroup of G which can be written as the stabiliser in G of a tensor $\xi_0 \in V^{r,s}$.*

Then, the tensor ξ_0 *defines a one-to-one-correspondence between the reductions* Q *of the* G*-structure* P *to the group* H *and global tensor fields* $\xi \in \Gamma(TM^{r,s})$ *with the property that there exists for all* $p \in M$ *a neighbourhood* U *of* p *and a local section* $s \in \Gamma(U, P)$ *such that* $s_q(\xi_q) = \xi_0$ *for all* $q \in U$.

We will say that the tensor fields ξ *with this property are* modelled by the tensor ξ_0 *and that* ξ_0 *is a* model tensor *for* ξ.

PROOF. We explain the main idea of the proof. Let the projection $P \to M$ be denoted by π. Given a global tensor field ξ with the property described in the lemma, we can define a non-empty subset Q of P fibrewise by setting $Q_p := \{u \in \pi^{-1}(p) \mid u(\xi_p) = \xi_0\}$. Since $H = \mathrm{Stab}_G(\xi_0)$, each fibre Q_p is invariant under the action of H and Q is a reduction of P to the group H.

Conversely, given a reduction Q of the G-structure P to the group H, we choose a covering $\{U_\alpha\}_{\alpha \in I}$ of M and sections $s_\alpha \in \Gamma(U_\alpha, Q)$ for all $\alpha \in I$. Now, we can define local tensor fields ξ_α by setting $(\xi_\alpha)_p = (s_\alpha)_p^{-1}(\xi_0)$. Using the assumption $H = \mathrm{Stab}_G(\xi_0)$, we can glue the local tensor fields to a well-defined global tensor field which has the desired property by construction. □

If there is a model tensor ξ_0 fixed such that $H = \mathrm{Stab}_{\mathrm{GL}(V)}(\xi_0)$, we will identify an H-structure with the corresponding defining tensor field ξ. We shall say that a G-structure P is defined by tensor fields ξ_i, $i = 1, \ldots, r$, if P is obtained by repeatedly reducing $\mathrm{GL}(M)$ with the help of the ξ_i.

EXAMPLE 1.3. For instance, the proposition can be used to identify $\mathrm{O}(p, q)$-structures and pseudo-Riemannian metrics g on a manifold M. For the standard inner product of $\mathbb{R}^{p,q}$ as model tensor, the $\mathrm{O}(p, q)$-structure corresponding to a metric g is the bundle of g-orthonormal frames which shall be denoted by $\mathrm{O}(M)$.

An example of a G-structure which cannot be defined by a tensor is a $\mathrm{GL}^+(V)$-structure which is equivalent to defining an orientation.

A linear connection ∇ on M is called *compatible* with a given G-structure $P \to M$, or a G*-connection*, if the associated connection on $\mathrm{GL}(M)$ reduces to a connection on P. This is by definition the case if the horizontal distribution defined by ∇ is contained in TP or equivalently, if the connection one-form has values in $\mathfrak{g}$. Using a partition of unity of M, a G-connection can always be constructed.

If the group G can be written as a stabiliser in $\mathrm{GL}(V)$ of some tensors on V, a G-connection can also be characterised as follows.

LEMMA 1.4. *A connection* ∇ *is compatible with a* G*-structure defined by tensor fields* ξ_i *with* $i = 1, \ldots, r$ *if and only if all defining tensor fields* ξ_i *are constant for* ∇, *i.e. satisfy* $\nabla \xi_i = 0$.

PROOF. See [**Sa1**], Lemma 1.3 of chapter 1. □

For instance, an $\mathrm{O}(p, q)$-connection of the $\mathrm{O}(p, q)$-structure $\mathrm{O}(M)$ defined by a pseudo-Riemannian metric is nothing else than a metric connection.

The *holonomy group* $\mathrm{Hol}_p(\nabla)$ of a linear connection ∇ in a point $p \in M$ is defined as the group of parallel translations along all piecewise smooth loops based at p. It is well-known that $\mathrm{Hol}_p(\nabla)$ is a Lie subgroup of $\mathrm{GL}(T_pM)$ and that its connected component is generated by all contractible loops at p. When identifying T_pM with a model vector space V, we can regard $\mathrm{Hol}_p(\nabla)$ as a subgroup of $\mathrm{GL}(V)$. As the holonomy groups in two different points are always conjugated, the holonomy group $\mathrm{Hol}(\nabla)$ without reference to a base point is well-defined as a subgroup of $\mathrm{GL}(V)$ up to conjugation.

By the following proposition, often called "holonomy principle", the holonomy representation can be used to determine the tensor fields which are parallel for a given connection.

Proposition 1.5. *Given a linear connection ∇ on a manifold M and a point $p \in M$, there is a one-to-one correspondence between*

(i) tensor fields $\xi \in \Gamma(TM^{r,s})$ which are parallel *for ∇, i.e. which are invariant under parallel transport,*
(ii) tensor fields $\xi \in \Gamma(TM^{r,s})$ which are constant for ∇ and
(iii) $\mathrm{Hol}_p(\nabla)$-invariant tensors $\xi_0 \in (T_pM)^{r,s}$.

Proof. See for instance [**Joy3**, Proposition 2.5.2]. □

Corollary 1.6. *Let ∇ be a linear connection on a manifold M and $p \in M$. Then, the holonomy group $\mathrm{Hol}_p(\nabla)$ is a subgroup of the group*

$$G = \{h \in GL(T_pM) \,|\, h(\xi_p) = \xi_p \text{ for all tensors fields } \xi \text{ with } \nabla\xi = 0\}.$$

We add the remark that the holonomy group $\mathrm{Hol}_p(\nabla)$ equals the group G defined in the corollary whenever $\mathrm{Hol}(\nabla)$ can be written as the stabiliser in $\mathrm{GL}(V)$ of some tensors on V.

Let us also recall the definition of the holonomy group of a connection ∇ on a principal bundle $P \to M$ with fibre G. For $f \in P$, this group is defined as

$$\mathrm{Hol}_f(P, \nabla) = \{h \in G \,|\, \text{There is a piecewise smooth horizontal curve in } P \text{ joining } f \text{ and } h \cdot f.\}$$

Again, the holonomy group $\mathrm{Hol}(P, \nabla)$ without reference to a base point can be regarded as a subgroup of G which is well-defined up to conjugation. Given a linear connection ∇ on a manifold M, it is well-known that the holonomy group $\mathrm{Hol}(\mathrm{GL}(M), \nabla) \subset \mathrm{GL}(V)$ of the associated connection on the frame bundle $\mathrm{GL}(M)$ equals the holonomy group $\mathrm{Hol}(\nabla)$ regarded as subgroup of $\mathrm{GL}(V)$ up to conjugation.

The following proposition can be viewed as a special case of the well-known reduction theorem of holonomy theory.

Proposition 1.7. *Let ∇ be a linear connection on a manifold M and let $f \in \mathrm{GL}(M)$. There exists a G-structure $P \to M$ for a given $G \subset \mathrm{GL}(V)$ which is compatible with ∇ and contains f if and only if $\mathrm{Hol}_f(\mathrm{GL}(M), \nabla) \subset G \subset \mathrm{GL}(V)$. If such a G-structure exists, then it is unique.*

Proof. See for instance [**Joy3**, Proposition 2.6.3]. □

2. Intrinsic torsion

Let the torsion tensor T^∇ of a linear connection ∇ be the tensor in $\Gamma(\Lambda^2(TM)^* \otimes TM)$ defined by

$$T^\nabla(X \wedge Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.1}$$

and let the curvature tensor R^∇ be the tensor in $\Gamma(\Lambda^2(TM)^* \otimes T^*M \otimes TM)$ defined by

$$R^\nabla(X \wedge Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z. \tag{2.2}$$

Given a G-structure P, the *adjoint bundle* $\mathfrak{g}(P)$ is the vector bundle $P \times_G \mathfrak{g}$ associated to P and the adjoint representation of $G \subset \mathrm{GL}(V)$ on $\mathfrak{g} \subset \mathfrak{gl}(V) = V^* \otimes V$. With this notation, the curvature tensor of a G-connection is in fact a section of $\Lambda^2(TM)^* \otimes \mathfrak{g}(P)$.

Given two G-connections ∇, ∇', the difference tensor $\nabla - \nabla'$ is a section of $T^*M \otimes \mathfrak{g}(P)$. The difference of the torsion tensors satisfies

$$T^\nabla - T^{\nabla'} = (\nabla - \nabla')_X Y - (\nabla - \nabla')_Y X, \tag{2.3}$$

and is thus also a section of $T^*M \otimes \mathfrak{g}(P) \subset T^*M^{\otimes 2} \otimes TM$. Let the bundle homomorphism

$$\sigma \,:\, T^*M \otimes \mathfrak{g}(P) \to \Lambda^2(TM)^* \otimes TM \tag{2.4}$$

be defined by skew-symmetrisation of the first two components.

Definition 2.1. The *intrinsic torsion* or *structure function* $\tau(P)$ of a G-structure P is defined by choosing an arbitrary G-connection ∇ and setting

$$\tau(P) = [T^\nabla] \in \Gamma(\frac{\Lambda^2(TM)^* \otimes TM}{\mathrm{im}(\sigma)}),$$

where the brackets denote the canonical fibrewise projection.

As $T^\nabla - T^{\nabla'} \in \mathrm{im}(\sigma)$, this definition does not depend on the choice of the G-connection ∇. Obviously, the intrinsic torsion of a G-structure vanishes if and only if there exists a torsionfree G-connection. If this is the case, the torsionfree G-connections are parametrised by the kernel of the mapping σ.

Definition 2.2.

(i) A G-structure P is called *flat* or *integrable* if there is around every point a local coordinate frame with values in P.
(ii) A G-structure P is called *torsionfree* if its intrinsic torsion $\tau(P)$ vanishes.
(iii) A local section $s \in \Gamma(U, P)$ in a G-structure P is called *k-flat* in a point $p \in U$ if and only if its k-jet in p coincides with the k-jet in p of a coordinate frame.
(iv) A G-structure P is called *k-flat* if it admits for every point p a local section defined on a neighbourhood of p which is k-flat in p.

In fact, it is well-known, see for instance [**Br1**], that a G-structure is 1-flat if and only if it is torsionfree and that a flat G-structure is k-flat for all k.

Example 2.3 ($\mathrm{O}(p,q)$-structures). We return to the example of pseudo-Riemannian metrics, i.e. $\mathrm{O}(p,q)$-structures. The existence of the unique torsionfree metric Levi-Civita connection ∇^g for a metric g implies that every $\mathrm{O}(p,q)$-structure is torsionfree and thus 1-flat. Moreover, it is well-known that an $\mathrm{O}(p,q)$-structure is flat if and only if the Riemannian curvature R^{∇^g} vanishes, i.e. if and only if it is 2-flat.

Example 2.4 ($\mathrm{Sp}(m,\mathbb{R})$-structures). A $\mathrm{Sp}(m,\mathbb{R})$-structure on a real manifold M of dimension $2m$ is defined by a global non-degenerate two-form ω and is also called an almost symplectic structure. A torsionfree $\mathrm{Sp}(m,\mathbb{R})$-structure is a symplectic structure, i.e. the defining two-form ω is closed. By the theorem of Darboux, a torsionfree $\mathrm{Sp}(m,\mathbb{R})$-structure is therefore always integrable.

In table 1, we list all G-structures appearing in this thesis. In particular, we give an equivalent characterisation for the vanishing of the intrinsic torsion, if possible in form of an exterior system. For a discussion of these characterisations, we refer to the chapter 3 except for the groups G_2 and Spin(7) which are discussed in section 3.1 of this chapter and section 3 of chapter 6.

Group G	dim M	equivalent structure	defining tensors	condition for $\tau = 0$
$\mathrm{GL}^+(\mathbb{R})$	n	orientation	-	always torsionfree
$\mathrm{SL}(n,\mathbb{R})$	n	volume form	$\nu \in \Omega^n M$, $\nu \neq 0$	always torsionfree
$\mathrm{O}(p,q)$	$p+q$	pseudo-Riemannian metric	non-degenerate, symmetric $g \in \Gamma(TM^{0,2})$	always torsionfree
$\mathrm{Sp}(m,\mathbb{R})$	2m	almost symplectic structure	$\omega \in \Omega^2 M$, $\omega^m \neq 0$	$d\omega = 0 \Leftrightarrow M$ is a symplectic manifold
$\mathrm{GL}(m,\mathbb{C})$	2m	almost complex structure	$J \in \Gamma(\mathrm{End}\,TM)$, $J^2 = -\mathrm{Id}$	$N^J = 0 \Leftrightarrow M$ is a complex manifold
$\mathrm{SL}(m,\mathbb{C})$	2m	complex volume form	non-degenerate, decomposable $\psi_+ + \mathrm{i}\psi_- \in \Omega^m_{\mathbb{C}} M$	$d\psi_+ = 0$, $d\psi_- = 0$
$\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R})$	2m	almost para-complex structure	$J \in \Gamma(\mathrm{End}\,TM)$, $J^2 = \mathrm{Id}$, $\dim \mathrm{Eig_J}(1) = m$	$N^J = 0 \Leftrightarrow M$ is a para-complex manifold
$\mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(m,\mathbb{R})$	2m	para-complex volume form	non-degenerate, decomposable $\psi_+ + \mathrm{i}_1\psi_- \in \Omega^m_C M$	$d\psi_+ = 0$, $d\psi_- = 0$
$\mathrm{U}(p,q)$	$2p+2q$	almost pseudo-Hermitian structure	(g, J, ω)	$d\omega = 0$, $N^J = 0 \Leftrightarrow$ pseudo-Kähler
$\mathrm{GL}(m,\mathbb{R})$[1]	$2m$	almost para-Hermitian structure	(g, J, ω)	$d\omega = 0$, $N^J = 0 \Leftrightarrow$ para-Kähler
$\mathrm{SU}(p,q)$	$2p+2q$	special almost pseudo- Hermitian structure	$(g, J, \omega, \psi_+, \psi_-)$	$d\omega = 0$, $d\psi_+ = 0$, $d\psi_- = 0 \Leftrightarrow$ Ricci-flat pseudo-Kähler
$\mathrm{SL}(m,\mathbb{R})$[1]	$2m$	special almost para-Hermitian structure	$(g, J, \omega, \psi_+, \psi_-)$	$d\omega = 0$, $d\psi_+ = 0$, $d\psi_- = 0 \Rightarrow$ Ricci-flat para-Kähler
G_2 or G_2^*	7	-	stable 3-form φ	$d\varphi = 0$, $d *_\varphi \varphi = 0$
Spin(7) or $\mathrm{Spin}_0(3,4)$	8	-	4-form Φ	$d\Phi = 0$

TABLE 1. G-structures appearing in this thesis and conditions for 1-flatness

[1]acting on $\mathbb{R}^{2m} = \mathbb{R}^m \oplus (\mathbb{R}^m)^*$

3. G-structures on pseudo-Riemannian manifolds

In the following, let (M, g) always be a pseudo-Riemannian manifold of signature (p, q) and let $\mathrm{O}(M)$ be the principal bundle with fibre $\mathrm{O}(p, q)$ consisting of the g-orthonormal frames. Reductions of $\mathrm{O}(M)$ enjoy a number of special properties which shall be discussed in this section.

The following lemma is a direct consequence of Proposition 1.7, Lemma 1.4 and the fact that the Levi-Civita connection of g, denoted by ∇^g, is the unique torsionfree and metric connection on (M, g).

LEMMA 3.1.

(i) The holonomy group $\mathrm{Hol}(g) = \mathrm{Hol}(\nabla^g)$ *of the metric is a subgroup of* $\mathrm{O}(p, q)$ *defined up to conjugation.*

(ii) The holonomy group $\mathrm{Hol}(g)$ *is contained in a subgroup* G *of* $\mathrm{O}(p, q)$ *if and only if there exists a torsionfree reduction of* $\mathrm{O}(M)$ *to the group* G*.*

(iii) Let G *be the stabiliser in* $\mathrm{O}(p, q)$ *of a model tensor* ξ_0*. Then, a reduction of* $\mathrm{O}(M)$ *to the group* G *is torsionfree if and only if the corresponding defining tensor field* ξ *modelled by* ξ_0 *is constant for the Levi-Civita connection* ∇^g*.*

REMARK 3.2. Since the defining tensor field of a torsionfree G-structure P with $G \subset \mathrm{O}(p, q)$ is thus parallel for ∇^g, such a G-structure is also called a *parallel* G-structure. Sometimes, also the notion of an integrable G-structure seems to be used synonymous with a torsionfree G-structure. However, our convention that an integrable G-structure is the same as a flat G-structure is well-established in the classical literature, see for instance [**Sa1**] and references therein.

Given a subgroup G of $\mathrm{O}(p, q)$, the Lie algebra of $\mathrm{O}(p, q)$ can be decomposed with respect to the Killing form into $\mathfrak{so}(p, q) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. When P is a reduction of $\mathrm{O}(M)$ to G, there is a corresponding splitting of the adjoint bundle $\mathfrak{so}(\mathrm{O}(M)) = \mathfrak{g}(P) \oplus \mathfrak{g}^\perp(P)$.

LEMMA 3.3. *For a reduction* P *of* $\mathrm{O}(M)$ *to a subgroup* G *of* $\mathrm{O}(p, q)$*, there are bundle isomorphisms*

$$\frac{\Lambda^2(TM)^* \otimes TM}{\sigma(T^*M \otimes \mathfrak{g}(P))} \cong \frac{T^*M \otimes \mathfrak{so}(\mathrm{O}(M))}{T^*M \otimes \mathfrak{g}(P)} \cong T^*M \otimes \mathfrak{g}^\perp(P).$$

In particular, the intrinsic torsion $\tau(P)$ *of the* G*-structure* P *can be viewed as a section of* $T^*M \otimes \mathfrak{g}^\perp(P)$*.*

PROOF. The isomorphism $TM \cong T^*M$ defined by the metric g induces an isomorphism

$$\mathfrak{so}(\mathrm{O}(M)) \cong \Lambda^2(TM)^*$$

such that also the mapping

$$\sigma \,:\, T^*M \otimes \mathfrak{so}(\mathrm{O}(M)) \to \Lambda^2(TM)^* \otimes TM,$$

defined in (2.4) is an isomorphism of vector bundles. Now, the first isomorphism of the lemma is given by the map induced by σ^{-1} on the quotients. The second isomorphism is obvious in each fibre. $\square$

COROLLARY 3.4. *Under the assumptions of the previous lemma, there is a unique G-connection $\bar{\nabla}$ on the G-structure P such that the difference tensor*

$$\bar{\nabla} - \nabla^g \in \Gamma(T^*M \otimes \mathfrak{so}(\mathrm{O}(M)))$$

equals the intrinsic torsion $\tau(P) \in \Gamma(T^*M \otimes \mathfrak{g}^\perp(P))$.

DEFINITION 3.5. The unique G-connection $\bar{\nabla}$ defined in the previous proposition is called the *minimal G-connection* of the G-structure P.

If the G-structure P is furthermore defined by a tensor, we have another identification.

PROPOSITION 3.6. *Let P be a reduction of $\mathrm{O}(M)$ to a subgroup $G = \mathrm{Stab}_{\mathrm{O}(p,q)}\xi_0$ where $\xi_0 \in V^{r,s}$ is a model tensor and let $\xi \in \Gamma(TM^{r,s})$ be the corresponding tensor field defined by P according to Proposition 1.2. Then, there is an injective vector bundle homomorphism*

$$\eta : T^*M \otimes \mathfrak{g}^\perp(P) \to T^*M \otimes TM^{r,s}$$

mapping the intrinsic torsion $\tau(P)$ to $\nabla^g\xi$.

PROOF. Let $\xi_0 \in V^{r,s}$ be a model tensor for ξ such that $G = \mathrm{Stab}_{\mathrm{SO}(p,q)}\xi_0$. We define the mapping

$$\tilde{\eta} : \mathfrak{so}(p,q) \to V^{r,s}, \tilde{\eta}(A) = A.\xi_0$$

where the dot denotes the induced action of $\mathfrak{so}(p,q)$ on $V^{r,s}$. The kernel of $\tilde{\eta}$ is exactly $\mathfrak{g}$. Moreover, let $\bar{\nabla}$ be the minimal G-connection of P. Then, the intrinsic torsion satisfies

$$\tau_X\xi = (\bar{\nabla} - \nabla^g)_X\xi = -\nabla^g_X\xi$$

by Corollary 3.4. The existence of the homomorphism η follows now immediately by extending the injective homomorphism $\tilde{\eta}|_{\mathfrak{g}^\perp}$ to the corresponding bundles. □

A standard way to classify G-structures on pseudo-Riemannian manifolds is to decompose the G-module $V^* \otimes \mathfrak{g}^\perp$ into irreducible components and define subclasses according to the vanishing of the components of the intrinsic torsion. If it is possible to define the G-structure by a tensor field $\xi \in \Gamma(TM^{r,s})$, it is equivalent, in view of the previous proposition, to decompose the subspace of $V^{r,s}$ with the same symmetries as $\nabla\xi$ and define the classes according to the vanishing of the components of $\nabla\xi$. In order to complete the classification, an example has to be constructed or the non-existence of examples has to be proven for each class.

3.1. Classification of G_2-structures. We illustrate this method of classification by reviewing the story of G_2-structures. By the considerations in chapter 1, section 3, a G_2-structure is defined by a global three-form φ which is everywhere stable and induces a Riemannian metric.

The decomposition of the subspace $\mathcal{X}$ of $V^* \otimes \Lambda^3V^*$ with the same symmetries as $\nabla^g\varphi$ into irreducible components

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \tag{3.1}$$

was first established in [**FG**]. Equivalent characterisations of the 16 resulting classes of G_2-manifolds, only in terms of $d\varphi$ and $d *_\varphi \varphi$, are given in [**MC1**]. The construction of examples was established by different authors, the final step, containing references to the previous work, is [**MMS**].

After all, the most important class remains the class of parallel G_2-structures or, equivalently, Riemannian manifolds with holonomy contained in G_2. As it is well-known, the existence of Riemannian manifolds with holonomy *equal* to G_2 was an open question for more than 30 years after G_2 appeared in Berger's list of possible Riemannian holonomy groups in 1955. The affirmative answer to this question was given first in [**Br1**], the first complete example is constructed in [**BrSa**] and the first compact example in [**Joy1**].

The non-compact form G_2^* is characterised by a stable three-form φ lying in the second open orbit of $\mathrm{GL}(7,\mathbb{R})$, see section 3 in chapter 1. Since the decomposition (3.1) is defined in terms of the invariant metric g_φ and the cross product $\times_\varphi$, which also exist for G_2^*, it can be defined literally for the corresponding G_2^*-module. Note that G_2 and G_2^* have the same complexification. Thus, by the representation theory argument given in section 2, it is easy to deduce the irreducibility of the G_2^*-module from that of the G_2-module. The same arguments applies to the characterisation in terms of $d\varphi$ and $d *_\varphi \varphi$ given in [**MC1**]. Of course, it remains to construct examples for all classes, if a classification is desired. We remark that the characterisation of torsionfree $\mathrm{G}_2^{(*)}$-structure by the property $d\varphi = d * \varphi = 0$ is already proved in [**G1**, Theorem 4.1] for both the compact and the non-compact form.

Since G_2- and G_2^*-structures constitute only a minor aspect of this thesis, we do not go into detail. The only classes we are interested in are the following which we define without reviewing the definition of all classes $\mathcal{X}_i$. We will use the notation $\mathrm{G}_2^{(*)}$ as a shorthand for "G_2 or G_2^*".

Definition 3.7. Let M be a real seven-manifold with a $\mathrm{G}_2^{(*)}$-structure φ.

(i) The $\mathrm{G}_2^{(*)}$-structure φ is called *nearly parallel* if
$$d\varphi = \mu *_\varphi \varphi$$
for a constant $\mu \in \mathbb{R}^*$.

(ii) The $\mathrm{G}_2^{(*)}$-structure φ is called *cocalibrated* if
$$d *_\varphi \varphi = 0.$$

Notice that a nearly parallel G_2-structure belongs to the class $\mathcal{X}_1$ and a cocalibrated one to the class $\mathcal{X}_1 \oplus \mathcal{X}_3$.

In the following chapter, we discuss the decomposition of the intrinsic torsion for G-structures with $G = \mathrm{U}(p,q)$ and $G = \mathrm{SU}(p,q)$.

CHAPTER 3

Special ε-Hermitian geometry

1. Almost complex and almost para-complex geometry

We recall that an *almost para-complex structure* on a $2m$-dimensional manifold M is an endomorphism field squaring to the identity such that both eigendistributions (for the eigenvalues ± 1) are m-dimensional. For an introduction to para-complex geometry we refer to [**AMT**], [**CMMS**] or [**CFG**].

Already in the algebraic part, we introduced a unified language describing almost complex and para-complex geometry simultaneously. Of course, we will also use this language for the corresponding structures on manifolds.

Definition 1.1. An *almost ε-complex manifold* is a manifold M of dimension $n = 2m$ endowed with an *almost ε-complex structure* which is defined as an almost complex structure if $\varepsilon = -1$ and an almost para-complex structure if $\varepsilon = 1$.

The vector space model for an almost ε-complex structure has been discussed in section 1.1 of chapter 1. In particular, an almost ε-complex structure is essentially the same as a $\mathrm{GL}(m, \mathbb{C}_\varepsilon)$-structure. All notions and identities for the model structures extend pointwise to the corresponding bundles. In analogy to the standard convention $\Omega^k M = \Gamma(\Lambda^k(TM)^*)$, we define

$$\Omega^{r,s} M = \Gamma(\Lambda^r(TM^{1,0})^* \otimes \Lambda^s(TM^{0,1})^*)$$

as the space of ε-complex (r, s)-forms on a manifold. Correspondingly, the symbols $[\![\Omega^{r,s}]\!]$ and $[\Omega^{r,r}]$ denote the *real forms of type* $(r, s) + (s, r)$ respectively *real forms of type* (r, r) on a manifold. The projection of a $(r + s)$-form α onto $\Omega^{r,s} M$ is denoted by $\alpha^{r,s}$. In particular, we can decompose the exterior differential d such that

$$d = d^{1,-2} + d^{1,0} + d^{0,1} + d^{-2,1}$$

where $d^{x,y}\alpha := (d\alpha)^{r+x,s+y}$ for $\alpha \in \Omega^{r,s} M$.

Definition 1.2. The Nijenhuis tensor $N = N^J$ of an almost ε-complex structure J is defined as the skew-symmetric covariant tensor field satisfying

$$N(X, Y) = -\varepsilon[X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] \tag{1.1}$$

for all $X, Y \in \mathfrak{X}(M)$.

For both values of ε, the Nijenhuis tensor can also be written as

$$N(X, Y) = -(\nabla_{JX} J)\, Y + (\nabla_{JY} J)\, X + J(\nabla_X J)\, Y - J(\nabla_Y J)\, X \tag{1.2}$$

for any torsionfree connection ∇. Thus, a $\mathrm{GL}(m, \mathbb{C}_\varepsilon)$-structure is torsionfree if and only if the Nijenhuis tensor vanishes. It is easy to see that the Nijenhuis tensor vanishes if and only if $d^{1,-2} = 0$ for both values of ε.

By the well-known theorem of Newlander-Nirenberg, a torsionfree $\mathrm{GL}(m,\mathbb{C})$-structure is integrable, i.e. defines a holomorphic atlas on M. In the para-complex world, a torsionfree $\mathrm{GL}(m,C)$-structure is also integrable by the Frobenius theorem and defines a para-holomorphic atlas. The notions of para-holomorphic atlas and para-complex manifold are for instance explained in [**CMMS**]. In particular, we can identify ε-complex manifolds and (smooth, real) manifolds endowed with an integrable almost ε-complex structure.

As also explained in [**CMMS**], a para-complex k-form α on a para-complex manifold (M,J) is para-holomorphic if and only $\bar{\partial}\alpha := d^{0,1}\alpha = 0$, completely analogous to the well-known case of a holomorphic form on a complex manifold. Thus, we call an ε-complex k-form α on an ε-complex manifold ε-holomorphic if and only $d^{0,1}\alpha = 0$.

Moreover, considering the discussion of the algebraic models in section 1.2 of chapter 1, it is obvious that an $\mathrm{SL}(m,\mathbb{C}_\varepsilon)$-structure is defined by a global ε-complex m-form Ψ which is everywhere non-degenerate and decomposable. We call such an m-form Ψ an *ε-complex volume form* on the manifold M. By Proposition 1.4, an ε-complex volume form Ψ induces an ε-complex structure J_Ψ.

PROPOSITION 1.3. *Let P be an $\mathrm{SL}(m,\mathbb{C}_\varepsilon)$-structure on a manifold M^{2m} defined by an ε-complex volume form $\Psi = \psi_+ + \mathrm{i}_\varepsilon\psi_-$.*

(i) The induced ε-complex structure J_Ψ is integrable if and only if $(d\Psi)^{m-1,2} = 0$.
(ii) Moreover, the following assertions are equivalent:
  - *(a) P is integrable (i.e. flat).*
  - *(b) P is torsionfree.*
  - *(c) $d\Psi = 0$.*
  - *(d) $d\psi_+ = 0$ and $d\psi_- = 0$.*
  - *(e) J_Ψ is integrable and Ψ is ε-holomorphic.*

PROOF. (i) As we already explained, the almost ε-complex structure J is integrable if and only if $d^{-1,2} = 0$ or, equivalently, $(d\xi)^{0,2} = 0$ for all $(1,0)$-forms ξ. However, an ε-complex one-form ξ is of type $(1,0)$ if and only if $\xi \wedge \Psi = 0$. Thus, for all $(1,0)$-forms ξ, the vanishing of $d^{-1,2}(\xi \wedge \Psi)$ yields

$$(d\xi)^{0,2} \wedge \Psi = \xi \wedge (d\Psi)^{m-1,2}$$

and the first assertion follows since wedging by Ψ is injective.

(ii) The implications $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$ are obvious. The equivalence of (c) and (e) is a direct consequence of part (i) since $\bar{\partial}\Psi = d^{0,1}\Psi = (d\Psi)^{m,1}$. Now, the implication $(e) \Rightarrow (a)$ follows since, given an ε-holomorphic $(m,0)$-form on an ε-complex manifold, there are ε-complex coordinates $(\frac{\partial}{\partial z_i})$ such that $\Psi = dz_1 \wedge \ldots \wedge dz_m$. □

REMARK 1.4. This proposition becomes particularly interesting in dimension $2m$, m odd, when all the data is already encoded in the real part ψ_+ due to Proposition 1.5. We will apply this proposition on the six-manifold $S^3 \times S^3$ in chapter 5.

2. Almost pseudo-Hermitian and almost para-Hermitian geometry

Recall that an *almost para-Hermitian structure* consists of a neutral metric and an antiorthogonal almost para-complex structure.

DEFINITION 2.1. An *almost ε-Hermitian manifold* is a manifold M of dimension $n = 2m$ endowed with an *almost ε-Hermitian structure* (g, J) which consists of a pseudo-Riemannian metric and an endomorphism field J satisfying $J^2 = \varepsilon \mathrm{Id}$ and $J^*g = -\varepsilon\, g$. The non-degenerate two-form $\omega := g(.,J.)$ is called the *fundamental two-form*.

Again, we refer to the corresponding model structure in section 1.3 and remark that all considerations and identities extend pointwise to the structure on the manifold. In particular, an almost ε-Hermitian structure can be identified with a $\mathrm{U}^\varepsilon(p,q)$-structure.

In the following, let ∇ always denote the Levi-Civita connection of the metric g of an almost ε-Hermitian manifold (M, g, J, ω). Differentiating the almost ε-complex structure, its square and the fundamental two-form yields for both values of ε the formulas

$$\begin{aligned}(\nabla_X J)\, Y &= \nabla_X(JY) - J(\nabla_X Y),\\ (\nabla_X J)\, JY &= -J(\nabla_X J)\, Y,\\ g(\nabla_X J)YZ &= -(\nabla_X \omega)(Y, Z),\end{aligned} \tag{2.1}$$

for all vector fields X, Y, Z. Using these formulas, it is easy to show that for any almost ε-Hermitian manifold, the tensor A defined by

$$A(X, Y, Z) = g(\nabla_X J)YZ = -(\nabla_X \omega)(Y, Z)$$

has the symmetries

$$A(X, Y, Z) = -A(X, Z, Y), \tag{2.2}$$

$$A(X, Y, Z) = \varepsilon A(X, JY, JZ) \tag{2.3}$$

for all vector fields X, Y, Z.

When (V, g, J, ω) is a model space for an almost ε-Hermitian structure, let $\mathcal{W}$ denote the $\mathrm{U}^\varepsilon(p,q)$-module of tensors with the same symmetries as A, i.e.

$$\mathcal{W} = \{\xi \in V^* \otimes \Lambda^2 V^* \mid \xi(X, Y, Z) = \varepsilon \xi(X, JY, JZ)\}$$

As explained in the previous chapter, the decomposition of this $\mathrm{U}^\varepsilon(p,q)$-module into irreducible components leads to a classification of almost ε-Hermitian manifolds.

For the compact form $\mathrm{U}(m)$, it is a classical result of Gray and Hervella [**GH**] that $\mathcal{W}$ splits into four irreducible components denoted by

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

A short proof for the irreducibility of the summands is also given in [**FFS**] where in fact the isomorphic space $V^* \otimes \mathfrak{u}(m)^\perp$ is decomposed. The special case $m = 3$ is discussed explicitly in [**AFS**]. Since the definitions of the $\mathcal{W}_i$ can be extended to the non-compact forms $\mathrm{U}(p,q)$, the application of the arguments given in section 2 yields that the analogous classes are irreducible as well. Instead of recalling the original definition of the classes $\mathcal{W}_i$, we restrict ourselves to listing in each case an equivalent characterisation in table 1.

Although the classes $\mathcal{W}_i$ are well-defined also in the para-complex case, i.e. for the group $\mathrm{GL}(m, \mathbb{R})$, the spaces are not irreducible since already $V = V^+ \oplus V^-$ decomposes as $\mathrm{GL}(m,\mathbb{R})$-module. In fact, it is shown in [**GaMa**] that each of the spaces $\mathcal{W}_i$ splits into exactly two irreducible summands.

REMARK 2.2. A discussion of the Gray-Hervella classes in the general case of almost ε-Hermitian structures is also given in [**Ki**]. The decomposition is given very explicitly, however, the irreducibility is not proven.

By the following useful formula, the classes $\mathcal{W}_i$ are completely determined by the Nijenhuis tensor and the exterior derivative of ω.

LEMMA 2.3. *On an almost ε-Hermitian manifold, the identity*

$$2(\nabla_X\omega)(Y,Z) = d\omega(X,Y,Z) + \varepsilon d\omega(X,JY,JZ) + \varepsilon g(N(Y,Z),JX) \tag{2.4}$$

holds for all vector fields X,Y,Z.

PROOF. For $\varepsilon = -1$, g Riemannian, the formula is classical, see [**KN**] or [**Na3**] for different direct proofs. Both computations hold literally for pseudo-Riemannian metrics and with sign modifications for $\varepsilon = 1$. For $\varepsilon = 1$, an explicit proof of this identity is also given [**CMMS**]. $\square$

In particular, an almost ε-Hermitian manifold is torsionfree if and only if it is ε-Kähler, i.e. if J is integrable and ω is a symplectic form.

Class	Characterisation	Name
0	$d\omega = 0$, $N^J = 0$	pseudo-Kähler
$\mathcal{W}_1$	∇J skew	nearly pseudo-Kähler
$\mathcal{W}_2$	$d\omega = 0$	almost pseudo-Kähler
$\mathcal{W}_3$	$N^J = 0$, $d\omega \in [\![\Omega_0^{2,1}]\!]$	
$\mathcal{W}_4$	$N^J = 0$, $d\omega = 2\theta \wedge \omega$	locally conformally pseudo-Kähler
$\mathcal{W}_1 \oplus \mathcal{W}_2$	$d\omega^{2,1} = 0$	quasi pseudo-Kähler
$\mathcal{W}_1 \oplus \mathcal{W}_3$	N^J skew, $d\omega \wedge \omega^{m-2} = 0$	
$\mathcal{W}_1 \oplus \mathcal{W}_4$	N^J skew, $d\omega = 2\theta \wedge \omega$	loc. conf. nearly pseudo-Kähler
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$d\omega \in [\![\Omega_0^{2,1}]\!]$	
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$d\omega = 2\theta \wedge \omega$	loc. conf. almost pseudo-Kähler
$\mathcal{W}_3 \oplus \mathcal{W}_4$	$N^J = 0$	pseudo-Hermitian
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$d\omega \wedge \omega^{m-2} = 0$	semi pseudo-Kähler
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$d\omega_0^{2,1} = 0$	loc. conf. quasi pseudo-Kähler
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	N^J skew	$\mathcal{G}_1$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d\omega^{3,0} = 0$	$\mathcal{G}_2$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	–	generic

TABLE 1. Gray-Hervella classes extended to almost pseudo-Hermitian structures, $\dim(M) = 2m$, $m \geq 3$

We discuss some properties of the following two interesting classes. Recall that a tensor field $B \in \Gamma((TM^*)^{\otimes 2} \otimes TM)$ is called totally skew-symmetric if the tensor $g(B(X,Y),Z)$ is a three-form. Moreover, we use the term *ε-Hermitian connection* on an almost ε-Hermitian manifold synonymous with $\mathrm{U}^{\varepsilon}(p,q)$-connection, i.e. a connection $\bar{\nabla}$ with $\bar{\nabla} J = \bar{\nabla} g = 0$.

DEFINITION 2.4. Let (M^{2m}, g, J, ω) be an almost ε-Hermitian manifold.

(i) The manifold M is called a *nearly ε-Kähler manifold* if its Levi-Civita connection ∇ satisfies the *nearly ε-Kähler condition*
$$(\nabla_X J)\, X = 0, \quad \forall X \in \Gamma(TM).$$
A nearly ε-Kähler manifold is called *strict* if $\nabla_X J \neq 0$ for all non-trivial vector fields X.

(ii) The manifold M is defined to be of type $\mathcal{G}_1$ if it admits an ε-Hermitian connection with skew-symmetric torsion.

As these structures are mainly studied for $\varepsilon = -1$ and g Riemannian, we explicitly prove for both cases a basic characterisation, which is well-known in the Riemannian context, in the more general setting of almost ε-Hermitian structures.

PROPOSITION 2.5. *An almost ε-Hermitian manifold (M^{2m}, g, J, ω) satisfies the nearly ε-Kähler condition if and only if $d\omega$ is of real type $(3,0)+(0,3)$ and the Nijenhuis tensor is totally skew-symmetric.*

PROOF. First of all, the nearly ε-Kähler condition is satisfied if and only if the tensor $A = -\nabla\omega$ is a three-form because of the antisymmetry (2.2).

Assume now that (g, J, ω) is a nearly ε-Kähler structure. A particular case of the identity (1.2), ch. 1, is the characterisation

$$[\![\Omega^{3,0}]\!] = \{\alpha \in \Omega^3 M \mid \alpha(X,Y,Z) = \varepsilon\alpha(X,JY,JZ)\} \tag{2.5}$$

of real forms of type $(3,0)+(0,3)$. Thus, the real three-form A is of type $(3,0)+(0,3)$ because of (2.3). Furthermore, since $d\omega$ is the alternation of $\nabla\omega$, we have

$$d\omega = 3\nabla\omega = -3A \in [\![\Omega^{3,0}]\!]. \tag{2.6}$$

If we apply the nearly ε-Kähler condition to the expression (1.2), the Nijenhuis tensor of a nearly ε-Kähler structure simplifies to

$$N(X,Y) = 4\, J(\nabla_X J)\, Y. \tag{2.7}$$

We conclude that the Nijenhuis tensor is skew-symmetric since

$$g(N(X,Y),Z) = -4A(X,Y,JZ) \overset{(2.3)}{=} -4\varepsilon J^* A(X,Y,Z). \tag{2.8}$$

The converse follows immediately from the identity (2.4) when considering (2.5). In order to be self-contained, we give a direct proof. Assume that $d\omega \in [\![\Omega^{3,0}]\!]$ and the Nijenhuis tensor is skew-symmetric. To begin with, we observe that
$$(\nabla_Y \omega)\,(X,X) = 0 = (\nabla_{JY}\omega)\,(X,JX)$$
by (2.2) and (2.3). With this identity, we have on the one hand
$$0 = \varepsilon g(N(JX,JY),JX) = g(N(X,Y),JX)$$

$$\begin{aligned}
&\overset{(1.2)}{=} -g((\nabla_{JX}J)\,Y, JX) + g((\nabla_{JY}J)\,X, JX) + g(J(\nabla_X J)\,Y, JX) - g(J(\nabla_Y J)\,X, JX)\\
&\overset{(2.1)}{=} (\nabla_{JX}\omega)\,(Y, JX) + \varepsilon(\nabla_X\omega)\,(Y, X)\\
&\overset{(2.2)}{=} (\nabla_{JX}\omega)\,(Y, JX) - \varepsilon(\nabla_X\omega)\,(X, Y),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
0 &= \varepsilon d\omega(X, X, Y) \overset{(2.5)}{=} d\omega(X, JX, JY)\\
&= (\nabla_X\omega)(JX, JY) + (\nabla_{JX}\omega)(JY, X) + (\nabla_{JY}\omega)(X, JX)\\
&\overset{(2.3)}{=} \varepsilon(\nabla_X\omega)(X, Y) + (\nabla_{JX}\omega)(Y, JX).
\end{aligned}$$

It follows that $(\nabla_X\omega)\,(X, Y) = 0$ which is equivalent to the nearly ε-Kähler condition. □

PROPOSITION 2.6. *An almost ε-Hermitian manifold (M^{2m}, g, J, ω) admits an ε-Hermitian connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor is totally skew-symmetric. If this is the case, the connection $\bar\nabla$ and its torsion T are uniquely defined by*

$$\begin{aligned}
g(\bar\nabla_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2}g(T(X, Y), Z),\\
g(T(X, Y), Z) &= \varepsilon g(N(X, Y), Z) - d\omega(JX, JY, JZ).
\end{aligned}$$

PROOF. The Riemannian case is proved in [**FI**], the para-complex case in [**IZ**]. In fact, the sketched proof in [**FI**] holds literally for the almost pseudo-Hermitian case with indefinite signature as well. For completeness, we give a direct proof for all cases simultaneously.

Note that a connection $\bar\nabla$ is ε-Hermitian if and only if $\bar\nabla J = 0$ and $\bar\nabla g = 0$. Let $T(X, Y) = \bar\nabla_X Y - \bar\nabla_Y X - [X, Y] = S_X Y - S_Y X$ be the totally skew-symmetric torsion of an ε-Hermitian connection $\bar\nabla$ where $S_X Y = \bar\nabla_X Y - \nabla_X Y$ is the difference tensor with respect to the Levi-Civita connection ∇ of g. Then, the Nijenhuis tensor is totally skew-symmetric as well, since we have

$$\begin{aligned}
g(N(X, Y), Z) = \varepsilon g(T(X, Y), Z) &+ g(T(JX, JY), Z)\\
&+ g(T(JX, Y), JZ) + g(T(X, JY), JZ),
\end{aligned} \tag{2.9}$$

using only $\bar\nabla J = 0$. Moreover, the difference tensor S_X is skew-symmetric with respect to g, for $\bar\nabla g = 0$. Combining this fact with the total skew-symmetry of the torsion, we find that $S_X Y = -S_Y X$ and consequently

$$g(\bar\nabla_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}g(T(X, Y), Z).$$

With this identity and $\bar\nabla\omega = 0$, the equation

$$2\nabla_{JX}\omega(Y, Z) = g(T(JX, Y), JZ) + g(T(JX, JY), Z) \tag{2.10}$$

follows. Finally, we verify the claimed formula for the torsion:

$$\begin{aligned}
d\omega(JX, JY, JZ) &\overset{(2.3)}{=} \varepsilon(\nabla_{JX}\omega(Y, Z) + \nabla_{JY}\omega(Z, X) + \nabla_{JZ}\omega(X, Y))\\
&\overset{(2.10)}{=} \varepsilon(g(T(JX, JY), Z) + g(T(JX, Y), JZ) + g(T(X, JY), JZ))\\
&\overset{(2.9)}{=} \varepsilon g(N(X, Y), Z) - g(T(X, Y), Z).
\end{aligned}$$

Conversely, if the Nijenhuis tensor is skew-symmetric, it is straightforward to verify that the defined connection is ε-Hermitian with skew-symmetric torsion. $\square$

3. Almost special pseudo- and para-Hermitian geometry

In this section, we discuss the G-structures for the groups $\mathrm{SU}^\varepsilon(p,q)$, $p+q=m$, which have been introduced in section 1.4. Again, all properties of the model structures extend pointwise to the corresponding structures on a manifold.

DEFINITION 3.1. An *almost special ε-Hermitian manifold* is a $2m$-dimensional manifold M endowed with an *almost special ε-Hermitian structure* which consists of an almost ε-Hermitian structure (g, J, ω) and an ε-complex volume form Ψ of constant length. An almost special ε-Hermitian structure is called *normalised* if it is normalised pointwise in the sense of Definition 1.14 of chapter 1.

REMARK 3.2. By rescaling Ψ by a constant, every almost special ε-Hermitian manifold can be normalised. Obviously, the properties of the structure do not change under this transformation. In contrast, the property of constant length is important. Although every ε-complex volume form could be normalised by multiplication by a function, the exterior derivative of Ψ would be changed under this transformation. However, as we will shortly see, the intrinsic torsion of the corresponding $\mathrm{SU}^\varepsilon(p,q)$-structure is encoded in the exterior derivative of Ψ and ω.

An important consequence of the considerations in section 1.4 of the algebraic preliminaries is the fact, that an almost special ε-Hermitian structure is completely determined by the two-form ω and the three-form Ψ. More precisely, given a non-degenerate two-form ω and a decomposable ε-complex m-form $\Psi = \psi_+ + i_\varepsilon \psi_-$ satisfying $\omega \wedge \Psi = 0$ and $\phi(\Psi) = c\phi(\omega)$ for a constant $c \in \mathbb{R}^*$, the pair (ω, Ψ) extends to a unique almost special ε-Hermitian structure (g, J, ω, Ψ). If $m = 2l-1$ is odd, even the two-form ω and the real part ψ_+ of Ψ suffice. For dimension six, this can be formulated very elegantly using the stable form formalism introduced in section 3 of chapter 1.

We call a differential form φ on a manifold *stable* if and only φ_p is stable for all $p \in M$. In particular, we call a pair $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ of stable forms on a six-manifold M *compatible* if

$$\omega \wedge \rho = 0 \qquad \Longleftrightarrow \qquad \omega(J_\rho ., .) = -\omega(., .J_\rho)$$

and *normalised* if

$$\phi(\rho) = 2\,\phi(\omega) \qquad \Longleftrightarrow \qquad J_\rho^* \rho \wedge \rho = \frac{2}{3}\,\omega^3. \tag{3.1}$$

As a special case of Proposition 1.13 of chapter 1, applied to every tangent space of a six-manifold M, a compatible and normalised pair $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ induces an almost ε-complex structure $J = J_\rho$ and a compatible pseudo-Riemannian metric $g = g_{(\omega,\rho)}$. As this construction is very important for the following parts of this thesis, we summarise the correspondences in dimension six in the following proposition.

PROPOSITION 3.3. *Let M be a six-manifold.*

(i) There is a one-to-one correspondence between normalised $\mathrm{SU}(3)$*-structures and compatible and normalised pairs $(\omega, \psi_+) \in \Omega^2 M \times \Omega^3 M$ of stable forms inducing a Riemannian metric.*

(ii) There is a one-to-one correspondence between normalised $\mathrm{SU}(1,2)$*-structures and compatible and normalised pairs $(\omega, \psi_+) \in \Omega^2 M \times \Omega^3 M$ of stable forms inducing a metric of signature* $(2,4)$.

(iii) There is a one-to-one correspondence between normalised $\mathrm{SL}(3,\mathbb{R})$*-structures with spacelike para-complex volume form Ψ and compatible and normalised pairs $(\omega, \psi_+) \in \Omega^2 M \times \Omega^3 M$ of stable forms with $\lambda(\psi_+) > 0$.*

Notice that the ε-complex volume form Ψ is spacelike in all three cases which can always be achieved in dimension six (and more generally in dimension $2m$, m odd) by multiplying the metric by -1.

3.1. The intrinsic torsion of $\mathrm{SU}(p,q)^\varepsilon$**-structures.** For large dimension $2m \geq 8$, the Gray-Hervella classification of $\mathrm{U}(m)$-structures has been refined in [**MC2**] to $\mathrm{SU}(m)$-structures as follows. The $\mathrm{SU}(m)$-module modelling the intrinsic torsion of an $\mathrm{SU}(m)$-structure decomposes into five irreducible components:

$$V^* \otimes \mathfrak{su}(m)^\perp = V^* \otimes (\mathfrak{u}(m)^\perp \oplus \mathbb{R}) \cong \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.$$

Here, the spaces $\mathcal{W}_i$, $i = 1, \ldots, 4$, are isomorphic to the spaces in the Gray-Hervella classification corresponding to $\nabla^g \omega$ and the space $\mathcal{W}_5 \cong V^*$ is coming from the covariant derivative $\nabla^g \Psi$. In fact, it is also shown in [**MC2**] that the five components are completely determined by $d\omega$ and $d\psi_+$ as long as $m \geq 4$.

The case of dimension six is different and has been studied first in [**ChSa**] for Riemannian signature. Here, the irreducible decomposition is

$$V^* \otimes \mathfrak{su}(p,q)^\perp \cong \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.$$

It is also shown that the intrinsic torsion is in fact completely determined by the exterior derivatives $d\omega$, $d\psi_+$ and $d\psi_-$. The relation between the components $\mathcal{W}_i$ and the components of the three exterior derivatives $d\omega$, $d\psi_+$ and $d\psi_-$ can be found for Riemannian signature in [**MC2**], and, including very explicit expressions, in [**Han**]. By the arguments given in section 2, the decomposition into irreducible components both for large dimension and dimension six extends literally to the non-compact forms $\mathrm{SU}(p,q)$, $p + q = m$.

In the following, we focus on dimension six. Instead of considering the classical definition of the classes $\mathcal{W}_i$, we focus directly on the characterisation in terms of the exterior derivative and will in fact define the classes in terms of the irreducible components of the resulting three-form and the two-four forms.

We recall that the irreducible decomposition of the $\mathrm{SU}(p,q)$-modules $\Lambda^3 V^*$ and $\Lambda^4 V^*$ induce the decompositions

$$\Omega^3 M = \underbrace{\mathbb{R}\psi_+ \oplus \mathbb{R}\psi_-}_{(3,0)+(0,3)} \oplus \underbrace{[\![\Omega_0^{2,1} M]\!] \oplus \Omega^1 M \wedge \omega}_{(2,1)+(1,2)},$$

$$\Omega^4 M = \underbrace{\Omega^0 M \wedge \omega^2 \oplus [\![\Omega_0^{1,1} M]\!] \wedge \omega}_{(2,2)} \oplus \underbrace{\Omega^1 M \wedge \psi_+}_{(3,1)+(1,3)}.$$

Corresponding to this decomposition, we denote the components of the exterior derivatives of the defining forms as

$$\begin{aligned} d\omega &= \tilde{W}_1^-\psi_+ + \tilde{W}_1^+\psi_- + W_3 + W_4 \wedge \omega, \\ d\psi_+ &= W_1^+\omega^2 + W_2^+ \wedge \omega + W_5 \wedge \psi_+, \\ d\psi_- &= W_1^-\omega^2 + W_2^- \wedge \omega + \tilde{W}_5 \wedge \psi_+. \end{aligned} \tag{3.2}$$

where $W_1^+, W_1^- \in \Omega^0 M$, $W_4, W_5 \in \Omega^1 M$, $W_2^+, W_2^- \in \Omega_0^{1,1} M$ and $W_3 \in \Omega_0^{2,1} M$. In fact, the same decomposition, although not irreducible, can be considered in the para-complex context, $\varepsilon = 1$.

PROPOSITION 3.4. *For a normalised* $\mathrm{SU}(p,q)^\varepsilon$*-structure,* $p + q = 3$*, defined by a normalised and compatible pair* (ω, ψ_+)*, the components of* $d\omega$*,* $d\psi_+$ *and* $d\psi_-$ *are related as follows:*
(i) $\tilde{W}_1^+ = -\frac{3}{2} W_1^+$
(ii) $\tilde{W}_1^- = \frac{3}{2} W_1^-$
(iii) $\tilde{W}_5 = -\varepsilon J^* W_5$

PROOF. We generalise the arguments given in the article [**ChSa**] which contain these formulas for Riemannian signature. Since $\omega \wedge \psi_+ = 0$ and $\omega \wedge \psi_- = 0$, the first two identities follow immediately from the normalisation (3.1):

$$\begin{aligned} W_1^+\omega^3 &= d\psi_+ \wedge \omega = \psi_+ \wedge d\omega = \tilde{W}_1^+\psi_+ \wedge \psi_- = -\frac{2}{3}\tilde{W}_1^+\omega^3, \\ W_1^-\omega^3 &= d\psi_- \wedge \omega = \psi_- \wedge d\omega = \tilde{W}_1^-\psi_- \wedge \psi_+ = \frac{2}{3}\tilde{W}_1^-\omega^3. \end{aligned}$$

In order to see the third identity, we observe that the exterior derivative of the $(0,3)$-form $\bar{\Psi}$ has no $(3,1)$-component and we obtain $d\psi_+^{3,1} = \mathrm{i}_\varepsilon d\psi_-^{3,1}$. We claim that $d\psi_+^{3,1} = (W_5)^{0,1} \wedge \Psi$. Indeed, using the identity $\alpha \wedge \psi_- = -\varepsilon J^*\alpha \wedge \psi_+$, see (1.10) in chapter 1, it follows $W_5 \wedge \psi_+ = \mathrm{Re}(W_5^{0,1} \wedge \Psi)$ which implies the claim by definition of W_5. Using the analogous identity for $d\psi_-^{3,1}$, the assertion follows from

$$\tilde{W}_5^{0,1} \wedge \Psi = d\psi_-^{3,1} = \varepsilon \mathrm{i}_\varepsilon d\psi_+^{3,1} = \varepsilon \mathrm{i}_\varepsilon (W_5)^{0,1} \wedge \Psi = (-\varepsilon J^* W_5)^{0,1} \wedge \Psi,$$

since wedging by Ψ is injective. □

In table 2, we have summarised the irreducible components of the intrinsic torsion including the dimensions.

3.2. The Nijenhuis tensor of an $\mathrm{SU}^\varepsilon(p,q)$**-structure.** This section contains a lemma and ideas from [**SSH**].

As we have already discussed in Proposition 1.3, the Nijenhuis tensor of an $\mathrm{SU}^\varepsilon(p,q)$-structure (g, J, ω, Ψ), $p+q = 3$, vanishes if and only if the $(2,2)$-component of Ψ vanishes, i.e. if and only if $W_1^- = W_1^+ = W_2^- = W_2^+ = 0$. In the following, we show explicitly that $W_2^\pm$ corresponds to the skew-symmetric part of the Nijenhuis tensor for all possible signatures.

The following choice of local frames, although differing from the standard basis defined in chapter 1, has been used in [**SSH**] and seems to be convenient for the calculations in an ε-complex basis in this section and section 5.

Class	irreducible $\mathrm{SU}(p,q)$-module	Dimension
$\mathcal{W}_1^+$	$\Lambda^0 V^*$	1
$\mathcal{W}_1^-$	$\Lambda^0 V^*$	1
$\mathcal{W}_2^+$	$\Lambda_0^{1,1} V^*$	8
$\mathcal{W}_2^-$	$\Lambda_0^{1,1} V^*$	8
$\mathcal{W}_3$	$\Lambda_0^{2,1} V^*$	12
$\mathcal{W}_4$	$\Lambda^1 V^*$	6
$\mathcal{W}_5$	$\Lambda^1 V^*$	6

TABLE 2. Irreducible classes of $\mathrm{SU}(p,q)$-structures, $p+q=3$

Given an almost ε-Hermitian structure (g,J,ω) on a six-manifold M, we choose a local orthonormal frame $\{e_1,\dots,e_{2m}\}$ such that $Je_i = e_{i+m}$ for $i=1,\dots,m$ and

$$\omega = \varepsilon \sum_{i=1}^{m} \sigma_i e^{i(i+m)},$$

where $\sigma_i := g(e_i,e_i)$ for $i=1,\dots,m$. The corresponding local ε-complex frame of $TM^{1,0}$ is

$$E_i = e_i^{1,0} = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon \varepsilon J e_i) = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon \varepsilon e_{i+m})$$

such that the ε-Hermitian metric $g_{\mathbb{C}_\varepsilon}$ satisfies

$$g_{\mathbb{C}_\varepsilon}(E_i, E_j) = \frac{1}{2}\sigma_i \delta_{ij} \qquad \text{and} \qquad g_{\mathbb{C}_\varepsilon}(E_i, E_j) = 0$$

in such a frame. The dual frame $\{E^1, E^2, E^3\}$ of $(TM^{1,0})^*$ is given by

$$E^i := (e^i + \mathrm{i}_\varepsilon \varepsilon J e^i) = (e^i + \mathrm{i}_\varepsilon e^{i+m})$$

for $i=1,2,3$ and we call it an *ε-unitary frame of $(1,0)$-forms* in the following.

The following lemma explicitly relates the Nijenhuis tensor to the exterior differential. For $\varepsilon=-1$, it gives a characterisation of Bryant's notion of a quasi-integrable $\mathrm{U}(p,q)$-structure, $p+q=3$, in dimension six [**Br2**].

LEMMA 3.5. *The Nijenhuis tensor of an almost ε-Hermitian six-manifold (M^6,g,J,ω) is totally skew-symmetric if and only if there exists a local $\mathbb{C}_\varepsilon$-valued function λ for every local ε-unitary frame $\{E^1,E^2,E^3\}$ of $(1,0)$-forms such that*

$$(dE^{\tau(1)})^{0,2} = \lambda\, \sigma_{\tau(1)}\, E^{\overline{\tau(2)}\,\overline{\tau(3)}} \tag{3.3}$$

for all even permutations τ of $\{1,2,3\}$.

PROOF. First of all, for vector fields $V=V^{1,0}$, $W=W^{1,0}$ of type $(1,0)$, the identities

$$N(\bar V,\bar W) = -4\varepsilon[\bar V,\bar W]^{1,0} \quad \text{and} \quad N(V,\bar W)=0$$

follow immediately from the definition of N. Using the first identity, we compute in an arbitrary local ε-unitary frame

$$\begin{aligned} dE^i(\bar{E}_j, \bar{E}_k) &= -E^i([\bar{E}_j, \bar{E}_k]) = -2\sigma_i\, g_{\mathbb{C}_\varepsilon}([\bar{E}_j, \bar{E}_k], E_i) \\ &= -2\sigma_i\, g_{\mathbb{C}_\varepsilon}([\bar{E}_j, \bar{E}_k]^{1,0}, E_i) = \frac{1}{2}\varepsilon\, \sigma_i\, g_{\mathbb{C}_\varepsilon}(N(\bar{E}_j, \bar{E}_k), E_i) \end{aligned}$$

for all possible indices $1 \leq i, j, k \leq 3$. If the Nijenhuis tensor is totally skew-symmetric, equation (3.3) follows by setting

$$\lambda = \frac{1}{2}\varepsilon\, g_{\mathbb{C}_\varepsilon}(N(\bar{E}_1, \bar{E}_2), E_3). \tag{3.4}$$

Conversely, the assumption (3.3) for every local ε-unitary frame implies that the Nijenhuis tensor is everywhere a three-form when considering the same computation and $N(V, \bar{W}) = 0$. □

If there is an $\mathrm{SU}^\varepsilon(p,q)$-reduction defined by an ε-complex volume form Ψ of constant length, this characterisation can be reformulated globally in the following sense. Obviously, the $(3,0)$-form Ψ satisfies $\Psi = zE^{123}$, $z \in \mathbb{C}_\varepsilon$, $z\bar{z} \neq 0$, in an ε-unitary frame of $(1,0)$-forms.

Proposition 3.6. *The Nijenhuis tensor of an* $\mathrm{SU}^\varepsilon(p,q)$*-structure,* $p+q=3$*, is totally skew-symmetric if and only if* $W_2^+ = 0$ *and* $W_2^- = 0$.

Proof. By definition of the components W_i in (3.2), we have

$$(d\Psi)^{2,2} = (d\psi_+)^{2,2} + \mathrm{i}_\varepsilon (d\psi_-)^{2,2} = (W_1^+ + \mathrm{i}_\varepsilon W_1^-)\,\omega^2 + (W_2^+ + \mathrm{i}_\varepsilon W_2^-) \wedge \omega.$$

As it suffices to proof the assertion locally, we choose an ε-unitary frame of $(1,0)$-forms $\{E^i\}$ such that $\Psi = \psi_+ + \mathrm{i}_\varepsilon \psi_- = zE^{123}$ for $z \in \mathbb{C}_\varepsilon$. The fundamental two-form is

$$\omega = -\frac{1}{2}\mathrm{i}_\varepsilon \sum_{k=1}^{m} \sigma_k\, E^{k\bar{k}}$$

in this frame such that we have on the one hand,

$$\begin{aligned} \omega \wedge \omega &= \frac{1}{2}\varepsilon(\sigma_2\sigma_3\, E^{2\bar{2}3\bar{3}} + \sigma_1\sigma_3\, E^{1\bar{1}3\bar{3}} + \sigma_1\sigma_2\, E^{1\bar{1}2\bar{2}}) \\ &= -\frac{1}{2}\varepsilon\sigma_1\sigma_2\sigma_3(\sigma_1\, E^{\bar{2}\bar{3}23} + \sigma_2\, E^{\bar{3}\bar{1}31} + \sigma_3\, E^{\bar{1}\bar{2}12}), \end{aligned}$$

and on the other hand

$$(d\Psi)^{2,2} = a\, \left((dE^1)^{0,2} \wedge E^{23} + (dE^2)^{0,2} \wedge E^{31} + (dE^3)^{0,2} \wedge E^{12}\right).$$

Comparing these expressions and considering Lemma 3.5, the assertion is immediate. □

Corollary 3.7. *Let* (ω, ψ_+) *be an* $\mathrm{SU}^\varepsilon(p,q)$*-structure,* $p+q=3$*, with* $W_2^+ = W_2^- = 0$. *In an* ε*-unitary frame* $\{E^1, E^2, E^3\}$ *of* $(1,0)$*-forms such that* $\Psi = zE^{123}$*,* $z \in \mathbb{C}_\varepsilon$*, and* $\sigma_i = 2g_{\mathbb{C}_\varepsilon}(E_i, E_i)$*, the identity*

$$W_1^+ + \mathrm{i}_\varepsilon W_1^- = -\sigma_1\sigma_2\sigma_3\, z\, g_{\mathbb{C}_\varepsilon}(N(\bar{E}_1, \bar{E}_2), E_3). \tag{3.5}$$

holds.

Proof. The assertion follows immediately by comparing the formulas in the proofs of Proposition 3.6 and Lemma 3.5. □

4. Half-flat structures

Finally, we come to the structures we are actually interested in.

DEFINITION 4.1. An almost special ε-Hermitian six-manifold (M, ω, ψ_+) is called

(i) *half-flat* if $W_1^+ = W_2^+ = W_4 = W_5 = 0$, i.e.

$$d\psi_+ = 0, \quad d\omega^2 = 0,$$

(ii) and *nearly half-flat* if $W_2^+ = W_4 = W_5 = 0$, i.e.

$$d\psi_+ = W_1^+ \, \omega \wedge \omega, \quad W_1^+ \in \mathbb{R}.$$

In order to shorten the notation, we shall use the term *half-flat structure* as a synonym for half-flat almost special ε-Hermitian structure.

Notice that ω^2 is proportional to the Hodge dual of ω, see Lemma 4.2 of chapter 1, such that the second equation is satisfied if and only if ω is coclosed.

EXAMPLE 4.2. A torsionfree $\mathrm{SU}^\varepsilon(p,q)$-structure, in particular a Calabi-Yau three-fold, is half-flat. Another important class of examples for half-flat structures is given by strict nearly ε-Kähler six-manifolds with $\|\nabla J\| \neq 0$, which are to be discussed in the following section.

Half-flat SU(3)-structures were first considered in [**Hi1**] as the natural class which can be evolved to a parallel G_2-structure via the Hitchin flow. This is discussed and generalised in chapter 6.

The name "half-flat" has been introduced in [**ChSa**]. It is chosen since the definition requires the vanishing of exactly half the intrinsic torsion in terms of dimension. However, considering that a torsionfree G-structure is not necessarily flat but only 1-flat, the name half 1-flat structure or half-torsionfree structure would describe the properties of this class more precisely.

The class of nearly half-flat structures is introduced in [**FIMU**]. The name has been chosen since these structures can be evolved similarly as half-flat structures and the resulting G_2-structures are in fact nearly parallel.

A very interesting subclass is the class characterised by the following lemma which can be viewed as an intersection of the classes of half-flat and nearly half-flat structures.

LEMMA 4.3. *A half-flat* $\mathrm{SU}^\varepsilon(p,q)$*-structure* (ω, ψ_+) *has totally skew-symmetric Nijenhuis tensor if and only if* (ω, ψ_-) *is nearly half-flat.*

PROOF. Since $d\psi_+ = 0$ for a half-flat $\mathrm{SU}^\varepsilon(p,q)$-structure, the exterior derivative of ψ_- is

$$d\psi^- = W_1^- \omega^2 + W_2^- \wedge \omega,$$

$W_1^- \in \Omega^0 M$, $W_2^- \in \Omega_0^{1,1} M$. By Proposition 3.6, the Nijenhuis tensor is skew-symmetric if and only W_2^- vanishes, i.e. if and only if

$$d\psi^- = W_1^- \omega^2. \tag{4.1}$$

Differentiating this equation yields $dW_1^- \wedge \omega^2 = 0$ since $d\omega^2 = 0$ and thus, W_1^- has to be constant since wedging by ω^2 is an isomorphism on one-forms. $\square$

Being at the same time half-flat, nearly half-flat and $\mathcal{G}_1$ should be sufficient motivation to introduce a name for this class.

DEFINITION 4.4. An $\mathrm{SU}^\varepsilon(p,q)$-structure (ω,ψ_+) is called *double half-flat* if $W_1^+ = W_2^+ = W_2^- = W_4 = W_5 = 0$, i.e.

$$d\psi_+ = 0, \quad d\psi_- = W_1^- \,\omega\wedge\omega, \quad W_1^- \in \mathbb{R}.$$

In summary, double half-flat structures can be evolved to both parallel and nearly parallel G_2-structures and admit an ε-unitary connection with totally skew-symmetric torsion. Additionally, as shown in [**ChSw**], these structures also induce an invariant G_2-structures with torsion on $N \times S^1$.

In the same article, [**ChSw**], a classification of the six-dimensional nilmanifolds N admitting a left-invariant double half-flat SU(3)-structure is achieved. Since the result is that six nilmanifolds admit such a structure, we conclude that these structures are not as scarce as nearly Kähler manifolds.

EXAMPLE 4.5. We give an example, see [**SSH**], of a normalised left-invariant double half-flat SU(3)-structure on $S^3\times S^3$. In fact, this example is not contained in a subclass, i.e. both W_3 and W_1^- are non-zero. In a global frame of left-invariant vector fields $\{e_1,e_2,e_3,f_1,f_2,f_3\}$ on $S^3\times S^3$ such that

$$de^1 = e^{23}\,,\quad de^2 = e^{31}\,,\quad de^3 = e^{12}\quad,\quad df^1 = f^{23}\,,\quad df^2 = f^{31}\,,\quad df^3 = f^{12},$$

we define the structure with $x = 2+\sqrt{3}$ by

$$\begin{aligned}
\omega &= e^1f^1 + e^2f^2 + e^3f^3,\\
\psi_+ &= -\frac{1}{2}x^2e^{123} + 2xe^{12}f^3 - 2xe^{13}f^2 - 2xe^1f^{23} + 2xe^{23}f^1\\
&\quad + 2xe^2f^{13} - 2xe^3f^{12} + (4x-8)f^{123},\\
\psi_- &= \frac{1}{2}xe^{123} - 2e^1f^{23} + 2e^2f^{13} - 2e^3f^{12} + 4f^{123},\\
g &= x\,(e^1)^2 + x\,(e^2)^2 + x\,(e^3)^2 + 4\,(f^1)^2 + 4\,(f^2)^2 + 4\,(f^3)^2\\
&\quad - 2x\,e^1\cdot f^1 - 2x\,e^2\cdot f^2 - 2xe^3\cdot f^3.
\end{aligned}$$

We close the section by a remark on the curvature of half-flat SU(3)-structures. In fact, the Ricci curvature of an arbitrary SU(3)-structure is computed in terms of the components W_i in [**BV**]. For half-flat structures, the Ricci curvature is also derived independently in [**AC1**]. However, as the computations are very involved and as we will need the curvature only in very special cases, we do not attempt here to generalise the results involving ε's and indefinite signature.

5. Nearly pseudo-Kähler and nearly para-Kähler six-manifolds

The main objective of this section, which is based on [**SSH**], is to generalise the characterisation of six-dimensional nearly Kähler manifolds by an exterior differential system, see [**RC**], to nearly pseudo-Kähler and nearly para-Kähler manifolds.

First of all, there is a unique ε-Hermitian connection with skew-symmetric torsion T on a nearly ε-Kähler manifold (M^{2m}, J, g, ω) by Proposition 2.6. In this case, the skew-symmetric torsion T simplifies to

$$T(X,Y) = \varepsilon J(\nabla_X J)Y = \frac{1}{4}\varepsilon N(X,Y)$$

due to the identities (2.6), (2.7) and (2.8). We call this connection the canonical ε-Hermitian connection of a nearly ε-Kähler manifold.

Proposition 5.1. *The canonical ε-Hermitian connection $\bar{\nabla}$ of a nearly ε-Kähler manifold (M^{2m}, J, g, ω) satisfies*

$$\bar{\nabla}(\nabla J) = 0 \quad \textit{and} \quad \bar{\nabla}(T) = 0.$$

Proof. The two assertions are equivalent since $\bar{\nabla} J = 0$. A short proof of the first assertion for the Hermitian case is given in [**BM**]. This proof generalises without changes to the pseudo-Hermitian case since it essentially uses the identity

$$2g((\nabla^2_{W,X} J)Y, Z) = -\sigma_{X,Y,Z}\, g((\nabla_W J)X, (\nabla_Y J)JZ),$$

which was proved in [**G1**] for Riemannian metrics and also holds true in the pseudo-Riemannian setting ([**Ka1**, Proposition 7.1]). The para-Hermitian version is proved in [**IZ**, Theorem 5.3]. □

Corollary 5.2. *On a nearly ε-Kähler manifold (M^{2m}, J, g, ω), the tensors ∇J and $N = 4\varepsilon T$ have constant length.*

Proof. This is an obvious consequence of Proposition 5.1 since both tensors are parallel with respect to the connection $\bar{\nabla}$ which preserves in particular the metric. □

Remark 5.3. In dimension six, the fact that ∇J has constant length is usually expressed by the equivalent assertion that a nearly ε-Kähler six-manifold is of constant type, i. e. there is a constant $\kappa \in \mathbb{R}$ such that

$$g((\nabla_X J)\, Y, (\nabla_X J)\, Y) = \kappa \{\, g(X,X)g(Y,Y) - g(X,Y)^2 + \varepsilon g(JX,Y)^2 \,\}.$$

In fact, the constant is $\kappa = \frac{1}{4}\|\nabla J\|^2$. Furthermore, it is well-known in the Riemannian case that strict nearly Kähler six-manifolds are Einstein manifolds with Einstein constant 5κ [**G1**]. The same is true in the para-Hermitian case [**IZ**] and in the pseudo-Hermitian case [**Sch4**].

The case $\|\nabla J\|^2 = 0$ for a strict nearly ε-Kähler six-manifold can only occur in the para-complex world. We give different characterisations of such structures which establish an obvious break in the analogy of nearly para-Kähler and nearly pseudo-Kähler manifolds. We recall from chapter 1 that, for $\varepsilon = 1$, there is a decomposition

$$[\![\Lambda^{3,0}(TM)^*]\!] \cong \Lambda^3(TM^+)^* \oplus \Lambda^3(TM^-)^*, . \tag{5.1}$$

In particular, the class $\mathcal{W}_1$ characterising nearly para-Kähler manifolds splits into two subclasses already as a $\mathrm{GL}(m,\mathbb{R})$-structure.

PROPOSITION 5.4. *The following assertions are equivalent on a six-dimensional strict nearly para-Kähler manifold* (M^6, g, J, ω)*:*

(i) $\|\nabla J\|^2 = \|A\|^2 = 0$
(ii) The three-form $A = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ *is either in* $\Gamma(\Lambda^3(TM^+)^*)$ *or in* $\Gamma(\Lambda^3(TM^-)^*)$.
(iii) The three-form $A = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ *is not stable.*
(iv) The metric g *is Ricci-flat.*

PROOF. Let (M^6, g, J, ω) be a strict nearly para-Kähler manifold. First of all, notice that $\|\nabla J\|^2 = \|A\|^2$ by (2.1) and that the three-form $A = -\nabla\omega = -\frac{1}{3}d\omega$ is of type $(3,0) + (0,3)$ by Proposition 2.5. Moreover, A vanishes nowhere by the definition of strictness.

A local proof suffices and we choose a local frame $\{e_1, \dots, e_6\}$ of eigenvectors of the para-complex structure J, e_1, e_2, e_3 for the eigenvalue $+1$, e_4, e_5, e_6 for the eigenvalue -1, such that the only non-vanishing components of the metric are given by $g(e_1, e_4) = g(e_2, e_5) = g(e_3, e_6) = 1$. According to the decomposition (5.1) of $[\![\Lambda^{3,0}(TM)^*]\!]$, there are local functions a and b such that $A = ae^{123} + be^{456}$. Due to the simple form of the metric in the chosen frame, it is easily verified, that $\|A\|^2 = g(A, A) = 2ab$. The first two assertions are thus both equivalent to $a = 0$ or $b = 0$. The third assertion is also equivalent to $a = 0$ or $b = 0$ by Proposition 3.5 of chapter 1. Finally, assertions (i) and (iv) are equivalent by [**IZ**, Theorem 5.5]. $\square$

Flat strict nearly para-Kähler manifolds (M, g, J, ω) are classified in [**CS2**]. It turns out that they always satisfy $\|\nabla J\|^2 = 0$. In [**GaMa**], almost para-Hermitian structures on tangent bundles TN of real three-dimensional manifolds N^3 are discussed. It is shown that the existence of nearly para-Kähler manifolds satisfying the second condition of Proposition 5.4 is equivalent to the existence of a certain connection on N^3. However, to the author's best knowledge, there does not exist a reference for an example of a Ricci-flat nearly para-Kähler structure which is not flat.

Finally, we come to the characterisation of six-dimensional nearly ε-Kähler manifolds by an exterior differential system generalising the classical result of [**RC**] which holds for $\varepsilon = -1$ and Riemannian metrics.

THEOREM 5.5. *Let* (M, g, J, ω) *be an almost* ε*-Hermitian six-manifold. Then* M *is a strict nearly* ε*-Kähler manifold with* $\|\nabla J\|^2 \neq 0$ *if and only if there is a reduction* $\Psi = \psi_+ + i_\varepsilon \psi_-$ *to* $\mathrm{SU}(p,q)^\varepsilon$ *which satisfies*

$$d\omega = 3\,\psi_+, \tag{5.2}$$

$$d\psi_- = 2\,\kappa\,\omega \wedge \omega, \tag{5.3}$$

where $\kappa := \frac{1}{4}\|\nabla J\|^2 = \frac{1}{2}W_1^-$ *is constant and non-zero. The corresponding pair* (ω, ψ_+) *of stable forms is normalised if and only if* $\kappa = 1$.

REMARK 5.6. Due to our sign convention $\omega = g(.,J.)$, the constant κ is positive in the Riemannian case and the second equation differs from that of other authors by a sign.

Proof. By Proposition 2.5, the manifold M is nearly ε-Kähler if and only if $d\omega$ is of type $(3,0)+(0,3)$ and the Nijenhuis tensor is totally skew-symmetric.

First, we assume that (g,J,ω) is strict nearly ε-Kähler such that $\|A\|^2 = \|\nabla J\|^2 \neq 0$. As $A = -\frac{1}{3}d\omega$ has constant length by Corollary 5.2, we can define a reduction $\Psi = \psi_+ + i_\varepsilon \psi_-$ by $\psi_+ = -A$ and $\psi_- = J^*\psi_+$ such that the first equation is satisfied. In fact, this reduction is half-flat, since ψ^+ is closed by definition and $d(\omega\wedge\omega) = 6\psi_+\wedge\omega = 0$. Moreover, it holds $W_2^- = 0$ as a consequence of Proposition 3.6. Considering also Proposition 3.4, the only non-vanishing component of the intrinsic torsion is W_1^- and the second equation is satisfied for $\kappa = \frac{1}{2}W_1^-$. Since $d\omega^2 = 0$, we obtain $d\kappa = 0$ and κ has to be constant.

It remains to prove $W_1^- = \frac{1}{2}\|\nabla J\|^2$. We want to apply Corollary 3.7 and choose an ε-unitary local frame such that

$$\Psi = -A - i_\varepsilon J^*A = zE^{123},$$

where $z \in \mathbb{C}_\varepsilon$ is constant. Since $g_{\mathbb{C}_\varepsilon}(E^i, E^i) = 2\sigma_i$, we have $4\kappa = \|\nabla J\|^2 = \|\psi_+\|^2 = \frac{1}{2}g_{\mathbb{C}_\varepsilon}(\Psi,\Psi) = 4\sigma_1\sigma_2\sigma_3 z\bar{z}$. Comparing this with

$$i_\varepsilon W_1^- \overset{(3.5)}{=} -\sigma_1\sigma_2\sigma_3 z g_{\mathbb{C}_\varepsilon}(N(\bar{E}_1,\bar{E}_2),E_3) \overset{(2.8)}{=} 4\varepsilon\sigma_1\sigma_2\sigma_3 z J^*A(\bar{E}_1,\bar{E}_2,\bar{E}_3) = 2\sigma_1\sigma_2\sigma_3 i_\varepsilon z\bar{z},$$

the assertion $W_1^- = \frac{1}{2}\|\nabla J\|^2$ follows.

Conversely, if a given $\mathrm{SU}(p,q)^\varepsilon$-structure satisfies the exterior system, the real three-form ψ_+ is obviously closed and the Nijenhuis tensor is totally skew-symmetric by Proposition 3.6. Considering that $d\omega = 3\nabla\omega$ is of type $(3,0)+(0,3)$ by the first equation, the structure is nearly ε-Kähler. Since $A = -\psi_+$ is stable, the structure is even strict nearly ε-Kähler and, by Proposition 5.4, $\|\nabla J\| = \|A\| \neq 0$. Now, the computation of the constants just carried out shows that in fact $\|\nabla J\| = 4\kappa$.

The additional assertion that the pair (ω,ψ_+) is normalised if and only if $\kappa = 1$ is immediate from Proposition 3.4. □

6. Automorphism groups of $\mathrm{SU}^\varepsilon(p,q)$-structures

This section is based on [**SSH**] as well.

An automorphism of an $\mathrm{SU}^\varepsilon(p,q)$-structure on a six-manifold M is an automorphism of principal fibre bundles or equivalently, a diffeomorphism of M preserving all tensors defining the $\mathrm{SU}^\varepsilon(p,q)$-structure. Since a normalised $\mathrm{SU}^\varepsilon(p,q)$-structure is characterised by a pair of compatible and normalised stable forms $(\omega,\rho) \in \Omega^2 M \times \Omega^3 M$ and since the construction of the remaining tensors J, ψ_- and g is invariant, a diffeomorphism preserving the two stable forms is already an automorphism of the $\mathrm{SU}^\varepsilon(p,q)$-structure and in particular an isometry.

This easy observation has the following consequences when combined with the exterior systems of the previous section and the naturality of the exterior derivative.

Proposition 6.1. *Let M be a six-manifold with an $\mathrm{SU}^\varepsilon(p,q)$-structure (ω,ρ).*

(i) If the exterior differential equation

$$d\omega = \mu\, \rho$$

is satisfied for a constant $\mu \neq 0$, then a diffeomorphism Φ of M preserving the two-form ω is an automorphism of the $\mathrm{SU}^\varepsilon(p,q)$-structure and in particular an isometry.

(ii) If the exterior differential equation

$$d(J_\rho^* \rho) = \nu\, \omega \wedge \omega$$

is satisfied for a constant $\nu \neq 0$, then a diffeomorphism Φ of M preserving

(a) the orientation and the three-form ρ,

(b) or the orientation and the three-form $J_\rho^*\rho$,

(c) or the ε-complex volume form $\Psi = \rho + \mathrm{i}_\varepsilon J_\rho^* \rho$,

is an automorphism of the $\mathrm{SU}^\varepsilon(p,q)$-structure and in particular an isometry.

We like to emphasise that both parts of the Proposition apply to strict nearly ε-Kähler structures with $\|\nabla J\|^2 \neq 0$.

Conversely, it is known for *complete* Riemannian nearly Kähler manifolds, that orientation-preserving isometries are automorphism of the almost Hermitian structure except for the round sphere S^6, see for instance [**Bu2**, Proposition 4.1]. However, this is not true if the metric is incomplete. In [**FIMU**, Theorem 3.6], a nearly Kähler structure is constructed on the incomplete sine-cone over a Sasaki-Einstein five-manifold $(N^5, \eta, \omega_1, \omega_2, \omega_3)$. In fact, the Reeb vector field dual to the one-form η is a Killing vector field which does not preserve ω_2 and ω_3. Thus, by the formulae given in [**FIMU**], the lift of this vector field to the nearly Kähler six-manifold is a Killing field for the sine-cone metric which does neither preserve Ψ nor ω nor J.

CHAPTER 4

Classification results for Lie groups admitting half-flat structures

This chapter deals with the existence of left-invariant half-flat structures on Lie groups. All results of this chapter are contained in [**SH**].

An $\mathrm{SU}^{\varepsilon}(p,q)$-structure (g, J, ω, Ψ) on a Lie group G is left-invariant if all defining tensors are invariant under the group multiplication from the left. When identifying the Lie algebra $\mathfrak{g}$ of G with the Lie algebra of left-invariant vector fields, a left-invariant $\mathrm{SU}^{\varepsilon}(p,q)$-structure is equivalently defined by a compatible and normalised pair $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ of stable forms on the Lie algebra.

Due to the formula

$$d\alpha(X,Y) = -\alpha([X,Y]), \quad \alpha \in \mathfrak{g}^*\,,\ X, Y \in \mathfrak{g},$$

the exterior derivative of G restricted to left-invariant one-forms contains the same information as the Lie bracket of $\mathfrak{g}$. In particular, an exterior system for left-invariant tensors on G reduces to a system of algebraic equations on $\mathfrak{g}$. Notice that the Jacobi identity is equivalent to $d^2 = 0$.

Let a *half-flat structure on a Lie algebra* $\mathfrak{g}$ be defined as a pair $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ of compatible stable forms which satisfy

$$d\rho = 0\,, \qquad d\omega^2 = 0.$$

Thus, the study of left-invariant half-flat structures on Lie groups reduces to studying half-flat structures on Lie algebras. As these structures are defined by stable forms on vector spaces, the normalisation can be ignored when considering the existence question.

Before we prove our main classification results concerning half-flat structures on direct sums of three-dimensional Lie algebras, we review the classification of three-dimensional Lie algebras.

1. Three-dimensional Lie algebras

Recall that a Lie algebra $\mathfrak{g}$ is called *unimodular* if the trace of the adjoint representation ad_{X} vanishes for all $X \in \mathfrak{g}$.

LEMMA 1.1. *The following conditions are equivalent for an n-dimensional Lie algebra.*

(i) $\mathfrak{g}$ *is unimodular*
(ii) All $(n-1)$-forms on $\mathfrak{g}$ are closed.
(iii) Let $\{c_{ij}^k\}$ denote the structure constants with respect to a basis $\{\mathrm{e}^i\}$ of $\mathfrak{g}^*$ which are defined by $d\mathrm{e}^k = \sum_{i<j} c_{ij}^k \mathrm{e}^{ij}$. Then, it holds $\sum_{k=1}^n c_{k,m}^k = 0$ for $1 \le m \le n$.
(iv) The associated connected Lie groups G are unimodular, i.e. their Haar measure is bi-invariant.

PROOF. Writing down the conditions (i) and (ii) with respect to a basis, we immediately see the equivalence to condition (iii) in both cases. The equivalence of (i) and (iv) is shown in [**Mi**]. □

Unimodularity is a necessary condition for the existence of a co-compact lattice, see for instance [**Mi**]. In fact, it is also sufficient in dimension three. Indeed, the closed three-manifolds of the form $\Gamma\backslash G$ where G is a Lie group with lattice Γ are classified in [**RV**]. Since a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of Lie algebras is unimodular if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular, a direct product $G_1 \times G_2$ of three-dimensional Lie groups admits a co-compact lattice if and only if it is unimodular.

LEMMA 1.2. *Let $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of two Lie algebras of dimension three. Moreover, let ω be a non-degenerate two-form in $\Lambda^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^* = \Lambda^2\mathfrak{g}_1{}^* \oplus (\mathfrak{g}_1{}^* \otimes \mathfrak{g}_2{}^*) \oplus \Lambda^2\mathfrak{g}_2{}^*$ such that the projections of ω on $\Lambda^2\mathfrak{g}_1{}^*$ and $\Lambda^2\mathfrak{g}_2{}^*$ vanish. Then ω^2 is closed if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular.*

PROOF. Since $\omega \in \mathfrak{g}_1{}^* \otimes \mathfrak{g}_2{}^*$ is non-degenerate, we can always choose bases $\{\mathrm{e}^i\}$ of $\mathfrak{g}_1{}^*$ and $\{\mathrm{f}^i\}$ of $\mathfrak{g}_2{}^*$ such that $\omega = \sum_{j=1}^3 \mathrm{e}^j\mathrm{f}^j$. Therefore, we have

$$\omega^2 = -2\sum_{i<j} \mathrm{e}^{ij}\mathrm{f}^{ij} \quad \Rightarrow \quad -\frac{1}{2}\, d\omega^2 = \sum_{i<j} d\,(\mathrm{e}^{ij}) \wedge \mathrm{f}^{ij} + \sum_{i<j} \mathrm{e}^{ij} \wedge d\,(\mathrm{f}^{ij}).$$

By Lemma 1.1, both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular if and only if all two-forms e^{ij} and f^{ij} are closed. Since the sum is a direct sum of Lie algebras, the assertion follows immediately. □

In the following chapter, we need to determine in which isomorphism class a given three-dimensional Lie algebra lies. All information we need, including proofs, can be found in [**Mi**]. We summarise the results in two propositions. Recall that a Euclidean cross product in dimension three is determined by a scalar product and an orientation.

PROPOSITION 1.3 (Unimodular case). *Let $\mathfrak{g}$ be a three-dimensional Lie algebra and choose a scalar product and an orientation.*

(a) There is a uniquely defined endomorphism L of $\mathfrak{g}$ such that $[u, v] = L(u \times v)$.
(b) The Lie algebra $\mathfrak{g}$ is unimodular if and only if L is self-adjoint.
(c) If $\mathfrak{g}$ is unimodular, the isomorphism class of $\mathfrak{g}$ is characterised by the signs of the eigenvalues of L. It can be achieved that there is at most one negative eigenvalue of L by possibly changing the orientation.

Recall that the unimodular kernel of a Lie algebra $\mathfrak{g}$ is the kernel of the Lie algebra homomorphism

$$\mathfrak{g} \to \mathbb{R}\ ,\ X \mapsto \mathrm{tr}(\mathrm{ad}_{\mathrm{X}}).$$

PROPOSITION 1.4 (Non-unimodular case). *Let $\mathfrak{g}$ be a three-dimensional Lie algebra which is not unimodular.*

(a) The unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}$ is two-dimensional and abelian.
(b) Let $X \in \mathfrak{g}$ such that $\mathrm{tr}(\mathrm{ad}_{\mathrm{X}}) = 2$ and let $\tilde{L} : \mathfrak{u} \to \mathfrak{u}$ be the restriction of ad_{X} to the unimodular kernel $\mathfrak{u}$. If $\tilde{L}$ is not the identity map, the isomorphism class of $\mathfrak{g}$ is characterised by the determinant D of $\tilde{L}$.

Name	Bianchi type	Eigenvalues of L	Standard Lie bracket
$\mathfrak{su}(2) \cong \mathfrak{so}(3)$	IX	(+,+,+)	$de^1 = e^{23},\ de^2 = e^{31},\ de^3 = e^{12}$
$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(1,2)$	VIII	(+,+,-)	$de^1 = e^{23},\ de^2 = e^{31},\ de^3 = e^{21}$
$\mathfrak{e}(2)$	VII_0	(+,+,0)	$de^2 = e^{31},\ de^3 = e^{12}$
$\mathfrak{e}(1,1)$	VI_0	(+,-,0)	$de^2 = e^{31},\ de^3 = e^{21}$
$\mathfrak{h}_3$	II	(+,0,0)	$de^3 = e^{12}$
$\mathbb{R}^3$	I	(0,0,0)	abelian

TABLE 1. Three-dimensional unimodular Lie algebras

Name	Bianchi type	Determinant D of $\tilde{L}$	Standard Lie bracket
$\mathfrak{r}_2 \oplus \mathbb{R}$	III	0	$de^2 = e^{21}$
$\mathfrak{r}_3$	IV	1 (and $\tilde{L} \neq \mathrm{id}$)	$de^2 = e^{21} + e^{31},\ de^3 = e^{31}$
$\mathfrak{r}_{3,1}$	V	1 (and $\tilde{L} = \mathrm{id}$)	$de^2 = e^{21},\ de^3 = e^{31}$
$\mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	VI	$D = \frac{4\mu}{(\mu+1)^2} < 0$	$de^2 = e^{21},\ de^3 = \mu e^{31}$
$\mathfrak{r}_{3,\mu}$ $(0 < \mu < 1)$	VI	$0 < D = \frac{4\mu}{(\mu+1)^2} < 1$	$de^2 = e^{21},\ de^3 = \mu e^{31}$
$\mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	VII	$D = 1 + \frac{1}{\mu^2} > 1$	$de^2 = \mu e^{21} + e^{13},\ de^3 = e^{21} + \mu e^{31}$

TABLE 2. Three-dimensional non-unimodular Lie algebras

We remark that all three-dimensional Lie algebras are solvable except for $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$, which are simple. The three-dimensional Heisenberg algebra $\mathfrak{h}_3$ represents the only non-abelian nilpotent isomorphism class. The two Lie algebras $\mathfrak{e}(2)$ and $\mathfrak{e}(1,1)$ correspond to the groups of rigid motions of the Euclidean plane $\mathbb{R}^2$ and of the Minkowskian plane $\mathbb{R}^{1,1}$, respectively. The names for the non-unimodular Lie algebras are taken from [**GOV**] and the Bianchi types are defined in the original classification by Bianchi from 1898, [**Bi1**], see [**Bi2**] for an English translation.

2. Classification of direct sums admitting a half-flat SU(3)-structure such that the summands are orthogonal

In this section, we consider half-flat SU(3)-structures such that the summands are orthogonal. The additional, rather strong assumption of orthogonality allows us to choose a basis which is very well adapted to the problem and find all solutions using an ansatz with arbitrary structure constants.

Recall that a Hermitian structure on a $2m$-dimensional Euclidean vector space (V, g) is given by an orthogonal complex structure J and that the fundamental two-form is $\omega = g(\,.\,, J.\,)$ by our conventions.

Lemma 2.1. *Let (V_1, g_1) and (V_2, g_2) be three-dimensional Euclidean vector spaces and let (g, J, ω) be a Hermitian structure on the orthogonal product $(V_1 \oplus V_2, g = g_1 + g_2)$. There are orthonormal bases $\{e_1, e_2, e_3\}$ of V_1 and $\{f_1, f_2, f_3\}$ of V_2 which can be joined to an orthonormal basis of $V_1 \oplus V_2$ such that*

$$\omega = a\,\mathrm{e}^{12} + \sqrt{1-a^2}\,\mathrm{e}^1\mathrm{f}^1 + \sqrt{1-a^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 - a\,\mathrm{f}^{12} \tag{2.1}$$

for a real number a with $-1 < a \leq 1$.

Proof. Let $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ be orthonormal bases of V_1 and V_2, respectively. The group $O(3) \times O(3)$ acts transitively on the pairs of orthonormal bases. Let Ω be the Gram matrix of the two-form ω with respect to our basis. Writing the upper right block of Ω as a product of an orthogonal and positive semi-definite matrix and acting with an appropriate pair of orthogonal matrices, we find an orthonormal basis and nine real parameters such that

$$\Omega = \begin{pmatrix} 0 & y_1 & y_2 & x_1 & 0 & 0 \\ -y_1 & 0 & y_3 & 0 & x_2 & 0 \\ -y_2 & -y_3 & 0 & 0 & 0 & x_3 \\ -x_1 & 0 & 0 & 0 & z_1 & z_2 \\ 0 & -x_2 & 0 & -z_1 & 0 & z_3 \\ 0 & 0 & -x_3 & -z_2 & -z_3 & 0 \end{pmatrix}$$

with $x_i \geq 0$ for all i and $\det(\Omega) \neq 0$.

Since $\omega = g(\,.\,, J.\,)$, the matrix Ω with respect to an orthonormal basis has to be a complex structure, i.e. $\Omega^2 = -\mathbb{1}$, where $\mathbb{1}$ denotes the identity matrix. In our basis, the square of Ω is

$$\begin{pmatrix} -y_1^2 - y_2^2 - x_1^2 & -y_2y_3 & y_1y_3 & 0 & y_1x_2 + x_1z_1 & y_2x_3 + x_1z_2 \\ -y_2y_3 & -y_1^2 - y_3^2 - x_2^2 & -y_1y_2 & -y_1x_1 - x_2z_1 & 0 & y_3x_3 + x_2z_3 \\ y_1y_3 & -y_1y_2 & -y_2^2 - y_3^2 - x_3^2 & -y_2x_1 - x_3z_2 & -y_3x_2 - x_3z_3 & 0 \\ 0 & -y_1x_1 - x_2z_1 & -y_2x_1 - x_3z_2 & -x_1^2 - z_1^2 - z_2^2 & -z_2z_3 & z_1z_3 \\ y_1x_2 + x_1z_1 & 0 & -y_3x_2 - x_3z_3 & -z_2z_3 & -x_2^2 - z_1^2 - z_3^2 & -z_1z_2 \\ y_2x_3 + x_1z_2 & y_3x_3 + x_2z_3 & 0 & z_1z_3 & -z_1z_2 & -x_3^2 - z_2^2 - z_3^2 \end{pmatrix}$$

We end up with a set of 18 quadratic equations (and one inequality) and determine all solutions modulo the action of $O(3) \times O(3)$ and an exchange of the summands.

On the one hand, assume $y_i = 0$ for all i. Then, we deduce $x_i = 1$ and $z_i = 0$ for all i and all equations are satisfied. In this case, the two-form ω is in the normal form (2.1) with $a = 0$.

On the other hand, assume that one of the y_i is different from zero, say $a := y_1 \neq 0$ without loss of generality. Inspecting the first two terms of the third line of Ω^2, we observe $y_2 = y_3 = 0$. Since $x_i \geq 0$, the first three elements on the diagonal enforce $x_3 = 1$, $x_1 = x_2 = \sqrt{-a^2+1}$ and $|a| \leq 1$. But $x_3 = 1$ and $y_2 = y_3 = 0$ imply that $z_2 = z_3 = 0$ due to row 3, terms 4 and 5. If $|a| < 1$ and thus $x_1 = x_2 > 0$, the term in row 1 and column 5 enforces $z_1 = -a$. Obviously, all equations are satisfied and ω is in the normal form (2.1). Finally, if $|a| = 1$, we have immediately $x_1 = x_2 = 0$ and $|z_1| = 1$ and all equations are satisfied again. Since changing the signs of the base vectors e_1 and f_1 is an orthogonal transformation which does not change x_3, we can obtain the normal form (2.1) for $a = 1$. Since we found all solutions to the 18 equations and the two-form ω is non-degenerate for all values of a, the Lemma is proven. □

We call the Hermitian structure of type I if it admits a basis with $a = 0$ and of type II if it admits a basis with $a \neq 0$.

THEOREM 2.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.*

The Lie algebra $\mathfrak{g}$ admits a half-flat $\mathrm{SU}(3)$*-structure such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are mutually orthogonal and such that the underlying Hermitian structure is of type I if and only if*

(i) $\mathfrak{g}_1 = \mathfrak{g}_2$ *and both are unimodular or*

(ii) $\mathfrak{g}_1$ *is non-abelian unimodular and* $\mathfrak{g}_2$ *abelian or vice versa.*

Moreover, the Lie algebra $\mathfrak{g}$ admits a half-flat $\mathrm{SU}(3)$*-structure such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are mutually orthogonal and such that the underlying Hermitian structure is of type II if and only if the pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ or $(\mathfrak{g}_2, \mathfrak{g}_1)$ is contained in the following list:*

$$\begin{aligned}
&(\mathfrak{e}(1,1), \mathfrak{e}(1,1)), && \\
&(\mathfrak{e}(2), \mathbb{R} \oplus \mathfrak{r}_2), && \\
&(\mathfrak{su}(2), \mathfrak{r}_{3,\mu}) && \textit{for } 0 < \mu \leq 1, \\
&(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{r}_{3,\mu}) && \textit{for } -1 < \mu < 0.
\end{aligned}$$

PROOF. Given a Hermitian structure (g, J, ω) such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal, we can use Lemma 2.1 and choose an orthonormal basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\{e_1, e_2, e_3\}$ spans $\mathfrak{g}_1$, $\{f_1, f_2, f_3\}$ spans $\mathfrak{g}_2$ and

$$\omega = a\,\mathrm{e}^{12} + \sqrt{1-a^2}\,\mathrm{e}^1\mathrm{f}^1 + \sqrt{1-a^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 - a\,\mathrm{f}^{12} \tag{2.2}$$

for a real number a with $-1 < a \leq 1$. The reductions from $\mathrm{U}(3)$ to $\mathrm{SU}(3)$ are parameterised by the space of complex-valued $(3,0)$-forms $\Psi = \psi + i\phi$ which is complex one-dimensional. We remark that, working on a vector space, the length normalisation of the $(3,0)$-form is not important for the existence question. The Lie bracket of the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is encoded in the 18 structure constants of $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$d\mathrm{e}^i = c^i_{j,k}\mathrm{e}^{jk} \quad \text{and} \quad d\mathrm{f}^i = c^{i+3}_{j+3,k+3}\mathrm{f}^{jk} \quad \text{with i,j,k} \in \{1,2,3\}.$$

Therefore, our ansatz includes 21 parameters consisting of 18 structure constants, two real parameters defining an arbitrary $\mathrm{SU}(3)$ reduction and the parameter a. Our strategy is to find all solutions of the equations defining half-flatness

$$d\omega^2 = 0 \quad \text{and} \quad d\psi = 0$$

and the Jacobi identity $d^2 = 0$ and to determine the isomorphism classes of the solutions if necessary.

Type I: Assume first that $a = 0$. Due to Lemma (1.2), the first half-flat equation $d\omega^2 = 0$ is satisfied if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular. It remains to solve the second half-flat equation for unimodular summands. Since we have $J(f_i) = e_i$ in our basis for $a = 0$, the dual vectors satisfy $\mathrm{e}^i \circ J = \mathrm{f}^i$. Therefore, the complex-valued form

$$\begin{aligned}\Psi_0 &= \psi_0 + \mathrm{i}\phi_0 = (\mathrm{e}^1 - \mathrm{i}\mathrm{e}^1 \circ J) \wedge (\mathrm{e}^2 - \mathrm{i}\mathrm{e}^2 \circ J) \wedge (\mathrm{e}^3 - \mathrm{i}\mathrm{e}^3 \circ J)\\ &= \mathrm{e}^{123} - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{i}(\mathrm{f}^{123} - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^1)\end{aligned}$$

is a $(3,0)$-form with respect to J. By multiplying Ψ_0 with a non-zero complex number $\xi_1 + \mathrm{i}\xi_2$, we obtain all $(3,0)$-forms. Their real part is $\psi = \xi_1\psi_0 - \xi_2\phi_0$. Considering that all two-forms on both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are closed, we compute the exterior derivative of ψ:

$$\begin{aligned}d\psi = &-(\xi_1 c^1_{1,2} - \xi_2 c^6_{5,6})\,\mathrm{e}^{12}\mathrm{f}^{23} - (\xi_1 c^1_{2,3} - \xi_2 c^4_{5,6})\,\mathrm{e}^{23}\mathrm{f}^{23} - (\xi_1 c^1_{3,1} - \xi_2 c^5_{5,6})\,\mathrm{e}^{31}\mathrm{f}^{23}\\ &-(\xi_1 c^2_{1,2} - \xi_2 c^6_{6,4})\,\mathrm{e}^{12}\mathrm{f}^{31} - (\xi_1 c^2_{2,3} - \xi_2 c^4_{6,4})\,\mathrm{e}^{23}\mathrm{f}^{31} - (\xi_1 c^2_{3,1} - \xi_2 c^5_{6,4})\,\mathrm{e}^{31}\mathrm{f}^{31}\\ &-(\xi_1 c^3_{1,2} - \xi_2 c^6_{4,5})\,\mathrm{e}^{12}\mathrm{f}^{12} - (\xi_1 c^3_{2,3} - \xi_2 c^4_{4,5})\,\mathrm{e}^{23}\mathrm{f}^{12} - (\xi_1 c^3_{3,1} - \xi_2 c^5_{4,5})\,\mathrm{e}^{31}\mathrm{f}^{12}.\end{aligned}$$

If ξ_1 or ξ_2 is zero we have obviously $d\psi = 0$ if and only if one of the summands is abelian. By Lemma (1.1), the unimodularity of $\mathfrak{g}_2$ is equivalent to $c^6_{6,4} = -c^5_{5,4}$, $c^6_{6,5} = -c^4_{4,5}$ and $c^5_{5,4} = -c^6_{6,4}$. Therefore, if both ξ_1 and ξ_2 are different from zero, $d\psi$ vanishes if and only if the structure constants of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ coincide up to the scalar $\frac{\xi_1}{\xi_2}$ and therefore $\mathfrak{g}_1 = \mathfrak{g}_2$. This comprises all solutions under the assumption $a = 0$.

Type II: Assume now that the U(3)-structure satisfies $a \neq 0$. To improve readability, the abbreviation $b := \sqrt{1 - a^2}$ is introduced.

With this notation, we compute

$$\begin{aligned}\frac{1}{2}\omega^2 &= a\,\mathrm{e}^{123}\mathrm{f}^3 - a\,\mathrm{e}^3\mathrm{f}^{123} - \mathrm{e}^{12}\mathrm{f}^{12} - b\,\mathrm{e}^{13}\mathrm{f}^{13} - b\,\mathrm{e}^{23}\mathrm{f}^{23},\\ \frac{1}{2}d(\omega^2) &= (c^4_{4,6} + c^5_{5,6} - ac^3_{1,2})\quad \mathrm{e}^{12}\mathrm{f}^{123} + (bc^6_{5,6} - bc^4_{4,5} - ac^3_{1,3})\quad \mathrm{e}^{13}\mathrm{f}^{123}\\ &\quad - (ac^3_{2,3} + bc^5_{4,5} + bc^6_{4,6})\quad \mathrm{e}^{23}\mathrm{f}^{123} + (c^1_{1,3} + c^2_{2,3} - ac^6_{4,5})\quad \mathrm{e}^{123}\mathrm{f}^{12}\\ &\quad - (bc^1_{1,2} - bc^3_{2,3} + ac^6_{4,6})\quad \mathrm{e}^{123}\mathrm{f}^{13} - (ac^6_{5,6} + bc^2_{1,2} + bc^3_{1,3})\quad \mathrm{e}^{123}\mathrm{f}^{23}.\end{aligned}$$

We reduce our ansatz to the space of solutions of $d\omega^2 = 0$ by substituting

$$\begin{aligned}c^2_{2,3} &= ac^6_{4,5} - c^1_{1,3} \quad , \quad c^3_{2,3} = b^2c^1_{1,2} - abc^5_{4,5} \quad , \quad c^3_{1,3} = -b^2c^2_{1,2} - abc^4_{4,5} \quad ,\\ c^5_{5,6} &= ac^3_{1,2} - c^4_{4,6} \quad , \quad c^6_{5,6} = b^2c^4_{4,5} - abc^2_{1,2} \quad \text{and} \quad c^6_{4,6} = -b^2c^5_{4,5} - abc^1_{1,2}.\end{aligned}$$

In our basis, we have $\mathrm{e}^1 \circ J = b\mathrm{f}^1 + a\mathrm{e}^2$, $\mathrm{e}^3 \circ J = \mathrm{f}^3$ and $\mathrm{f}^2 \circ J = -b\mathrm{e}^2 + a\mathrm{f}^1$. Using these identities, we find the (3,0)-form $\Psi_0 = \psi_0 + \mathrm{i}\phi_0$ with

$$\begin{aligned}\psi_0 &= +b\,\mathrm{f}^{123} - b\,\mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^1 + a\,\mathrm{e}^1\mathrm{f}^{13} + a\,\mathrm{e}^2\mathrm{f}^{23},\\ \phi_0 &= -b\,\mathrm{e}^{123} + b\,\mathrm{e}^3\mathrm{f}^{12} - \mathrm{e}^2\mathrm{f}^{13} + \mathrm{e}^1\mathrm{f}^{23} - a\,\mathrm{e}^{13}\mathrm{f}^1 - a\,\mathrm{e}^{23}\mathrm{f}^2.\end{aligned}$$

In the following, we work with the real part $\psi = \xi_1\psi_0 - \xi_2\phi_0$ of an arbitrary (3,0)-form. By possibly changing the roles of the two summands, we can assume that ξ_1 is non-zero and we normalise our (3,0)-form such that $\xi_1 = 1$. The exterior derivative of ψ is, after inserting the above substitutions:

$$
\begin{aligned}
d\psi = {} & ab\, c^6_{4,5}\, \mathrm{e}^{123}\mathrm{f}^3 \quad -\xi_2 ab\, c^3_{1,2}\, \mathrm{e}^3\mathrm{f}^{123} \quad -b\,(\, c^6_{4,5} + \xi_2\, c^3_{1,2})\, \mathrm{e}^{12}\mathrm{f}^{12} \\
& + (\; -\xi_2 ab\, c^1_{1,2} \quad -a^2 b\, c^2_{1,2} \quad -a^3 c^4_{4,5} \quad +\xi_2 a^2 c^5_{4,5})\, \mathrm{e}^1\mathrm{f}^{123} \\
& + (\; a^2 b\, c^1_{1,2} \quad -\xi_2 ab\, c^2_{1,2} \quad -\xi_2 a^2 c^4_{4,5} \quad -a^3 c^5_{4,5})\, \mathrm{e}^2\mathrm{f}^{123} \\
& + (\; \xi_2 a^3 c^1_{1,2} \quad -a^2 c^2_{1,2} \quad +ab\, c^4_{4,5} \quad +\xi_2 a^2 b\, c^5_{4,5})\, \mathrm{e}^{123}\mathrm{f}^1 \\
& + (\; a^2 c^1_{1,2} \quad +\xi_2 a^3 c^2_{1,2} \quad -\xi_2 a^2 b\, c^4_{4,5} \quad +ab\, c^5_{4,5})\, \mathrm{e}^{123}\mathrm{f}^2 \\
& + (\; a(2-a^2)\, c^1_{1,2} \quad +\xi_2\, c^2_{1,2} \quad +b^3\, c^5_{4,5})\, \mathrm{e}^{12}\mathrm{f}^{13} \\
& + (\; -\xi_2\, c^1_{1,2} \quad +a(2-a^2)\, c^2_{1,2} \quad -b^3\, c^4_{4,5} \;)\, \mathrm{e}^{12}\mathrm{f}^{23} \\
& + (\; +\xi_2 b^3\, c^2_{1,2} \quad \xi_2 a(2-a^2)\, c^4_{4,5} \quad +c^5_{4,5})\, \mathrm{e}^{13}\mathrm{f}^{12} \\
& + (\; -\xi_2 b^3\, c^1_{1,2} \quad -c^4_{4,5} \quad +\xi_2 a(2-a^2)\, c^5_{4,5})\, \mathrm{e}^{23}\mathrm{f}^{12} \\
& + (\; a\, c^1_{1,3} \quad +\xi_2\, c^2_{1,3} \quad +\xi_2 a\, c^4_{4,6} \quad +c^5_{4,6} \;)\, \mathrm{e}^{13}\mathrm{f}^{13} \\
& + (\xi_2 a^2 c^3_{1,2} \quad -a\, c^1_{1,3} \quad -\xi_2 c^1_{2,3} \quad -\xi_2 a\, c^4_{4,6} \quad -c^4_{5,6} \quad +a^2 c^6_{4,5})\, \mathrm{e}^{23}\mathrm{f}^{23} \\
& + (\; a\, c^3_{1,2} \quad -\xi_2\, c^1_{1,3} \quad +a\, c^2_{1,3} \quad -c^4_{4,6} \quad +\xi_2 a\, c^4_{5,6} \;)\, \mathrm{e}^{13}\mathrm{f}^{23} \\
& + (\; -\xi_2\, c^1_{1,3} \quad +a\, c^1_{2,3} \quad -c^4_{4,6} \quad +\xi_2 a\, c^5_{4,6} \quad +\xi_2 a\, c^6_{4,5})\, \mathrm{e}^{23}\mathrm{f}^{13}
\end{aligned}
$$

We need to determine all solutions of the coefficient equations of $d\psi = 0$. First of all, we observe that the variables $c^1_{1,2}$, $c^2_{1,2}$, $c^4_{4,5}$ and $c^5_{4,5}$ are subject to eight linear equations and claim that there is no non-trivial solution of this linear system. Indeed, the determinant of the four by four coefficient matrix of the first four equations is $a^4(a^2\xi_2^2+1)(a^2+\xi_2^2)(a^2+b^2)^2 = a^4(a^2\xi_2^2+1)(a^2+\xi_2^2)$ and thus never vanishes for $a \neq 0$. To deal with the remaining eight structure constants, subject to seven equations, we treat three cases separately.

(a) Assume first that $b \neq 0$ and $\xi_2 \neq 0$, i.e. $0 < |a| < 1$. Obviously, we have $c^3_{1,2} = 0$ and $c^6_{4,5} = 0$ by the vanishing of the first three coefficients. Moreover, applying easy row transformations to the remaining four equations, we observe that it holds necessarily $c^2_{1,3} = c^1_{2,3}$ and $c^5_{4,6} = c^4_{5,6}$. Considering this, $d\psi = 0$ is finally satisfied if and only if

$$s := c^4_{4,6} = \frac{a\,(\xi_2^2-1)\,c^4_{5,6} - (a^2+\xi_2^2)\,c^1_{1,3}}{\xi_2(a^2+1)}, \quad t := c^1_{2,3} = -\frac{(\xi_2^2 a^2+1)\,c^4_{5,6} + a(1-\xi_2^2)\,c^1_{1,3}}{\xi_2(a^2+1)}.$$

Applying all substitutions, the set of solutions of the two half-flat equations is parameterised by the four parameters a, ξ_2,

$$p := c^1_{1,3} \quad \text{and} \quad q := c^4_{5,6}.$$

In order to determine the isomorphism class of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ for all solutions, we apply Propositions 1.3 and 1.4. We choose orientations on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $e_1 \times e_2 = -e_3$ and $e_4 \times e_5 = -e_6$. Let $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ denote the matrices representing the endomorphisms defined in Proposition 1.3 with respect to our bases. On the set of solutions, they simplify to

$$L_{\mathfrak{g}_1} = \begin{pmatrix} t & -p & 0 \\ -p & -t & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{\mathfrak{g}_2} = \begin{pmatrix} q & -s & 0 \\ -s & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Jacobi identity is already satisfied. Both $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ are symmetric and in consequence, both summands are unimodular. The eigenvalues of $L_{\mathfrak{g}_1}$ are 0 and $\pm\sqrt{p^2+t^2}$ and those of $L_{\mathfrak{g}_2}$ are 0 and $\pm\sqrt{s^2+q^2}$. Hence, if $p \neq 0$ or $q \neq 0$, the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is isomorphic to $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ with two remaining parameters $\xi_2 \neq 0$ and $0 < |a| < 1$. If $p = 0$ and $q = 0$, the Lie algebra is abelian.

(b) Now assume $b \neq 0$ and $\xi_2 = 0$. In this case, the equations simplify considerably and the only solution of $d\psi = 0$ is given by

$$c^6_{4,5} = 0\,, \quad c^4_{4,6} = ac^1_{2,3}\,, \quad c^5_{4,6} = -ac^1_{1,3}\,, \quad c^4_{5,6} = -ac^1_{1,3}\,, \quad c^3_{1,2} = -c^2_{1,3} + c^1_{2,3}.$$

As before, we rename the remaining parameters

$$p := c^2_{1,3} \quad , \quad q := c^1_{2,3} \quad \text{and} \quad r := c^1_{1,3},$$

and have a closer look at

$$L_{\mathfrak{g}_1} = \begin{pmatrix} q & -r & 0 \\ -r & -p & 0 \\ 0 & 0 & -p+q \end{pmatrix} \quad \text{and} \quad L_{\mathfrak{g}_2} = \begin{pmatrix} -ar & -aq & 0 \\ -ap & ar & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, the Jacobi identity is already satisfied. The first summand is always unimodular, the second summand is unimodular if and only if $p = q$. If $p = q$, both matrices are of the same type as in case (a) and $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is isomorphic to $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ or abelian.

It remains to apply Proposition 1.4 to identify the isomorphism class of the solutions with $p \neq q$. Without changing the isomorphism class, we can normalise such that $p = q + 1$. We need to find a vector $X \in \mathfrak{g}_2$ with $\mathrm{tr}(\mathrm{ad_X}) = 2$. Since $\mathrm{tr}(\mathrm{ad_{f_3}}) = \mathrm{c}^4_{4,6} + \mathrm{c}^5_{5,6} = -\mathrm{a}$, we choose $X = -\frac{2}{a} f_3$. The unimodular kernel $\mathfrak{u}$ is spanned by f_1 and f_2 and the restriction of $\mathrm{ad_X}$ on $\mathfrak{u}$ is represented by the matrix

$$\tilde{L}_{\mathfrak{g}_2} = \begin{pmatrix} -2q & 2r \\ 2r & 2(q+1) \end{pmatrix} \qquad \text{with} \quad D = \det(\tilde{L}_{\mathfrak{g}_2}) = -4(q(q+1) + r^2) \leq 1.$$

If $\tilde{L}_{\mathfrak{g}_2}$ is not the identity matrix, the value of D determines the isomorphism class of $\mathfrak{g}_2$. However, the corresponding class of the unimodular summand $\mathfrak{g}_1$ varies with the value of D. In fact, with $r^2 = -q(q+1) - \frac{1}{4}D$, the eigenvalues of $L_{\mathfrak{g}_1}$ are -1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1-D}$. Comparing with the lists in chapter 1, we find the remaining classes listed in the theorem.

(c) The last case to be discussed is $b = 0$ which corresponds to $a = 1$. Now, the equation $d\psi = 0$ is equivalent to

$$c^1_{2,3} = -\xi_2 c^4_{5,6} + \xi_2 c^1_{1,3} + c^4_{4,6}\,, \quad c^3_{1,2} = \xi_2 c^1_{1,3} + c^4_{4,6} - \xi_2 c^4_{5,6} - c^2_{1,3},$$
$$c^5_{4,6} = -\xi_2 c^2_{1,3} - \xi_2 c^4_{4,6} - c^1_{1,3}\,, \quad c^6_{4,5} = \xi_2 c^4_{4,6} + \xi_2 c^2_{1,3} + c^4_{5,6} + c^1_{1,3}.$$

Considering these substitutions, the Jacobi identity is satisfied if and only if

$$\xi_2 c^4_{4,6} + \xi_2 c^2_{1,3} + c^4_{5,6} + c^1_{1,3} = 0 \quad \text{or} \quad \xi_2 c^1_{1,3} + c^4_{4,6} - \xi_2 c^4_{5,6} - c^2_{1,3} = 0.$$

Writing down the matrices $L_{\mathfrak{g}_1}$ and $L_{\mathfrak{g}_2}$ for both cases, it is easy to see that they are of the same form as in case (b). Therefore, the possible isomorphism classes of Lie algebras are exactly the same as in case (b).

Since we have discussed all solutions of the half-flat equations, the theorem is proved. $\square$

3. Classification of direct sums admitting a half-flat SU(3)-structure

From now on, we will drop the additional assumption of orthogonality. In fact, we have to develop a completely different method in order to solve the existence problem of half-flat SU(3)-structures on direct sums of three-dimensional Lie algebras.

3.1. Obstructions to the existence of half-flat SU(3)-structures. To begin with, we prove an obstruction to the existence of a half-flat SU(3)-structure on a Lie algebra following the idea of [**Con**, Theorem 2].

We denote by Z^p the space of closed p-forms on a Lie algebra and by W^0 the annihilator of a subspace W.

LEMMA 3.1. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra and $\mathfrak{g}^* = V \oplus W$ a (vector space) decomposition such that V is two-dimensional and such that*

$$Z^3 \subset \Lambda^2 V \wedge W \oplus V \wedge \Lambda^2 W. \tag{3.1}$$

Then, the subspace V is J_ρ-invariant for all closed stable three-forms ρ.

PROOF. Let $\rho \in Z^3$ be stable and $\alpha \in V$. Since $\dim V = 2$, the assumption (3.1) implies for all $v \in V^0$

$$v \lrcorner \rho \in \Lambda^2 V \oplus V \wedge W\,, \qquad \alpha \wedge \rho \in \Lambda^3 V \wedge W \oplus \Lambda^2 V \wedge \Lambda^2 W.$$

Therefore, by the formula

$$J_\rho^* \alpha(v)\, \phi(\rho) = \alpha \wedge (v \lrcorner \rho) \wedge \rho, \qquad v \in V,\ \alpha \in V^*, \tag{3.2}$$

which is proved in Proposition 1.5, it holds

$$0 = \alpha \wedge (v \lrcorner \rho) \wedge \rho = J_\rho^* \alpha(v)\, \phi(\rho)$$

for all $v \in V^0$ and, by definition of the annihilator, the subspace V is J_ρ-invariant. □

PROPOSITION 3.2. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra and $\mathfrak{g}^* = V \oplus W$ a decomposition such that V is two-dimensional and such that*

$$Z^3 \subset \Lambda^2 V \wedge W \quad \oplus V \wedge \Lambda^2 W, \tag{3.3}$$

$$Z^4 \subset \Lambda^2 V \wedge \Lambda^2 W \oplus V \wedge \Lambda^3 W. \tag{3.4}$$

Then, the subspace V is isotropic and J_ρ-invariant for every half-flat structure (ω, ρ). In particular, the Lie algebra $\mathfrak{g}$ does not admit a half-flat SU(3)*-structure.*

PROOF. Suppose that (ω, ρ) is a half-flat structure on $\mathfrak{g}$, in particular $\rho \in Z^3$ and $\omega^2 \in Z^4$ by definition. By Lemma 3.1, the subspace V is J_ρ-invariant. Moreover, the identity

$$\alpha \wedge J_\rho^* \beta \wedge \omega^2 = \frac{1}{3} g(\alpha, \beta) \omega^3 \tag{3.5}$$

for all $\alpha, \beta \in V$ was shown in Lemma 1.8. Thus, the assumption (3.4) and $\dim V = 2$ imply that V has to be an isotropic subspace of $\mathfrak{g}^*$. This is of course impossible for definite metrics and there cannot exist a half-flat SU(3)-structure. □

DEFINITION 3.3. Let $\mathfrak{g}$ be a Lie algebra. A decomposition $\mathfrak{g}^* = V \oplus W$ is called a *coherent splitting* if

$$dV \subset \Lambda^2 V, \tag{3.6}$$

$$dW \subset \Lambda^2 V \oplus V \wedge W. \tag{3.7}$$

REMARK 3.4. The definition can be reformulated into an equivalent dual condition:

$$(3.6) \iff 0 = d\sigma(X, .) = -\sigma([X, .]) \text{ for all } X \in V^0, \sigma \in V \iff [V^0, \mathfrak{g}] \subset V^0,$$

$$(3.7) \iff 0 = d\sigma(X, Y) = -\sigma([X, Y]) \text{ for all } X, Y \in V^0, \sigma \in W \iff [V^0, V^0] \subset W^0$$

In other words, a coherent splitting corresponds to a decomposition of $\mathfrak{g}$ into an abelian ideal and a vector space complement.

As elaborated in [**Con**], a coherent splitting with $\dim V = 2$ allows the introduction of a double complex such that the obstruction conditions (3.3), (3.4) can be formulated in terms of the cohomology of this double complex. However, in the situation we are interested in, it turns out to be more practical to avoid homological algebra. Indeed, the verification of the obstruction conditions can be simplified as follows.

LEMMA 3.5. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras.*

(i) Let $\alpha_1 \in {\mathfrak{g}_1}^*$ and $\alpha_2 \in {\mathfrak{g}_2}^*$ be one-forms defining $V = \mathrm{span}(\alpha_1, \alpha_2)$. Then $\mathfrak{g}^* = V \oplus W$ is a coherent splitting for any complement W of V if and only if the two one-forms α_i are closed and satisfy

$$\mathrm{im}(d : \mathfrak{g}_i^* \to \Lambda^2 \mathfrak{g}_i^*) \subset \alpha_i \wedge \mathfrak{g}_i^* \quad \textit{for both } i. \tag{3.8}$$

(ii) If both summands are non-abelian, every *coherent splitting with* $\dim V = 2$ *is defined by closed one-forms $\alpha_1 \in {\mathfrak{g}_1}^*$ and $\alpha_2 \in {\mathfrak{g}_2}^*$ satisfying* (3.8).

(iii) There exists a coherent splitting with $\dim V = 2$ *on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is solvable.*

(iv) If $\mathfrak{g}$ is unimodular, there is no decomposition $\mathfrak{g}^* = V \oplus W$ with two-dimensional V satisfying both obstruction conditions (3.3) and (3.4).

PROOF. (i) Since both the exterior algebras $\Lambda^* \mathfrak{g}_i^*$ are d-invariant, the condition (3.6) is satisfied if and only if both generators are closed and (3.7) is equivalent to (3.8).

(ii) Assume that both summands $\mathfrak{g}_i$ are not abelian and let a coherent splitting be defined by an abelian four-dimensional ideal V^0 and a complement. In consequence, both the intersection $V^0 \cap \mathfrak{g}_i$ and the projection of V^0 on $\mathfrak{g}_i$ are abelian subalgebras of $\mathfrak{g}_i$ for both i and thus at most two-dimensional. Since a one-dimensional intersection $V^0 \cap \mathfrak{g}_i$ would require the projection on the other summand to be three-dimensional, it follows that the intersections $V^0 \cap \mathfrak{g}_i$ have to be exactly two-dimensional. Equivalently, the two-dimensional space V is generated by two one-forms $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. Now, the assertion follows from part (i).

(iii) On the one hand, if $\mathfrak{g}$ is not solvable, one of the summands has to be simple, say $\mathfrak{g}_1$. However, the intersection of a four-dimensional abelian ideal with $\mathfrak{g}_1$ would be zero or $\mathfrak{g}_1$, both of which is not possible since $\dim \mathfrak{g} = 6$ and since $\mathfrak{g}_1$ is not abelian. On the other hand, inspecting the list of standard bases in tables 1 and 2 reveals that any three-dimensional solvable Lie algebra $\mathfrak{h}$ contains a closed one-form α such that $\mathrm{im}\, d \subset \alpha \wedge \mathfrak{h}^*$. Therefore, if $\mathfrak{g}$ is solvable, i.e. both summands are solvable, a coherent splitting exists by part (i).

(iv) Assume that $\mathfrak{g}$ is unimodular and let W be an arbitrary four-dimensional subspace of $\mathfrak{g}^*$. It suffices to show that there always exists a closed three-form with non-zero projection on $\Lambda^3 W$ *or* a closed four-form with non-zero projection on $\Lambda^4 W$. If the projection of W on one of the summands $\mathfrak{g}_i$ is surjective, every non-zero element of $\Lambda^3\mathfrak{g}_i^*$ is closed and has non-zero projection on $\Lambda^3 W$. Otherwise, the image of the projection of W on either of the summands has to be two-dimensional for dimensional reasons. In this case, there is always a closed four-form with non-zero projection on $\Lambda^4 W$ since all four-forms in $\Lambda^2\mathfrak{g}_1^* \wedge \Lambda^2\mathfrak{g}_2^*$ are closed by unimodularity. This finishes the proof of the lemma.

□

LEMMA 3.6. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of three-dimensional Lie algebras and let $\mathfrak{g}^* = V \oplus W$ be a coherent splitting such that $V = \mathrm{span}(\alpha_1, \alpha_2)$ is defined by closed one-forms $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$ satisfying* (3.8). *Then, the obstruction conditions* (3.3) *and* (3.4) *are equivalent to the condition that d is injective when restricted to $\Lambda^3 W$ and $\Lambda^4 W$.*

PROOF. The injectivity of d on $\Lambda^3 W$ and $\Lambda^4 W$ is obviously necessary for (3.3) and (3.4). With the assumptions, it is also sufficient since the coherent splitting satisfies $dW \subset V \wedge W$ and $dV = 0$ such that the images of $\Lambda^3 W$ and $\Lambda^4 W$ are linearly independent from the images of the complements $\Lambda^2 V \wedge W \oplus V \wedge \Lambda^2 W$ and $\Lambda^2 V \wedge \Lambda^2 W \oplus V \wedge \Lambda^3 W$, respectively. □

3.2. The classification. Using the obstruction established in the previous section, we obtain the following classification result.

THEOREM 3.7. *A direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of three-dimensional Lie algebras admits a half-flat* $\mathrm{SU}(3)$*-structure if and only if*

(i) $\mathfrak{g}$ is unimodular or
(ii) $\mathfrak{g}$ is not solvable or
(iii) $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ or $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.

PROOF. A standard basis of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ will always denote the union of a standard basis $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ of $\mathfrak{g}_1$ and a standard basis $\{\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3\}$ of $\mathfrak{g}_2$ as defined in tables 1 and 2. For all Lie algebras admitting a half-flat SU(3)-structure, such a structure is explicitly given in a standard basis in the three tables at the end of this section. We remark that most examples are constructed exploiting the stable form formalism and with extensive computer support. The non-existence on the remaining Lie algebras is settled as follows.

In most of the cases, the obstructions of section 3.1 can be applied directly.

LEMMA 3.8. *The Lie algebra $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and all Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ solvable and $\mathfrak{g}_2$ one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0 < |\mu| \leq 1$, $\mathfrak{r}'_{3,\mu}$, $\mu > 0$, do not admit a half-flat* $\mathrm{SU}(3)$*-structure.*

PROOF. We want to apply the obstruction established in Proposition 3.2 and, given any of the Lie algebras $\mathfrak{g}$ in question, we define a decomposition

$$V = \mathrm{span}\{\mathrm{e}^1, \mathrm{f}^1\}\ , \quad W = \mathrm{span}\{\mathrm{e}^2, \mathrm{e}^3, \mathrm{f}^2, \mathrm{f}^3\}\ ,$$

in a standard basis of $\mathfrak{g}^*$. By Lemma 3.6, it suffices to show that this is a coherent splitting such that the restrictions $d_{|\Lambda^3 W}$ and $d_{|\Lambda^4 W}$ are injective. In fact, the coherence can be verified directly by comparing the conditions of Lemma 3.5, (i), with the standard bases of the solvable three-dimensional Lie algebras.

If $\mathfrak{g}_2$ is one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0<|\mu|\leq 1$ or $\mathfrak{r}'_{3,\mu}$, $\mu>0$, the standard bases satisfy

$$d\mathrm{f}^2 \neq 0,\ \nexists\, c\in\mathbb{R} : d\mathrm{f}^3 = c\, d\mathrm{f}^2,\ d\mathrm{f}^{23}\neq 0.$$

Thus, considering again that the exterior algebras $\Lambda^*\mathfrak{g}_i^*$ of the summands are d-invariant, the image

$$d(\Lambda^3 W) = \mathrm{span}\{d(\mathrm{e}^{23}\mathrm{f}^2),\, d(\mathrm{e}^{23}\mathrm{f}^3),\, d(\mathrm{e}^2\mathrm{f}^{23}),\, d(\mathrm{e}^3\mathrm{f}^{23})\}$$

is four-dimensional and the image

$$d(\Lambda^4 W) = \mathrm{span}\{d(\mathrm{e}^{23}\mathrm{f}^{23})\}$$

is one-dimensional. The same restrictions are injective for $\mathfrak{g}=\mathfrak{r}_2\oplus\mathbb{R}\oplus\mathfrak{r}_2\oplus\mathbb{R}$, since in this case $d\mathrm{e}^{23}\neq 0$ and $d\mathrm{f}^{23}\neq 0$. This finishes the proof. □

The obstruction theory cannot be applied directly to the two remaining Lie algebras, although they admit coherent splittings and we have to deal with them separately.

LEMMA 3.9. *The Lie algebra* $\mathfrak{g}=\mathfrak{h}_3\oplus\mathfrak{r}_2\oplus\mathbb{R}$ *does not admit a half-flat* SU(3)*-structure. Furthermore, there is no decomposition* $\mathfrak{g}^*=V\oplus W$ *with two-dimensional* V *which satisfies the obstruction condition* (3.3).

PROOF. We start by proving the second assertion. Let $W\subset\mathfrak{g}^*$ be an arbitrary four-dimensional subspace. It suffices to show that there is always a closed three-form with non-zero projection on $\Lambda^3 W$. If the projection of W on one of the summands $\mathfrak{g}_i^*$ is surjective, a generator of $\Lambda^3\mathfrak{g}_i^*$ is closed and has non-zero projection on $\Lambda^3 W$. For dimensional reasons, the only remaining possibility is that both projections have two-dimensional image in W. However, since all two-forms in $\Lambda^2\mathfrak{h}_3^*$ are closed and the kernel of d is two-dimensional on $\mathfrak{r}_2\oplus\mathbb{R}$, there is necessarily a closed three-form in $\Lambda^2\mathfrak{h}_3^*\wedge(\mathfrak{r}_2\oplus\mathbb{R})^*$ with non-zero projection on $\Lambda^3 W$. Therefore, the obstruction condition (3.3) is never satisfied.

However, we can prove that there is no half-flat SU(3)-structure by refining the idea of the obstruction condition as follows. Suppose that (ρ,ω) is a half-flat SU(3)-structure, i.e. $\rho\in Z^3$ and $\sigma=\frac{1}{2}\omega^2\in Z^4$ and let $\{\mathrm{e}_1,\ldots,\mathrm{f}_3\}$ denote a standard basis of $\mathfrak{h}_3\oplus\mathfrak{r}_2\oplus\mathbb{R}$. We claim that

$$\mathrm{f}^1\wedge J_\rho^*\mathrm{f}^1\wedge\sigma = 0$$

which suffices to prove the non-existence since f_1 would be isotropic by (3.5). First of all, an easy calculation reveals that

$$\mathrm{f}_1\wedge\sigma\in\mathrm{span}\{\mathrm{f}^1\mathrm{e}^{12}\mathrm{f}^{23},\,\mathrm{f}^1\mathrm{e}^{123}\mathrm{f}^3\}$$

for an arbitrary closed four-form σ. Thus, it remains to show that $J_\rho^*\mathrm{f}^1$ has no component along e^3 and f^2 or equivalently that

$$J_\rho^*\mathrm{f}^1(v)\phi(\rho)\overset{(3.2)}{=}\mathrm{f}^1\wedge(v\lrcorner\,\rho)\wedge\rho$$

vanishes for $v\in\{\mathrm{e}_3,\mathrm{f}_2\}$. This assertion is straightforward to verify for an arbitrary closed three-form ρ. □

For the last remaining Lie algebra, we apply a different argument.

LEMMA 3.10. *The Lie algebra* $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathbb{R}^3$ *does not admit a closed stable form* ρ *with* $\lambda(\rho) < 0$*, in particular it does not admit a half-flat* SU(3)*-structure. Furthermore, there is no decomposition* $\mathfrak{g}^* = V \oplus W$ *with two-dimensional* V *which satisfies the obstruction condition* (3.3)*.*

PROOF. Suppose that ρ is a closed stable form inducing a complex or a para-complex structure J_ρ. Let $\{\mathrm{e}_1, \mathrm{e}_2\}$ be a basis of $\mathfrak{r}_2$ such that $d\mathrm{e}^2 = \mathrm{e}^{21}$. Since ρ is closed, there are a one-form $\beta \in (\mathbb{R}^4)^*$, a two-form $\gamma \in \Lambda^2(\mathbb{R}^4)^*$ and a three-form $\delta \in \Lambda^3(\mathbb{R}^4)^*$, such that

$$\rho = \mathrm{e}^{12} \wedge \beta + \mathrm{e}^1 \wedge \gamma + \delta.$$

Therefore, we have

$$K_\rho(\mathrm{e}_2) = \kappa((\mathrm{e}_2 \lrcorner\, \rho) \wedge \rho) = \kappa(-\mathrm{e}^1 \wedge \beta \wedge \delta)$$

with $\beta \wedge \delta \in \Lambda^4(\mathbb{R}^4)^*$. However, this implies that $J(\mathrm{e}_2)$ is proportional to e_2 by (3.7) which is only possible if $\lambda(\rho) > 0$ and the first assertion is proven.

To prove the second assertion, it suffices to show that for every four-dimensional subspace $W \subset \mathfrak{g}^*$, there is a closed three-form with non-zero projection on $\Lambda^3 W$. This follows immediately from the observation that $\dim(\ker d) = 5$ which implies that

$$\dim(\ker d \cap W) \geq 3$$

for every four-dimensional subspace W. □

The lemma finishes the proof of the theorem as all possible direct sums according to the classification of three-dimensional Lie algebras have been considered. □

We remark that the lemmas 3.9 and 3.10 give two examples of solvable Lie algebras which show that the condition of [**Con**, Theorem 5], which characterises six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure, cannot be generalised without further restrictions to solvable Lie algebras.

Table 3: Unimodular direct sums of three-dimensional Lie algebras

Lie algebra	Half-flat SU(3)-structure with $\omega = \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$
$\mathfrak{h} \oplus \mathfrak{h}$, $\mathfrak{h}$ unimodular	$\rho = \frac{1}{2}\sqrt{2}\,\{\mathrm{e}^{123} - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{f}^{123}\}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{h} \oplus \mathbb{R}^3$, $\mathfrak{h}$ unimodular	$\rho = \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{su}(2) \oplus \mathfrak{sl}(2,\mathbb{R})$,	$\rho = 2^{\frac{1}{4}}\,\{\frac{1}{2}\,\mathrm{e}^{123} + \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} - 2\,\mathrm{f}^{123}\}$ $g = \sqrt{2}\,\{\frac{3}{2}\,(\mathrm{e}^1)^2 + \frac{3}{2}\,(\mathrm{e}^2)^2 + \frac{1}{2}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + 3\,(\mathrm{f}^3)^2$ $+ 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1 + 2\,\mathrm{e}^2\!\cdot\!\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\!\mathrm{f}^3\}$
$\mathfrak{su}(2) \oplus \mathfrak{e}(2)$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\!\cdot\!\mathrm{f}^1$

Table 3 – continued on next page

Table 3 – continued

Lie algebra	Half-flat SU(3)-structure with $\omega = \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}(2)$	$\rho = -2\,\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + 2\,(\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 + 2\,\mathrm{e}^2\cdot\mathrm{f}^2 - 2\,\mathrm{e}^3\cdot\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$\rho = -2\,\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + 2\,(\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 + 2\,\mathrm{e}^2\cdot\mathrm{f}^2 - 2\,\mathrm{e}^3\cdot\mathrm{f}^3$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}(1,1)$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2\,(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\cdot\mathrm{f}^1$
$\mathfrak{su}(2) \oplus \mathfrak{h}_3$, $\mathfrak{e}(2) \oplus \mathfrak{h}_3$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \frac{5}{4}\,\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = \frac{5}{4}\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{5}{4}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + \frac{5}{4}\,(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $-\mathrm{e}^1\cdot\mathrm{f}^1 - \mathrm{e}^2\cdot\mathrm{f}^2 + \mathrm{e}^3\cdot\mathrm{f}^3$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}_3$, $\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$	$\rho = -\mathrm{e}^{23}\mathrm{f}^1 - \frac{5}{4}\,\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}$ $g = \frac{5}{4}\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{5}{4}\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + \frac{5}{4}\,(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $+\,\mathrm{e}^1\cdot\mathrm{f}^1 + \mathrm{e}^2\cdot\mathrm{f}^2 - \mathrm{e}^3\cdot\mathrm{f}^3$

Table 4: Solvable, non-unimodular direct sums admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^3\mathrm{f}^1 - \mathrm{f}^{23}$ $\rho = \mathrm{e}^{23}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{21} + \mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^1\mathrm{f}^{31}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$
$\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = -\mathrm{e}^1\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^2 + \mathrm{e}^2\mathrm{f}^1 - \mathrm{f}^{23}$ $\rho = \mathrm{e}^{23}\mathrm{f}^3 - 2\,\mathrm{e}^{31}\mathrm{f}^1 + \mathrm{e}^{12}\mathrm{f}^2 - 3\,\mathrm{e}^1\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + 2\,\mathrm{f}^{123}$ $g = 2\,(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2\,(\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + 5\,(\mathrm{f}^3)^2 - 2\,\mathrm{e}^1\cdot\mathrm{f}^2 - 6\,\mathrm{e}^3\cdot\mathrm{f}^3$

Table 5: Direct sums which are neither solvable nor unimodular

Lie algebra	Half-flat SU(3)-structure
$\mathfrak{su}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$\omega = \mathrm{e}^1\mathrm{f}^1 - \mathrm{f}^{23} + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{12} - \mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{f}^1)^2 + 2(\mathrm{f}^2)^2 + (\mathrm{f}^3)^2 - 2\mathrm{e}^3\cdot\mathrm{f}^2$
$\mathfrak{su}(2) \oplus \mathfrak{r}_3$	$\omega = \mathrm{f}^{23} + \mathrm{e}^{23} + 2\mathrm{e}^1\mathrm{f}^1$

Table 5 – continued on next page

Table 5 – continued

Lie algebra	Half-flat SU(3)-structure
	$\rho = \frac{2}{3}3^{\frac{3}{4}}\{\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{31} + \mathrm{e}^2\mathrm{f}^{12}\}$ $g = \frac{2}{3}\sqrt{3}\,\{2(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + (\mathrm{f}^3)^2$ $+2\mathrm{e}^1{\cdot}\mathrm{f}^1 - \mathrm{e}^2{\cdot}\mathrm{e}^3 + \mathrm{f}^2{\cdot}\mathrm{f}^3\}$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_3$	$\omega = \mathrm{e}^1\mathrm{f}^1 - 2\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2$ $\rho = \frac{1}{3}\mathrm{e}^{23}\mathrm{f}^1 + 3\mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{31}\mathrm{f}^3 + \mathrm{e}^{12}\mathrm{f}^2 + \frac{4}{3}\mathrm{e}^{12}\mathrm{f}^3 - 4\mathrm{e}^2\mathrm{f}^{31} + \frac{7}{3}\mathrm{e}^3\mathrm{f}^{31}$ $+3\mathrm{e}^2\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{12} - 26\mathrm{f}^{123}$ $g = 3(\mathrm{e}^1)^2 + \frac{4}{9}(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \frac{17}{3}(\mathrm{f}^1)^2 + 94(\mathrm{f}^2)^2 + \frac{328}{9}(\mathrm{f}^3)^2$ $-8\mathrm{e}^1{\cdot}\mathrm{f}^1 - \frac{2}{3}\mathrm{e}^2{\cdot}\mathrm{e}^3 + \frac{34}{3}\mathrm{e}^2{\cdot}\mathrm{f}^2 + \frac{16}{9}\mathrm{e}^2{\cdot}\mathrm{f}^3 - 16\mathrm{e}^3{\cdot}\mathrm{f}^2 - \frac{34}{3}\mathrm{e}^3{\cdot}\mathrm{f}^3 + \frac{224}{3}\mathrm{f}^2{\cdot}\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{r}_{3,\mu}$ $(0 < \mu \leq 1)$	$\omega = \frac{1}{\mu+1}\mathrm{e}^{12} + \mathrm{e}^3\mathrm{f}^1 - \mathrm{f}^{32}$ $\rho = \mu^{-\frac{1}{4}}(\mu+1)^{-\frac{1}{2}}\{\mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^{23}\mathrm{f}^3 - \mu\mathrm{e}^1\mathrm{f}^{13} - \mathrm{e}^2\mathrm{f}^{12}\}$ $g = \mu^{-\frac{1}{2}}\{\frac{\mu}{\mu+1}(\mathrm{e}^1)^2 + \frac{1}{\mu+1}(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \mu(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 + \mu(\mathrm{f}^3)^2\}$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	$\omega = \frac{1}{\mu+1}\mathrm{e}^{23} + \mathrm{e}^1\mathrm{f}^1 + \mathrm{f}^{32}$ $\rho = (-\mu)^{-\frac{1}{4}}(\mu+1)^{-\frac{1}{2}}\{\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^{13}\mathrm{f}^2 + \mathrm{e}^2\mathrm{f}^{12} - \mu\mathrm{e}^3\mathrm{f}^{13}\}$ $g = (-\mu)^{-\frac{1}{2}}\{(\mathrm{e}^1)^2 + \frac{1}{\mu+1}(\mathrm{e}^2)^2 - \frac{\mu}{\mu+1}(\mathrm{e}^3)^2 - \mu(\mathrm{f}^1)^2 + (\mathrm{f}^2)^2 - \mu(\mathrm{f}^3)^2\}$
$\mathfrak{su}(2) \oplus \mathfrak{r}_{3,\mu}$ $(-1 < \mu < 0)$	$\omega = \mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^1 - \frac{\mu(2\mu+3)}{2(\mu+1)^2}\mathrm{e}^{23} - \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^1\mathrm{f}^3 + \frac{\mu(2\mu+3)}{2(\mu+1)^2}\mathrm{e}^{12}$ $-\frac{2\mu^2+\mu-2}{2(\mu+1)^2}\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = -\frac{2\mu^2+3\mu+2}{2(\mu+1)^2}\mathrm{e}^{23}\mathrm{f}^1 - \frac{1}{\mu}\mathrm{e}^{23}\mathrm{f}^3 - 2\mathrm{e}^{13}\mathrm{f}^2 + \frac{2\mu^2+3\mu+2}{2(\mu+1)^2}\mathrm{e}^{12}\mathrm{f}^1$ $-\frac{1}{\mu}\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^1\mathrm{f}^{13} - \mathrm{e}^3\mathrm{f}^{13} + 2\mathrm{e}^2\mathrm{f}^{12} + 2\mathrm{f}^{123}$ $g = -\frac{\mu^2+\mu+1}{\mu(\mu+1)}(\mathrm{e}^1)^2 - \frac{4\mu^4+20\mu^3+29\mu^2+16\mu+4}{4\mu(\mu+1)^3}(\mathrm{e}^2)^2 - \frac{\mu^2+\mu+1}{\mu(\mu+1)}(\mathrm{e}^3)^2$ $-\frac{\mu}{\mu+1}(\mathrm{f}^1)^2 + \frac{4+3\mu}{\mu+1}(\mathrm{f}^2)^2 - \frac{\mu+1}{\mu}(\mathrm{f}^3)^2$ $+\frac{2(\mu^2+1+3\mu)}{\mu(\mu+1)}\mathrm{e}^1{\cdot}\mathrm{e}^3 + \frac{2(\mu+2)}{\mu+1}\mathrm{e}^1{\cdot}\mathrm{f}^2 - \frac{2\mu^2+5\mu+2}{\mu(\mu+1)}\mathrm{e}^2{\cdot}\mathrm{f}^3 + \frac{2(\mu+2)}{\mu+1}\mathrm{e}^3{\cdot}\mathrm{f}^2$
$\mathfrak{sl}(2) \oplus \mathfrak{r}_{3,\mu}$ $(0 < \mu \leq 1)$	$\omega = \frac{2(2\mu+1)^{\frac{1}{2}}}{(\mu+1)^2}\mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^1 + \mathrm{f}^{23} + \frac{\mu}{\mu+1}\mathrm{e}^{13} + \mathrm{e}^1\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$ $\rho = 2\frac{2(2\mu+1)^{\frac{1}{2}}}{(\mu+1)^2}\mathrm{e}^{123} + \mathrm{e}^{23}\mathrm{f}^2 - \mathrm{e}^{13}\mathrm{f}^1 + \frac{1}{\mu}\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{13} + \mathrm{e}^1\mathrm{f}^{12} + \frac{\mu+1}{\mu}\mathrm{f}^{123}$ $g = \frac{\mu^3+11\mu^2+7\mu+1}{\mu(\mu+1)^3}(\mathrm{e}^1)^2 + \frac{\mu+1}{\mu}(\mathrm{e}^2)^2 + (2\mu+1)(\mathrm{e}^3)^2 + \frac{\mu+1}{\mu}(\mathrm{f}^1)^2$ $+\frac{\mu+1}{\mu^2}(\mathrm{f}^3)^2 + \frac{1+3\mu+2\mu^2}{\mu}(\mathrm{f}^2)^2 + \frac{6(2\mu+1)^{\frac{1}{2}}}{\mu+1}\mathrm{e}^1{\cdot}\mathrm{e}^3 + \frac{2(2\mu+1)^{\frac{1}{2}}(3\mu+1)}{\mu(\mu+1)}\mathrm{e}^1{\cdot}\mathrm{f}^2$ $+\frac{4(2\mu+1)}{\mu(\mu+1)^2}\mathrm{e}^1{\cdot}\mathrm{f}^3 + \frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\mathrm{e}^2{\cdot}\mathrm{f}^1 + (4+4\mu)\mathrm{e}^3{\cdot}\mathrm{f}^2 + \frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\mathrm{e}^3{\cdot}\mathrm{f}^3$ $+\frac{2(2\mu+1)^{\frac{1}{2}}}{\mu}\mathrm{f}^2{\cdot}\mathrm{f}^3$
$\mathfrak{su}(2) \oplus \mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	$\omega = \mathrm{e}^2\mathrm{f}^2 - 2\mu\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^1\mathrm{f}^1$ $\rho = \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{31} - \mu\mathrm{e}^3\mathrm{f}^{31} + \mu\mathrm{e}^2\mathrm{f}^{12} + \mathrm{e}^3\mathrm{f}^{12}$

Table 5 – continued on next page

Table 5 – continued

Lie algebra	Half-flat SU(3)-structure
	$+(\mu^2-1)\mathrm{f}^{123}$ $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{f}^1)^2 + (\mu^2+1)(\mathrm{f}^2)^2 + (\mu^2+1)(\mathrm{f}^3)^2$ $+2\mathrm{e}^1\cdot\mathrm{f}^1 + 2\mu\mathrm{e}^2\cdot\mathrm{f}^3 - 2\mu\mathrm{e}^3\cdot\mathrm{f}^2$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}'_{3,\mu}$ $(\mu > 0)$	$\omega = \mathrm{e}^2\mathrm{f}^2 - 2\mu\mathrm{f}^{23} + \mathrm{e}^3\mathrm{f}^3 + \mathrm{e}^1\mathrm{f}^1$ $\rho = \frac{1}{2}\mathrm{e}^{23}\mathrm{f}^1 + 2\mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{12}\mathrm{f}^3 + 2\mathrm{e}^2\mathrm{f}^{31} + \mu\mathrm{e}^3\mathrm{f}^{31}$ $+2\mu\mathrm{e}^2\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{12} - (4\mu^2 + \frac{29}{4})\mathrm{f}^{123}$ $g = 2(\mathrm{e}^1)^2 + \frac{1}{2}(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + \frac{13}{8}(\mathrm{f}^1)^2 + (16\mu^2 + \frac{29}{2})(\mathrm{f}^2)^2$ $+(2\mu^2 + \frac{29}{4})(\mathrm{f}^3)^2 + 3\mathrm{e}^1\cdot\mathrm{f}^1 - 5\mathrm{e}^2\cdot\mathrm{f}^2 - 2\mu\mathrm{e}^2\cdot\mathrm{f}^3 - 8\mu\mathrm{e}^3\cdot\mathrm{f}^2 + 5\mathrm{e}^3\cdot\mathrm{f}^3$ $-10\mu\mathrm{f}^2\cdot\mathrm{f}^3$

4. Half-flat SU(1, 2)-structures on direct sums

An interesting question is whether the results proved in the previous sections for half-flat structures inducing Riemannian metrics also hold for half-flat $\mathrm{SU}(p,q)$-structures, $p+q=3$, with indefinite metrics. It suffices to consider $\mathrm{SU}(1,2)$-structures after possibly multiplying the metric by minus one.

First of all, the obstruction condition of Proposition 3.2 does not apply since isotropic subspaces are of course possible for metrics of signature $(2,4)$. For instance, the Lie algebra $\mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ does admit a half-flat $\mathrm{SU}(1,2)$-structure but no half-flat $\mathrm{SU}(3)$-structure. Indeed, the structure defined in the standard basis by

$$\begin{aligned}
\rho &= -\mathrm{e}^{123} - \mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^{12}\mathrm{f}^2 + 2\,\mathrm{e}^{13}\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{13} + \mathrm{f}^{123},\\
\omega &= \mathrm{e}^{13} - \mathrm{e}^1\mathrm{f}^2 + \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^3 - \mathrm{f}^{12},\\
g &= -(\mathrm{e}^2)^2 - 2\,(\mathrm{f}^3)^2 + 2\,\mathrm{e}^1\cdot\mathrm{e}^3 + 2\,\mathrm{e}^1\cdot\mathrm{f}^2 + 2\,\mathrm{e}^1\cdot\mathrm{f}^3 - 2\,\mathrm{e}^2\cdot\mathrm{f}^3 + 2\,\mathrm{e}^3\cdot\mathrm{f}^1 + 2\,\mathrm{f}^1\cdot\mathrm{f}^3,
\end{aligned}$$

is a half-flat $\mathrm{SU}(1,2)$-structure with $V = \mathrm{span}\{\mathrm{e}^1, \mathrm{f}^1\}$ J_ρ-invariant and isotropic.

In fact, the obstruction established in Lemma 3.10 is stronger and also shows the non-existence of a half-flat $\mathrm{SU}(1,2)$-structure on $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathbb{R}^3$. It can be generalised to the following Lie algebras.

Lemma 4.1. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie algebra such that $\mathfrak{g}_1$ is one of the algebras $\mathbb{R}^3$, $\mathfrak{h}_3$ or $\mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{g}_2$ is one of the algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\mu}$, $0 < |\mu| \leq 1$, $\mathfrak{r}'_{3,\mu}$, $\mu > 0$.*

Every closed three-form ρ on one of these Lie algebras $\mathfrak{g}$ satisfies $\lambda(\rho) \geq 0$. In particular, these Lie algebras do not admit a half-flat $\mathrm{SU}(p,q)$-structure for any signature (p,q) with $p+q=3$.

Proof. The proof is straightforward, but tedious without computer support. In a fixed basis, the condition $d\rho = 0$ is linear in the coefficients of an arbitrary three-form ρ and can be solved directly. When identifying $\Lambda^6 V^*$ with $\mathbb{R}$ with the help of a volume form ν, one can calculate the quartic invariant $\lambda(\rho) \in \mathbb{R}$, for instance in a standard basis.

Carrying this out with Maple and factorising the resulting expression, we verified $\lambda(\rho) \geq 0$ for an arbitrary closed three-form on any of the Lie algebras in question.

As a half-flat $\mathrm{SU}(p,q)$-structure is defined by a pair (ρ, ω) of stable forms which satisfy in particular $\lambda(\rho) < 0$ and $d\rho = 0$, such a structure cannot exist and the lemma is proven. $\square$

We add the remark, that a result analogous to Lemma 2.1 for a pseudo-Hermitian structure of indefinite signature would involve a considerably more complicated normal form for ω. Therefore, a generalisation of the proof of Theorem 2.2 to indefinite metrics seems to be difficult.

5. Half-flat $\mathrm{SL}(3,\mathbb{R})$-structures on direct sums

Finally, we turn to the para-complex case of $\mathrm{SL}(3,\mathbb{R})$-structures. Recall that a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure is defined by a pair (ρ, ω) of stable forms such that J_ρ is an almost para-complex structure and

$$\omega \wedge \rho = 0, \quad d\omega^2 = 0, \quad d\rho = 0.$$

As the induced metric is always neutral and $\lambda(\rho) > 0$, neither Proposition 3.2 nor Lemma 4.1 obstruct the existence of such a structure. For instance, the Lie algebra $\mathfrak{r}_2 \oplus \mathbb{R} \oplus \mathfrak{r}_3$ does not admit a half-flat $\mathrm{SU}(p,q)$-structure for any signature (p,q) with $p+q=3$, but

$$\begin{aligned}
\rho &= -2\,\mathrm{e}^{12}\mathrm{f}^3 - 2\,\mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^3\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{31} + \mathrm{f}^{123},\\
\omega &= \mathrm{e}^{13} - \mathrm{e}^{23} + \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2 - \mathrm{e}^3\mathrm{f}^1 + 2\,\mathrm{f}^{13},\\
g &= -2\,(\mathrm{e}^1\!\cdot\!\mathrm{e}^3 - \mathrm{e}^2\!\cdot\!\mathrm{e}^3 + \mathrm{e}^1\!\cdot\!\mathrm{f}^3 + \mathrm{e}^2\!\cdot\!\mathrm{f}^2 + \mathrm{e}^3\!\cdot\!\mathrm{f}^1\,),
\end{aligned}$$

is an example of a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure.

When trying to generalise Theorem 2.2 to the para-complex situation, we find an astonishingly similar result if we additionally require the metric to be definite when restricted to one of the summands. We omit the proofs which are very similar to the original ones due to the analogies explained in section 1.4.

Lemma 5.1. *Let (V_1, g_1) and (V_2, g_2) be Euclidean vector spaces and let (g, J, ω) be a para-Hermitian structure on the orthogonal product $(V_1 \oplus V_2, g = -g_1 + g_2)$. There are orthonormal bases $\{e_1, e_2, e_3\}$ of V_1 and $\{f_1, f_2, f_3\}$ of V_2 which can be joined to a pseudo-orthonormal basis of $V_1 \oplus V_2$ such that*

$$\omega = a\,\mathrm{e}^{12} + \sqrt{1+a^2}\,\mathrm{e}^1\mathrm{f}^1 + \sqrt{1+a^2}\,\mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 + a\,\mathrm{f}^{12} \tag{5.1}$$

for a real number a.

In analogy to the Hermitian case, we call the para-Hermitian structure of type I if $a = 0$ and of type II if $a \neq 0$.

Theorem 5.2. *A direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of three-dimensional Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ admits a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure (g, J, ω, Ψ) such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal with respect to the metric g and the restriction of g to both summands is definite if and only if the pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ is contained in the following list:*

type I : a) $\mathfrak{g}_1 = \mathfrak{g}_2$ *unimodular,* *b)* $\mathfrak{g}_1$ *non-abelian unimodular and* $\mathfrak{g}_2$ *abelian or vice versa,*

$$\begin{aligned}
\textit{type II} \quad : \quad & (\mathfrak{e}(1,1), \mathfrak{e}(1,1)), \\
& (\mathfrak{e}(2), \mathbb{R} \oplus \mathfrak{r}_2), \\
& (\mathfrak{su}(2), \mathfrak{r}_{3,\mu}) \quad \textit{for} \quad 0 < \mu \leq 1, \\
& (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{r}_{3,\mu}) \quad \textit{for} \quad -1 < \mu < 0.
\end{aligned}$$

If we require, instead of orthogonality, that the $\mathrm{SL}(3,\mathbb{R})$-structure is adapted to the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the sense that the summands $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the eigenspaces of J, we find the following interesting relation to unimodularity.

PROPOSITION 5.3. *A direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *of three-dimensional Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *admits a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure* (g, J, ω, Ψ) *such that* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are the* ± 1*-eigenspaces of* J *if and only if both* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are unimodular.*

PROOF. Let (g, J, ω, Ψ) be an $\mathrm{SL}(3,\mathbb{R})$-structure on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_1$ is the $+1$-eigenspace of J and $\mathfrak{g}_2$ is the -1-eigenspaces of J. Since $\psi^+ = \mathrm{Re}\Psi$ is a stable form inducing the para-complex structure J, we can choose bases $\{\mathrm{e}^i\}$ of ${\mathfrak{g}_1}^*$ and $\{\mathrm{f}^i\}$ of ${\mathfrak{g}_2}^*$ such that $\psi^+ = \mathrm{e}^{123} + \mathrm{f}^{123}$ is in the normal form (3.8). Thus, the real part ψ^+ is closed as we are dealing with a direct sum of Lie algebras. Due to the simple form of ψ^+, it is easy to verify that the relation $\omega \wedge \psi^+ = 0$ holds for an arbitrary non-degenerate ω if and only if ω has only terms in ${\mathfrak{g}_1}^* \otimes {\mathfrak{g}_2}^*$. Now we are in the situation of Lemma 1.2 and conclude that the only remaining equation $d\omega^2 = 0$ is satisfied if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular. $\square$

CHAPTER 5

Description of all half-flat structures on certain Lie groups

After solving the problem on which direct sums of three-dimensional Lie groups there do exist left-invariant half-flat structures, the question arises how many there are on each Lie group. A reasonable restriction in the left-invariant case is to describe all half-flat structures modulo Lie group automorphisms. For a simply connected Lie group, it is equivalent to describe all half-flat structures on the corresponding Lie algebra modulo Lie algebra automorphisms. If the Lie group is not simply connected, it has to be distinguished between inner Lie algebra automorphisms, which always lift to a Lie group automorphism, and outer Lie algebra automorphisms, which do not necessarily lift to the group level.

In this chapter, the uniqueness problem for half-flat structures is studied in detail for the compact Lie algebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and the nilpotent Lie algebra $\mathfrak{h}_3\oplus\mathfrak{h}_3$ including also the case of nearly half-flat structures. As this analysis turns out to be quite technical, we do not attempt to complete the classification of *all* half-flat structures on direct sums of three-dimensional Lie groups which had to be done for each of the 78 cases of direct sums individually.

For the Lie algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, we consider the following uniqueness problem which can be solved by a similar method. As explained in the introduction, there is a left-invariant nearly pseudo-Kähler structure on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ and we prove the uniqueness of this structure in analogy to the case of the nearly Kähler structure on $S^3\times S^3$, see [**Bu1**].

1. Half-flat structures on $S^3\times S^3$

Let $\mathfrak{g}$ be a Lie algebra. By definition, the exterior differential

$$d\,:\,\mathfrak{g}^*\to\Lambda^2\mathfrak{g}^*,\ \theta\mapsto d\theta\,,\qquad d\theta(X,Y)=-\theta([X,Y]), \tag{1.1}$$

commutes with the action of the group of Lie algebra automorphisms $\mathrm{Aut}(\mathfrak{g})$. For every simple Lie algebra, d is injective on one-forms since the annihilator of the kernel of d, the derived subalgebra $[\mathfrak{g},\mathfrak{g}]$, would be a proper ideal if the kernel would not vanish. Thus, on the two three-dimensional simple Lie algebras $\mathfrak{su}(2)\cong\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$, the mapping (1.1) is a natural equivariant isomorphism of $\mathrm{Aut}(\mathfrak{g})$-modules.

Recall that we defined a standard basis of $\mathfrak{su}(2)$ as a basis $\{\mathrm{e}_1,\,\mathrm{e}_2,\,\mathrm{e}_3\}$ such that the Lie bracket is given on the dual basis by

$$d(\mathrm{e}^i)=\mathrm{e}^{(i+1)(i+2)}$$

with $i\in\{1,2,3\}$ and indices taken modulo 3. Given a standard basis, we will always identify $\Lambda^2\mathfrak{g}^*=\mathbb{R}^3$ using the induced basis $\{e_{23},e_{31},e_{12}\}$. With this convention, the isomorphism (1.1) is represented by the identity matrix with respect to standard basis.

Moreover, the automorphism group $\mathrm{Aut}(\mathfrak{su}(2))$ with respect to this basis equals the standard matrix group $\mathrm{SO}(3,\mathbb{R})$ and acts by usual matrix multiplication on both $\mathfrak{g}^* = \mathbb{R}^3$ and $\Lambda^2\mathfrak{g}^* = \mathbb{R}^3$.

From now on, we consider $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with a standard basis which is defined as the union of standard bases $\{\mathrm{e}_i\}$ and $\{\mathrm{f}_i\}$ of the summands. The group of Lie algebra automorphisms with respect to this basis equals

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{SO}(3)\times\mathrm{SO}(3) \,\cup\, \xi(\mathrm{SO}(3)\times\mathrm{SO}(3)) \tag{1.2}$$

where $\mathrm{SO}(3) \times \mathrm{SO}(3)$ is embedded diagonally in $\mathrm{GL}(\mathfrak{g})$ and $\xi = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}$ is an outer automorphism exchanging the factors.

To begin with, we establish a normal form for a coclosed stable two-form ω on $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Lemma 1.1. *On a Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ which is a direct sum of three-dimensional simple Lie algebras, a non-degenerate two-form ω satisfies $d\omega^2 = 0$ if and only if $\omega \in \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$.*

Proof. Since d is an isomorphism on one-forms on both summands, all two-forms in $\Lambda^2\mathfrak{g}_1^*$ and $\Lambda^2\mathfrak{g}_2^*$ are exact and thus closed. In particular, all four-forms in $\Lambda^2\mathfrak{g}_1^* \otimes \Lambda^2\mathfrak{g}_2^*$ are closed proving that ω^2 is closed if $\omega \in \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$. Conversely, no non-trivial four-form in $(\Lambda^3\mathfrak{g}_1^* \otimes \mathfrak{g}_2^*) \oplus (\mathfrak{g}_1^* \otimes \Lambda^3\mathfrak{g}_2^*)$ is closed. Thus, a two-form ω with $d\omega^2 = 0$ satisfies either $\omega \in \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$ or $\omega \in \Lambda^2\mathfrak{g}_1^* \oplus \Lambda^2\mathfrak{g}_2^*$. However, the second case can be excluded since ω is non-degenerate and proof is finished. $\square$

Lemma 1.2. *On the Lie algebra $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, consider the action of $\mathrm{Aut}(\mathfrak{g})$ on the set of non-degenerate two-forms on $\mathfrak{g}$ with $d\omega^2 = 0$. In any standard basis, a minimal system of representatives of the orbits under this action is given by*

$$\left\{\omega = \alpha\, \mathrm{e}^1\mathrm{f}^1 + \beta\, \mathrm{e}^2\mathrm{f}^2 + \gamma\, \mathrm{e}^3\mathrm{f}^3 \;\middle|\; 0 < \alpha \leq \beta \leq \gamma\right\}. \tag{1.3}$$

Proof. By Lemma 1.1, every non-degenerate two-form ω with $d\omega^2 = 0$ can be written in a standard basis as $\omega = s_{ij}\mathrm{e}^i\mathrm{f}^j$ for an invertible 3×3 matrix $S = (s_{ij})$. An element $(X, Y) \in \mathrm{SO}(3) \times \mathrm{SO}(3)$ acts on ω by the transformation $S \mapsto X^tSY$ and the outer automorphism ξ acts by $S \mapsto -S$. Now, the polar decomposition of S and the diagonalisation of the symmetric part yield $S = O_1O_2^tDO_2$ for two orthogonal matrices $O_1, O_2 \in \mathrm{O}(3)$ and a unique diagonal matrix $D = \mathrm{diag}(\alpha, \beta, \gamma)$ with $0 < \alpha \leq \beta \leq \gamma$. Thus, every ω is mapped to a unique representative of the form (1.3) by applying the Lie algebra automorphism defined by $X^t = \pm O_2O_1^t \in \mathrm{SO}(3)$ and $Y = \pm O_2^t \in \mathrm{SO}(3)$ followed by ξ if necessary. $\square$

A similar result can be obtained for three-forms as follows. An arbitrary three-form can be written in a standard basis as

$$\rho \;=\; a_1\mathrm{e}^{123} + \sum_{i,k=1}^{3} b_{ik}\mathrm{e}^{(i+1)(i+2)}\mathrm{f}^k + \sum_{i,k=1}^{3} c_{ik}\mathrm{e}^i\mathrm{f}^{(k+1)(k+2)} + a_2\mathrm{f}^{123} \tag{1.4}$$

and we arrange the 20 parameters in a column vector $A = (a_1, a_2)^t$ and two 3×3 matrices $B = (b_{ik})$ and $C = (c_{ik})$. The action of an inner Lie algebra automorphism $(X, Y) \in \mathrm{SO}(3) \times \mathrm{SO}(3)$ on a three-form ρ defined by the triple (A, B, C) is given by

$$(A, B, C) \mapsto (A,\; X^tBY,\; X^tCY). \tag{1.5}$$

Indeed, the invariance of A is due to the fact that both X and Y have determinant 1. Moreover, as the equivariant isomorphism (1.1) is given by the identity matrix, both B and C transform like a two-form in $\mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$ and the transformation rule for B and C follows.

Due to the simplicity of this transformation rule, we are able to derive normal forms for closed three-forms and compatible pairs of stable forms satisfying the half-flat equations.

LEMMA 1.3. *Let the space of three-forms on* $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ *be identified with the space of triples* (A, B, C) *according to (1.4) in a standard basis. Then, a three-form* (A, B, C) *is closed if and only* $C = -B$*. A minimal system of representatives of closed three-forms under the action of* $\mathrm{Aut}(\mathfrak{g})$ *is given by the closed subset*

$$\{\,(A, B, -B) \mid B = \mathrm{diag}(b_1, b_2, b_3),\ b_i \geq 0,\ i = 1, 2, 3\,\}$$

of $\mathbb{R}^5$.

PROOF. The assertion that a three-form (A, B, C) is closed if and only if $C = -B$ is obvious by definition of the standard basis. Since $\mathrm{Aut}(\mathfrak{g})$ acts on (A, B, C) by the formula (1.5), the matrix B transforms exactly as the invertible matrix S in the proof of Lemma 1.2. Considering the normal form of S and the fact that B is not required to be invertible, the assertion on the normal form of $(A, B, -B)$ is immediate. □

The following theorem is the main result of this section and can be viewed as the explicit description of the "moduli space" of left-invariant half-flat structures on $S^3 \times S^3$.

THEOREM 1.4. *On the Lie algebra* $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$*, consider the action of* $\mathrm{Aut}(\mathfrak{g})$ *on the compatible pairs* $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ *of stable forms satisfying the half-flat equations*

$$d\rho = 0\,, \qquad d\omega^2 = 0.$$

In any standard basis, a minimal system of representatives under this action is given by

$$\begin{aligned}(1.6)\qquad \{\,(\omega,\rho) \mid\ & \omega = \alpha\,\mathrm{e}^1\mathrm{f}^1 + \beta\,\mathrm{e}^2\mathrm{f}^2 + \gamma\,\mathrm{e}^3\mathrm{f}^3,\\ & \rho = a_1\mathrm{e}^{123} + \sum_i^3 b_i\,(\mathrm{e}^{(i+1)(i+2)}\mathrm{f}^i - \mathrm{e}^i\mathrm{f}^{(i+1)(i+2)}) + a_2\mathrm{f}^{123},\\ & (a_1, a_2, b_1, b_2, b_3, \alpha, \beta, \gamma)^t \in \mathbb{R}^8 \text{ such that } \lambda(\rho) \neq 0 \text{ and either}\\ & 0 < \alpha < \beta < \gamma \text{ or}\\ & 0 < \alpha = \beta,\ 0 < \gamma,\ b_1 \leq b_2 \text{ or}\\ & 0 < \alpha = \beta = \gamma,\ b_1 \leq b_2 \leq b_3\,\}.\end{aligned}$$

When identifying $\Lambda^6\mathfrak{g}^*$ *with* $\mathbb{R}$ *via the volume form* $\nu = \mathrm{e}^{123456}$*, the quartic invariant* $\lambda(\rho)$ *associated to a stable three-form* ρ *in this minimal system is the homogeneous polynomial*

$$(1.7)\quad \lambda = b_1^4 + b_2^4 + b_3^4 + 4b_1b_2b_3(a_2 - a_1) - 2b_1^2b_2^2 - 2b_2^2b_3^2 - 2b_3^2b_1^2 + 2a_1a_2(b_1^2 + b_2^2 + b_3^2) + a_1^2a_2^2$$

of order four in the five variables a_1, a_2, b_1, b_2, b_3*. A pair* (ω, ρ) *in the set (1.6) is normalised such that* $\phi(\rho) = 2\phi(\omega)$ *if and only if*

$$(1.8)\qquad 2\alpha\beta\gamma = \sqrt{\varepsilon\lambda}.$$

PROOF. We choose a standard basis and assume that ω is in the normal form (1.3) with $0 < \alpha \leq \beta \leq \gamma$ which represents exactly the orbits of the coclosed two-forms under $\mathrm{Aut}(\mathfrak{g})$.

It remains to determine, for each triple $(\alpha,\beta,\gamma)^t$, a minimal system of representatives of closed compatible stable three-forms under the action of $H := \mathrm{Stab}_{\mathrm{Aut}(\mathfrak{g})}(\omega)$.

When parametrising a closed three-forms by a triple $(A, B, -B)$ according to (1.4), the six coefficients of the five-form $\omega \wedge \rho$ are

$$\alpha b_{12} - \beta b_{21}\,, \quad \alpha b_{21} - \beta b_{12}\,, \quad \alpha b_{13} - \gamma b_{31}\,, \quad \alpha b_{31} - \gamma b_{13}\,, \quad \beta b_{23} - \gamma b_{32}\,, \quad \beta b_{32} - \gamma b_{23}.$$

Thus, the compatibility $\omega \wedge \rho = 0$ is satisfied if and only if

$$b_{12} = \frac{\beta}{\alpha} b_{21} = \frac{\alpha}{\beta} b_{21}\,, \quad b_{31} = \frac{\alpha}{\gamma} b_{13} = \frac{\gamma}{\alpha} b_{13}\,, \quad b_{23} = \frac{\gamma}{\beta} b_{32} = \frac{\beta}{\gamma} b_{32}\,. \tag{1.9}$$

By separating three cases, we determine all solutions modulo the action of H.

First of all, if $0 < \alpha < \beta < \gamma$, the equations are satisfied if and only if all b_{ij} with $i \neq j$ vanish. Since the stabiliser H of ω in $\mathrm{Aut}(\mathfrak{g})$ is trivial in this case, there is no further restriction on the parameters $a_1, a_2, b_{11} = b_1, b_{22} = b_2, b_{33} = b_3$ except for $\lambda \neq 0$.

Secondly, if two of the three parameters are equal and the third is different, we can achieve $\alpha = \beta$ by reordering with the help of the $\mathrm{Aut}(\mathfrak{g})$-action. After this reordering, it still holds $\alpha > 0$, $\gamma > 0$, but not necessarily $\alpha < \gamma$. In this case, the equations are satisfied if and only if $b_{12} = b_{21}$ and $b_{13} = b_{31} = b_{23} = b_{32} = 0$. In this case, the stabiliser H of ω in $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{SO}(2)$ embedded in the upper left corner of $\mathrm{SO}(3)$ and then diagonally into $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Hence, we can diagonalise the symmetric upper left corner of B with the action of H. The resulting eigenvalues $b_1 \leq b_2$ uniquely characterise these types of orbits.

Thirdly, if $\alpha = \beta = \gamma$, the equations are satisfied if and only if B is symmetric. However, in this case, the stabiliser H equals $\mathrm{SO}(3)$ embedded diagonally into $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Recalling the transformation rule of B, (1.5), it is again possible to diagonalise B as it is symmetric. The eigenvalues b_1, b_2, b_3 of B are possibly negative, but can be ordered such that $b_1 \leq b_2 \leq b_3$ uniquely determine the remaining orbits.

This shows that the set (1.6) is a minimal system of the orbits under the $\mathrm{Aut}(\mathfrak{g})$-action as we have discussed all solutions of $\omega \wedge \rho = 0$. The computation of the corresponding invariant $\lambda(\rho)$ is straightforward. The normalisation condition is immediate since $\phi(\rho) = \sqrt{|\lambda(\rho)|}$ by definition and $\phi(\omega) = \frac{1}{6}\omega^3 = \alpha\beta\gamma \mathrm{e}^{123}\mathrm{f}^{123}$. $\square$

For applications, it will be useful to have explicit formulas for the tensors induced by such a pair (ω, ρ).

Corollary 1.5. *The tensors $J = J_\rho$, $\hat{\rho} = J^*\rho$ and $g = g_{(\omega,\rho)}$ induced by a pair (ω,ρ) defined by (1.6) are given with respect to the eight parameters $(a_1, a_2, b_1, b_2, b_3, \alpha, \beta, \gamma)$ by the following formulas. We use the abbreviation $\kappa = \frac{1}{\sqrt{\varepsilon\lambda}}$ and the formula $J_\rho^*\rho = -\varepsilon\rho(J_\rho\cdot, \cdot, \cdot)$.*

$$\begin{aligned}
J(\mathrm{e}_1) &= \kappa(a_1a_2 - b_1^2 + b_2^2 + b_3^2)\, \mathrm{e}_1 + \kappa(2a_1b_1 + 2b_2b_3)\, \mathrm{f}_1\\
J(\mathrm{e}_2) &= \kappa(a_1a_2 + b_1^2 - b_2^2 + b_3^2)\, \mathrm{e}_2 + \kappa(2a_1b_2 + 2b_1b_3)\, \mathrm{f}_2\\
J(\mathrm{e}_3) &= \kappa(a_1a_2 + b_1^2 + b_2^2 - b_3^2)\, \mathrm{e}_3 + \kappa(2a_1b_3 + 2b_1b_2)\, \mathrm{f}_3\\
J(\mathrm{f}_1) &= \kappa(2b_1a_2 - 2b_2b_3)\, \mathrm{e}_1 + \kappa(-a_1a_2 + b_1^2 - b_2^2 - b_3^2)\, \mathrm{f}_1\\
J(\mathrm{f}_2) &= \kappa(2b_2a_2 - 2b_1b_3)\, \mathrm{e}_2 + \kappa(-a_1a_2 - b_1^2 + b_2^2 - b_3^2)\, \mathrm{f}_2\\
J(\mathrm{f}_3) &= \kappa(2b_3a_2 - 2b_1b_2)\, \mathrm{e}_3 + \kappa(-a_1a_2 - b_1^2 - b_2^2 + b_3^2)\, \mathrm{f}_3
\end{aligned}$$

$$\begin{aligned}
J^*\rho = -\varepsilon\kappa \, \{ \, & (-a_1(a_1a_2 + b_1^2 + b_2^2 + b_3^2) - 2b_1b_2b_3)\, \mathrm{e}^{123} \\
& + (a_2(a_1a_2 + b_1^2 + b_2^2 + b_3^2) - 2b_1b_2b_3)\, \mathrm{f}^{123} \\
& - (a_1a_2b_1 \quad + b_1^3 - b_1b_2^2 - b_1b_3^2 + 2a_2b_2b_3)\, \mathrm{e}^1\mathrm{f}^{23} \\
& + (a_1a_2b_2 - b_2b_1^2 \quad + b_2^3 - b_2b_3^2 + 2a_2b_1b_3)\, \mathrm{e}^2\mathrm{f}^{13} \\
& - (a_1a_2b_3 - b_3b_1^2 - b_3b_2^2 \quad + b_3^3 + 2a_2b_1b_2)\, \mathrm{e}^3\mathrm{f}^{12} \\
& - (a_1a_2b_1 \quad + b_1^3 - b_1b_2^2 - b_1b_3^2 - 2a_1b_3b_2)\, \mathrm{e}^{23}\mathrm{f}^1 \\
& + (a_1a_2b_2 - b_2b_1^2 \quad + b_2^3 - b_2b_3^2 - 2a_1b_3b_1)\, \mathrm{e}^{13}\mathrm{f}^2 \\
& - (a_1a_2b_3 - b_3b_1^2 - b_3b_2^2 \quad + b_3^3 - 2a_1b_2b_1)\, \mathrm{e}^{12}\mathrm{f}^3 \, \}
\end{aligned}$$

$$\begin{aligned}
g = -2\varepsilon\kappa \, \{ \, & \alpha(b_2b_3 + a_1b_1)\,(\mathrm{e}^1)^2 + \beta(b_1b_3 + a_1b_2)\,(\mathrm{e}^2)^2 + \gamma(b_1b_2 + a_1b_3)\,(\mathrm{e}^3)^2 \\
& + \alpha(b_2b_3 - a_2b_1)\,(\mathrm{f}^1)^2 + \beta(b_1b_3 - a_2b_2)\,(\mathrm{f}^2)^2 + \gamma(b_1b_2 - a_2b_3)\,(\mathrm{f}^3)^2 \\
& - \alpha(a_1a_2 - b_1^2 + b_2^2 + b_3^2)\, \mathrm{e}^1 \cdot \mathrm{f}^1 - \beta(a_1a_2 + b_1^2 - b_2^2 + b_3^2)\, \mathrm{e}^2 \cdot \mathrm{f}^2 \\
& - \gamma(a_1a_2 + b_1^2 + b_2^2 - b_3^2)\, \mathrm{e}^3 \cdot \mathrm{f}^3 \, \}.
\end{aligned}$$

Similarly, we are also able to parametrise all left-invariant nearly half-flat structures on $S^3 \times S^3$. Recall that an $\mathrm{SU}^\varepsilon(p,q)$-structure (ω, ρ) is nearly half-flat if and only if there is a non-zero constant ν such that $d\rho = \nu\omega^2$.

Proposition 1.6. *There is a one-to-one correspondence between left-invariant half-flat structures and left-invariant nearly half-flat structures on* $S^3 \times S^3$. *More explicitly, given a standard basis of* $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ *such that the half-flat structures* (ω, ρ) *modulo* $\mathrm{Aut}(\mathfrak{g})$ *are parametrised by the minimal system* (1.6)*, the image of the mapping*

$$\omega \mapsto \omega\,, \quad (A, B, -B) \mapsto (A, B, -B - \lambda\, \mathrm{diag}(\beta\gamma, \alpha\gamma, \alpha\beta))$$

is a minimal system of representatives for the action of $\mathrm{Aut}(\mathfrak{g})$ *on the nearly half-flat structures for the constant* λ.

Proof. We start by describing the orbits of nearly half-flat structure under $\mathrm{Aut}(\mathfrak{g})$. Since $d\omega^2 = 0$ for a nearly half-flat structure (ω, ρ), we can choose, as before, a standard basis such that ω belongs to the minimal system of representatives given by (1.3) and such that

$$\omega^2 = -2\alpha\beta\, \mathrm{e}^{12}\mathrm{f}^{12} - 2\alpha\gamma\, \mathrm{e}^{13}\mathrm{f}^{13} - 2\beta\gamma\, \mathrm{e}^{23}\mathrm{f}^{23}. \tag{1.10}$$

Is is easy to see, that a three-form ρ, given by a triple (A, B, C) with respect to this standard basis by formula (1.4), satisfies $d\rho = \frac{\lambda}{2}\omega^2$ if and only if $C = -B - \lambda\, \mathrm{diag}(\beta\gamma, \alpha\gamma, \alpha\beta)$. However, the diagonal entries of B do not appear in the coefficients of the five-form $\omega \wedge \rho$. Thus, the compatibility condition $\omega \wedge \rho = 0$ is satisfied if and only the six equations (1.9) are satisfied. In particular, the solutions modulo $\mathrm{Aut}(\mathfrak{g})$ are exactly described by the image of the mapping given in the proposition and the assertion follows. □

Recall that a double half-flat structure (ω, ρ) has been defined as a half-flat structure satisfying additionally

$$d(J_\rho^*\rho) = \mu\omega^2 \tag{1.11}$$

for a constant $\mu \in \mathbb{R}^*$.

PROPOSITION 1.7. *A half-flat structure of the form (1.6) is double half-flat if and only if there is a constant* $\mu \in \mathbb{R}^*$ *such that*

$$\begin{aligned} \mu\alpha\beta &= \varepsilon\kappa\,(b_3 b_1^2 + b_3 b_2^2 - b_3^3 + b_1 b_2(a_1 - a_2) - a_1 a_2 b_3), \\ \mu\alpha\gamma &= \varepsilon\kappa\,(b_2 b_1^2 - b_2^3 + b_2 b_3^2 + b_1 b_3(a_1 - a_2) - a_1 a_2 b_2), \\ \mu\beta\gamma &= \varepsilon\kappa\,(-b_1^3 + b_1 b_2^2 + b_1 b_3^2 + b_2 b_3(a_1 - a_2) - a_1 a_2 b_1). \end{aligned}$$

Notice that, since α, β and γ are different from zero and neither of them appears on the right hand side, the three equations always eliminate three parameters.

PROOF. An explicit formula for the induced three-form $J^*\rho$ is given in Corollary 1.5 such that it is straightforward to compute the exterior derivative

$$\begin{aligned} d(J_\rho^*\rho) &= -2\varepsilon\kappa\,(b_3 b_1^2 + b_3 b_2^2 - b_3^3 + b_1 b_2(a_1 - a_2) - a_1 a_2 b_3)\,\mathrm{e}^{12}\mathrm{f}^{12} \\ &\quad -2\varepsilon\kappa\,(b_2 b_1^2 - b_2^3 + b_2 b_3^2 + b_1 b_3(a_1 - a_2) - a_1 a_2 b_2)\,\mathrm{e}^{13}\mathrm{f}^{13} \\ &\quad -2\varepsilon\kappa\,(-b_1^3 + b_1 b_2^2 + b_1 b_3^2 + b_2 b_3(a_1 - a_2) - a_1 a_2 b_1)\,\mathrm{e}^{23}\mathrm{f}^{23}. \end{aligned} \tag{1.12}$$

Comparing this four-form with ω^2, given by formula (1.10), the assertion follows. $\square$

As another application of the stable form formalism and the formula for the action of $\mathrm{Aut}(\mathfrak{g})$ on three-forms, we solve the following problem concerning left-invariant complex structures on $S^3 \times S^3$.

PROPOSITION 1.8. *There is no left-invariant integrable* $\mathrm{SL}(3,\mathbb{C})$*-structure on* $S^3 \times S^3$*. Equivalently, a left-invariant complex structure* J *on* $S^3 \times S^3$ *does not admit a left-invariant holomorphic* $(3,0)$*-form.*

REMARK 1.9. However, there do exist left-invariant complex structures J on $S^3 \times S^3$ constructed in [**CE**] using the Hopf fibration $S^1 \to S^3 \to S^2$, see also [**KN**], Ex. 2.5 in ch. IX. It is shown in [**Dau**] that these structures exhaust all left-invariant complex structures on $S^3 \times S^3$.

PROOF. By Proposition 1.3, both assertions are equivalent and it suffices to prove that there is no closed left-invariant complex volume form. Due to the properties of stable three-forms in dimension six explained in chapter 1, section 3, we can equivalently show that there is no left-invariant real three-form ρ which satisfies $\lambda(\rho) < 0$ and

$$d\rho = 0\,, \qquad d(J_\rho^*\rho) = 0.$$

Let ρ be an arbitrary closed stable three-form on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We choose a standard basis and express ρ as a triple $(A, B, -B)$ according to formula (1.4). As $\mathrm{Aut}(\mathfrak{g})$ preserves the exterior system, we can assume that $B = \mathrm{diag}(b_1, b_2, b_3)$ for three non-negative real numbers by applying Lemma 1.3. Due to the necessary stability of ρ, we can moreover assume that $\lambda(\rho) \neq 0$, i.e. $\kappa = \frac{1}{\sqrt{\varepsilon\lambda}} \neq 0$. The induced almost complex structure J_ρ and the three-form $J_\rho^*\rho$ are exactly those computed in Corollary 1.5. The exterior derivative $d(J_\rho^*\rho)$, see (1.12), vanishes if and only if

$$b_3(\ b_1^2 + b_2^2 - b_3^2 - a_1 a_2) + b_1 b_2(a_1 - a_2) = 0, \tag{1.13}$$

$$b_2(\ b_1^2 - b_2^2 + b_3^2 - a_1 a_2) + b_1 b_3(a_1 - a_2) = 0, \tag{1.14}$$

$$b_1(-b_1^2+b_2^2+b_3^2-a_1a_2)+b_2b_3(a_1-a_2) \;=\; 0. \tag{1.15}$$

We claim that $\lambda(\rho) \geq 0$ for all solutions of this system of algebraic equations which is obviously symmetric in b_1, b_2 and b_3. First of all, if at least two of the three b_i vanish, say b_1 and b_2, then $\lambda(\rho) = (b_3^2+a_1a_2)^2 \geq 0$ by factorising the expression (1.7) for λ. Secondly, if exactly one of them vanishes, say b_1, we can add the first equation divided by b_3 and the second equation divided by b_2. The resulting equation is $a_1a_2 = 0$, which implies that $\lambda(\rho) = (b_2-b_3)^2(b_2+b_3)^2 \geq 0$.

Finally, we assume that all three b_i are different from zero. Multiplying (1.13) by b_2 and subtracting (1.14) multiplied by b_3, we obtain

$$(b_2^2-b_3^2)(2b_1b_2b_3+b_1^2(a_1-a_2)) \;=\; 0.$$

Similar manipulations yield

$$\begin{aligned}(b_3^2-b_1^2)(2b_1b_2b_3+b_2^2(a_1-a_2)) &= 0,\\ (b_1^2-b_2^2)(2b_1b_2b_3+b_3^2(a_1-a_2)) &= 0.\end{aligned}$$

We claim that it always holds $b_1^2 = b_2^2$ after possibly permuting the b_i. Indeed, if $b_1^2 \neq b_2^2$ and $b_1^2 \neq b_3^2$, subtracting the last two equations implies that $b_2^2 = b_3^2$ since $a_1 = a_2$ would contradict the assumption that the b_i are different from zero. Thus, we can assume that $b_1 = \pm b_2$. When we insert this into (1.15), we find

$$b_1(b_3^2-a_1a_2 \pm b_3(a_1-a_2)) = b_1(b_3 \pm a_1)(b_3 \mp a_2) = 0.$$

Therefore, all solutions of the system are given by $b_3 = \mp a_1$ or $b_3 = \pm a_2$. The first solution simplifies the expression (1.7) for λ such that

$$\lambda = a_1^4 - 4b_2^2a_1(a_2-a_1) - 4b_2^2a_1^2 + 2a_1a_2(2b_2^2+a_1^2) + a_1^2a_2^2 = a_1^2(a_1+a_2)^2 \geq 0.$$

Similarly, the second solution, i.e. $b_1 = \pm b_2$ and $b_3 = \pm a_2$, yields $\lambda = a_2^2(a_1+a_2)^2 \geq 0$.

As we have shown $\lambda(\rho) > 0$ for all stable forms inducing an integrable J_ρ, the proposition is proved. □

Example 1.10. In contrast to $\mathrm{SL}(3,\mathbb{C})$-structures, there is a trivial example of a left-invariant integrable $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$-structure on $S^3 \times S^3$. In a standard basis, we can choose for instance $\rho = \mathrm{e}^{123} + \mathrm{f}^{123}$ such that the induced almost para-complex structure J_ρ is obviously integrable since the two S^3-factors are real eigenspaces for J_ρ.

In fact, a similar method has been used in [**Bu1**] to prove the uniqueness of the nearly Kähler structure on $S^3 \times S^3$. This result is extended in [**SSH**] as follows.

Proposition 1.11. *On the Lie groups $G \times H$ with $Lie(G) = Lie(H) = \mathfrak{su}(2)$, there is a unique left-invariant strict nearly Kähler structure up to homothety and equivalence of left-invariant* $\mathrm{U}(3)$*-structures. Moreover, there is neither a left-invariant strict nearly para-Kähler structure with $\|\nabla J\|^2 \neq 0$ nor a left-invariant strict nearly pseudo-Kähler structure with an indefinite metric.*

Proof. By Theorem 5.5, chapter 3, and the usual Lie theory arguments, left-invariant nearly ε-Kähler structures on $G \times H$ with $\|\nabla J\|^2 = 4$ are in one-to-one correspondence with half-flat structures (ω, ρ) on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ satisfying the algebraic exterior system

$$
\begin{aligned}
d\omega &= 3\,\rho, && (1.16)\\
d(J_\rho^*\rho) &= 2\,\omega^2. && (1.17)
\end{aligned}
$$

Applying a homothety, i.e. rescaling the metric by a possibly negative number, $\|\nabla J\|^2 = 4$ can always be achieved for a strict nearly ε-Kähler structure (with $\|\nabla J\|^2 \neq 0$). Furthermore, since an inner Lie algebra automorphism always lifts to a Lie group automorphism which is in particular an automorphism of left-invariant U(3)-structures, it suffices to show that this exterior system has a unique solution on the Lie algebra.

Because of $d\omega^2 = 0$, we can choose a standard basis such that

$$\omega = \alpha \mathrm{e}^1\mathrm{f}^1 + \beta \mathrm{e}^2\mathrm{f}^2 + \gamma \mathrm{e}^3\mathrm{f}^3$$

as usually. Thus, the first equation (1.16) is satisfied if and only if

$$3\rho = d\omega = \alpha(\mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^1\mathrm{f}^{23}) + \beta(\mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^2\mathrm{f}^{31}) + \gamma(\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{12}).$$

As this pair (ω, ρ) happens to be in our normal form (1.6), we can simplify the expressions for ω^2, (1.10), and $d(J^*\rho)$, (1.12), in order to explicitly write down the second equation (1.17). It is explicitly shown in [**Bu1**], see also the English version [**Bu2**], that the resulting algebraic system has only one solution given by $\alpha = \beta = \gamma = \frac{1}{18}\sqrt{3}$. By substituting this into our formula (1.7) or into formula (18) in [**Bu2**], we obtain that the quartic invariant λ is negative for this solution and the non-existence of nearly para-Kähler structures is proved. A nearly pseudo-Kähler structure with an indefinite metric cannot exist either, since the induced metric is positive definite, see the second part of Lemma 2.3 in [**Bu2**], or substitute the solution into the formula given in Corollary 1.5. This finishes also the proof of the uniqueness assertion. □

REMARK 1.12. We like to point out that there are half-flat SU(3)-structures (ω, ρ) on $S^3 \times S^3$ inducing non-isometric metrics $g_{(\omega,\rho)}$. On the one hand, the metric induced by the half-flat structure (ω, ρ) defined in a standard basis by

$$
\begin{aligned}
\omega &= \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3\,,\\
\rho &= \frac{1}{2}\sqrt{2}\,(\,\mathrm{e}^{123} - \mathrm{e}^1\mathrm{f}^{23} - \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{f}^{123}\,)\,,
\end{aligned}
$$

in chapter 4 is such that the standard basis is orthonormal. In other words, the induced metric is the Riemannian product of two copies of the bi-invariant Einstein metric on S^3. In particular, this Einstein metric is not isometric to the nearly Kähler Einstein metric induced by the half-flat structure given in the proof of Proposition 1.11.

2. Half-flat structures on $H_3 \times H_3$

In this section, we focus on the Lie group $H_3 \times H_3$ with Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ where H_3 denotes the three-dimensional Heisenberg group. Apart from describing all half-flat structures on the Lie algebra, we give various explicit examples and prove a strong rigidity result concerning the induced metric of a large subclass of half-flat structures. Finally, we show that there are no nearly half-flat structures on this Lie algebra. Except for the observation on nearly half-flat structures, the results of this section are contained in [**CLSS**].

In analogy to the case of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the starting point is a normal form modulo Lie algebra automorphisms for stable coclosed two-forms $\omega \in \Lambda^2\mathfrak{g}^*$. Recall that we call a basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ for $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ a standard basis if the only non-vanishing Lie brackets are given by

$$de^3 = e^{12}, \qquad df^3 = f^{12}.$$

The connected component of the automorphism group of the Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ in the standard basis is

$$\text{(2.1)}\quad \mathrm{Aut}_0(\mathfrak{h}_3 \oplus \mathrm{h}_3) = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ a^t & \det(A) & c^t & 0 \\ 0 & 0 & B & 0 \\ d^t & 0 & b^t & \det(B) \end{pmatrix}, \ \mathrm{A, B} \in \mathrm{GL}(2, \mathbb{R}),\ \mathrm{a, b, c, d} \in \mathbb{R}^2 \right\}.$$

We denote by $\mathfrak{g}_i$, $i = 1, 2$, the two summands, by $\mathfrak{z}_i$ their centres and by $\mathfrak{z}$ the centre of $\mathfrak{g}$. The annihilator of the centre is $\mathfrak{z}^0 = \ker \mathrm{d}$ and similarly for the summands by restricting d. With this notation, we have the decompositions

$$\begin{aligned} \mathfrak{g}^* &\cong \mathfrak{z}_1^0 \oplus \mathfrak{z}_2^0 \oplus \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0} \oplus \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0}, \\ \Lambda^2\mathfrak{g}^* &\cong \Lambda^2(\mathfrak{z}^0) \oplus \underbrace{(\frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0} \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_1} \oplus \underbrace{(\mathfrak{z}_1^0 \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_2} \oplus \underbrace{(\mathfrak{z}_2^0 \wedge \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0})}_{\mathfrak{k}_3} \oplus \underbrace{(\mathfrak{z}_1^0 \wedge \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0}) \oplus (\mathfrak{z}_2^0 \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_4}. \end{aligned}$$

By $\omega^{\mathfrak{k}_i}$ we denote the projection of a two-form ω onto one of the spaces $\mathfrak{k}_i$, $i = 1, 2, 3, 4$, defined as indicated in the decomposition. We observe that $\mathfrak{k}_1 = \Lambda^2(\frac{\mathfrak{g}^*}{\mathfrak{z}^0})$ and $\omega^{\mathfrak{k}_1} = 0$ if and only if $\omega(\mathfrak{z}, \mathfrak{z}) = 0$.

Lemma 2.1. *Consider the action of* $\mathrm{Aut}(\mathfrak{h}_3 \oplus \mathfrak{h}_3)$ *on the set of non-degenerate two-forms* ω *on* $\mathfrak{g}$ *with* $d\omega^2 = 0$. *The orbits modulo rescaling are represented in a standard basis by the following two-forms:*

$$\begin{aligned}
\omega_1 &= \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3, && \textit{if } \omega^{\mathfrak{k}_1} \neq 0, \\
\omega_2 &= \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^{13} + \mathrm{f}^{13}, && \textit{if } d\omega = 0 \iff \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} = 0,\ \omega^{\mathfrak{k}_3} = 0, \\
\omega_3 &= \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^1, && \textit{if } \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} \neq 0,\ \omega^{\mathfrak{k}_3} \neq 0,\ \omega^{\mathfrak{k}_4} = 0, \\
\omega_4 &= \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^1 + \mathrm{e}^{13} + \beta\mathrm{f}^{13}, && \textit{if } \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} \neq 0,\ \omega^{\mathfrak{k}_3} \neq 0,\ \omega^{\mathfrak{k}_4} \neq 0, \\
\omega_5 &= \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^{13} + \mathrm{f}^{13} && \textit{otherwise},
\end{aligned}$$

where $\beta \in \mathbb{R}$ *and* $\beta \neq -1$.

PROOF. Let

$$\omega = \sum \alpha_i \mathrm{e}^{(i+1)(i+2)} + \sum \beta_i \mathrm{f}^{(i+1)(i+2)} + \sum \gamma_{i,j} \mathrm{e}^i \mathrm{f}^j$$

be an arbitrary non-degenerate two-form expressed in a standard basis. We will give in each case explicitly a change of standard basis by an automorphism of the form (2.1) with the notation

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ a^t = (a_5, a_6),\ B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},\ b^t = (b_5, b_6),\ c^t = (c_1, c_2),\ d^t = (d_1, d_2).$$

First of all, if $\omega^{\mathfrak{k}_1} \neq 0$, the term $\gamma_{3,3}\mathrm{e}^3\mathrm{f}^3$ is different from zero and we rescale such that $\gamma_{3,3} = 1$. Then, the application of the change of basis

$$\begin{aligned}
a_1 &= 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1,\ a_5 = -\gamma_{1,3},\ a_6 = -\gamma_{2,3},\\
b_1 &= \gamma_{2,2} - \gamma_{2,3}\gamma_{3,2} - \alpha_1\beta_1,\ b_2 = -\gamma_{1,2} - \beta_1\alpha_2 + \gamma_{3,2}\gamma_{1,3},\ b_3 = -\gamma_{2,1} + \gamma_{3,1}\gamma_{2,3} - \beta_2\alpha_1,\\
b_4 &= \gamma_{1,1} - \alpha_2\beta_2 - \gamma_{1,3}\gamma_{3,1},\ b_5 = -\gamma_{3,1}\gamma_{2,2} + \gamma_{3,1}\alpha_1\beta_1 + \gamma_{3,2}\gamma_{2,1} + \gamma_{3,2}\beta_2\alpha_1,\\
b_6 &= \gamma_{3,1}\gamma_{1,2} + \gamma_{3,1}\beta_1\alpha_2 - \gamma_{3,2}\gamma_{1,1} + \gamma_{3,2}\alpha_2\beta_2,\\
c_1 &= \beta_2\gamma_{2,2} - \beta_2\gamma_{2,3}\gamma_{3,2} + \beta_1\gamma_{2,1} - \beta_1\gamma_{3,1}\gamma_{2,3},\ d_1 = -\alpha_2,\\
c_2 &= -\beta_2\gamma_{1,2} + \beta_2\gamma_{3,2}\gamma_{1,3} - \beta_1\gamma_{1,1} + \beta_1\gamma_{1,3}\gamma_{3,1},\ d_2 = \alpha_1,
\end{aligned}$$

transforms ω into $\tilde{\omega} = \tilde{\gamma}_{1,1}(\mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3) + \tilde{\alpha}_3\mathrm{e}^{12} + \tilde{\beta}_3\mathrm{f}^{12}$, $\tilde{\gamma}_{1,1} \neq 0$. This two-form satisfies $d\tilde{\omega}^2 = 0$ if and only if $\tilde{\alpha}_3 = 0, \tilde{\beta}_3 = 0$ and the normal form ω_1 is achieved by rescaling.

Secondly, the vanishing of $d\omega$ corresponds to $\omega^{\mathfrak{k}_1} = 0$, $\omega^{\mathfrak{k}_2} = 0$, $\omega^{\mathfrak{k}_3} = 0$ or $\gamma_{3,3} = \gamma_{1,3} = \gamma_{2,3} = \gamma_{3,1} = \gamma_{3,2} = 0$ in a standard basis. By non-degeneracy, at least one of α_1 and α_2 is not zero and we can always achieve $\alpha_1 = 0, \alpha_2 \neq 0$. Indeed, if $\alpha_1 \neq 0$, we apply the transformation (2.1) with $a_1 = 1$, $a_2 = 1$, $a_4 = \frac{\alpha_2}{\alpha_1}$, $B = \mathbb{1}$ and all remaining entries zero. With an analogous argument, we can assume that $\beta_1 = 0$, $\beta_2 \neq 0$. Since $\gamma_{2,2} \neq 0$ by non-degeneracy, we can rescale ω such that $\gamma_{2,2} = 1$. Now, the transformation of the form (2.1) given by

$$\begin{gathered}
a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\beta_2,\ b_1 = 1,\ b_2 = 0,\ b_3 = 0,\ b_4 = -\alpha_2,\ a_5 = 0,\\
a_6 = -\frac{\alpha_3\beta_2}{\alpha_2},\ b_5 = 0,\ b_6 = -\frac{\alpha_2\beta_3}{\beta_2},\ c_1 = \frac{\gamma_{1,1}}{\alpha_2},\ c_2 = -\gamma_{1,2},\ d_1 = 0,\ d_2 = \gamma_{2,1},
\end{gathered}$$

maps ω to a multiple of the normal form ω_2.

Thirdly, we assume that ω is non-degenerate with $\omega^{\mathfrak{k}_1} = 0$, i.e. $\gamma_{3,3} = 0$ and both $\omega^{\mathfrak{k}_2} \neq 0$, i.e. $\gamma_{1,3}$ or $\gamma_{2,3} \neq 0$, and $\omega^{\mathfrak{k}_3} \neq 0$, i.e. $\gamma_{3,1}$ or $\gamma_{3,2} \neq 0$. Similar as before, we can achieve $\gamma_{2,3} = 0$, $\gamma_{1,3} \neq 0$ by applying, if $\gamma_{2,3} \neq 0$, the transformation (2.1) with $a_1 = 1$, $a_2 = 1$, $a_4 = -\frac{\gamma_{1,3}}{\gamma_{2,3}}$, $B = \mathbb{1}$ and all remaining entries zero. Analogously, we can assume $\gamma_{3,2} = 0$, $\gamma_{3,1} \neq 0$ and rescaling yields $\gamma_{2,2} = 1$, which is non-zero by non-degeneracy. After this simplification, the condition $d\omega^2 = 0$ implies that $\alpha_1 = \beta_1 = 0$ and the transformation

$$\begin{aligned}
&a_1 = 1,\ a_2 = 0,\ a_3 = \frac{\alpha_2\beta_3 - \gamma_{3,1}\gamma_{1,2}}{\gamma_{3,1}},\ a_4 = \gamma_{1,3},\ a_5 = 0,\ a_6 = 0,\ b_1 = 1,\ b_2 = 0,\\
&b_3 = \frac{\beta_2\alpha_3 - \gamma_{1,3}\gamma_{2,1}}{\gamma_{1,3}},\ b_4 = \gamma_{3,1},\ b_5 = \frac{\gamma_{1,2}\gamma_{1,3}\gamma_{2,1}\gamma_{3,1} - \gamma_{1,1}\gamma_{1,3}\gamma_{3,1} - \alpha_2\alpha_3\beta_2\beta_3}{\gamma_{1,3}^2\gamma_{3,1}},\ b_6 = 0,\\
&c_1 = 0,\ c_2 = \beta_3,\ d_1 = 0,\ d_2 = -\alpha_3,
\end{aligned}$$

maps ω to $\tilde{\omega} = \mathrm{e}^1\mathrm{f}^3 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^1 + \tilde{\alpha}_2\mathrm{e}^{31} + \tilde{\beta}_2\mathrm{f}^{31}$. The condition $\omega^{\mathfrak{k}_4} = 0$ corresponds to $\tilde{\alpha}_2 = 0$, $\tilde{\beta}_2 = 0$, i.e. normal form ω_3. If $\omega^{\mathfrak{k}_4} \neq 0$, we can achieve $\tilde{\alpha}_2 \neq 0$ by possibly changing the summands. Now, the transformation

$$\begin{aligned} a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{1}{\tilde{\alpha}_2},\ a_5 = 0,\ a_6 = 0,\ c_1 = 0,\ c_2 = 0, \\ b_1 = -\frac{1}{\tilde{\alpha}_2},\ b_2 = 0,\ b_3 = 0,\ b_4 = -\frac{1}{\tilde{\alpha}_2},\ b_5 = 0,\ b_6 = 0,\ d_1 = 0,\ d_2 = 0, \end{aligned}$$

maps $\tilde{\omega}$ to the fourth normal form ω_4.

The cases that remain are $\omega^{\mathfrak{k}_1} = 0$ and either $\omega^{\mathfrak{k}_2} \neq 0, \omega^{\mathfrak{k}_3} = 0$ or $\omega^{\mathfrak{k}_3} = 0, \omega^{\mathfrak{k}_2} \neq 0$. After changing the summands if necessary, we can assume $\omega^{\mathfrak{k}_3} = 0$ and $\omega^{\mathfrak{k}_2} \neq 0$, i.e. $\gamma_{3,1} = \gamma_{3,2} = \gamma_{3,3} = 0$ and at least one of $\gamma_{1,3}$ or $\gamma_{2,3}$ non-zero. As before, we can achieve $\gamma_{2,3} = 0$ by the transformation $a_1 = 1, a_2 = 1, a_4 = -\frac{\gamma_{1,3}}{\gamma_{2,3}}$. Evaluating $d\omega^2 = 0$ yields $\alpha_1 = 0$. Now, non-degeneracy enforces that $\beta_1 \neq 0$ or $\beta_2 \neq 0$, and after another similar transformation $\beta_1 = 0$. Finally, the simplified ω is non-degenerate if and only if $\gamma_{2,2}\alpha_2\beta_2 \neq 0$ and, after rescaling such that $\gamma_{2,2} = 1$, the transformation

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{\gamma_{1,3}^2}{\beta_2},\ a_5 = 0,\ a_6 = \frac{\gamma_{1,3}^2(\gamma_{1,3}\gamma_{2,1} - \alpha_3\beta_2)}{\alpha_2\beta_2^2}, \\ b_1 &= -\frac{\gamma_{1,3}}{\beta_2},\ b_2 = 0,\ b_3 = 0,\ b_4 = -\alpha_2,\ b_5 = 0,\ b_6 = -\frac{\beta_3\alpha_2}{\beta_2}, \\ c_1 &= 0,\ c_2 = -\frac{\gamma_{1,2}\beta_2 + \gamma_{1,3}\beta_3}{\beta_2},\ d_1 = -\frac{\gamma_{1,1}}{\beta_2},\ d_2 = \frac{\gamma_{1,3}^2\gamma_{2,1}}{\beta_2^2}, \end{aligned}$$

maps ω to a multiple of the fifth normal form ω_5. $\square$

Using this lemma, it is possible to describe all half-flat structures (ω, ρ) on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ as follows. In a fixed standard basis such that ω is in one of the normal forms, the equations $d\rho = 0$ and $\omega \wedge \rho = 0$ are linear in the coefficients of an arbitrary three-form ρ. Thus, it is straightforward to write down all compatible closed three-forms for each normal form which depend on nine parameters in each case. The stable forms in this nine-dimensional space are parametrised by the complement of the zero-set of the polynomial $\lambda(\rho)$ of order four. One parameter is eliminated when we moreover require the pair (ω, ρ) to be normalised. We remark that the computation of the induced tensors J_ρ, $\hat{\rho}$ and $g_{(\omega,\rho)}$ may require computer support, in particular, the signature of the metric is not obvious. However, stability is an open condition: If a single half-flat structure (ω_0, ρ_0) is explicitly given such that ω_0 is one of the normal forms, then the eight-parameter family of normalised compatible closed forms defines a deformation of the given half-flat structure (ω_0, ρ_0) in some neighbourhood of (ω_0, ρ_0).

For instance, the closed three-forms which are compatible with the first normal form

$$\omega = \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3 \tag{2.2}$$

in a standard basis can be parametrised as follows:

$$\begin{aligned} \rho = \rho(a_1, \dots, a_9) &= a_1\,\mathrm{e}^{123} + a_2\,\mathrm{f}^{123} + a_3\,\mathrm{e}^1\mathrm{f}^{23} + a_4\,\mathrm{e}^2\mathrm{f}^{13} + a_5\,\mathrm{e}^{23}\mathrm{f}^1 + a_6\,\mathrm{e}^{13}\mathrm{f}^2 \\ &+ a_7\,(\mathrm{e}^2\mathrm{f}^{23} - \mathrm{e}^1\mathrm{f}^{13}) + a_8\,(\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^3\mathrm{f}^{12}) + a_9\,(\mathrm{e}^{23}\mathrm{f}^2 - \mathrm{e}^{13}\mathrm{f}^1). \end{aligned} \tag{2.3}$$

The quartic invariant $\lambda(\rho)$ depending on the nine parameters is

$$\begin{aligned}\lambda(\rho) &= (2a_6a_4a_8^2 + 2a_1a_2a_8^2 + 2a_8^2a_3a_5 - 4a_5a_7^2a_6 - 4a_9^2a_4a_3 - 4a_9^2a_2a_8 + 4a_7^2a_8a_1 \\ &+ 4a_7a_8^2a_9 + a_1^2a_2^2 + a_6^2a_4^2 + a_3^2a_5^2 + a_8^4 - 2a_6a_4a_3a_5 + 4a_5a_7a_9a_3 + 4a_9a_4a_6a_7 \\ &- 4a_5a_2a_6a_8 + 4a_4a_8a_1a_3 - 4a_9a_2a_1a_7 - 2a_1a_2a_6a_4 - 2a_1a_2a_3a_5)\,(\mathrm{e}^{123}\mathrm{f}^{123})^{\otimes 2}.\end{aligned}$$

EXAMPLE 2.2. For each possible signature, we give an explicit normalised half-flat structure with fundamental two-form (2.2). The first and the third example are taken from chapter 4. To begin with, the closed three-form

$$\rho = \frac{1}{\sqrt{2}}(\mathrm{e}^{123} - \mathrm{f}^{123} - \mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^2\mathrm{f}^{31} + \mathrm{e}^{31}\mathrm{f}^2 - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{e}^{12}\mathrm{f}^3) \tag{2.4}$$

induces a half-flat SU(3)-structure (ω, ρ) such that the standard basis is orthonormal. Similarly, the closed three-form

$$\rho = \frac{1}{\sqrt{2}}(\mathrm{e}^{123} - \mathrm{f}^{123} - \mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^{23}\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^{31} - \mathrm{e}^{31}\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^{12} - \mathrm{e}^{12}\mathrm{f}^3) \tag{2.5}$$

induces a half-flat $\mathrm{SU}(1,2)$-structure (ω, ρ) such that the standard basis is pseudo-orthonormal with e_1 and e_4 being spacelike. Finally, the closed three-form

$$\rho = \sqrt{2}\,(\mathrm{e}^{123} + \mathrm{f}^{123}), \tag{2.6}$$

induces a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure (ω, ρ) such that the two $\mathfrak{h}_3$-summands are the eigenspaces of the para-complex structure J_ρ, which is integrable since also $d\hat{\rho} = 0$. The induced metric is

$$g = 2\left(\mathrm{e}^1 \cdot \mathrm{e}^4 + \mathrm{e}^2 \cdot \mathrm{e}^5 + \mathrm{e}^3 \cdot \mathrm{e}^6\right).$$

In fact, half-flat structures with Riemannian metrics are only possible if ω belongs to the orbit of the first normal form.

LEMMA 2.3. *Let (ω, ρ) be a half-flat SU(3)-structure on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$. Then it holds $\omega^{\mathfrak{k}_1} \neq 0$. In particular, there is a standard basis such that $\omega = \omega_1 = \mathrm{e}^1\mathrm{f}^1 + \mathrm{e}^2\mathrm{f}^2 + \mathrm{e}^3\mathrm{f}^3$.*

PROOF. Suppose that (ω, ρ) is a half-flat SU(3)-structure on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ with $\omega^{\mathfrak{k}_1} = 0$. Thus, we can choose a standard basis such that ω is in one of the normal forms $\omega_2, \dots, \omega_5$ of Lemma 2.1 and ρ belongs to the corresponding nine-parameter family of compatible closed three-forms. We claim that the basis one-form e^1 is isotropic in all four cases which yields a contradiction since the metric of an SU(3)-structure is positive definite. The quickest way to verify the claim is the direct computation of the induced metric, which depends on nine parameters, with the help of a computer. In order to verify the assertion by hand, the following formulas shorten the calculation considerably. For all one-forms α, β and all vectors v, the ε-complex structure J_ρ and the metric g induced by a compatible pair (ω, ρ) of stable forms satisfy

$$\begin{aligned}\alpha \wedge J_\rho^*\beta \wedge \omega^2 &= g(\alpha, \beta)\frac{1}{3}\omega^3, \\ J_\rho^*\alpha(v)\phi(\rho) &= \alpha \wedge \rho \wedge (v \lrcorner \rho),\end{aligned}$$

by Lemma 1.8 and Proposition 1.4 of chapter 1. For instance, for the second normal form ω_2, it holds $\mathrm{e}^1 \wedge \omega_2^2 = -2\mathrm{e}^{12}\mathrm{f}^{123}$. Thus, by the first formula, it suffices to show that $J_\rho^*\mathrm{e}^1(e_3) = \mathrm{e}^1(J_\rho e_3) = 0$ which is in turn satisfied if

$e^1 \wedge \rho \wedge (e_3 \lrcorner \rho) = 0$ due to the second formula. A similar simplification applies to the other normal forms and we omit the straightforward calculations. $\square$

Moreover, the geometry turns out to be very rigid if $\omega^{\mathfrak{k}_1} = 0$. We recall that simply connected para-hyper-Kähler symmetric spaces with abelian holonomy are classified in [**ABCV**], [**Cor**]. In particular, there exists a unique simply connected four-dimensional para-hyper-Kähler symmetric space with one-dimensional holonomy group, which is defined in [**ABCV**], Section 4. We denote the underlying pseudo-Riemannian manifold as (N^4, g_{PHK}).

Proposition 2.4. *Let (ω, ρ) be a left-invariant half-flat structure with $\omega^{\mathfrak{k}_1} = 0$ on $H_3 \times H_3$ and let g be the pseudo-Riemannian metric induced by (ω, ρ). Then, the pseudo-Riemannian manifold $(H_3 \times H_3, g)$ is either flat or isometric to the product of (N^4, g_{PHK}) and a two-dimensional flat factor. In particular, the metric g is Ricci-flat.*

Proof. Due to the assumption $\omega^{\mathfrak{k}_1} = 0$, we can choose a standard basis such that ω is in one of the normal forms $\omega_2, \ldots, \omega_5$. In each case separately, we do the following. We write down all compatible closed three-forms ρ depending on nine parameters. With computer support, we calculate the induced metric g. For the curvature considerations, it suffices to work up to a constant such that we can ignore the rescaling by $\lambda(\rho)$ which is different from zero by assumption. Now, we transform the left-invariant co-frame $\{\mathrm{e}^1, \ldots, \mathrm{f}^3\}$ to a coordinate co-frame $\{dx_1, \ldots, dy_3\}$ by applying the transformation defined by

$$\mathrm{e}^1 = dx_1,\ \mathrm{e}^2 = dx_2,\ \mathrm{e}^3 = dx_3 + x_1 dx_2,\ \mathrm{f}^1 = dy_1,\ \mathrm{f}^2 = dy_2,\ \mathrm{f}^3 = dy_3 + y_1 dy_2, \tag{2.7}$$

such that the metric is accessible for any of the numerous packages computing curvature. The resulting curvature tensor $R \in \Gamma(\mathrm{End}\,\Lambda^2 TM)$, $M = H^3 \times H^3$, has in each case only one non-trivial component

$$R(\partial_{x_1} \wedge \partial_{y_1}) = c\, \partial_{x_3} \wedge \partial_{y_3} \tag{2.8}$$

for a constant $c \in \mathbb{R}$ and R is always parallel. Thus, the metric is flat if $c = 0$ and symmetric with one-dimensional holonomy group if $c \neq 0$, for $H_3 \times H_3$ is simply connected and a naturally reductive homogeneous metric is complete.

Furthermore, it turns out that the metric restricted to $TN := span\{\partial_{x_1}, \partial_{x_3}, \partial_{y_1}, \partial_{y_3}\}$ is non-degenerate and of signature $(2, 2)$ for all parameter values. Thus, the manifold splits in a four-dimensional symmetric factor with neutral metric and curvature tensor (2.8) and the two-dimensional orthogonal complement which is flat. Since a simply connected symmetric space is completely determined by its curvature tensor and the four-dimensional para-hyper-Kähler symmetric space (N^4, g_{PHK}) has the same signature and curvature tensor, the four-dimensional factor is isometric to (N^4, g_{PHK}). Finally, the metric g is Ricci-flat since g_{PHK} is Ricci-flat. $\square$

Example 2.5. The following examples define half-flat normalised $\mathrm{SU}(1,2)$-structures with $\omega^{\mathfrak{k}_1} = 0$ in a standard basis. None of the examples is flat. Thus, the four structures are equivalent as $\mathrm{SO}(2,4)$-structures due to Proposition 2.4, but the examples show that the geometry of the reduction to $\mathrm{SU}(1,2)$ is not as rigid.

$$\begin{aligned} \omega = \omega_2\,, \quad & \rho = \mathrm{e}^{12}\mathrm{f}^3 + \sqrt{2}\mathrm{e}^{13}\mathrm{f}^2 + \mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^3\mathrm{f}^{12} + \sqrt{2}\mathrm{f}^{123}, \\ & g = -(\mathrm{e}^2)^2 - (\mathrm{f}^2)^2 + 2\,\mathrm{e}^1{\cdot}\mathrm{e}^3 - 2\sqrt{2}\,\mathrm{e}^1{\cdot}\mathrm{f}^3 + 2\sqrt{2}\,\mathrm{e}^3{\cdot}\mathrm{f}^1 - 2\,\mathrm{f}^1{\cdot}\mathrm{f}^3, \end{aligned}$$

(Ricci-flat pseudo-Kähler since $d\omega = 0$, $d\hat{\rho} = 0$);

$$\begin{aligned}
\omega = \omega_3\,, \quad & \rho = \mathrm{e}^{123} + \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{13}\mathrm{f}^2 + \mathrm{e}^1\mathrm{f}^{12} - 2\mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^2\mathrm{f}^{13} - \mathrm{e}^3\mathrm{f}^{12},\\
& g = -(\mathrm{e}^2)^2 - 2\,(\mathrm{f}^2)^2 + 2\,\mathrm{e}^1\!\cdot\mathrm{f}^1 + 2\,\mathrm{e}^1\!\cdot\mathrm{f}^3 + 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\mathrm{f}^1 - 2\,\mathrm{f}^1\!\cdot\mathrm{f}^3,\\
& (d\omega \neq 0,\ J_\rho \text{ integrable since } d\hat{\rho} = 0);\\
\omega = \omega_4\,, \quad & \rho = \beta\,\mathrm{e}^{12}\mathrm{f}^3 - \beta\,\mathrm{e}^{13}\mathrm{f}^2 + \beta\,\mathrm{e}^1\mathrm{f}^{23} + \frac{\beta+1}{\beta^3}\,\mathrm{e}^{23}\mathrm{f}^1 + \frac{\beta^4-\beta-1}{\beta^3}\,\mathrm{e}^2\mathrm{f}^{13}\\
& \qquad - \beta\,\mathrm{e}^3\mathrm{f}^{12} - (\beta^2+2\beta)\,\mathrm{f}^{123}, \qquad\qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
& g = -\frac{1}{\beta^2}\,(\mathrm{e}^2)^2 - \beta^2\,(\mathrm{f}^2)^2 + 2\beta^2\,\mathrm{e}^1\!\cdot\mathrm{f}^3 - \frac{2}{\beta^2(\beta+1)}\,\mathrm{e}^3\!\cdot\mathrm{f}^1 - \frac{2(\beta^4+\beta+1)}{\beta^2}\,\mathrm{f}^1\!\cdot\mathrm{f}^3;\\
\omega = \omega_5\,, \quad & \rho = \mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{13}\mathrm{f}^2 - \mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^{23}\mathrm{f}^1 - \mathrm{e}^3\mathrm{f}^{12} + \mathrm{f}^{123}, \qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
& g = -(\mathrm{e}^2)^2 - 2\,(\mathrm{f}^2)^2 + 2\,\mathrm{e}^1\!\cdot\mathrm{e}^3 + 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 + 2\,\mathrm{f}^1\!\cdot\mathrm{f}^3.
\end{aligned}$$

EXAMPLE 2.6. Moreover, we give examples of half-flat normalised $\mathrm{SL}(3,\mathbb{R})$-structures with $\omega^{\mathfrak{k}_1} = 0$. Again, none of the structures is flat.

$$\begin{aligned}
\omega = \omega_2\,, \quad & \rho = \sqrt{2}\,(\,\mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^{23}\mathrm{f}^1\,), \qquad (d\omega = 0,\ d\hat{\rho} = 0),\\
& g = 2\,\mathrm{e}^1\!\cdot\mathrm{e}^3 - 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 - 2\,\mathrm{f}^1\!\cdot\mathrm{f}^3;\\
\omega = \omega_3\,, \quad & \rho = \sqrt{2}\,(\,\mathrm{e}^{12}\mathrm{f}^3 + \mathrm{e}^{13}\mathrm{f}^2 + \mathrm{e}^1\mathrm{f}^{12} - \mathrm{e}^3\mathrm{f}^{12}\,), \qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
& g = -2\,(\mathrm{e}^1)^2 + 2\,\mathrm{e}^1\!\cdot\mathrm{e}^3 - 2\,\mathrm{e}^1\!\cdot\mathrm{f}^3 + 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 - 2\,\mathrm{f}^1\!\cdot\mathrm{f}^3;\\
\omega = \omega_4\,, \quad & \rho = -\sqrt{2\beta+2}\,(\,\mathrm{e}^{12}\mathrm{f}^3 - \mathrm{e}^1\mathrm{f}^{23} + \mathrm{e}^2\mathrm{f}^{13} - \mathrm{e}^3\mathrm{f}^{12}), \qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
& g = -2\,(\mathrm{f}^2)^2 + 2\,\mathrm{e}^1\!\cdot\mathrm{e}^3 + 2\,\mathrm{e}^1\!\cdot\mathrm{f}^3 + 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 - 2\,\mathrm{e}^3\!\cdot\mathrm{f}^1 - (2\beta+4)\,\mathrm{f}^1\!\cdot\mathrm{f}^3;\\
\omega = \omega_5\,, \quad & \rho = \sqrt{2}\,(\,\mathrm{e}^{123} + \mathrm{f}^{123}\,), \qquad (d\omega \neq 0,\ d\hat{\rho} = 0),\\
& g = 2\,\mathrm{e}^1\!\cdot\mathrm{f}^3 + 2\,\mathrm{e}^2\!\cdot\mathrm{f}^2 + 2\,\mathrm{e}^3\!\cdot\mathrm{f}^1.
\end{aligned}$$

Finally, we deal with the case of left-invariant nearly half-flat structures on $H_3 \times H_3$.

LEMMA 2.7. *There are no nearly half-flat structures on* $\mathfrak{h}_3 \oplus \mathfrak{h}_3$.

PROOF. By definition, an $\mathrm{SU}^\varepsilon(p,q)$-structure (ω,ρ) is nearly half-flat if and only $d\rho = \nu\omega^2$ for a non-zero constant ν. However, none of the five normal forms of Lemma 2.1 is exact which is easily proved in a standard basis. Alternatively, it is not hard to verify in a standard basis $(e_1,\ldots,e_6)$ that every exact four-form is degenerate. For instance, the six-vector $(d\rho \lrcorner e_{1\ldots6})^3$ is quickly computed by a computer and turns out to vanish for every three-form ρ. □

3. Nearly ε-Kähler structures on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

The main result of [**SSH**] is the proof of the uniqueness of the nearly pseudo-Kähler structure on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Although the main idea of the proof is essentially the same as that of the proof of the uniqueness of the nearly Kähler structure on $S^3 \times S^3$, see also Proposition 1.11, the technical difficulties turn out to be much more delicate for the non-compact form than for the compact form. For this reason, we do not attempt a complete analysis of the left-invariant half-flat structures on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. In fact, it is not hard to show that there exist large families of half-flat structures analogous to those existing on $S^3 \times S^3$, however, the exact description of the orbit space modulo Lie algebra automorphisms is expected to be very technical.

In the following we present the proof given in [**SSH**] with minimal modifications.

Theorem 3.1. *Let G be a Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Up to homothety and equivalence of left-invariant $\mathrm{U}(3)$-structures, there is a unique left-invariant nearly ε-Kähler structure with $\|\nabla J\|^2 > 0$ on $G \times G$. The nearly ε-Kähler metric is of signature $(2,4)$. In particular, there is no left-invariant nearly para-Kähler structure.*

Remark 3.2. The proof also shows that there there is a left-invariant nearly ε-Kähler structure with $\|\nabla J\|^2 \neq 0$ on $G \times H$ with $Lie(G) = Lie(H) = \mathfrak{sl}(2,\mathbb{R})$ if $G \neq H$ which is unique up to homothety *and* exchanging the orientation.

Proof. Completely analogous to the proof of Proposition 1.11, it suffices to show the existence of a solution of the algebraic exterior system (1.16), (1.17) on the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$ which is unique up to inner Lie algebra automorphisms and exchanging the summands. Notice that the outer automorphism exchanging the summands of course lifts to the corresponding Lie group, if the two three-dimensional factors are the same.

Again, we search for a normal form of coclosed stable two-form ω modulo Lie algebra automorphisms in a fixed standard Lie bracket. In order to improve the readability of the technical proof, we break the main part into three lemmas, step by step simplifying ω. Recall that we denoted a basis $\{e_1, e_2, e_3\}$ as a standard basis of $\mathfrak{so}(1,2)$ if the Lie bracket satisfies

$$de^1 = -\mathrm{e}^{23}\,, \quad de^2 = \mathrm{e}^{31}\,, \quad de^3 = \mathrm{e}^{12}.$$

With respect to this basis, an inner automorphism in $\mathrm{SO}_0(1,2)$ acts by usual matrix multiplication on $\mathfrak{so}(1,2)$.

Lemma 3.3. *Denote by $(\mathbb{R}^{1,2}, \langle\cdot,\cdot\rangle)$ the vector space $\mathbb{R}^3$ endowed with its standard Minkowskian scalar-product and denote by $\mathrm{SO}_0(1,2)$ the connected component of the identity of its group of isometries. Consider the action of $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ on the space of real 3×3 matrices $\mathrm{Mat}(3,\mathbb{R})$ given by*

$$\begin{array}{rrcl} \Phi\, :\, \mathrm{SO}_0(1,2) \times \mathrm{Mat}(3,\mathbb{R}) \times \mathrm{SO}_0(1,2) & \to & \mathrm{Mat}(3,\mathbb{R}) \\ (A, C, B) & \mapsto & A^t C B. \end{array}$$

Then any invertible element $C \in \mathrm{Mat}(3,\mathbb{R})$ lies in the orbit of an element of the form

$$\begin{pmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \beta & z \\ \alpha & x & y \\ 0 & 0 & \gamma \end{pmatrix}$$

with $\alpha, \beta, \gamma, x, y, z \in \mathbb{R}$ and $\alpha\beta\gamma \neq 0$.

Proof. Let an arbitrary invertible element $C \in \mathrm{Mat}(3,\mathbb{R})$ be given. Denote by $\{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^{1,2}$. There are three different cases:

1.) Suppose, that the first column c of C has negative length. We extend c to a Lorentzian basis $\{l_1 = c/\alpha, l_2, l_3\}$ with $\alpha := \sqrt{|\langle c, c\rangle|}$. The linear map L defined by extension of $L(l_i) = e_i$ is by definition a Lorentz transformation. The transformation L can be chosen time-oriented (by replacing l_1 by $\pm l_1$) and oriented (by replacing l_3 by $\pm l_3$). With this definition we obtain

$$\Phi(L^t, C, \mathbb{1}) = \begin{pmatrix} \alpha & * \\ 0 & C' \end{pmatrix} \text{ with an element } C' \in \mathrm{Mat}(2,\mathbb{R}).$$

Using the polar decomposition we can express $C' = O_1 S$ as a product of $O_1 \in SO(2)$ and a symmetric matrix S in $\mathrm{Mat}(2,\mathbb{R})$ and diagonalise S by $O_2 \in SO(2)$. If we put

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & O_2^{-1} O_1^{-1} \end{pmatrix} \text{ and } L_2 = \begin{pmatrix} 1 & 0 \\ 0 & O_2 \end{pmatrix}$$

we obtain

$$\Phi(L_1^t, \Phi(L^t, C, \mathbb{1}), L_2) = \begin{pmatrix} \alpha & x & y \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

2.) Next suppose, that the first column c of C has positive length. Again, we extend c to a Lorentzian basis $\{l_1, l_2 = c/\alpha, l_3\}$ with $\alpha := \sqrt{|\langle c, c\rangle|}$. The linear map L defined by extension of $L(l_i) = e_i$ is by definition a Lorentz transformation. The transformation L can be chosen time-oriented (by replacing l_1 by $\pm l_1$) and oriented (by replacing l_3 by $\pm l_3$). We get

$$\Phi(L^t, C, \mathbb{1}) = \begin{pmatrix} 0 & * \\ \alpha & C' \\ 0 & \end{pmatrix} \text{ with an element } C' \in \mathrm{Mat}(2,\mathbb{R}).$$

The first column of this matrix is stable under the right-operation of

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & O_1 \end{pmatrix} \text{ with } O_1 \in SO(2)$$

and there exists an element $O_1 \in SO(2)$ such that it holds

$$\Phi(\mathbb{1}, \Phi(L^t, C, \mathbb{1}), L_1) = \begin{pmatrix} 0 & \beta & z \\ \alpha & x & y \\ 0 & 0 & \gamma \end{pmatrix}.$$

3.) Finally suppose, that it holds $\langle c, c\rangle = 0$. Then there exists an oriented and time-oriented Lorentz transformation L such that $L(c) = \kappa(e_1 + e_2)$ with $\kappa \neq 0$. Afterwards one finds as in point 2.) an element $O \in SO(2)$, such that it holds

$$C' := \Phi(L^t, C, O) = \begin{pmatrix} \kappa & c_1 & * \\ \kappa & c_2 & * \\ 0 & 0 & * \end{pmatrix}.$$

Let

$$B(q) := \begin{pmatrix} \cosh(q) & \sinh(q) & 0 \\ \sinh(q) & \cosh(q) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim: There exist $q_1, q_2 \in \mathbb{R}$ such that

$$\Phi\left(B(q_1)^t\,,\, C'\,,\, B(q_2)\right) = \begin{pmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{pmatrix}.$$

To prove this claim let us first consider the right-action of $B(q)$ on $C'' := \Phi\left(B(q_1)^t, C', \mathbb{1}\right)$

$$\Phi\left(\mathbb{1}, C'', B(q)\right) = \begin{pmatrix} c''_{11}\cosh(q) + c''_{12}\sinh(q) & * & * \\ c''_{21}\cosh(q) + c''_{22}\sinh(q) & * & * \\ 0 & 0 & * \end{pmatrix} \text{ for } q \in \mathbb{R}.$$

We choose q_2 such that $c''_{21}\cosh(q_2) + c''_{22}\sinh(q_2)$ vanishes. This is only possible if $-c''_{22}/c''_{21}$ is in the range of coth, i.e. $|c''_{22}/c''_{21}| > 1$.
In the sequel we show, that this can always be achieved by the left-action of an element $B(q_1)$ on C' and that $c''_{21} \neq 0$. In fact, it is

$$\begin{aligned} c''_{22} &= c_1\sinh(q_1) + c_2\cosh(q_1) \\ c''_{21} &= \kappa(\sinh(q_1) + \cosh(q_1)) = \kappa \mathrm{e}^{q_1} \\ \frac{c''_{22}}{c''_{21}} &= \frac{c_1 + c_2}{2\kappa} + \frac{c_2 - c_1}{2\kappa}\mathrm{e}^{-2q_1}. \end{aligned}$$

We observe, that $c_1 \neq c_2$, since the matrix C is invertible. Therefore we can always achieve $|c''_{22}/c''_{21}| > 1$. This proves the claim and finishes the proof of the lemma.

□

Lemma 3.4. *Let $\mathfrak{g} = \mathfrak{h} = \mathfrak{so}(1,2)$ and let $\{\mathrm{e}^1, \mathrm{e}^2, \mathrm{e}^3\}$ be a basis of $\mathfrak{g}^*$ and $\{\mathrm{e}^4, \mathrm{e}^5, \mathrm{e}^6\}$ a basis of $\mathfrak{h}^*$ such that the Lie brackets are given by*

$$(3.1) \quad d\mathrm{e}^1 = -\mathrm{e}^{23}, \quad d\mathrm{e}^2 = \mathrm{e}^{31}, \quad d\mathrm{e}^3 = \tau\mathrm{e}^{12} \quad \textit{and} \quad d\mathrm{e}^4 = -\mathrm{e}^{56}, \quad d\mathrm{e}^5 = \mathrm{e}^{64}, \quad d\mathrm{e}^6 = \mathrm{e}^{45}$$

for some $\tau \in \{\pm 1\}$. Then, every non-degenerate two-form ω on $\mathfrak{g} \oplus \mathfrak{h}$ satisfying $d\omega^2 = 0$ can be written

$$(3.2) \qquad \omega = \alpha\,\mathrm{e}^{14} + \beta\,\mathrm{e}^{25} + \gamma\,\mathrm{e}^{36} + x\,\mathrm{e}^{15} + y\,\mathrm{e}^{16} + z\,\mathrm{e}^{26}$$

for $\alpha, \beta, \gamma \in \mathbb{R} - \{0\}$ and $x, y, z \in \mathbb{R}$ modulo an automorphism in $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$.

Proof. We choose standard bases $\{\mathrm{e}^1, \mathrm{e}^2, \mathrm{e}^3\}$ for $\mathfrak{g}$ and $\{\mathrm{e}^4, \mathrm{e}^5, \mathrm{e}^6\}$ for $\mathfrak{h}$. Due to the assumption $d\omega^2 = 0$ and Lemma 1.1, we may write $\omega = \sum_{i,j=1}^3 c_{ij}\mathrm{e}^{i(j+3)}$ for an invertible matrix $C = (c_{ij}) \in \mathrm{Mat}(3, \mathbb{R})$. When a pair $(A, B) \in \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ acts on the two-form ω, the matrix C is transformed to A^tCB. Applying Lemma 3.3, we can achieve by an inner automorphism that C is in one of the normal forms given in that lemma. However, an exchange of the base vectors e_1 and e_2 corresponds exactly to exchanging the first and the second row of C. Therefore, we can always write ω in the claimed normal form by adding the sign τ in the Lie bracket of the first summand $\mathfrak{g}$. □

LEMMA 3.5. *Let* $\{\mathrm{e}^1, \ldots, \mathrm{e}^6\}$ *be a basis of* $\mathfrak{so}(1,2) \times \mathfrak{so}(1,2)$ *such that*

$$d\mathrm{e}^1 = -\mathrm{e}^{23}, \quad d\mathrm{e}^2 = \mathrm{e}^{31}, \quad d\mathrm{e}^3 = \mathrm{e}^{12} \quad \text{and} \quad d\mathrm{e}^4 = -\mathrm{e}^{56}, \quad d\mathrm{e}^5 = \mathrm{e}^{64}, \quad d\mathrm{e}^6 = \mathrm{e}^{45}. \tag{3.3}$$

Then the only $\mathrm{SU}^{\varepsilon}(p,q)$*-structure* (ω, ψ^+) *modulo inner automorphisms and modulo exchanging the summands, which solves the two nearly* ε*-Kähler equations* (1.16) *and* (1.17)*, is determined by*

$$\omega = \frac{\sqrt{3}}{18}(\mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{36}). \tag{3.4}$$

PROOF. Since $d\omega^2 = 0$ by the second equation (1.17), we can choose a basis satisfying (3.1) such that ω is in the normal form (3.2). In order to satisfy the first equation (1.16), we have to set

$$\begin{aligned} 3\psi^+ = d\omega \;&=\; -\alpha\,\mathrm{e}^{234} + \alpha\,\mathrm{e}^{156} - x\,\mathrm{e}^{235} + x\,\mathrm{e}^{146} - y\,\mathrm{e}^{236} - y\,\mathrm{e}^{145} \\ &-\; \beta\,\mathrm{e}^{135} + \beta\,\mathrm{e}^{246} - z\,\mathrm{e}^{136} - z\,\mathrm{e}^{245} + \tau\gamma\,\mathrm{e}^{126} - \gamma\,\mathrm{e}^{345}. \end{aligned}$$

The compatibility $\omega \wedge \psi^+ = 0$ is equivalent to $d(\omega^2) = 0$. It remains to determine all solutions of the second nearly ε-Kähler equation (1.17) modulo automorphisms.

For the sake of readability, we identify $\Lambda^6(\mathfrak{g}\oplus\mathfrak{h})^*$ with $\mathbb{R}$ by means of e^{123456}. Supported by Maple, we compute

$$\begin{aligned}
K_{\psi^+}(e_1) &= (x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)e_1 - (2x\beta+2yz)e_2 \\
&- 2\tau\gamma y e_3 + 2\tau\gamma\beta e_4, \\
K_{\psi^+}(e_2) &= (2x\beta+2yz)e_1 + (-x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)e_2 \\
&- 2\tau\gamma z e_3 + 2\tau\gamma x e_4 - 2\tau\alpha\gamma e_5, \\
K_{\psi^+}(e_3) &= 2y\gamma e_1 - 2z\gamma e_2 + (-x^2-y^2+z^2+\alpha^2+\beta^2-\tau\gamma^2)e_3 \\
&+ (2y\beta-2xz)e_4 + 2\alpha z e_5 - 2\alpha\beta e_6, \\
K_{\psi^+}(e_4) &= -2\beta\gamma e_1 + 2x\gamma e_2 + (2y\beta-2xz)e_3 \\
&+ (x^2+y^2-z^2+\alpha^2-\beta^2-\tau\gamma^2)e_4 - 2\alpha x e_5 - 2\alpha y e_6, \\
K_{\psi^+}(e_5) &= 2\alpha\gamma e_2 - 2\alpha z e_3 + 2\alpha x e_4 \\
&+ (-x^2+y^2-z^2-\alpha^2+\beta^2-\tau\gamma^2)e_5 + (2\beta z-2xy)e_6, \\
K_{\psi^+}(e_6) &= 2\alpha\beta e_3 + 2\alpha y e_4 \\
&+ (2\beta z-2xy)e_5 + (x^2-y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)e_6.
\end{aligned}$$

We assume that $\lambda(\psi^+) \neq 0$ and check this a posteriori for the solutions we find. Hence, we can set $k := \frac{1}{\pm\sqrt{|\lambda(\psi^+)|}}$ and $J_{\psi^+} = kK_{\psi^+}$. Since $\psi^+ + \mathrm{i}_{\varepsilon}J^*_{\psi^+}\psi^+$ is a $(3,0)$-form with respect to J_{ψ^+}, we have $\psi^- = J^*_{\psi^+}\psi^+ = \varepsilon\psi^+(J_{\psi^+}.,.,.)$ which turns out to be

$$\begin{aligned}
\varepsilon\frac{27}{k}\,\psi^- \;&=\; 2\tau\alpha\beta\gamma\,\mathrm{e}^{123} + 2\tau y\alpha\gamma\,\mathrm{e}^{124} + 2\tau\gamma(xy-\beta z)\,\mathrm{e}^{125} - 2(x\beta+yz)\alpha\,\mathrm{e}^{134} \\
&+\; \tau\gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\,\mathrm{e}^{126}
\end{aligned}$$

$$\begin{aligned}
&- \{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\,\mathrm{e}^{135}\\
&+ \{z(x^2-y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\,\mathrm{e}^{136}\\
&- \{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\,\mathrm{e}^{145}\\
&- \{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\,\mathrm{e}^{146}\\
&- \alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\,\mathrm{e}^{156}\\
&- \alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\,\mathrm{e}^{234}\\
&- \{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\,\mathrm{e}^{235}\\
&+ \{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\,\mathrm{e}^{236}\\
&- \{z(x^2-y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\,\mathrm{e}^{245}\\
&- \{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\,\mathrm{e}^{246}\\
&+ \gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\,\mathrm{e}^{345}\\
&- 2(x\beta+yz)\alpha\,\mathrm{e}^{256}-2\gamma(xy-\beta z)\,\mathrm{e}^{346}-2y\alpha\gamma\,\mathrm{e}^{356}+2\alpha\beta\gamma\,\mathrm{e}^{456}.
\end{aligned}$$

Furthermore, we compute the exterior derivative

$$\begin{aligned}
\varepsilon\frac{27}{k}d\psi^- \;=\;& -4\tau\gamma\alpha y\,\mathrm{e}^{1256}\;-4\tau\gamma(xy-\beta z)\,\mathrm{e}^{1246}\;+4\alpha(x\beta+yz)\,\mathrm{e}^{1356}\\
&+\;2\tau\gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\,\mathrm{e}^{1245}\\
&+\;2\{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\,\mathrm{e}^{1346}\\
&+\;2\{z(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\,\mathrm{e}^{1345}\\
&+\;2\{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\,\mathrm{e}^{2345}\\
&+\;2\{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\,\mathrm{e}^{2346}\\
&+\;2\alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\,\mathrm{e}^{2356}
\end{aligned}$$

and

$$\omega^2 \;=\; 2((y\beta-xz)\,\mathrm{e}^{1256}-\alpha z\,\mathrm{e}^{1246}-x\gamma\,\mathrm{e}^{1356}-\alpha\beta\,\mathrm{e}^{1245}-\alpha\gamma\,\mathrm{e}^{1346}-\beta\gamma\,\mathrm{e}^{2356}).$$

The second nearly Kähler equation (1.17) is therefore equivalent to the following nine coefficient equations:

$$\begin{aligned}
(\alpha\beta-27\varepsilon k^{-1}\gamma)\,x &= -\alpha\,yz, && (\mathrm{e}^{1356})\\
(\tau\gamma\alpha-27\varepsilon k^{-1}\beta)\,y &= -27\varepsilon k^{-1}xz, && (\mathrm{e}^{1256})\\
(\tau\beta\gamma-27\varepsilon k^{-1}\alpha)\,z &= \tau\gamma xy, && (\mathrm{e}^{1246})\\
x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2 &= 54\varepsilon k^{-1}\tfrac{\beta\gamma}{\alpha}, && (\mathrm{e}^{2356})\\
z(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2) &= -2\beta yx, && (\mathrm{e}^{1345})\\
x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2 &= 54\varepsilon k^{-1}\tfrac{\alpha\gamma}{\beta}-2\tfrac{xyz}{\beta}, && (\mathrm{e}^{1346})\\
y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2) &= -2\beta zx, && (\mathrm{e}^{2345})\\
-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2 &= 54\tau\varepsilon k^{-1}\tfrac{\alpha\beta}{\gamma}, && (\mathrm{e}^{1245})\\
x(-x^2-y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2) &= -2\beta yz. && (\mathrm{e}^{2346})
\end{aligned}$$

Recall that $\alpha,\beta,\gamma\neq 0$ because ω is non-degenerate. We claim that there is no solution if any of x, y or z is different from zero.

On the one hand, assume that one of them is zero. Using one of the first three equations respectively, we find that at least one of the other two has to be zero as well. However, in all three cases, we may easily deduce that the third one has to be zero as well by comparing equations 4 and 5 respectively 6 and 7 respectively 8 and 9.

On the other hand, if we assume that all three of them are different from zero, the bracket in the first equation is necessarily different from zero and we may express x by a multiple of yz. Substituting this expression into equations 2 and 3, yields expressions for y^2 and z^2 in terms of α,β, γ and k. But if we insert all this into equation 4 (or 6 or 8 alternatively), we end up with a contradiction after a slightly tedious calculation.

To conclude, we can set $x = y = z = 0$ without losing any solutions of the second nearly Kähler equation which simplifies to the equations

$$\begin{aligned}
\alpha^3 - \alpha\beta^2 - \tau\alpha\gamma^2 - 54\varepsilon k^{-1}\beta\gamma &= 0,\\
\beta^3 - \tau\beta\gamma^2 - \beta\alpha^2 - 54\varepsilon k^{-1}\gamma\alpha &= 0,\\
\gamma^3 - \tau\gamma\alpha^2 - \tau\gamma\beta^2 - 54\varepsilon k^{-1}\alpha\beta &= 0.
\end{aligned}$$

Setting $c_1 = \alpha^2 + \beta^2 + \tau\gamma^2$ and $c_2 = 54\varepsilon k^{-1}\alpha\beta\gamma$, these are equivalent to

$$\begin{aligned}
2\alpha^4 - c_1\alpha^2 - c_2 &= 0,\\
2\beta^4 - c_1\beta^2 - c_2 &= 0,\\
2\gamma^4 - c_1\tau\gamma^2 - c_2 &= 0.
\end{aligned} \tag{3.5}$$

To finish the proof, we have to show that all real solutions of the system (3.5) are isomorphic under $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ to

$$\alpha = \beta = \gamma = \frac{\sqrt{3}}{18}, \qquad \tau = 1.$$

Since α^2, β^2 and $\tau\gamma^2$ satisfy the same quadratic equation, at least two of them have to be identical, say $\alpha^2 = \beta^2$. However, if $\tau\gamma^2$ was the other root of the quadratic equation, we would have $\alpha^2 + \tau\gamma^2 = \frac{1}{2}c_1$ and by definition of c_1 at the same time $2\alpha^2 + \tau\gamma^2 = c_1$. This would only be possible, if γ was zero, a contradiction to the non-degeneracy of ω. Therefore τ has to be $+1$ and α,β and γ have to be identical up to sign. By applying one of the proper and orthochronous Lorentz transformations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

on, say, the second summand, it is always possible to achieve that the signs of α, β and γ are identical.

So far, we found a basis satisfying (3.3) such that

$$\omega = \alpha(\mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{36}).$$

It is straightforward to check that the quartic invariant in this basis is

$$\lambda(\frac{1}{3}d\omega) = -\frac{1}{27}\alpha^4. \tag{3.6}$$

Therefore, there cannot exist a nearly para-Kähler structure and we can set $\varepsilon = -1$. Inserting $k = \pm\frac{1}{\sqrt{-\lambda}} = \pm 3\sqrt{3}\alpha^{-2}$ into equations (3.5) yields

$$2\alpha^4 - 3\alpha^4 \pm \frac{54}{3\sqrt{3}}\alpha^5 = 0 \iff \alpha = \pm\frac{1}{18}\sqrt{3}.$$

Finally, we can achieve that α is positive by applying the Lie algebra automorphism exchanging the two summands, i.e. $e_i \mapsto e_{i+3 \bmod 6}$ and the lemma is proven. □

In fact, the uniqueness, existence and non-existence statements claimed in the theorem follow directly from this lemma and formula (3.6). □

We summarise the data of the unique nearly pseudo-Kähler structure in the basis (3.3), in particular the signature of the induced metric is easily seen to be $(2,4)$:

$$\begin{aligned}
\omega &= \frac{1}{18}\sqrt{3}\,(\mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{36})\\
\psi^+ &= \frac{1}{54}\sqrt{3}\,(\mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{156} - \mathrm{e}^{234} + \mathrm{e}^{246} - \mathrm{e}^{345})\\
\psi^- &= -\frac{1}{54}(2\,\mathrm{e}^{123} + \mathrm{e}^{126} - \mathrm{e}^{135} - \mathrm{e}^{156} - \mathrm{e}^{234} - \mathrm{e}^{246} + \mathrm{e}^{345} + 2\,\mathrm{e}^{456})\\
J(e_1) &= -\frac{1}{3}\sqrt{3}\,e_1 - \frac{2}{3}\sqrt{3}\,e_4\,, \quad J(e_4) = \frac{2}{3}\sqrt{3}\,e_1 + \frac{1}{3}\sqrt{3}\,e_4\\
J(e_2) &= -\frac{1}{3}\sqrt{3}\,e_2 + \frac{2}{3}\sqrt{3}\,e_5\,, \quad J(e_5) = -\frac{2}{3}\sqrt{3}\,e_2 + \frac{1}{3}\sqrt{3}\,e_5\\
J(e_3) &= -\frac{1}{3}\sqrt{3}\,e_3 + \frac{2}{3}\sqrt{3}\,e_6\,, \quad J(e_6) = -\frac{2}{3}\sqrt{3}\,e_3 + \frac{1}{3}\sqrt{3}\,e_6\\
g &= \frac{1}{9}\,(\,(\mathrm{e}^1)^2 - (\mathrm{e}^2)^2 - (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 - (\mathrm{e}^5)^2 - (\mathrm{e}^6)^2 - \mathrm{e}^1\cdot\mathrm{e}^4 - \mathrm{e}^2\cdot\mathrm{e}^5 - \mathrm{e}^3\cdot\mathrm{e}^6).
\end{aligned}$$

CHAPTER 6

Hitchin flow

In the final chapter, we discuss the evolution of half-flat and nearly half-flat structures. All results in this chapter are part of [**CLSS**].

1. Half-flat structures and parallel $\mathrm{G}_2^{(*)}$-structures

For compatibility reasons, we use the following notation introduced in [**CLSS**]. By $H^{\varepsilon,\tau}$, we denote the real form of $\mathrm{SL}(3,\mathbb{C})$ corresponding to the standard basis defined in section 4.1 of chapter 1, i.e. $H^{-1,1} = \mathrm{SU}(3) \subset \mathrm{SO}(6)$, $H^{-1,-1} = \mathrm{SU}(1,2) \subset \mathrm{SO}(2,4)$, $H^{1,1} = \mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SO}(3,3)$. Let $G^{\varepsilon,\tau}$ denote the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ in which $H^{\varepsilon,\tau}$ is embedded, i.e. $G^{-1,1} = \mathrm{G}_2 \subset \mathrm{SO}(7)$, and $G^{-1,-1} = G^{1,1} = \mathrm{G}_2^* \subset \mathrm{SO}(3,4)$.

Due to Proposition 3.3, chapter 3, a normalised $H^{\varepsilon,\tau}$-structure on a six-manifold M can be identified with a pair $(\omega,\rho) \in \Omega^2 M \times \Omega^3 M$ of stable forms which are compatible,

$$\omega \wedge \rho = 0 \quad \iff \quad \omega(\,.\,, J_\rho\,.) = -\omega(J_\rho\,.\,,\,.), \tag{1.1}$$

and normalised,

$$J_\rho^* \rho \wedge \rho = \frac{2}{3}\,\omega^3 \quad \iff \quad \phi(\rho) = 2\phi(\omega). \tag{1.2}$$

In this chapter, we will assume that all $H^{\varepsilon,\tau}$-structures are normalised.

Now, let M be an orientable hypersurface in a seven-manifold N with $G^{\varepsilon,\tau}$-structure φ and induced metric g_φ. There is a natural $H^{\varepsilon,\tau}$-structure (ω,ρ) on M which can be defined by applying pointwise the algebraic construction of 4.5, chapter 1, as follows. In the Riemannian case, this relation seems to appear first in [**Cal**]. Choosing a unit normal vector field $\xi \in \mathfrak{X}(N)$ with $g_\varphi(\xi,\xi) = -\varepsilon$, the pair (ω,ρ) given by

$$\omega = -\varepsilon(\xi \lrcorner\, \varphi)_{|TM}\,, \qquad \rho = \varphi_{|TM} \tag{1.3}$$

defines indeed an $H^{\varepsilon,\tau}$-structure on the hypersurface M which is normalised for the right choice of ξ. Obviously, the stable three-form φ satisfies

$$\varphi_{|TM} = \xi^\flat \wedge \omega + \rho \tag{1.4}$$

such that Lemma 4.4 of chapter 1 yields

$$(*_\varphi \varphi)_{|TM} = -\varepsilon(\xi^\flat \wedge \hat\rho + \hat\omega) = -\varepsilon(\xi^\flat \wedge J_\rho^*\rho + \frac{1}{2}\omega^2). \tag{1.5}$$

In consequence, if the $G^{\varepsilon,\tau}$-structure φ is parallel, i.e. $d\varphi = 0$ and $d * \varphi = 0$, the induced $H^{\varepsilon,\tau}$-structure on the hypersurface M is half-flat. For the compact forms, a complete analysis of the relation of the intrinsic torsions of the G_2-structure and the $\mathrm{SU}(3)$-structure on the hypersurface is carried out in [**MC3**].

Conversely, certain one-parameter families of half-flat structures define parallel $\mathrm{G}_2^{(*)}$-structures.

Proposition 1.1. *Let $H^{\varepsilon,\tau}$ be a real form of* $\mathrm{SL}(3,\mathbb{C})$, *$G^{\varepsilon,\tau}$ the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ and (ρ,ω) a one-parameter family of $H^{\varepsilon,\tau}$-structures on a six-manifold M with a parameter t from an interval I. Then, the three-form*

$$\varphi = \omega \wedge dt + \rho$$

defines a parallel $G^{\varepsilon,\tau}$-structure on $M \times I$ if and only if the $H^{\varepsilon,\tau}$-structure (ρ,ω) is half-flat for all t and satisfies the following evolution equations

$$\dot{\rho} = d\omega \tag{1.6}$$

$$\dot{\sigma} = d\hat{\rho} \tag{1.7}$$

with $\sigma = \frac{1}{2}\omega^2$.

Proof. Let (ρ,ω) be an $H^{\varepsilon,\tau}$-structure and $\varphi = \omega \wedge dt + \rho$ a stable three-form on $\check{M} := M \times I$. By Lemma 4.4, ch. 1, the Hodge-dual of φ is given by

$$*\varphi = \varepsilon\,(\hat{\rho} \wedge dt - \sigma)\,.$$

Denoting by $\check{d}$ the differential on $\check{M}$ and by d the differential on M we calculate

$$\check{d}\varphi = d\omega \wedge dt + dt \wedge \dot{\rho} + d\rho = (d\omega - \dot{\rho}) \wedge dt + d\rho \tag{1.8}$$

$$\check{d} * \varphi = \varepsilon\,(d\hat{\rho} \wedge dt - dt \wedge \dot{\sigma} - d\sigma) = \varepsilon(d\hat{\rho} - \dot{\sigma}) \wedge dt - \varepsilon d\sigma \tag{1.9}$$

Thus, φ defines a parallel $G^{\varepsilon,\tau}$-structure if and only if the evolution equations (1.6) and (1.7) and the half-flat equations are satisfied. $\square$

The evolution equations (1.6) and (1.7) are the *Hitchin flow equations*, as found in [**Hi1**] for SU(3)-structures, applied to $H^{\varepsilon,\tau}$-structures. Their solutions (ρ,ω), called *Hitchin flow*, have to satisfy possibly dependent conditions in order to yield a parallel $\mathrm{G}_2^{(*)}$-structure: the evolution equations and the compatibility equations for the family of half-flat structures. The following theorem shows that the evolution equations together with an initial condition already ensure that the family consists of half-flat structures. A special version of this theorem was proved in [**Hi1**] under the assumption that M is compact and that $H = \mathrm{SU}(3)$.

Theorem 1.2. *Let (ρ_0,ω_0) be a half-flat $H^{\varepsilon,\tau}$-structure on a six-manifold M. Furthermore, let $(\rho,\omega) \in \Omega^3 M \times \Omega^2 M$ be a one-parameter family of stable forms with parameters from an interval I satisfying the evolution equations* (1.6) *and* (1.7). *If $(\rho(t_0),\omega(t_0)) = (\rho_0,\omega_0)$ for a $t_0 \in I$, then (ρ,ω) is a family of half-flat $H^{\varepsilon,\tau}$-structures. In particular, the three-form*

$$\varphi = \omega \wedge dt + \rho \tag{1.10}$$

defines a parallel $G^{\varepsilon,\tau}$-structure on $M \times I$ and the induced metric

$$g_\varphi = g(t) - \varepsilon dt^2, \tag{1.11}$$

has holonomy contained in $G^{\varepsilon,\tau}$, where $g = g(t)$ is the family of metrics on M associated to (ρ,ω).

Proof. Differentiating the evolution equations (1.6) and (1.7) gives $d\dot{\rho} = d\dot{\sigma} = 0$. As the initial structure (ρ_0,ω_0) is half-flat, this implies

$$d\rho = 0\,, \qquad d\sigma = 0 \tag{1.12}$$

for all $t \in I$. Hence, in order to obtain a family of half-flat structures we have to verify that the compatibility condition (1.1) holds for all $t \in I$.

LEMMA 1.3. *Let M be a six-manifold with $H^{\varepsilon,\tau}$-structure (ρ, ω), $\phi : \Omega^3 M \to \Omega^6 M$ defined pointwise by the map $\phi : \Lambda^3 T_p^* M \to \Lambda^6 T_p^* M$ given in Proposition 3.4, chapter 1, and $\hat{\rho}$ defined by $d\phi_\rho(\xi) = \hat{\rho} \wedge \xi$ for all $\xi \in \Omega^3 M$. If $\mathcal{L}_X$ denotes the Lie derivative, then*

$$\mathcal{L}_X(\phi(\rho)) = \hat{\rho} \wedge \mathcal{L}_X \rho.$$

PROOF. First note that the $\mathrm{GL}(n, \mathbb{R})$-equivariance of the map $\phi : \Lambda^3 T_p^* M \to \Lambda^6 T_p^* M$ implies that the corresponding map $\phi : \Omega^3 M \to \Omega^6 M$ is equivariant under diffeomorphisms. Indeed, if ψ is a (local) diffeomorphism of M we get that

$$\psi^*(\phi(\rho)) = \phi(\psi^* \rho).$$

Let ψ_t be the flow of the vector field X. Then the Lie derivative is given by

$$\mathcal{L}_X(\phi(\rho)) = \frac{d}{dt} \left(\psi_t^* \phi(\rho)\right)|_{t=0} = \frac{d}{dt} \phi(\psi_t^* \rho)|_{t=0} = d\phi_\rho(\mathcal{L}_X \rho),$$

implying the statement. $\square$

LEMMA 1.4. *A stable three-form $\rho \in \Omega^3 M$ on a six-manifold satisfies for any $X \in \mathfrak{X}(M)$*

$$\hat{\rho}_X \wedge \rho = -\hat{\rho} \wedge \rho_X \,, \tag{1.13}$$

$$(d\hat{\rho})_X \wedge \rho = \hat{\rho} \wedge (d\rho)_X \,, \tag{1.14}$$

where ρ_X denotes the interior product of X with the form ρ.

PROOF. The first identity is in fact (1.11), ch. 1, when considering that $\rho + \mathrm{i}_\varepsilon \hat{\rho}$ is an ε-complex volume form due to Lemma 3.6, ch. 1. In order to prove the second identity, we compute, using Lemma 1.3 in the second step,

$$\begin{aligned}
(d\hat{\rho})_X \wedge \rho - \hat{\rho} \wedge (d\rho)_X &= -d\hat{\rho} \wedge \rho_X + \hat{\rho} \wedge d(\rho_X) - \hat{\rho} \wedge \mathcal{L}_X \rho \\
&= -d(\hat{\rho} \wedge \rho_X) - \mathcal{L}_X(\phi(\rho)) \\
&= -d(\hat{\rho} \wedge \rho_X + \phi(\rho)_X) \\
&\overset{(3.2)}{=} -\frac{1}{2} d(\hat{\rho} \wedge \rho_X + \hat{\rho}_X \wedge \rho).
\end{aligned}$$

Hence, the first identity (1.13) implies (1.14). $\square$

Using this lemma, we calculate the t-derivative of the six-form $\omega_X \wedge \omega \wedge \rho = \sigma_X \wedge \rho$ for any vector field X:

$$\begin{aligned}
\frac{\partial}{\partial t}(\sigma_X \wedge \rho) &= \dot{\sigma}_X \wedge \rho + \sigma_X \wedge \dot{\rho} \\
&\overset{(1.6),(1.7)}{=} (d\hat{\rho})_X \wedge \rho + \sigma_X \wedge d\omega \\
&\overset{(1.14)}{=} \hat{\rho} \wedge (d\rho)_X + \omega_X \wedge d(\omega^2) \\
&\overset{(1.12)}{=} 0.
\end{aligned}$$

Together with the initial condition $\omega_0 \wedge \rho_0 = 0$ this implies that $\sigma_X \wedge \rho = 0$ for all $t \in I$ and for all vector fields X. Since ω is non degenerate, the product of any one-form with $\omega \wedge \rho$ vanishes and thus, the compatibility condition $\omega \wedge \rho = 0$ holds for all t.

The preservation of the normalisation (1.2) in time is shown in [**Hi1**], in the final part of the proof of Theorem 8. The idea is to compute the second derivative of the volume form assigned to a stable three-form. In fact, the proof holds literally for all signatures since all it uses is the first compatibility condition we have just proved. $\square$

COROLLARY 1.5. *Let M be a real analytic six-manifold with a half-flat $H^{\varepsilon,\tau}$-structure that is given by a pair of analytic stable forms (ω_0, ρ_0).*

(i) Then, there exists a unique maximal solution (ω, ρ) of the evolution equations (1.6), (1.7) *with initial value (ω_0, ρ_0), which is defined on an open neighbourhood $\Omega \subset \mathbb{R}\times M$ of $\{0\} \times M$. In particular, there is a parallel $G^{\varepsilon,\tau}$-structure on Ω.*

(ii) Moreover, the evolution is natural in the sense that, given a diffeomorphism f of M, the pullback $(f^*\omega, f^*\rho)$ of the solution with initial value (ω_0, ρ_0) is the solution of the evolution equations for the initial value $(f^*\omega_0, f^*\rho_0)$.

In particular, if f is an automorphism of the initial structure (ω_0, ρ_0), then, for all $t \in \mathbb{R}$, f is an automorphism of the solution $(\omega(t), \rho(t))$ defined on the (possibly empty) open set $U_t = \{p \in M \mid (t, p) \in \Omega$ and $(t, f(p)) \in \Omega\}$.

(iii) Furthermore, assume that M is compact or a homogeneous space $M = G/K$ such that the $H^{\varepsilon,\tau}$-structure is G-invariant. Then there is a unique maximal interval $I \ni 0$ and a unique solution (ω, ρ) of the evolution equations (1.6), (1.7) *with initial value (ω_0, ρ_0) on $I \times M$. In particular, there is a parallel $G^{\varepsilon,\tau}$-structure on $I \times M$.*

PROOF. If the manifold and the initial structure (ω_0, ρ_0) are analytic, there exists a unique maximal solution of the evolution equations on a neighbourhood Ω of $M \times \{0\}$ in $M \times \mathbb{R}$ by the Cauchy-Kovalevskaya theorem. The naturality of the solution is an immediate consequence of the uniqueness due to the naturality of the exterior derivative. If M is compact, there is a maximal interval I such that the solution is defined on $M \times I$. The same is true for a homogeneous half-flat structure (ω_0, ρ_0) as it is determined by $(\omega_0, \rho_0)_{|p}$ for any $p \in M$. $\square$

We remark that, for a homogeneous half-flat structure (ω_0, ρ_0), the evolution equations reduce to a system of ordinary differential equations due to the naturality assertion of the corollary. This simplification will be used in Section 5.2 to construct metrics with holonomy equal to G_2 and G_2^*.

1.1. Remark on completeness: geodesically complete conformal G_2-metrics. The $\mathrm{G}_2^{(*)}$-metrics arising from the Hitchin flow on a six-manifold N are of the form

$$(I \times N, dt^2 + g_t)$$

with an open interval $I = (a, b)$ and a family of Riemannian metrics g_t depending on $t \in I$ (formula (1.11) in Theorem 1.2). As curves of the form $t \mapsto (t, x)$ are geodesics for this metric, they are obviously geodesically incomplete if a or $b \in \mathbb{R}$.

For the *Riemannian* case and *compact* manifolds N, we shall explain how one easily obtains *complete* metrics by a conformal change of the G_2-metric.

LEMMA 1.6. *Let N be a compact manifold with a family g_r of Riemannian metrics. Then the Riemannian metric on $\mathbb{R} \times N$ defined by $h = dr^2 + g_r$ is geodesically complete.*

PROOF. Denote by d the distance on $\mathbb{R} \times N$ induced by the Riemannian metric $h = dr^2 + g_r$ and by d_r the distance on N induced by g_r. For a curve γ in $M = \mathbb{R} \times N$ we have that the length of $\gamma(t) = (r(t), x(t))$ satisfies

$$\ell(\gamma) = \int_0^1 \sqrt{\dot{r}(t)^2 + g_{r(t)}(\dot{x}(t), \dot{x}(t))}dt \ \geq \ \int_0^1 |\dot{r}(t)|dt \ \geq \ |r(1) - r(0)|.$$

As the distance of two points $p = (r, x)$ and $q = (s, y)$ is defined as the infimum of the lengths of all curves joining them, this inequality implies that

$$d(p, q) \geq |r - s|. \tag{1.15}$$

Note also that a curve $\gamma(t) = ((s - r)t + r, x)$ joining $p = (r, x)$ and $q = (s, x)$ in $\mathbb{R} \times \{x\}$ has length $\ell(\gamma) = |r - s|$ and thus, for such p, q we get that $d(p, q) = |r - s|$. On the other hand, for $p = (r, x)$ and $q = (r, y)$ with the same $\mathbb{R}$-projection r we only get that $d(p, q) \leq d_r(x, y)$.

Since h has Riemannian signature we can use the Hopf-Rinow Theorem and consider a Cauchy sequence $p_n = (r_n, x_n) \in \mathbb{R} \times N$ w.r.t. the distance d. Equation (1.15) then implies that the sequence r_n is a Cauchy sequence in $\mathbb{R}$. Hence, r_n converges to $r \in \mathbb{R}$. Since N is compact, the sequence x_n has a subsequence x_{n_k} converging to $x \in N$. For $p = (r, x)$ and $q_{n_k} := (r, x_{n_k})$ the triangle inequality implies that

$$d(p, p_{n_k}) \ \leq \ d\left(p, q_{n_k}\right) + d\left(q_{n_k}, p_{n_k}\right) \ \leq \ d_r(x, x_{n_k}) + d\left(q_{n_k}, p_{n_k}\right) \ = \ d_r(x, x_{n_k}) + |r - r_{n_k}|.$$

Hence, p_{n_k} converges to p. As p_n was a Cauchy sequence, we have found p as a limit for p_n. By the Theorem of Hopf and Rinow, M is geodesically complete. □

The lemma can be used to obtain the desired result.

PROPOSITION 1.7. *Let $(M = I \times N, h = dt^2 + g_t)$ be a Riemannian metric on a product of an open interval I and a compact manifold N. Then (M, h) is globally conformally equivalent to a metric on $\mathbb{R} \times N$ that is geodesically complete. The scaling factor depends only on $t \in I$ and is determined by a diffeomorphism $\varphi : \mathbb{R} \to I$.*

PROOF. Let $\varphi : \mathbb{R} \to I$ be a diffeomorphism with inverse $r = \varphi^{-1}$. Changing the coordinate t to r, the metric h on $I \times N$ can be written as

$$h \ = \ \left(\varphi'(r)dr\right)^2 + g_{\varphi(r)} \ = \ \varphi'(r)^2 \left(dr^2 + \frac{1}{\varphi'(r)^2} g_{\varphi(r)}\right).$$

Hence, h is globally conformally equivalent to the metric $dr^2 + \frac{1}{\varphi'(r)^2} g_{\varphi(r)}$ on $\mathbb{R} \times N$. By the lemma, this metric is geodesically complete. □

Regarding the solution of the Hitchin flow equations, we obtain the following consequence of Theorem 1.2, Corollary 1.5, and Proposition 1.7.

COROLLARY 1.8. *Let M be a compact analytic six-manifold with half-flat $\mathrm{SU}(3)$-structure given by analytic stable forms (ρ_0, ω_0). Then there is a complete metric on $\mathbb{R} \times M$ that is globally conformal to the parallel G_2-metric obtained by the Hitchin flow.*

In Example 5.9 of Section 5.2 we will construct explicit examples of this type. Finally, note that due to the Cheeger-Gromoll splitting Theorem, see for example [**Bes**, Theorem 6.79], one cannot expect to obtain by the Hitchin flow irreducible G_2-metrics that are complete without allowing degenerations of g_t.

2. Nearly half-flat structures and nearly parallel $\mathrm{G}_2^{(*)}$ -structures

Recall that a $\mathrm{G}_2^{(*)}$-structure φ on a seven-manifold N is called nearly parallel if

$$d\varphi = \mu *_\varphi \varphi \tag{2.1}$$

for a constant $\mu \in \mathbb{R}^*$. Nearly parallel G_2- and G_2^*-structures are also characterised by the existence of a Killing spinor, refer [**FKMS**] respectively [**Ka1**].

Given an orientable hypersurface M in an almost seven-manifold N with nearly parallel $\mathrm{G}_2^{(*)}$-structure φ, the induced $H^{\varepsilon,\tau}$-structure (1.3) satisfies the equation $d\rho = -\frac{1}{2}\varepsilon\mu\omega^2 = -\varepsilon\mu\hat{\omega}$ due to the formulas (1.4) and (1.5). In order to remain compatible with [**CLSS**], we call a $H^{\varepsilon,\tau}$-structure satisfying

$$d\rho = \frac{\lambda}{2}\omega^2 = \lambda\sigma \tag{2.2}$$

nearly half-flat for the constant $\lambda \in \mathbb{R}^*$ (and not for the constant $\frac{\lambda}{2}$ as in chapter 3). In other words, the induced $H^{\varepsilon,\tau}$-structure on the hypersurface is nearly half-flat for the constant $-\varepsilon\mu$.

In [**FIMU**], nearly half-flat SU(3)-structures have been introduced in the context of evolution equations on six-manifolds M leading to nearly parallel G_2-structures on the product of M and an interval. For compact manifolds M, it is shown in [**St**] that a solution of these evolution equations which is a nearly half-flat SU(3)-structure for a time $t = t_0$ already defines a nearly parallel G_2-structure. In the following, we extend the evolution equations to all possible signatures and give a simplified proof for the properties of the solutions which also holds for non-compact manifolds.

Proposition 2.1. *Let $H^{\varepsilon,\tau}$ be a real form of $\mathrm{SL}(3,\mathbb{C})$, $G^{\varepsilon,\tau}$ the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ and (ρ,ω) a one-parameter family of $H^{\varepsilon,\tau}$-structures on a six-manifold M with a parameter t from an interval I. Then, the three-form*

$$\varphi = \omega \wedge dt + \rho$$

defines a nearly parallel $G^{\varepsilon,\tau}$-structure for the constant $\mu \neq 0$ on $M \times I$ if and only if the $H^{\varepsilon,\tau}$-structure (ρ,ω) is nearly half-flat for the constant $-\varepsilon\mu$ for all $t \in I$ and satisfies the evolution equation

$$\dot{\rho} \;=\; d\omega - \varepsilon\mu\hat{\rho}. \tag{2.3}$$

Proof. The assertion follows directly from the following computation, analogously to the proof of Proposition 1.1:

$$\begin{aligned} \check{d}\varphi &= d\omega \wedge dt + dt \wedge \dot{\rho} + d\rho \;=\; (d\omega - \dot{\rho}) \wedge dt + d\rho, \\ \mu * \varphi &= \varepsilon\mu\left(\hat{\rho} \wedge dt - \sigma\right). \end{aligned}$$

□

The main theorem for the parallel case generalises as follows. Recall (3.3) that for a stable four-form $\sigma = \frac{1}{2}\omega^2 = \hat{\omega}$, the application of the operator $\sigma \mapsto \hat{\sigma}$ yields the stable two-form

$$\hat{\hat{\omega}} = \hat{\sigma} = \frac{1}{2}\omega.$$

THEOREM 2.2. *Let (ρ_0, ω_0) be a nearly half-flat $H^{\varepsilon,\tau}$-structure for the constant $\lambda \neq 0$ on a six-manifold M. Let M be oriented such that $\omega_0^3 > 0$. Furthermore, let $\rho \in \Omega^3 M$ be a one-parameter family of stable forms with parameters coming from an interval I such that $\rho(t_0) = \rho_0$ and such that the evolution equation*

$$\dot{\rho} \;=\; \frac{2}{\lambda} d(\widehat{d\rho}) + \lambda\, \hat{\rho} \tag{2.4}$$

is satisfied for all $t \in I$. Then $(\rho, \omega = \frac{2}{\lambda}\widehat{d\rho})$ is a family of nearly half-flat $H^{\varepsilon,\tau}$-structures for the constant λ. In particular, the three-form

$$\varphi = \omega \wedge dt + \rho$$

defines a nearly parallel $G^{\varepsilon,\tau}$-structure for the constant $-\varepsilon\lambda$ on $M \times I$.

PROOF. First of all, we observe that $d\rho$ is stable in a neighbourhood of the stable form $d\rho_0 = \lambda\sigma_0$, since stability is an open condition. Furthermore, the operator $d\rho \mapsto \widehat{d\rho}$ is uniquely defined by the orientation induced from ω_0. Therefore, the evolution equation is locally well-defined and we assume that ρ is a solution on an interval I. The only possible candidate for a nearly half-flat structure for the constant λ is $(\rho, \omega = \frac{2}{\lambda}\widehat{d\rho})$ since only this two-form ω satisfies the nearly half-flat equation $\sigma = \hat{\omega} = \frac{1}{\lambda} d\rho$. Obviously, it holds

$$d\sigma = 0 = d\omega \wedge \omega. \tag{2.5}$$

By Proposition 2.1, it only remains to show that this pair of stable forms defines an $H^{\varepsilon,\tau}$-structure, or equivalently, that the compatibility conditions (1.1) and (1.2) are preserved in time. By taking the exterior derivative of the evolution equation, we find

$$\dot{\sigma} = \frac{1}{\lambda} d\dot{\rho} = d\hat{\rho} \tag{2.6}$$

which is in fact the second evolution equation of the parallel case. Completely analogous to the parallel case, the following computation implies the first compatibility condition:

$$\begin{aligned} \frac{\partial}{\partial t}(\sigma_X \wedge \rho) &= \dot{\sigma}_X \wedge \rho + \sigma_X \wedge \dot{\rho} \\ &\overset{(2.4),(2.6)}{=} (d\hat{\rho})_X \wedge \rho + \sigma_X \wedge d\omega + \lambda\, \sigma_X \wedge \hat{\rho} \\ &\overset{(1.14),(2.2)}{=} \hat{\rho} \wedge (d\rho)_X + \omega_X \wedge \omega \wedge d\omega + (d\rho)_X \wedge \hat{\rho} \\ &\overset{(2.5)}{=} 0. \end{aligned}$$

The proof of the second compatibility condition in [**Hi1**] again holds literally since the term $\hat{\rho} \wedge \dot{\rho} = \hat{\rho} \wedge d\omega$ is the same as in the case of the parallel evolution. ☐

The system (2.4) of second order in ρ can easily be reformulated into a system of first order in (ω, ρ) to which we can apply the Cauchy-Kovalevskaya theorem. Indeed, a solution (ω, ρ) of the system

$$\dot{\rho} = d\omega + \lambda\hat{\rho}\,, \qquad \dot{\sigma} = d\hat{\rho}, \tag{2.7}$$

with nearly half-flat initial value $(\omega(t_0), \rho(t_0))$ is nearly half-flat for all t and also satisfies the system (2.4). Conversely, (2.4) implies (2.7) with $\sigma = \hat{\omega} = \frac{1}{\lambda} d\rho$.

Therefore, for an initial nearly half-flat structure which satisfies assumptions analogous to those of Corollary 1.5, we obtain existence, uniqueness and naturality of a solution of the system (2.7), or, equivalently, of (2.4).

3. Cocalibrated $\mathrm{G}_2^{(*)}$-structures and parallel $\mathrm{Spin}(7)$- and $\mathrm{Spin}_0(3,4)$-structures

In [**Hi1**], another evolution equation is introduced which relates cocalibrated G_2-structures on compact seven-manifolds M to parallel Spin(7)-structures. As before, we generalise the evolution equation to non-compact manifolds and indefinite metrics.

As we have already seen in Section 4.3 of chapter 1, the stabiliser in $\mathrm{GL}(V)$ of a four-form Φ_0 on an eight-dimensional vector space V is Spin(7) or $\mathrm{Spin}_0(3,4)$ if and only if it can be written as in (4.12) of chapter 1 for a stable three-form φ on a seven-dimensional subspace with stabiliser G_2- or G_2^*, respectively. Thus, a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on an eight-manifold M is defined by a four-form $\Phi \in \Omega^4 M$ such that $\Phi_p \in \Lambda^4 T_p^* M$ has this property for all p. By formula (4.13) of chapter 1 for the metric g_Φ induced by Φ, an oriented hypersurface in (M, Φ) with spacelike unit normal vector field n with respect to g_Φ carries a natural G_2- or G_2^*-structure, respectively, defined by $\varphi = n \lrcorner \Phi$.

A Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure Φ is *parallel* if and only if $d\Phi = 0$. We remark that the proof for the Riemannian case given in [**Sa2**, Lemma 12.4] is not hard to transfer to the indefinite case when considering [**Br1**, Proposition 2.5] and using the complexification of the two spin groups.

Due to this fact, the induced $\mathrm{G}_2^{(*)}$-structure φ on an oriented hypersurface in an eight-manifold M with parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure Φ is *cocalibrated*, i.e. it satisfies

$$d *_\varphi \varphi = 0. \tag{3.1}$$

Conversely, a cocalibrated $\mathrm{G}_2^{(*)}$-structure can be embedded in an eight-manifold with parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure as follows.

THEOREM 3.1. *Let M be a seven-manifold and $\varphi \in \Omega^3 M$ be a one-parameter family of stable three-forms with a parameter t in an interval I satisfying the evolution equation*

$$\frac{\partial}{\partial t}(*_\varphi \varphi) \;=\; d\varphi. \tag{3.2}$$

If φ is cocalibrated at $t = t_0 \in I$, then φ defines a family of cocalibrated G_2- *or* $\mathrm{G}_2^{(*)}$-*structures for all $t \in I$. Moreover, the four-form*

$$\Phi = dt \wedge \varphi + *_\varphi \varphi \tag{3.3}$$

defines a parallel Spin(7)- *or* $\mathrm{Spin}_0(3,4)$-*structure on $M \times I$, respectively, which induces the metric*

$$g_\Phi = g_\varphi + dt^2. \tag{3.4}$$

PROOF. Since the time derivative of $d * \varphi$ vanishes when inserting the evolution equation, the family stays cocalibrated if it is cocalibrated at an initial value. As before, we denote by $\check{d}$ the exterior differential on $\check{M} := M \times I$ and differentiate the four-form (3.3):

$$\check{d}\Phi = -dt \wedge d\varphi + d(*\varphi) + dt \wedge \frac{\partial}{\partial t}(*\varphi).$$

Obviously, this four-form is closed if and only the evolution equation is satisfied and the family is cocalibrated. The formula for the induced metric corresponds to formula (4.13) of chapter 1. $\square$

As before, the Cauchy-Kovalevskaya theorem guarantees existence and uniqueness of solutions if assumptions analogous to those of Corollary 1.5 are satisfied.

REMARK 3.2. We observe that nearly parallel G_2- and G_2^*-structures are in particular cocalibrated such that analytic nearly half-flat structures in dimension six can be embedded in parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structures in dimension eight by evolving them twice with the help of the Theorems 2.2 and 3.1.

4. Evolution of nearly ε-Kähler manifolds

In order to illustrate the results of the previous sections, we discuss the evolution of nearly ε-Kähler manifolds. The explicit solution of the Hitchin flow yields a simple and unified proof for the correspondence of nearly ε-Kähler manifolds and parallel $\mathrm{G}_2^{(*)}$-structures on cones. We complete the picture by considering similarly the evolution of nearly Kähler structures to nearly parallel $\mathrm{G}_2^{(*)}$-structures on (hyperbolic) sine cones and the evolution of nearly parallel $\mathrm{G}_2^{(*)}$-structures to parallel Spin(7)- and $\mathrm{Spin}_0(3,4)$-structures on cones. Our presentation in terms of differential forms unifies various results in the literature, which were originally obtained using spinorial methods, and applies to all possible real forms of the relevant groups.

4.1. Cones over nearly ε-Kähler manifolds. Recall that we proved in chapter 3, Theorem 5.5, that a nearly ε-Kähler six-manifold with $\|\nabla J\|^2 = 4$ is equivalent to a normalised $H^{\varepsilon,\tau}$-structure (ω,ρ) which satisfies

$$d\omega = 3\rho\,, \tag{4.1}$$

$$d\hat{\rho} = 4\hat{\omega}. \tag{4.2}$$

In particular, nearly ε-Kähler structures (ω,ρ) in dimension six are half-flat and the structure $(\omega,\hat{\rho})$ is nearly half-flat (for the constant $\lambda = 4$).

PROPOSITION 4.1. *Let (M, h_0) be a pseudo-Riemannian six-manifold of signature $(6,0)$, $(4,2)$ or $(3,3)$ and let $(\bar{M} = M \times \mathbb{R}^+, \bar{g}_\varepsilon = h_0 - \varepsilon dt^2)$ be the timelike cone for $\varepsilon = 1$ and the spacelike cone for $\varepsilon = -1$. There is a one-to-one correspondence between nearly ε-Kähler structures (h_0, J) with $\|\nabla J\|^2 = 4$ on (M, h_0) and parallel G_2- and G_2^*-structures φ on $\bar{M}$ which induce the cone metric $\bar{g}_\varepsilon$.*

PROOF. This well-known fact is usually proved using Killing spinors, see [**Bär**], [**Gru**] and [**Ka2**]. We give a proof relying exclusively on the framework of stable forms and the Hitchin flow. For Riemannian signature, this point of view is also adopted in [**ChSa**] and [**Bu2**].

The $H^{\varepsilon,\tau}$-structures inducing the given metric h_0 are the reductions of the bundle of orthonormal frames of (M, h_0) to the respective group $H^{\varepsilon,\tau}$. Given any $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0, we consider for $t \in \mathbb{R}^+$ the one-parameter family

$$\omega = t^2\omega_0\,,\ \rho = t^3\rho_0, \tag{4.3}$$

which induces the family of metrics $h = t^2 h_0$. By formula (1.11), the metric g_φ on $\bar{M}$ induced by the stable three-form $\varphi = \omega \wedge dt + \rho$ is exactly the cone metric $\bar{g}_\varepsilon$.

It is easily verified that the family (4.3) consists of half-flat structures satisfying the evolution equations if and only if the initial value $(\omega(1), \rho(1)) = (\omega_0, \rho_0)$ satisfies the exterior system (4.1), (4.2). Therefore, the stable three-form φ on the cone $(\bar{M}, \bar{g}_\varepsilon)$ is parallel if and only if the $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0 is a nearly ε-Kähler structure with $\|\nabla J\|^2 = 4$.

Conversely, let φ be a stable three-form on $\bar{M}$ which induces the cone metric $\bar{g}_\varepsilon$. Since ∂_t is a normal vector field for the hypersurface $M = M \times \{1\}$ satisfying $\bar{g}(\partial_t, \partial_t) = -\varepsilon$, we obtain an $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0 defined by

$$\omega_0 = (\partial_t \lrcorner\, \varphi)_{|TM}\,, \qquad \rho_0 = \varphi_{|TM} \tag{4.4}$$

with the help of Proposition 4.5 of chapter 1. Since the two constructions are inverse to each other, the proposition follows. □

Example 4.2. Consider the flat $(\mathbb{R}^{(3,4)} \setminus \{0\}, \langle.,.\rangle)$ which is isometric to the cone $(M^\varepsilon \times \mathbb{R}^+, t^2 h_\varepsilon - \varepsilon dt^2)$ over the pseudo-spheres $M^\varepsilon := \{p \in \mathbb{R}^{(3,4)} \mid \langle p, p\rangle = -\varepsilon\}$, $\varepsilon = \pm 1$, with the standard metrics h_ε of constant sectional curvature $-\varepsilon$ and signature $(2,4)$ for $\varepsilon = -1$ and $(3,3)$ for $\varepsilon = 1$. Obviously, a stable three-form φ inducing the flat metric $\langle.,.\rangle$ is parallel if and only if it is constant. Thus, the previous discussion and Proposition 4.5 of chapter 1, in particular formula (4.11), yield a bijection

$$\begin{array}{rcl} \mathrm{SO}(3,4)/\,\mathrm{G}_2^* & \to & \{\varepsilon\text{-complex structures } J \text{ on } M^\varepsilon \text{ such that } (h_\varepsilon, J) \text{ is nearly } \varepsilon\text{-Kähler}\} \\ \varphi & \mapsto & J \qquad \text{with} \quad J_p(v) = -p \times v, \qquad \forall\, p \in M^\varepsilon \end{array}$$

where the cross-product $\times$ induced by φ is defined by formula (3.14) of chapter 1. In other words, the pseudo-spheres $(M^\varepsilon, h_\varepsilon)$ admit a nearly ε-Kähler structure which is unique up to conjugation by the isometry group $\mathrm{O}(3,4)$ of h_ε. In fact, these ε-complex structures on the pseudo-spheres are already considered in [**Li**] and the nearly para-Kähler property for $\varepsilon = 1$ is for instance shown in [**Bej**].

4.2. Sine cones over nearly ε-Kähler manifolds. For Riemannian signature, it has been shown in [**FIMU**] that the evolution of a nearly Kähler SU(3)-structure to a nearly parallel G_2-structure induces the Einstein sine cone metric. This result can be extended as follows. We prefer to consider (hyperbolic) cosine cones since they are defined on all of $\mathbb{R}$ in the hyperbolic case.

Proposition 4.3. *Let (M, h_0) be a pseudo-Riemannian six-manifold.*

(i) If h_0 is Riemannian, or has signature $(2,4)$, respectively, there is a one-to-one correspondence between nearly (pseudo-)Kähler structures (h_0, J) on M with $\|\nabla J\|^2 = 4$ and nearly parallel G_2-structures, or G_2^*-structures, respectively, for the constant $\mu = -4$ on the spacelike cosine cone

$$(M \times (-\frac{\pi}{2}, \frac{\pi}{2}),\ \cos^2(t) h_0 + dt^2).$$

(ii) If h_0 has signature $(3,3)$, there is a one-to-one correspondence between nearly para-Kähler structures (h_0, J) on M with $\|\nabla J\|^2 = 4$ and nearly parallel G_2^*-structures

for the constant $\mu = 4$ *on the timelike hyperbolic cosine cone*

$$(M \times \mathbb{R}, -\cosh^2(t)h_0 - dt^2).$$

PROOF. (i) Starting with any SU(3)- or SU(1, 2)-reduction (ω_0, ρ_0) of h_0, the one-parameter family

$$\omega = \cos^2(t)\omega_0\,,\ \rho = -\cos^3(t)(\sin(t)\rho_0 + \cos(t)\hat{\rho}_0)$$

with $(\omega(0), \rho(0)) = (\omega_0, -\hat{\rho}_0)$ defines a stable three-form $\varphi = \omega \wedge dt + \rho$ on $M \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $z\Psi_0 = z(\rho_0 + i\hat{\rho}_0)$ is a $(3, 0)$-form w.r.t. the induced almost complex structures $J_{\mathrm{Re}\,(z\Psi_0)}$ for all $z \in \mathbb{C}^*$, the structure $J_\rho = J_{\rho_0}$ is constant in t. Thus, the metric g_φ induced by φ is the cosine cone metric. Moreover, it holds $\hat{\rho} = -\cos^3(t)(\sin(t)\hat{\rho}_0 - \cos(t)\rho_0)$ due to Corollary 3.7.

It takes a short calculation to verify that the one-parameter family is nearly half-flat (for the constant $\lambda = -4$) and satisfies the evolution equation (2.3) if and only (ω_0, ρ_0) satisfies the exterior system (4.1), (4.2). Thus, applying Proposition 2.1, the three-form $\varphi = \omega \wedge dt + \rho$ defines a nearly parallel $G^{\varepsilon,\tau}$-structure on $M \times (-\frac{\pi}{2}, \frac{\pi}{2})$ (for the constant $\mu = -4$) if and only if (h_0, J_{ρ_0}) is nearly ε-Kähler with $\|\nabla J\|^2 = 4$.

The inverse construction is given by (4.4) in analogy to the case of the ordinary cone.

(ii) The proof in the para-complex case is completely analogous if we consider the one-parameter family

$$\omega = \cosh^2(t)\omega_0\,,\ \rho = -\cosh^3(t)(\sinh(t)\rho_0 + \cosh(t)\hat{\rho}_0)$$

which is defined for all $t \in \mathbb{R}$. We note the following subtleties regarding signs. By Proposition 3.4 of chapter 1, we know that the mapping $\rho \mapsto \hat{\rho}$ is homogeneous of degree 1, but not linear. Indeed, by applying Corollary 3.7 of chapter 1, we find

$$\sinh(t)\widehat{\rho_0 + \cosh}(t)\hat{\rho}_0 = -\sinh(t)\hat{\rho}_0 - \cosh(t)\rho_0.$$

Using this formula, one can check that $J_\rho = J_{\hat{\rho}_0} = -J_{\rho_0}$ is constant in t such that the metric induced by (ω, ρ) is in fact $h = -\cosh^2(t)h_0$.

$\square$

The fact that the (hyperbolic) cosine cone over a six-manifold carrying a Killing spinor carries again a Killing spinor was proven in [**Ka1**]. By relating spinors to differential forms, these results also imply the existence of a nearly parallel $\mathrm{G}_2^{(*)}$-structures on the (hyperbolic) cosine cone over a nearly ε-Kähler manifold.

EXAMPLE 4.4. The (hyperbolic) cosine cone of the pseudo-spheres $(M^\varepsilon, h_\varepsilon)$ of Example 4.2 has constant sectional curvature 1, for instance due to [**ACGL**, Corollary 2.3], and is thus (locally) isometric to the pseudo-sphere $S^{3,4} = \{p \in \mathbb{R}^{(4,4)} \,|\, \langle p, p\rangle = 1\} = \mathrm{Spin}_0(3, 4)/\,\mathrm{G}_2^*$.

4.3. Cones over nearly parallel $\mathrm{G}_2^{(*)}$-structures. By Lemma 9 in [**Bär**], there is a one-to-one correspondence on a Riemannian seven-manifold (M, g_0) between nearly parallel G_2-structures and parallel Spin(7)-structures on the Riemannian cone. In order to illustrate the evolution equations for nearly parallel G_2^*-structures, we extend this result to the indefinite case by applying Theorem 3.1. This is possible since nearly parallel

G_2^*-structures are in particular cocalibrated. Again, the fact that the cone over a nearly parallel G_2^*-manifold admits a parallel spinor can be derived from the connection to Killing spinors as observed in [**Ka1**].

PROPOSITION 4.5. *Let (M, g_0) be a pseudo-Riemannian seven-manifold of signature $(3,4)$. There is a one-to-one correspondence between nearly parallel G_2^*-structures for the constant 4 which induce the given metric g_0 and parallel $\mathrm{Spin}_0(3,4)$-structures on $M \times \mathbb{R}^+$ inducing the cone metric $\bar{g} = t^2 g_0 + dt^2$.*

PROOF. Let φ_0 be any cocalibrated G_2^*-structure on M inducing the metric g_0. The one-parameter family of three-forms defined by $\varphi = t^3\varphi_0$ for $t \in \mathbb{R}^+$ induces the family of metrics $g = t^2 g_0$ such that the Hodge duals are $*_\varphi\varphi = t^4 *_{\varphi_0}\varphi_0$. By (3.4), the $\mathrm{Spin}_0(3,4)$-structure $\Psi = dt \wedge \varphi + *_\varphi\varphi$ on $M \times \mathbb{R}^+$ induces the cone metric $\bar{g}$. Conversely, given a $\mathrm{Spin}_0(3,4)$-structure Ψ on the cone $(M \times \mathbb{R}^+, \bar{g})$, we have the cocalibrated G_2^*-structure $\varphi_0 = \partial_t \lrcorner \Psi$ on M, which also induces the given metric g_0. Since the evolution equation (3.2) is satisfied if and only if the initial value φ_0 is nearly parallel for the constant 4 and since the two constructions are inverse to each other, the assertion follows from Theorem 3.1. $\square$

EXAMPLE 4.6. We consider again the easiest example, i.e. the flat $\mathbb{R}^{(4,4)} \setminus \{0\}$ which is isometric to the cone over the pseudo-sphere $S^{3,4}$. Analogous to Example 4.2, the proposition just proved yields a proof of the fact that the nearly parallel G_2^*-structures for the constant 4 on $S^{3,4}$ are parametrised by $\mathrm{SO}(4,4)/\mathrm{Spin}_0(3,4)$, i.e. by the four homogeneous spaces (4.14). In particular, these structures are conjugated by the isometry group $\mathrm{O}(4,4)$ of $S^{3,4}$.

Summarising the application of the three Propositions 4.1, 4.3 and 4.5 to pseudo-spheres, we find a mutual one-to-one correspondence between

(1) nearly pseudo-Kähler structures with $\|\nabla J\|^2 \neq 0$ on $(S^{2,4}, g_{can})$,
(2) nearly para-Kähler structures with $\|\nabla J\|^2 \neq 0$ on $(S^{3,3}, g_{can})$,
(3) parallel G_2^*-structures on $(\mathbb{R}^{(3,4)}, g_{can})$,
(4) nearly parallel G_2^*-structures on the spacelike cosine cone over $(S^{2,4}, g_{can})$,
(5) nearly parallel G_2^*-structures on the timelike hyperbolic cosine cone over $(S^{3,3}, g_{can})$,
(6) nearly parallel G_2^*-structures on $(S^{3,4}, g_{can})$ and
(7) parallel $\mathrm{Spin}_0(3,4)$-structures on $(\mathbb{R}^{(4,4)}, g_{can})$.

This geometric correspondence is reflected in the algebraic fact that the four homogeneous spaces (4.14) are isomorphic.

5. Evolution of half-flat structures on nilmanifolds $\Gamma \setminus H_3 \times H_3$

In this section, we will develop a method to explicitly determine the parallel $G_2^{(*)}$-structure induced by an arbitrary invariant half-flat structure on a compact nilmanifold $\Gamma \setminus H_3 \times H_3$ without integrating. In particular, this method is applied to construct three explicit large families of metrics with holonomy equal to G_2 or G_2^*, respectively.

5.1. Evolution of invariant half-flat structures on nilmanifolds. Given as initial value a half-flat structure on a Lie algebra, the evolution equations

$$\dot{\rho} = d\omega \ , \qquad \dot{\sigma} = d\hat{\rho} \ , \tag{5.1}$$

reduce to a system of ordinary differential equations and a unique solution exists on a maximal interval I. Due to the structure of the equation, the solution differs from the initial values by adding exact forms to σ_0 and ρ_0. In other words, an initial value (σ_0, ρ_0) evolves within the product $[\sigma_0] \times [\rho_0]$ of their respective Lie algebra cohomology classes.

Every nilpotent Lie group N with rational structure constants admits a cocompact lattice Γ and the resulting compact quotients $\Gamma\backslash N$ are called nilmanifolds. Recall that a geometric structure on a nilmanifold $\Gamma\backslash N$ is called *invariant* if is induced by a left-invariant geometric structure on N.

Explicit solutions of the Hitchin flow equations on several nilpotent Lie algebras can be found for instance in [**CF**] and [**AS**]. In both cases, a metric with holonomy contained in G_2 has been constructed before by a different method and this information is used to obtain the solution. For a symplectic half-flat initial value, another explicit solution on one of these Lie algebras is given in [**CT**]. In all cases, the solution depends only on one variable.

At least for four nilpotent Lie algebras including $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, a reason for the simple structure of the solutions has been observed in [**AS**]. Indeed, the following lemma shows that the evolution of σ takes place in a one-dimensional space. As it is common practice in the literature, we define a nilpotent Lie algebra by giving the image of a basis of one-forms under the exterior derivative, see for instance [**Sa2**]. The same reference also contains a list of all six-dimensional nilpotent Lie algebras.

LEMMA 5.1. *Let ρ be a closed stable three-form with dual three-form $\hat{\rho}$ on a six-dimensional nilpotent Lie algebra $\mathfrak{g}$.*

(i) If $\mathfrak{g}$ is one of the three Lie algebras

$$(0,0,0,0,e^{12},e^{34})\,, \quad (0,0,0,0,e^{13}+e^{42},e^{14}+e^{23})\,, \quad (0,0,0,0,e^{12},e^{14}+e^{23}),$$

then $d\hat{\rho} \in \Lambda^4 U$ for the four-dimensional kernel U of $d : \Lambda^1\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$.

(ii) If $\mathfrak{g}$ is the Lie algebra

$$(0,0,0,0,0,e^{12}+e^{34}),$$

then $d\hat{\rho} \in \Lambda^4 U$ for the four-dimensional subspace $U = \mathrm{span}\{e^1, e^2, e^3, e^4\}$ of $\ker d$.

REMARK 5.2. The assertion of the lemma is not true for the remaining six-dimensional nilpotent Lie algebras with $b_1 = \dim(\ker d) = 4$ or $b_1 = 5$. In each case, we have constructed a closed stable ρ such that $d\hat{\rho}$ is not contained in $\Lambda^4(\ker d)$.

In fact, this lemma can also be viewed as a corollary of the following lemma which we will prove first.

LEMMA 5.3. *Let ρ be a closed stable three-form on one of the four Lie algebras of Lemma 5.1 and let U be the four-dimensional subspace of $\ker d$ defined there. In all four cases, the space U is J_ρ-invariant where J_ρ denotes the almost (para-)complex structure induced by ρ.*

PROOF. For $\lambda(\rho) < 0$, the assertion is similar to that of [**AS**, Lemma 2]. However, since the only proof seems to be given for the Iwasawa algebra for integrable J in [**KeS**, Theorem 1.1], we give a complete proof.

Let $\mathfrak{g}$ be one of the three Lie algebras given in part (i) of Lemma 5.1 and $U = \ker d$. Obviously, the two-dimensional image of d lies within $\Lambda^2 U$ in all three cases. By $J = J_\rho$ we denote the almost ε-complex structure associated to the closed stable three-form ρ.

We define the J-invariant subspace $W := U \cap J^* U$ of $\mathfrak{g}$ such that $2 \leq \dim W \leq 4$. In fact, $\dim W = 4$ is equivalent to the assertion. The other two cases are not possible, which can be seen as follows. To begin with, assume that W is two-dimensional. When choosing a complement W' of W in U, we have by definition of W that

$$V = W \oplus W' \oplus J^* W'.$$

We observe that, for $\varepsilon = 1$, the ± 1-eigenspaces of J restricted to $W' \oplus J^* W'$ are both two-dimensional. Therefore, we can choose for both values of ε a basis

$$\{e^1, e^2, e^3, e^4 = J^* e^1, e^5 = J^* e^2, e^6 = J^* e^3\}$$

of V such that e^1, e^2, e^3 and e^4 are closed and $de^5, de^6 \in \Lambda^2 U$. Since $\rho + \mathrm{i}_\varepsilon J_\rho^* \rho$ is a $(3,0)$-form in both cases, it is possible to change the basis vectors e^1, e^4 within $W \subset \ker d$ such that

$$\rho + \mathrm{i}_\varepsilon J_\rho^* \rho = (e^1 + \mathrm{i}_\varepsilon e^4) \wedge (e^2 + \mathrm{i}_\varepsilon e^5) \wedge (e^3 + \mathrm{i}_\varepsilon e^6)$$

and thus

$$\rho = e^{123} + \varepsilon e^{156} - \varepsilon e^{246} + \varepsilon e^{345}.$$

By construction of the basis, we have that

$$0 = d\rho = -\varepsilon e^1 \wedge de^5 \wedge e^6 + \varepsilon e^1 \wedge e^5 \wedge de^6 + \alpha$$

with $\alpha \in \Lambda^4 U$. As the first two summands are linearly independent and not in $\Lambda^4 U$, we conclude that both $e^1 \wedge de^5$ and $e^1 \wedge de^6$ vanish. Thus, the closed one-form e^1 has the property that the wedge product of e^1 with any exact two-form vanishes. However, an inspection of the standard basis of each of the three Lie algebras in question reveals that such a one-form does not exist on these Lie algebras and we have a contradiction to $\dim W = 2$.

Since a J-invariant space cannot be three-dimensional for $\varepsilon = -1$, the proof is finished for this case. However, if $\varepsilon = 1$, the case $\dim W = 3$ cannot be excluded that easy. Assuming that it is in fact $\dim W = 3$, we choose again a complement W' of W in U and find a decomposition

$$V = W \oplus W' \oplus J^* W' \oplus W''$$

with $J^* W'' = W''$. Without restricting generality, we can assume that J acts trivially on W''. Then, we find a basis for V such that the $+1$-eigenspace of J is spanned by $\{e^1, e^4 + e^5, e^6\}$ and the -1-eigenspace by $\{e^2, e^3, e^4 - e^5\}$, where e^1, e^2, e^3 and e^4 are closed and $e^5 = J^* e^4$. Since the given closed three-form ρ generates this J, it has to be of the form

$$\rho = a e^1 \wedge (e^4 + e^5) \wedge e^6 + b e^{23} \wedge (e^4 - e^5)$$

for two real constants a, b. The vanishing exterior derivative

$$d\rho = a e^1 \wedge d(e^{56}) \qquad \text{mod} \quad \Lambda^4 U$$

leads to the same contradiction as in the first case and part (i) is shown.

In fact, the same arguments apply to the Lie algebra of part (ii). The four-dimensional space $U \subset \ker d$ spanned by $\{e^1, ..., e^4\}$ also satisfies $\mathrm{im} d \subset \Lambda^2 U$. Going through the above arguments, the only difference is that e^5 or e^6 may be closed. However, at least one of them is not closed and its image under d generates the exact two-forms. Again, there is no one-form $\beta \in U$ such that $\beta \wedge \gamma = 0$ for all exact two-forms γ and the arguments given in part (i) lead to contradictions for both $\dim W = 2$ and $\dim W = 3$. □

Proof of Lemma 5.1. Let ρ be a closed stable three-form on one of the four nilpotent Lie algebras and $U \subset \ker d$ as defined in the lemma. For both values of ε, we can apply Lemma 5.3 and choose two linearly independent closed $(1,0)$-forms E^1 and E^2 within the J_ρ-invariant space $U \otimes \mathbb{C}_\varepsilon$. Considering that $\rho + \mathrm{i}_\varepsilon \hat{\rho}$ is a $(3,0)$-form for both values of ε, there is a third $(1,0)$-form E^3 such that $\rho + \mathrm{i}_\varepsilon \hat{\rho} = \mathrm{E}^{123}$. Since $d\rho = 0$ and $\mathrm{im} d \subset \Lambda^2 U$, it follows that the exterior derivative

$$d\hat{\rho} = \varepsilon \mathrm{i}_\varepsilon d(E^{123}) = \varepsilon \mathrm{i}_\varepsilon E^{12} \wedge dE^3$$

is an element of $\Lambda^4 U$. □

5.2. Solving the evolution equations on $H_3 \times H_3$. Due to the preparatory work of Lemma 2.1, chapter 5, and Lemma 5.1 of the previous section, it turns out to be possible to explicitly evolve every half-flat structure on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ without integrating.

Proposition 5.4. *Let (ω_0, ρ_0) be any half-flat $H^{\varepsilon,\tau}$-structure on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ with $\omega_0^{\mathfrak{k}_1} = 0$. Then, the solution of the evolution equations* (5.1) *is affine linear in the sense that*

$$\sigma(t) = \sigma_0 + t\, d\hat{\rho}_0\,, \qquad \rho(t) = \rho_0 + t\, d\omega_0 \tag{5.2}$$

and is well-defined for all $t \in \mathbb{R}$.

Proof. Let $\{e_1, \ldots, f_3\}$ be a standard basis such that ω_0 is in one of the normal forms $\omega_2, \ldots, \omega_5$ of Lemma 2.1 which satisfy $\omega_0^{\mathfrak{k}_1} = 0$. By Lemma 5.1 and the second evolution equation, we know that there is a function $y(t)$ with $y(0) = 0$ such that

$$\sigma(t) = \sigma_0 + y(t) e^{12} f^{12} = \frac{1}{2}\omega_0^2 + y(t) e^{12} f^{12}.$$

For each of the four normal forms, the unique two-form $\omega(t)$ with $\frac{1}{2}\omega(t)^2 = \sigma(t)$ and $\omega(0) = \omega_0$ is

$$\omega(t) = \omega_0 - y(t) e^1 f^1.$$

However, the two-form $e^1 f^1$ is closed such that the exterior derivative $d\omega(t) = d\omega_0$ is constant. Therefore, we have $\rho(t) = \rho_0 + t\, d\omega_0$ by the first evolution equation. Moreover, the two-form $\omega(t)$ is stable for all $t \in \mathbb{R}$ since it holds $\phi(\omega(t)) = \phi(\omega_0)$ for each of the normal forms and for all $t \in \mathbb{R}$. It remains to show that $d\hat{\rho}(t)$ is constant in all four cases which implies that the function $y(t)$ is linear by the second evolution equation.

As explained in section 2 of chapter 5, it is easy to write down, for each normal form ω_0 separately, all compatible, closed three-forms ρ_0, which depend on nine parameters. For $\rho(t) = \rho_0 + t\, d\omega_0$, we verify with the help of a computer that $\lambda(\rho(t)) = \lambda(\rho_0)$ is constant such that $\rho(t)$ is stable for all $t \in \mathbb{R}$ since ρ_0 is stable. When we also calculate $J_{\rho(t)}$ and $\hat{\rho}(t) = J^*_{\rho(t)} \rho(t)$, it turns out in all four cases that $d\hat{\rho}(t)$ is constant. This finishes the proof. □

We cannot expect that this affine linear evolution of spaces which have one-dimensional holonomy, due to Proposition 2.4, ch. 5, yields metrics with full holonomy G_2^*. Indeed, due to the following result the geometry does not change significantly compared to the six-manifold.

COROLLARY 5.5. *Let* (ω_0, ρ_0) *be a half-flat* $H^{\varepsilon,\tau}$*-structure on* $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ *with* $\omega_0^{\mathfrak{k}_1} = 0$ *and let* g_φ *be the Ricci-flat metric induced by the parallel stable three-form* φ *on* $M \times \mathbb{R}$ *defined by the solution* (5.2) *of the evolution equations with initial value* (ω_0, ρ_0)*. Then, the pseudo-Riemannian manifold* $(M \times \mathbb{R}, g_\varphi)$ *is either flat or isometric to the product of the four-dimensional para-hyper-Kähler symmetric space* (N^4, g_{PHK}) *and a three-dimensional flat factor.*

PROOF. By formula (1.11), the metric g_φ is determined by the time-dependent metric $g(t)$ induced by $(\omega(t), \rho(t))$. All assertions follow from the analysis of the curvature of g_φ completely analogous to the proof of Proposition 2.4, ch. 5. □

The situation changes completely when we consider the first normal form ω_1 of Lemma 2.1, ch. 5.

PROPOSITION 5.6. *Let* (ω_0, ρ_0) *be any normalised half-flat* $H^{\varepsilon,\tau}$*-structure on* $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ *with* $\omega_0^{\mathfrak{k}_1} \neq 0$*. There is always a standard basis* $\{e_1, \dots, f_3\}$ *such that* $\omega_0 = e^1 f^1 + e^2 f^2 + e^3 f^3$*. In such a basis, we define* $(\omega(x), \rho(x))$ *by*

$$\begin{aligned} \rho(x) &= \rho_0 + x(e^{12} f^3 - e^3 f^{12}), \\ \omega(x) &= 2\,(\varepsilon\kappa(x))^{-\frac{1}{2}} \left(\frac{1}{4}\varepsilon\kappa(x)\, e^1 f^1 + \frac{1}{4}\varepsilon\kappa(x)\, e^2 f^2 + e^3 f^3 \right), \end{aligned}$$

where $\kappa(x)\,(e^{123} f^{123})^{\otimes 2} = \lambda(\rho(x))$*. Furthermore, let* I *be the maximal interval containing zero such that the polynomial* $\kappa(x)$ *of order four does not vanish for any* $x \in I$*. The parallel stable three-form* (1.10) *on* $M \times I$ *obtained by evolving* (ω_0, ρ_0) *along the Hitchin flow* (5.1) *is*

$$\varphi = \frac{1}{2}\sqrt{\varepsilon\kappa(x)}\,\omega(x) \wedge dx + \rho(x).$$

The metric induced by φ*, which has holonomy contained in* $G^{\varepsilon,\tau}$*, is by* (1.11) *given as*

$$g_\varphi = g(x) - \frac{1}{4}\kappa(x) dx^2, \tag{5.3}$$

where $g(x)$ *denotes the metric associated to* $(\omega(x), \rho(x))$*. The variable* x *is related to the parameter* t *of the Hitchin flow by the ordinary differential equation* (5.6)*.*

PROOF. Since $\omega_0^{\mathfrak{k}_1} \neq 0$, we can always choose a standard basis such that $\omega_0 = e^1 f^1 + e^2 f^2 + e^3 f^3$ is in the first normal form of Lemma 2.1. Then ρ_0 is of the form (2.3), ch. 5.

Moreover, by Lemma 5.1, there is a function $y(t)$ which is defined on an interval containing zero and satisfies $y(0) = 0$ such that the solution of the second evolution equation can be written

$$\sigma(t) = \sigma_0 + y(t) e^{12} f^{12}.$$

The unique $\omega(t)$ that satisfies $\omega(0) = \omega_0$ and $\frac{1}{2}\omega(t)^2 = \sigma(t)$ for all t is

$$\omega(t) = \sqrt{1 - y(t)}\; e^1 f^1 + \sqrt{1 - y(t)}\; e^2 f^2 + \frac{1}{\sqrt{1 - y(t)}}\; e^3 f^3.$$

Since

$$d\omega(t) = \frac{1}{\sqrt{1-y(t)}}(e^{12}f^3 - e^3 f^{12}), \tag{5.4}$$

there is another function $x(t)$ with $x(0) = 0$ such that the solution of the first evolution equation can be written

$$\rho(t) = \rho_0 + x(t)(e^{12}f^3 - e^3 f^{12}). \tag{5.5}$$

This three-form is compatible with $\omega(t)$ for all t, as one can easily see from (2.3), ch. 5. Furthermore, the solution is normalised by Theorem 1.2, which implies

$$\sqrt{\varepsilon\lambda(\rho(t))} = \phi(\rho(t)) = 2\phi(\omega(t)) = -2\sqrt{1-y(t)}\, e^{123} f^{123}.$$

Hence, we can eliminate $y(t)$ by

$$y(t) = 1 - \frac{1}{4}\varepsilon\kappa(x(t)).$$

We remark that the normalisation of $\rho_0 = \rho(0)$ corresponds to $\kappa(0) = 4\varepsilon$. Comparing (5.4) and (5.5), the evolution equations are equivalent to the single ordinary differential equation

$$\dot{x} = \frac{2}{\sqrt{\varepsilon\kappa(x(t))}} \tag{5.6}$$

for the only remaining parameter $x(t)$. In fact, we do not need to solve this equation in order to compute the parallel $\mathrm{G}_2^{(*)}$-form when we substitute the coordinate t by x via the local diffeomorphism $x(t)$ satisfying $dt = \frac{1}{2}\sqrt{\varepsilon\kappa(x(t))}\, dx$. Inserting all substitutions into the formulas (1.10) and (1.11) for the stable three-form φ on $M \times I$ and the induced metric g_φ, all assertions of the proposition follow immediately from Theorem 1.2. $\square$

Example 5.7. The invariant $\kappa(x)$ and the induced metric $g(x)$ for the three explicit half-flat structures of Example 2.2, ch. 5, are the following.
If (ω_0, ρ_0) is the SU(3)-structure (2.4), it holds

$$\begin{aligned}
\kappa(x) &= (x-\sqrt{2})^3(x+\sqrt{2}), \qquad\qquad I = (-\sqrt{2}, \sqrt{2}),\\
g(x) &= (1 - \tfrac{1}{2}\sqrt{2}\,x)\left((e^1)^2 + (e^2)^2 - 4\kappa(x)^{-1}(e^3)^2 + (e^4)^2 + (e^5)^2 - 4\kappa(x)^{-1}(e^6)^2\right)\\
&+ \sqrt{2}\,x(1 - \tfrac{1}{2}\sqrt{2}\,x)\left(e^1\!\cdot e^4 + e^2\!\cdot e^5 + 4\kappa(x)^{-1}e^3\!\cdot e^6\right).
\end{aligned}$$

If (ω_0, ρ_0) is the SU(1,2)-structure (2.5), we have

$$\begin{aligned}
\kappa(x) &= (x-\sqrt{2})(x+\sqrt{2})^3, \qquad\qquad I = (-\sqrt{2}, \sqrt{2}),\\
g(x) &= (1 + \tfrac{1}{2}\sqrt{2}\,x)\left((e^1)^2 - (e^2)^2 + 4\kappa(x)^{-1}(e^3)^2 + (e^4)^2 - (e^5)^2 + 4\kappa(x)^{-1}(e^6)^2\right)\\
&- \sqrt{2}\,x(1 + \tfrac{1}{2}\sqrt{2}\,x)\left(e^1\!\cdot e^4 + e^2\!\cdot e^5 + 4\kappa(x)^{-1}e^3\!\cdot e^6\right).
\end{aligned}$$

And for the $\mathrm{SL}(3,\mathbb{R})$-structure (2.6), it holds

$$\begin{aligned}
\kappa(x) &= (2+x^2)^2, \qquad\qquad I = \mathbb{R},\\
g(x) &= (2+x^2)\left(e^1\!\cdot e^4 + e^2\!\cdot e^5\right) + 4(2-x^2)\kappa(x)^{-1}e^3\!\cdot e^6 + 4\sqrt{2}\,x\kappa(x)^{-1}\left((e^3)^2 - (e^6)^2\right).
\end{aligned}$$

Theorem 5.8. *Let $(\omega(x), \rho(x))$ be the solution of the Hitchin flow with one of the three half-flat structures (ω_0, ρ_0) of Example 2.2, ch. 5, as initial value (see Proposition 5.6 for the explicit solution and Example 5.7 for the corresponding metric $g(x)$, defined for $x \in I$).*

Then, the holonomy of the metric g_φ on $M \times I$ defined by formula (5.3) equals G_2 for the $\mathrm{SU}(3)$-structure (ω_0, ρ_0) and G_2^* for the other two structures.

Moreover, restricting the eight-parameter family of half-flat structures given by (2.3), ch. 5, to a small neighbourhood of the initial value (ρ_0, ω_0) yields in each case an eight-parameter family of metrics of holonomy equal to G_2 or G_2^*.

Proof. For all three cases, we transform the left-invariant frame into a coordinate frame as explained in the proof of Lemma 2.4, chapter 4, and calculate the curvature R of the metric g_φ defined by (5.3). Carrying this out with the package "tensor" contained in Maple 10, we obtain that the rank of the curvature viewed as endomorphism on two-vectors is 14. This implies that the holonomy of g_φ in fact equals G_2 or G_2^*.

The assertion for the eight-parameter family is an immediate consequence. Indeed, by construction, the rank of the curvature endomorphism is bounded from above by 14 and being of maximal rank is an open condition. □

To conclude this section we address the issue of completeness and use the Riemannian family in Example 5.7 and Corollary 1.8 to construct a complete conformally parallel G_2-metric on $\mathbb{R} \times (\Gamma\backslash H_3 \times H_3)$.

Example 5.9. Let H_3 be the Heisenberg group and $N = \Gamma\backslash H_3 \times H_3$ be a compact nilmanifold given by a lattice Γ. Let us denote by $x : I \to (-\sqrt{2}, \sqrt{2})$ the maximal solution to the equation

$$\dot{x}(t) = \frac{2}{\sqrt{(\sqrt{2} - x(t))^3 (x(t) + \sqrt{2})}},$$

with initial condition $x(0) = 0$, defining the t-dependent family of Riemannian metrics

$$\begin{aligned} g_t \;&=\; \frac{\sqrt{2} - x(t)}{\sqrt{2}} \left((e^1)^2 + (e^2)^2 + (e^4)^2 + (e^5)^2 \right) + x(t) \left(\sqrt{2} - x(t) \right) \left(e^1 \cdot e^4 + e^2 \cdot e^5 \right) \\ &\quad + \frac{2\sqrt{2}}{(\sqrt{2} - x(t))^2 (x(t) + \sqrt{2})} \left((e^3)^2 + (e^6)^2 \right) - \frac{4x(t)}{(\sqrt{2} - x(t))^2 (x(t) + \sqrt{2})} e^3 \cdot e^6. \end{aligned}$$

If $\varphi : \mathbb{R} \to I$ is a diffeomorphism, then the metric

$$dr^2 + \frac{1}{\varphi'(r)^2} g_{\varphi(r)}$$

is globally conformally parallel G_2 and geodesically complete.

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Zusammenfassung

Eine SU(3)-Struktur auf einer sechsdimensionalen Mannigfaltigkeit ist definiert durch ein Paar von globalen Differentialformen $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$, die mehrere Kompatibilitätsbedingungen erfüllen. Eine SU(3)-Struktur heißt halbflach, falls die beiden definierenden Formen das äußere Differentialgleichungssystem

$$d\omega^2 = 0, \; d\rho = 0$$

lösen. Halbflache Strukturen wurden 2001 von N. Hitchin als natürliche Startwerte einer geometrischen Evolutionsgleichung eingeführt, wobei die Lösung der Evolution Metriken mit Holonomie G_2 induziert. In der Physik werden halbflache Strukturen als interne Mannigfaltigkeiten bei Kompaktifizierungen von zehndimensionalen Superstringtheorien studiert.

In der vorliegenden Dissertation werden linksinvariante halbflache SU(3)-Strukturen auf sechsdimensionalen Liegruppen studiert, insbesondere auf Produkten von dreidimensionalen Liegruppen. Das Hauptergebnis der Arbeit ist die vollständige Klassifikation derjenigen Produkte von zwei dreidimensionalen Liegruppen, die eine linksinvariante halbflache SU(3)-Struktur zulassen. Eine ähnliche Klassifizierung wird unter der Zusatzbedingung bewiesen, dass die beiden Faktoren orthogonal zueinander sind.

Desweiteren wird auf den Liegruppen $S^3 \times S^3$ und $H_3 \times H_3$ das Problem studiert, wieviele linksinvariante halbflache SU(3)-Strukturen modulo Liegruppenautomorphismen existieren. In beiden Fällen wird das Problem vollständig durch die explizite Parametrisierung dieser Strukturen gelöst. Auf der Liegruppe $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ wird mit ähnlichen Methoden die Eindeutigkeit der bekannten linksinvarianten nearly pseudo-Kählerstruktur nachgewiesen, die insbesondere auch halbflach ist.

Im letzten Abschnitt der Arbeit wird ausführlich der Hitchinfluss behandelt. Unter anderem wird ein neuer Beweis für Hitchins Hauptresultat gegeben, der auch für indefinite Metriken und nichtkompakte Mannigfaltigkeiten angewendet werden kann. Auf $H_3 \times H_3$ wird der Hitchinfluss explizit gelöst für alle zuvor parametrisierten halbflachen SU(3)-Strukturen. Dabei werden auch Beispiele von neuen Metriken mit Holonomie gleich G_2 und gleich G_2^* konstruiert.

Lebenslauf

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