Zoo of Lie n-Algebras

Urs Schreiber and Jim Stasheff

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Abstract

We present a menagerie of examples for Lie n-algebras, study their morphisms and discuss applications to higher order connections, in particular String 2-connections and Chern-Simons 3-connections.

Contents

1 Introduction

We investigate several concrete examples, some of them of relevance in quantum field theory, of higher order generalizations of Lie algebras and of connections taking values in these.

Higher Lie algebras have been conceived as, equivalently, Lie n-algebras, L_∞-algebras, or, dually, quasi-free differential graded commutative algebras (quasi-"FDA"s, of "qfDGCA"s).

As Lie n-algebras, they arise through a process of categorification, as pioneered by Baez and his school. From their point of view, a Lie group is a Lie groupoid with a single object. Accordingly, a Lie n-group is a Lie n-groupoid with a single object.

Just as Lie groups have Lie algebras, Lie n-groups have Lie n-algebras, but in both cases, the algebra can be studied without recourse to the groups. Baez and Crans [2] have discussed how semistrict Lie n-algebras are the same as L_∞-algebras that are concentrated in the first n degrees.

An L_∞-algebra L can be described (see Definition **??**) as a graded cocommutative coassociative coalgebra $S^c sL$ with a coderivation D of degree -1 that squares to 0.

Dually, on the space $\bigwedge^\bullet (sL)^*$, an L_∞-algebra L induces a differential graded commutative algebra which is free as a graded commutative algebras (we say "quasi-free DGCA" for short, but notice that in the physics literature these are known as "free differential algebras" or "FDA"s), whose derivation differential of degree 1 is given by

$$d\omega = -\omega(D(\cdot))\,.$$

All these descriptions of higher Lie algebras have their advantages:

- the coalgebra picture is the most convenient one for many computations;
- the DCGA picture is most directly related to connections, curvatures and Bianchi identities with values in the given Lie n-algebra;
- the Lie n-algebra picture is conceptually the most powerful one.

We hope this work will be of interest to somewhat disparate readers: applied n-category theorists, homotopy theorists and cohomological physicists. Hopefully the table of contents will help each to find the parts most appealing to their individual tastes.

2 Main results

2.1 Non-fake-flat 2-Connections

For G a Lie group, parallel transport in a G-bundle with connection is, locally, equivalent [27] to a smooth functor

$$\mathrm{tra} : \mathcal{P}_1(Y) \to \Sigma G\,,$$

where $\mathcal{P}_1(Y)$ is the groupoid of thin homotopy classes of paths in the cover Y, and where ΣG is the Lie group, regarded as a 1-object groupoid.

The curvature of this transport is a 2-functor

$$\mathrm{curv} : \Pi_2(Y) \to \Sigma(\mathrm{INN}(G))$$

from the fundamental 2-groupoid of Y to the inner automorphism 2-group of G.

We discuss how the infinitesimal formulation of this is given by

$$
\begin{tikzcd}
\text{n=1} & \text{n=2} \\
\mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A)"]
\end{tikzcd}
\,,
$$

for $A \in \Omega^1(X, \mathfrak{g})$.

In [4] 2-connections on 2-bundles were similarly realized locally as parallel transport 2-functors

$$\mathrm{tra}_2 : \mathcal{P}_2(X) \to \Sigma G_{(2)}$$

with values in a strict 2-group, whose descent data reproduced that proposed in [9] up to a constraint known as “fake flatness”. The unconstrained data is obtained instead from a smooth 3-functor [25]

$$\mathrm{curv} : \Pi_3(Y) \to \Sigma(\mathrm{INN}(G_{(2)}))\,.$$

Being a morphism of n-groupoids, such an n-functor

$$\mathrm{curv} : \Pi_n(Y) \to \Sigma(\mathrm{INN}(G_{(n-1)}))$$

should have a differential version as a morphism of Lie n-algebroids

$$d\mathrm{curv} : \mathrm{Lie}(\Pi_n(Y)) \to \mathrm{Lie}(\mathrm{INN}(G_{(n-1)}))\,.$$

While we do not discuss this differential version in general, we claim that the resulting morphism is, in its dual incarnation, a qfDGCA morphism

$$f^* : (\mathrm{inn}(\mathfrak{g}_{(2)}))^* \to \Omega^\bullet(Y)$$

from the qfDGCA corresponding to the given Lie n-algebra to the deRham complex of Y, which we may interpret as an algebroid morphism

$$f : \mathrm{Vect}(X) \to \mathrm{inn}(\mathfrak{g}_{(2)})\,,$$

where $\mathrm{Vect}(X)$ denotes the algebroid with identity anchor map over TX.

For the special case that

$$\mathfrak{g}_{(2)} := (t : \mathfrak{h} \to \mathfrak{g})$$

is a strict Lie 2-algebra, coming from a crossed module of ordinary Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, we present several results demonstrating that these qfDGCA morphisms indeed capture all the corresponding data found in

[27, 4, 25, 9] in that connections with values in the strict Lie algebra $(\mathfrak{g} \to \mathfrak{h})$ itself are given by pairs of forms (A, B) which satisfy the "fake flatness" constraint $F_A + t \circ B = 0$, while connections with values in $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$ come from arbitrary pairs of such forms:

n=2 n=3

$$
\begin{tikzcd}
(\mathfrak{h} \to \mathfrak{g}) \arrow[r, hook] & \mathrm{inn}(\mathfrak{h} \to \mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "{(A,B)}" above left, "{\substack{F_A + t_* B = 0 \\ d_A B = 0}}" below left] & \mathrm{Vect}(X) \arrow[u, "{(A,B)}"]
\end{tikzcd}
\tag{1}
$$

for

$$
(A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h}) \,.
$$

Moreover, we show that derivation homotopies of these morphisms

$$
\begin{tikzcd}
\mathrm{Vect}(X) \arrow[r, bend left, "{(A,B)}"] \arrow[r, bend right, "{(A',B')}"'] \arrow[r, phantom, "{(\lambda,a)\Downarrow}"] & \mathrm{inn}(\mathfrak{g} \to \mathfrak{h})
\end{tikzcd}
$$

capture the corresponding *linearized* (or "infinitesimal") gauge (and higher gauge) transformations.

All this pertains to n-connections on trivial n-bundles or, hence, to that on nontrivial n-bundles after these have been trivialized over a cover. The general case of qfDGCA n-connections on nontrivial n-bundles is postponed to [26].

With the relation between n-connections and qfDGCA-morphisms thus established, we move on to investigate n-connections directly in the qfDGCA-picture, which would presumeably be much harder to investigate using other pictures. This is the content of 2.2.

2.2 String 2-Connection and Chern-Simons 3-Connection

Baez and Crans [2], later also [32], had defined, for any semisimple Lie algebra $\mathfrak{g}$, a 1-parameter family of Lie 2-algebras called $\mathfrak{g}_k$. This was shown [3] to be equivalent to the strict Kac-Moody Lie 2-algebras $\mathrm{string}_k(\mathfrak{g})$, which sit in the exact sequence [3]

$$
0 \longrightarrow (\hat{\Omega}_k \mathfrak{g} \to \Omega \mathfrak{g}) \longrightarrow \mathrm{string}_k(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow 0 \ . \tag{2}
$$

Both integrate to a 2-group [20, 3], the realization of whose nerve, as a category, is a model for the topological group known as the String group associated with the semisimple closed compact Lie group G at level k [30].

For $\mathrm{String}_k(G)$-bundles to exist, the first Pontryagin class of the base has to vanish [30]. If it doesn't, then the obstruction classifies a Chern-Simons 2-gerbe – a 3-bundle – whose connection 3-form is a Chern-Simons form [28].

(On general grounds n-gerbes are the $(n+1)$-sheaves of $(n+1)$-sections of $(n+1)$-bundles. For instance a 0-gerbe is just the sheaf of sections of a (1-)bundle. An explicit proof of the equivalence of 1-gerbes with 2-bundles is given in [7]. An analogous discussion for *G-bundle gerbes* is in [5].)

Therefore, understanding connections with values in Lie n-algebras related to $\mathrm{string}_k(\mathfrak{g})$ is relevant both for understanding connections on String 2-bundles, as well as for understanding the proper 3-categorical nature of the Chern-Simons functional.

It turns out that there is an isomorphism of Lie 3-algebras

$$\mathrm{inn}(\mathfrak{g}_k) \simeq \mathrm{cs}_k(\mathfrak{g})$$

of the inner derivation Lie 3-algebra of the semistrict version of the String Lie 2-algebra with the Chern-Simons Lie 3-algebra from **??**. This in particular implies that $\mathrm{cs}_k(\mathfrak{g})$ is also trivializable. But the nontrivial information is extracted by an epimorphism

$$\mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g})$$

to the *Chern* Lie 3-algebra $\mathrm{ch}_k(\mathfrak{g})$. This remembers only the curvature 4-form of the Chern-Simons connection, not the "3-form potential" that it came from.

This means that the situation we find is this:

$$\begin{tikzcd}
n=1 & n=2 & n=3 & n=3 & \\
\mathfrak{g} \arrow[d, equal] & \mathrm{string}_k(\mathfrak{g}) \arrow[l, two heads] \arrow[r, hook] \arrow[d, equal, "\sim"] & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) \arrow[d, equal, "\sim"] & & \text{as strict as possible} \\
\mathfrak{g} & \mathfrak{g}_k \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) & \text{as small as possible} \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A,B)", "{F_A=0,\ dB+k\mathrm{CS}(A)=0}"'] & \mathrm{Vect}(X) \arrow[u, "(A,B,C)", "C=dB+k\mathrm{CS}(A)"'] & \mathrm{Vect}(X) \arrow[u, "(A,C)", "dC=\langle F_A\wedge F_A\rangle"'] &
\end{tikzcd} ,$$

for

$$(A,B,C) \in \Omega^1(X,\mathfrak{g}) \times \Omega^2(X) \times \Omega^3(X)\,.$$

The epimorphism on the left here is a remnant of the exact sequence (2), while the inclusion in the middle is that of (1).

Again, derivation homotopies of these maps provide the relevant linearized gauge transformations and hence the linearized gluing data for these connection p-forms.

We may interpret this as saying that

- String-bundles with connection are 2-bundles with structure 2-group $\mathrm{String}_k(G)$.
- The Chern-Simons functional is a 3-connection on a 3-bundle with structure 3-group $\mathrm{INN}(\mathrm{String}_k(G))$.

Notice that, by the above, a Chern-Simons 3-connection

$$(A, B, C) : \mathrm{Vect}(X) \to \mathrm{cs}_k(\mathfrak{g})$$

with curvature 4-form

$$dC = dk\mathrm{CS}(A) = k\langle F_A \wedge F_A \rangle \,,$$

where F_A is the curvature 2-form of A, factors through the String Lie 2-algebra

$$(A, B, C) : \; \mathrm{Vect}(X) \xrightarrow{(A,B)} \mathfrak{g}_k \hookrightarrow \mathrm{cs}_k(\mathfrak{g})$$

if and only if the 4-form

$$\langle F_A \wedge F_A \rangle = 0$$

vanishes.

This should be the n-connection perspective on the statement in [30], that

"*String connections trivialize Chern-Simons theory.*"

2.3 Supergravity as a higher gauge theory

Notably due to the remarkable old work [1], physicists have long been aware of the fact that qfDGCAs (or "FDA"s as they are, imprecisely, called in the respective literature) are a remarkably powerful tool for the description of supergravity theories in various dimensions.

The right interpretation of this fact, however, seems to have remained mysterious. Frequently the term "soft group manifold" is used to motivate what, from our perspective, is nothing but an n-connection with values in a certain Lie n-algebra.

In fact, as shown in [1] the entire field content of 11-dimensional supergravity may be regarded, from our perspective, as a 4-connection on spacetime with values in the inner derivations of the supergravity Lie 3-algebra

$$\mathrm{inn}(\mathrm{sugra}(10, 1)) \,.$$

While technically this is a rather trivial point – due to the power of the statement of the equivalence of semistrict Lie n-algebras with n-term qfDGCAs and using the result on inner derivations discussed in **??** – it does matter conceptually:

if supergravity is really a higher gauge theory of certain 4-bundles with connection, this indicates that and how the usual formulation is really just in terms of the local connection on a possibly globally nontrivial 4-bundle. The notion of higher morphisms of qfDGCAs which we discuss here provides the structure for gluing such local data to a global object in a way that has, as far as we are aware, not been considered so far.

But even locally, we think that our conception of supergravity field configurations as n-connections with values in Lie n-algebras clarifies some notions that haunt the supergravity literature without yet having found their proper home there.

3 Lie Algebras

An ordinary Lie algebra, or "Lie 1-algebra" in our context, provides a particularly simple example of the structures that we are studying here. Despite its simplicity, the coalgebra and qfDGCA description of Lie algebras and their morphisms exhibit already many of the phenomena which are relevant for the more sophisticated examples.

In particular, the expression of a Lie algebra valued connection 1-form in terms of a morphism of DGCAs already exhibits the peculiar flatness constraint and the linearized gauge transformation behaviour that play a crucial role as we move up to higher n.

3.1 Examples

3.1.1 Ordinary Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. Define a codifferential $D = d_2 : S^c s\mathfrak{g} \to S^c s\mathfrak{g}$ (with $\mathfrak{g}$ regarded as of degree 0) by

$$d_2(sX \vee sY) = s[X, Y]$$

for all $X \in \mathfrak{g}$ and extended as a coderivation. This means

$$D(sX_1 \vee sX_2 \vee \cdots \vee sX_p) = \sum \pm s[X_i, X_j] \vee sX_1 \vee ..... \vee sX_p\,,$$

where the sum is over $i < j$, X_i and X_j are omitted in the factors further to the right if they are bracketed up front, and the sign is that obtained from commuting X_i and X_j from their original position to the the first two positions.

To check $D^2 = 0$, we need only check it on $sX_1 \vee sX_2 \vee sX_3$, where it is readily seen to correspond to the Jacobi identity.

This is the *Chevalley-Eilenberg* chain complex for computing Lie algebra homology.

In terms of a basis $\{X_a\}$ for $\mathfrak{g}$ and structure constants $C^c{}_{ab}$ so that

$$[X_a, X_b] = C^c{}_{ab} X_c,$$

we have $D(sX_a \vee sX_b) = C^c{}_{ab} sX_c$.

Alternatively, to see the usual cochain algebra in terms of a basis $\{q^a\}$ of the vector space dual $\mathfrak{g}^*$, consider the free graded commutative algebra $\bigwedge^\bullet \mathfrak{g}^*$, where $\mathfrak{g}^*$ is in degree 1. Using

$$dq^a(sX_b \vee sX_c) = -q^a(D(sX_b \vee sX_c)) = -q^a(s[X_b, X_c])$$

we find that the differential on that algebra is given by

$$dq^a = -\frac{1}{2} C^a{}_{bc} q^b \wedge q^c\,. \qquad (3)$$

This is the *Chevalley-Eilenberg* cochain complex for computing Lie algebra cohomology with values in the ground field.

We will often suppress the "$\wedge$" and write $q^a q^b$ for $q^a \wedge q^b$.

3.1.2 Concrete example: $\mathfrak{gl}(N)$

The following concrete example of a qfDGCA coming from a Lie algebra will be useful later on:

Proposition 1 *Let $\{t^i{}_j\}$ be the canonical basis for $(s\mathfrak{gl}(N))^*$. Then the differential on*

$$\bigwedge^\bullet (s\mathfrak{gl}(N))^*$$

describing the Lie algebra structure on $\mathfrak{gl}(N)$ acts as

$$dt^i{}_j = -t^i{}_k t^k{}_j \,.$$

3.1.3 Super Lie algebras

Our Lie n-algebras are n-categories internal to the category Vect of vector spaces. If we instead consider categories internal to the category sVect of *super* vector spaces, we obtain *super Lie n-algebras*.

For us, a super vector space is a $\mathbb{Z}_2$-graded vector space

$$V = V_0 \oplus V_1 \,.$$

The crucial property of sVect is the nontrivial braiding isomorphism

$$\sigma : \; V \otimes W \xrightarrow{\sim} W \otimes V$$

which introduces a sign whenever two odd graded spaces exchange position: if $\sigma_0 : \; V \otimes W \xrightarrow{\sim} W \otimes V$ denotes the ordinary braiding isomorphism in Vect, then

$$\sigma|_{V_i \otimes W_j} := (-1)^{ij} \sigma_0|_{V_i \otimes W_j} \,.$$

In particular, an $\mathbb{N}$-*graded* vector space internal to super vector spaces is an $\mathbb{N} \times \mathbb{Z}_2$-bigraded vector space.

Super Lie n-algebras are equivalent to L_∞ algebras and to qfDGCAs built from graded super vector spaces.

For an element $v \in V$ of such a bigraded vector space, we shall continue to write

$$|v| \in \mathbb{N}$$

for its degree in $\mathbb{N}$. Then we write

$$[v] \in \mathbb{Z}_2$$

for the super degree in $\mathbb{Z}_2$. Combining our Koszul grading with the grading inherited from sVect, we then get the graded commutation relation

$$v \vee w = (-1)^{(|v||w| + [v][w])} w \vee v$$

for all $v, w \in V$. Similarly for elements $\omega, \lambda \in V^*$.

Notice in particular that our graded differentials d have Koszul degree $+1$ but super degree 0, so that their graded Leibniz rule is still

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^{|w|} \omega \wedge d\lambda \,.$$

The qfDGCA of the super-Poincaré Lie algebra. Denote by

$$\mathfrak{iso}(10,1) = \mathfrak{so}(10,1) \ltimes \mathbb{R}^{11}$$

the Poincaré Lie algebra on $\mathbb{R}^{11}$, and by

$$\mathfrak{iso}^s(10,1)$$

the corresponding super Lie algebra. This is by itself $\mathbb{Z}_2$-graded, such that

$$\bigwedge^\bullet(s\mathfrak{iso}^s(10,1))^*$$

is $\mathbb{N}$-$\mathbb{Z}_2$-bigraded. The graded symmetry is with respect to the total degree. For $\psi_\alpha \in \mathfrak{iso}^s(10,1)$ odd graded and $\psi^\alpha \in (s\mathfrak{iso}^s(10,1))^*$ dual to that, we now have

$$\psi^\alpha \wedge \psi^\beta = +\psi^\beta \wedge \psi^\alpha \,.$$

After choosing a basis $\{v^a\}$ of $(s\mathbb{R})^*$, a basis $\{\omega^{ab}\}$ of $(s\mathfrak{so}(10,1))^*$, there is a choice of generators $\{\psi^\alpha\}$ of this kind such that the differential on $\bigwedge^\bullet(s\mathfrak{iso}^s(10,1))^*$ encoding the super Lie algebra structure is determined on these generators by

$$\begin{aligned} d\omega^{ab} &= \omega^{ac} \wedge \omega^{cb} \\ dv^a &= \omega^{ab} \wedge v^b + \frac{i}{2}(\Gamma^a)^\alpha{}_\beta \bar\psi_\alpha \wedge \psi^\beta \\ d\psi^\alpha &= \frac{1}{4}(\Gamma^{ab})^\alpha{}_\beta \omega^{ab} \wedge \psi^\beta \,. \end{aligned}$$

(Here the $\{\Gamma^a\}$ are representation matrices of the Clifford algebra generators, and $\bar\psi := \psi C$ for the corresponding charge conjugation matrix C.)

3.2 Morphisms

Lie algebra morphisms

$$f : \mathfrak{h} \to \mathfrak{g}$$

correspond bijectively with morphisms of the corresponding qfDGCA:

$$f^* : (\Lambda^\bullet \mathfrak{g}^*, d) \to (\Lambda^\bullet \mathfrak{h}^*, d) \,.$$

Being algebra morphisms, these morphisms of complexes are entirely specified by their action on the generators

$$f^* : \mathfrak{g}^* \to \mathfrak{h}^* \,,$$

which is nothing but the dual morphism to the original morphism of Lie algebras.

In order for this to be a morphism of complexes, f^* has to satisfy the chain map condition, which says that the diagram

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "d"] \arrow[d, "f^*"'] & \mathfrak{g}^* \wedge \mathfrak{g}^* \arrow[d, "f^* \wedge f^*"] \\ \mathfrak{h}^* \arrow[r, "d"] & \mathfrak{h}^* \wedge \mathfrak{h}^* \end{tikzcd}$$

has to commute. This indeed says that the map intertwines the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$.

The dual picture is straightforward. The morphism

$$f : \mathfrak{h} \to \mathfrak{g}$$

extends to a map of graded coalgebras

$$f : S^c s\mathfrak{h} \to S^c s\mathfrak{g}$$

which is a chain map if and only if

$$\begin{tikzcd} \mathfrak{h} \vee \mathfrak{h} \arrow[r, "{[\ ,\ ]}"] \arrow[d, "f \vee f"'] & \mathfrak{h} \arrow[d, "f"] \\ \mathfrak{g} \vee \mathfrak{g} \arrow[r, "{[\ ,\ ]}"] & \mathfrak{g} \end{tikzcd}$$

commutes.

4 Lie Algebroids

By definition, the vector space sL corresponding to the L_∞-algebra $(S^c(sL), D)$ which corresponds to some Lie n-algebra is concentrated in degree $n \geq 1$.

The natural generalization to spaces (sL) concentrated in degree $n \geq 0$ generalizes Lie n-algebras to Lie n-algebroids.

To the best of our knowledge an n-categorical conception of Lie n-algebroids which would parallel that of Lie n-algebras and their relation to L_∞-algebras has not been considered yet.

Here we shall simply adopt this as a definition.

Definition 1 *A* Lie n-algebroid *is an L_∞-algebra $(S^c_A(sL), D)$ which is a cocommutative coalgebra over a commutative algebra A, or equivalently the dual qfDGCA $(\bigwedge^\bullet{}_A (sL)^*, d)$.*

Remark. Notice this means that $(S^c_A(sL), D)$, respectively $(\bigwedge^\bullet{}_A(sL)^*, d)$ in degree 0 is a copy of A as in the following basic example.

4.1 Examples

4.1.1 The tangent algebroid Vect(X)

In his doctoral thesis [23], George Rinehart effectively identified de Rham cohomology of a smooth manifold X with the Lie algebra cohomology of the space of vector fields $\Gamma(X)$ on X. Note that, for this to make sense, he regards $\Gamma(X)$ as a Lie algebra over $C^\infty(X)$ rather than over a ground field. Then the usual exterior algebra

$$\Omega^\bullet(X)$$

is generated by the dual of $\Gamma(X)$ over $C^\infty(X)$. For finite dimensional X, $\Gamma(X)$ is of finite rank over $C^\infty(X)$ where that rank is generally greater than the dimension of X.

Thus this exterior algebra can be regarded as

$$\Omega^\bullet(X) = \mathrm{Hom}_{C^\infty(X)}(S^c sVect(X), C^\infty(X)) .$$

In terms of a generating set of vector fields, structure constants become structure functions belonging to $C^\infty(M)$.

The same situation may be described in terms of the tangent algebroid with trivial anchor map

$$\rho = \mathrm{id} : TX \to TX .$$

In our language of L_∞-algebras we phrase this as follows.

Definition 2 *Given a manifold X, the L_∞-algebra*

$$\mathrm{Vect}(X) = (S^c(sL), D)$$

is that defined on the space

$$sL := C^\infty(X) \oplus s\Gamma(TX) ,$$

concentrated in degree 0 and 1, where the codifferential $D = d_1 + d_2$ is defined by

$$\begin{aligned} d_1(sV) &= 0 \\ d_2(sV \vee sW) &= s[V,W] \\ d_2(sV \vee f) &= V(f) \\ d_2(f \vee g) &= fg , \end{aligned}$$

for all $V, W \in \Gamma(TX)$ and all $f, g \in C^\infty(X)$.

4.2 Morphisms

Here we shall study morphisms

$$f : \mathrm{Vect}(X) \to \mathfrak{g}_{(n)}$$

in terms of the dual DGCA morphism

$$f^* : (\mathfrak{g}_{(n)})^* \to \Omega^\bullet(X) .$$

4.2.1 Flat connections

Let $\bigwedge^\bullet \mathfrak{g}^*$ be the qfDGCA encoding a Lie algebra $\mathfrak{g}$ as above.

By inspection of the definition, one finds

Proposition 2 *DGCA-morphisms*

$$f^* : \bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet(X)$$

are in bijection with $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(X, \mathfrak{g})$ such that

$$F_A = 0 .$$

We write

$$f_A^*$$

for the DGCA morphism corresponding to the 1-form A under this correspondence.

Here $F_A := dA + A \wedge A$ is the curvature 2-form of A. For $\{X_a\}$ a basis of $\mathfrak{g}$ as before, with $A = A^a X_a$ this reads, more explicitly,

$$F_A^a = dA^a + \frac{1}{2} C^a{}_{bc} A^b \wedge A^c \,.$$

Notice the simple but, in its generalization, very useful fact that the expression on the right directly mimics the appearance (3) of the differential on $\bigwedge^\bullet \mathfrak{g}^*$.

Another simple but, in its generalization, very useful fact is that we may regard such a 1-form A as a flat connection on a trivial G-bundle over X, where $\mathfrak{g} \simeq \mathrm{Lie}(G)$.

Gauge Transformations. A gauge transformation between two such 1-forms A and A' is a G-valued function $g \in \Omega^0(X, G)$ such that

$$A' = gAg^{-1} + gdg^{-1} \,.$$

Here and henceforth, we abuse notation: dg^{-1} means $d(g^{-1})$.

We shall need an equivalent version of this, which is more symmetric in A and A'. To that end, let

$$g = \exp(-s\lambda)$$

for some $\lambda \in \mathfrak{g}$ and some $s \in \mathbb{R}$. The above is equivalent to

$$e^{\frac{s}{2}\lambda} A' e^{-\frac{s}{2}\lambda} = e^{-\frac{s}{2}} A e^{\frac{s}{2}} + e^{-\frac{s}{2}} (de^{s\lambda}) e^{\frac{s}{2}\lambda} \,.$$

Differentiating both sides with respect to s and evaluating the result at $s = 0$ yields

$$A' - \frac{1}{2}[A', \lambda] = A + \frac{1}{2}[A, \lambda] + d\lambda \,.$$

This is what physicists call an "infinitesimal gauge transformation"

$$\lambda : A \to A' \,.$$

Proposition 3 *These infinitesimal gauge transformations $\lambda : A \to A'$ are in bijective correspondence with derivation homotopies*

$$\begin{tikzcd} \bigwedge^\bullet \mathfrak{g}^* \arrow[rr, bend left, "f_A^*"] \arrow[rr, bend right, "f_{A'}^*"'] \arrow[rr, phantom, "\Downarrow h_\lambda"] & & \Omega^\bullet(X) \end{tikzcd}$$

Proof. This follows straightforwardly from inspection of Def. **??**. λ is the value of the derivation homotopy on generators in $(s\mathfrak{g})^*$. $\square$

Remark. We will find this same phenomenon also for all gauge transformations of higher order connections that we consider later on: derivation homotopies of DGCA morphisms know only about the linearized form of the ordinary gauge transformation.

As we will discuss elsewhere, by using a higher notion of Cartan connection, this turns out to be enough information to construct connections on nontrivial n-bundles.

5 Lie 2-Algebras

5.1 Examples

5.1.1 Strict Lie 2-algebra/crossed module $(\mathfrak{h} \to \mathfrak{g})$

Let $(\mathfrak{g}, \mathfrak{h})$ be an infinitesimal crossed module, also known as a Lie algebra crossed module [17] or a strict Lie 2-algebra [2]. That is, a morphism of Lie algebras $t : \mathfrak{h} \to \mathfrak{g}$ together with an action $\alpha : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ of the Lie algebra of derivations of $\mathfrak{h}$, such that

(1) α is a Lie morphism, i.e. $\mathfrak{h}$ is a Lie module over $\mathfrak{g}$, which is to say, for $X, Y \in \mathfrak{g}$ and $A, B \in \mathfrak{h}$:

$$\alpha([X, Y]) = \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X)$$

(2)

$$t(\alpha(X))(A) = [X, tA]$$

(3)

$$\alpha(tA)(B) = [A, B].$$

An alternative notation which will be helpful is to write

$$[X, A] := \alpha(X)(A)\,.$$

If we extend t to $\mathfrak{g}$ as the zero map, then trivially we have $t^2 = 0$, but we do *not* have a DG Lie algebra since t is not a derivation.

However, we can still transfer the definition to a coderivation differential $S^c(sg \oplus ssh)$, namely

Proposition 4 *The codifferential of degree -1*

$$D = d_1 + d_2 : S^c(sg \oplus ssh) \to S^c(sg \oplus ssh)$$

defined by

$$d_1(ssA) = s(tA)$$

$$d_2(sX \vee sY) = s[X, Y]$$

$$d_2(sX \vee ssA) = ss[X, A]\,.$$

squares to zero

$$D^2 = 0$$

if and only if conditions (1) and (2) above hold.

Proof. The fact that $D^2 = 0$ must be checked only on terms v for which $D^2 v$ has a component in $s\mathfrak{g} \oplus ss\mathfrak{h}$. These are $sX \vee sY \vee sZ$ and $sX \vee sY \vee ssA$, but also on $sX \vee ssA$ and $ssA \vee ssB$.

On $sX \vee sY \vee sZ$, $D^2 = 0$ is equivalent to the Jacobi identity on $\mathfrak{g}$. In $D(sX \vee sY \vee ssA)$ and $D(sX \vee ssA \vee ssB)$, there are terms which still have two instances of $\vee$ and so will have no component in $s\mathfrak{g} \oplus ss\mathfrak{h}$ after another application of D. The remaining terms of $D(sX \vee sY \vee ssA)$ map to 0 under D precisely if (1) holds.

Then

$$D^2(sX \vee ssA) = D(ss[X, A] - sX \vee stA) = st([X, A]) - s[X, tA] = 0$$

if and only if (2) holds.

Finally $D(ssA \vee ssB) = D(ssA) \vee ssB + ssA \vee D(ssB) = stA \vee ssB + ssA \vee stB$. Applying D again gives $ss[A, B]$ twice with opposite signs, hence 0. □

Dually, consider the free graded-commutative algebra $\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*)$.

Choose a basis $\{a^a\}$ of $\mathfrak{g}^*$ and a basis $\{b^i\}$ of $\mathfrak{h}^*$. In this basis, let the structure constants of $\mathfrak{g}$ be $C^a{}_{bc}$. Let the action of $\mathfrak{g}$ on $\mathfrak{h}$ have structure constants $\alpha^i{}_{aj}$. Let the morphism from $\mathfrak{h}$ to $\mathfrak{g}$ have components $t^a{}_i$.

Our differential $d\omega = (-1)^{|\omega|}\omega(D(\cdot))$ on this algebras is given on basis elements by

$$\begin{aligned} da^a &= -\frac{1}{2} C^a{}_{bc} a^b a^c - t^a{}_i b^i \\ db^i &= -\alpha^i{}_{aj} a^a b^j \,. \end{aligned}$$

The nature of the form of these equations will become more evident when we consider DGCA morphisms from $\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*)$ to $\Omega^\bullet(X)$ in 5.2.1.

5.1.2 The Weil Algebra $W(G)$ and $\Omega^\bullet(G/H)$

A major example of H. Cartan's algebraicization of differential geometry, especially principal bundles, is his analysis [11] of the Weil algebra $W(G)$. It plays the role of differential forms on the universal principal G-bundle, which did not exist in those days.

In our notations, the underlying algebra of $W(G)$is

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$

where $\mathfrak{g}$ is the Lie algebra of the finite dimensional Lie group G. From our revisionist point of view, consider the infinitesimal crossed module $t = id : \mathfrak{g} \to \mathfrak{g}$ with the action of the target $\mathfrak{g}$ on the source $\mathfrak{g}$ given by the adjoint action, i.e. by the Lie bracket. According to proposition 4, the coderivation differential on $S^c(s\mathfrak{g} \oplus ss\mathfrak{g})$ is given by

$$D = d_1 + d_2$$

defined by

$$\begin{aligned} d_1(ssA) &= s(tA) \\ d_2(sX \vee sY) &= s[X, Y] \end{aligned}$$

$$d_2(sX \vee ssA) = ss[X, A]\,,$$

for all $X, Y, A \in \mathfrak{g}$.

Dually, the differential on $\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$ is given on basis elements $a^a \in s\mathfrak{g}^*$ and $b^a = sa^a \in ss\mathfrak{g}^*$ by

$$\begin{aligned} da^a &= -\frac{1}{2}C^a{}_{bc}a^b a^c - b^a \\ db^a &= -C^a{}_{bc}a^b b^c\,. \end{aligned}$$

The comparison with Cartan's defintion is effectuated in terms of the corresponding dual basis $x_a \in \mathfrak{g}$ (so that $a^a = e(x_a$ in his notation) and the operations

$$h : a^a \mapsto b^a,$$

$$\theta(x_c) : a^a \mapsto -\frac{1}{2}C^a{}_{bc}a^b$$

and

$$\theta(x_c) : b^a \mapsto -\frac{1}{2}C^a{}_{bc}b^b.$$

Another Lie 2-algebra coming from an infinitesimal crossed module that Cartan considered (implicitly) is the special case that $t : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the inclusion of a Lie sub-algebra, $(\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*), d)$. This is used by Cartan to calculate the cohomology of the corresponding homogeneous space G/H in the case in which G is compact connected and H is a connected closed subgroup.

Remark. We reencounter Cartan's strict Lie 2-algebra $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$ as the sub Lie-2-algebra $\mathrm{inn}(\mathfrak{g})$ of the Lie 2-algebra $\mathrm{DER}(\mathfrak{g})$ in 5.1.3.

5.1.3 Lie 2-algebra $\mathrm{DER}(\mathfrak{g})$ of (inner) derivations ($\mathrm{inn}(\mathfrak{g})$)

One classical example for a class of infinitesimal crossed modules is

Definition 3 (derivation Lie 2-algebra) *The infinitesimal crossed module*

$$(t : \mathfrak{h} \to \mathfrak{g}) = (\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}))$$

with the obvious action of derivatins $\mathrm{Der}(\mathfrak{g})$ *on* $\mathfrak{g}$ *gives rise, under proposition 5.1.1, to the derivation Lie 2-algebra*

$$\mathrm{DER}(\mathfrak{g})$$

of the Lie 1-algebra $\mathfrak{g}$.

Remark. This is the infinitesimal version of the automorphism 2-group $\mathrm{AUT}(G)$ of any Lie group G, which can be defined literally as the automorphism category of G, when the latter is regarded as a 1-object groupoid. While the automorphism $(n+1)$-group of any n-groupoid (an n-group, for instance) has an obvious definition, a general notion of derivation Lie $(n+1)$-algebra of a given Lie n-algebra should also exist, but is less obvious. Here we won't go any further into the general definition of $\mathrm{DER}(\cdot)$. But see [29].

Inside $\mathrm{DER}(\mathfrak{g})$ we have the sub-Lie-2-algebra of inner derivations.

Definition 4 (inner derivation Lie 2-algebra) *The infinitesimal crossed module*

$$(t : \mathfrak{h} \to \mathfrak{g}) = (\mathrm{id} : \mathfrak{g} \to \mathfrak{g})$$

with the obvious adjoint action of $\mathfrak{g}$ *on itself gives rise, under proposition 5.1.1, to the inner derivation Lie 2-algebra*

$$\mathrm{inn}(\mathfrak{g})$$

of the Lie 1-algebra $\mathfrak{g}$.

Remark. This is the Lie 2-algebra of the strict 2-group

$$\mathrm{INN}(G)$$

coming from the crossed module of groups $(\mathrm{id} : G \to G)$. The property of $\mathrm{INN}(G)$ which is important in the context of connections is that $\mathrm{INN}(G)$ is the codiscrete category over G. This means that there is precisely one morphism in $\mathrm{INN}(G)$ for any ordered pair of elements in G. We can hence understand the category $\mathrm{INN}(G)$ as obtained from the 0-category G by killing the 0-th homotopy group.

This has two important consequences. First:

Proposition 5 *For any Lie algebra* $\mathfrak{g}$*,the Lie-2-algebra* $\mathrm{inn}(\mathfrak{g})$ *is, as an object in the Baez-Crans 2-category of Lie 2-algebras, equivalent to the trivial Lie 2-algebra.*

Proof. The nature of morphisms in this 2-category is discussed in **??**. Using these definitions, the proof is an easy exercise. The proof has also been given in [3]. □

While $\mathrm{inn}(\mathfrak{g})$ is trivializable, hence apparently uninteresting, there is non-trivial information in *how* it trivializes. We see this in 5.2.2, where we shown that 2-connections with values in $\mathrm{inn}(\mathfrak{g})$ are trivial as 2-connections, but nontrivial as 1-connections.

In fact, passing from $\mathfrak{g}$ to $\mathrm{inn}(\mathfrak{g})$ removes the flatness constraint 4.2.1 on 1-connections . It does also introduce another flatness constraint, now one level higher: that is the Bianchi identity (see also table **??**).

5.1.4 Concrete example: $\mathrm{inn}(\mathfrak{gl}(N))$

As an illustrative example, consider the inner derivation Lie 2-algebra of the qfDGCA considered in 3.1.2.

Proposition 6 *Let* $\{t^i{}_j\}$ *be the canonical basis for* $(s\mathfrak{gl}(N))^*$ *and* $\{r^i{}_j\}$ *that for* $(ss\mathfrak{gl}(N))$.

Then the differential on

$$\bigwedge^\bullet((s\mathfrak{gl}(N))^* \oplus (ss\mathfrak{gl}(N))^*)$$

describing the Lie 2-algebra structure on $\mathrm{inn}(\mathfrak{gl}(N))$ *acts as*

$$dt^i{}_j = -t^i{}_k t^k{}_j + r^i{}_j$$

$$dr^i{}_j = -t^i{}_k r^k{}_j + r^i{}_k t^k{}_j \,.$$

5.1.5 Semistrict Lie 2-algebra

Baez-Crans defined strict Lie-2-algebras, as above, and generalized them to *semi-strict* Lie-2-algebras for which the strict Jacobi identity is replaced by a homotopy relation. They show that these are equivalent to 2-term L_∞-algebras (those concentrated in the first two degrees, $V = V_0 \oplus V_1$).

The differential $D = d_1 + d_2 + d_3$ is determined by

$$d : V_1 \to V_0$$

$$\begin{aligned} l_2 &: V_0 \vee V_0 \to V_0 \\ l_2 &: V_0 \vee V_1 \to V_1 \\ l_3 &: V_0 \vee V_0 \vee V_0 \to V_1 \,. \end{aligned}$$

The ternary bracket d_3 is the **Jacobiator** of the Lie 2-algebra. If D fails to define a strict Lie 2-algebra as in example 5.1.1, then the boundary of the Jacobiator cancels the discrepancy. That is, the binary bracket d_2 satisfies the Jacobi relation on $V_0 \vee V_0 \vee V_0$ only modulo the image of d_3 under d_1:

$$dl_3(X \vee Y \vee Z) = -l_2(l_2(X \vee Y) \vee Z) + l_2(l_2(X \vee Z) \vee Y) - l_2(l_2(Y \vee Z) \vee X)$$

and on $V_0 \vee V_0 \vee V_1$

$$l_3(X \vee Y \vee tA) = -l_2(l_2(X \vee Y) \vee A) + l_2(l_2(X \vee A) \vee Y) - l_2(l_2(Y \vee A) \vee X).$$

Notice that this means that when d_1 is trivial, we can still have a nontrivial Jacobiator even though d_2 satisfies the Jacobi identity on the nose. This will be important in example 5.1.6 below.

For the dual qfDGCAformulation, again choose a basis $\{a^a\}$ of sV_0^* and $\{b^i\}$ of sV_1^*,. The most general differential on $\bigwedge^\bullet((sV_0)^* \oplus (sV_1)^*)$ is defined by

$$da^a = -\frac{1}{2}C^a{}_{bc}a^b a^c - t^a{}_i b^i$$

and

$$db^i = -\alpha^i{}_{aj}a^a b^j - \frac{1}{6}r^i{}_{abc}a^a a^b a^c \,.$$

The components here encode the above maps as follows (where $\{a_a\}$ and $\{b_i\}$ are the bases dual to $\{a^a\}$ and $\{b^i\}$, respectively):

$$\frac{1}{2}C^a{}_{bc}a_a = l_2(a_b, a_c)$$

$$\begin{aligned} \frac{1}{2}\alpha^i{}_{aj}b_i &= l_2(a_b, b_i) \\ t^a{}_i a_a &= d(b_i) \\ \frac{1}{6}r^i{}_{abc} &= \frac{1}{2}l_3(a_a, a_b, a_c) \,. \end{aligned}$$

5.1.6 The String Lie 2-algebra $\mathfrak{g}_k$ / $\mathrm{string}_k(\mathfrak{g})$

A very simple but also very interesting example of a semistrict Lie 2-algebra is the skeletal version of the String Lie 2-algebra. It is equivalent to a strict but infinite-dimensional Lie 2-algebra.

Skeletal version of the String Lie 2-algebra. Baez and Crans noticed that a semisimple Lie algebra $\mathfrak{g}$ together with the k-fold multiple of its canonical 3-cocycle $\langle[\cdot,\cdot],.\rangle$, where $\langle,\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ is the canonical inner product, may be regarded as a Lie 2-algebra

$$\mathfrak{g}_k$$

which is mildly non-strict.

In terms of (co)differential (co)algebra, $\mathfrak{g}_k$ looks as follows. For $\mathfrak{g}$ any Lie semisimple algebra and $\mathfrak{h} = \mathrm{Lie}(\mathbb{R})$, consider $S^c(s\mathfrak{g}\oplus ss\mathfrak{h})$ with differential $D = d_2 + d_3$ given by

$$d_2(sX \vee sY) = s[X,Y]$$

$$d_3(sX \vee sY \vee sZ) = k\langle[X,Y],Z\rangle ssB$$

where B is a choice of basis for h. This squares to zero by the invariance of $\langle\ ,\ \rangle$ and the Jacobi identity.

In terms of a basis $\{a^a\}$ of $(s\mathfrak{g})^*$ and $\{b\}$ of $(ss\mathfrak{h})^*$, define a differential by

$$\begin{aligned} da^a &= -\frac{1}{2}C^a{}_{bc}a^b a^c \\ db &= k\frac{1}{6}C_{abc}a^a a^b a^c\,, \end{aligned}$$

where $C_{abc} = k_{aa'}C^{a'}{}_{bc}$ with k_{ab} the components of $\langle\cdot,\cdot\rangle$ in the chosen basis.

Remark. This Lie 2-algebra is called *skeletal*, since, when we regard it as a linear category following Baez-Crans, it is a category where all isomorphic objects are actually equal, i.e. where every isomorphism has source the same as its target. Here this translates into the property that $d_1 = 0$, since d_1 measures the difference between source and target of a morphism in the Lie 2-algebra.

In words, being skeletal means that a Lie 2-algebra is "as small as possible". The following Lie 2-algebra is much larger and not skeletal. But it is strict, meaning that d_3 vanishes. Still, it is equivalent, as a Lie 2-algebra, to the one described above.

The semistrict Lie 2-algebra from example 5.1.6 turns out to be equivalent to

Definition 5 (Strict version of String Lie 2-algebra) *The strict String Lie 2-algebra is the strict Lie 2-algebra (example 5.1.1) coming from the (infinite-dimensional) crossed module*

$$\mathrm{string}_k(\mathfrak{g}) := (t : \hat{\Omega}_k\mathfrak{g} \to P\mathfrak{g})\,.$$

Here $\mathfrak{g}$ is the semisimple Lie algebra from example 5.1.6, $P\mathfrak{g}$ is the Lie algebra of based paths in $\mathfrak{g}$ and $\hat{\Omega}_k\mathfrak{g}$ is the Kac-Moody central extension of the Lie algebra of based loops in $\mathfrak{g}$.

The morphism $t : \hat{\Omega}_k\mathfrak{g} \to P\mathfrak{g}$ simply forgets the central part and embeds loops into paths.

In more detail, elements of $P\mathfrak{g}$ are maps

$$p : [0, 2\pi] \to \mathfrak{g}$$

such that $p(0) = 0$ and equipped with the pointwise Lie bracket

$$[p_1, p_2](\sigma) = [p_1(\sigma), p_2(\sigma)] \, .$$

Similarly, elements of $\Omega\mathfrak{g}$ are maps

$$l : [0, 2\pi] \to \mathfrak{g}$$

such that $l(0) = 0$ and $l(2\pi) = 0$.

Elements of the central extension $\hat{\Omega}_k\mathfrak{g}$ are pairs

$$(l, c) \in \Omega\mathfrak{g} \times \mathbb{R} \, .$$

Paths act on loops by

$$[p, (l, c)] = ([p, l], 2k \int_0^{2\pi} \langle p, dl \rangle) \, .$$

Notice that this then also defines the bracket on $\hat{\Omega}_k\mathfrak{g}$.

If $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of a semisimple, compact simply connected Lie group G, we assume the invariant inner product $\langle \cdot, \cdot \rangle$ to be normlaized such that the 3-form $\langle \cdot, [\cdot, \cdot] \rangle$ on G generates the third integral cohomology of G. But as long as we do not consider integrating our Lie 2-algebras to Lie 2-groups, this normalization condition is pure convention.

The notion of equivalence here is the 2-categorical one coming from the notion of 1- and 2-morphisms of Lie 2-algebras as described in the next section.

Remark. We refer to this Lie 2-algebra as $\mathrm{string}_k(\mathfrak{g})$, because, as shown in [20, 3], it integrates to a Lie 2-group whose nerve (whith the 2-group regarded as a category), geometrically realized, is a model for the topological (1-)group known as $\mathrm{String}_k(G)$.

5.2 Morphisms

5.2.1 Flat 2-connections

Let $(t : \mathfrak{h} \to \mathfrak{g})$ be an infinitesimal crossed module and let $(\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*), d)$ be the corresponding qfDGCA according to 5.1.1.

Proposition 7 *qfDGCA morphisms from $(\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*), d)$ to the deRham complex of some manifold X*

$$f^* : (\bigwedge{}^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*), d) \to \Omega^\bullet(X)$$

are in bijective correspondence with pairs consisting of a 1-form $A \in \Omega^1(X, \mathfrak{g})$ and a 2-form $B \in \Omega^2(X, \mathfrak{h})$ satisfying

$$t \circ B + F_A = 0 \, , \tag{4}$$

and

$$d_A H = 0 \, .$$

Here

$$H := d_A B$$

is the 3-form curvature.

Proof. This is a straightforward computation. A is the value of f^* on $(s\mathfrak{g})^*$ and B that on $(s\mathfrak{h})^*$. The condition $t \circ B + F_A = 0$ is the chain map condition evaluated on $(s\mathfrak{g})^*$ and $d_A B = 0$ is the chain map condition evaluated on $(ss\mathfrak{h})^*$. □

Remark. This observation is also discussed in [31].

Remark. Morphisms

$$f^* : (t : \mathfrak{h} \to \mathfrak{g})^* \to \Omega^\bullet(X)$$

as above may be regarded as flat connections on trivial principal $G_{(2)}$-2-bundles, where $G_{(2)}$ is a 2-group that integrates the crossed module $(t : \mathfrak{h} \to \mathfrak{g})$.

In the theory of connections on principal 2-bundles and on nonabelian gerbes, the 2-form $\beta = t \circ B + F_A$ has been addressed as the *fake curvature*, although *2-form curvature* would be a more appropriate name. Similarly, H is addressed as the 3-form curvature. [9, 4].

5.2.2 Non-flat 1-connections as flat 2-connections

Consider the special case that the infinitesimal crossed module is $(\mathrm{id} : \mathfrak{g} \to \mathfrak{g})$, corresponding to the Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ from definition 4. Then we find that a morphism

$$f^* : (\mathrm{inn}(\mathfrak{g}))^* \to \Omega^\bullet(X)$$

is entirely determined by a 1-form $A \in \Omega^1(X, \mathfrak{g})$. The 2-form in this case is constrained to be (up to a sign) the curvature 2-form of A:

$$B = -F_A\,.$$

Moreover, the flatness of this 2-connection is now nothing but the Bianchi identity of the 2-form curvature:

$$d_A F_A = 0\,.$$

We hence have a bijection between general $\mathfrak{g}$-connection 1-forms on (trivial G-bundles over) X, and flat $(\mathrm{id} : \mathfrak{g} \to \mathfrak{g})$-2-connections on (trivial $\mathrm{INN}(G_2)$-2-bundles) over X. (The generalization to nontrivial bundles will be discussed elsewhere.)

Analogously, in the next section we will find a Lie 3-algebra obtained from the Lie 2-algebra $(t : \mathfrak{h} \to \mathfrak{g})$ which is such that morphisms from it to $\Omega^\bullet(X)$ are exactly as above, but without the constraint (4).

The following two examples should be compared with example 4.2.1.

5.2.3 Gauge transformations

Let (A,B) and (A',B') be two pairs of p-forms as in proposition 7.

For G a Lie group integrating $\mathfrak{g}$, a gauge transformation

$$(g,a):(A,B)\to(A',B')$$

between these is a function $g\in\Omega^0(X,G)$ and a 1-form $a\in\Omega^1(X,\mathfrak{h})$ such that

$$A' = gAg^{-1} + gdg^{-1} - t\circ a$$

and

$$B' = \alpha_g(B) + d_A a + a\wedge a\,.$$

Here $\alpha_g:\mathfrak{h}\to\mathfrak{h}$ is the action of G on $\mathfrak{h}$ whose differential yields the action of $\mathfrak{g}$ on $\mathfrak{h}$ that comes with the given infinitesimal crossed module.

This one derives for instance from looking at pseudonatural transformation between 2-functors from 2-paths to the strict 2-group that integrates the infinitesimal crossed module $(t:\mathfrak{h}\to\mathfrak{g})$ [4].

The composition of two such gauge transformations turns out to be

$$(g',a')\circ(g,a) = (g'g, a+\alpha_g(a'))\,,$$

so that

$$(g,a)^{-1} = (g^{-1}, -\alpha_{g^{-1}}(a))$$

As before in 4.2.1, we need an equivalent reformulation of this which is more symmetric in (A,B) and (A',B') in order to relate this to derivation homotopies.

To this end, again, set

$$g(s) = \exp(-s\lambda)$$

for $\lambda\in\mathfrak{g}$ such that $g(1)=g$. Then

$$(A,B)\xrightarrow{(e^{-s\lambda},sa)}(A',B') \;=\; (A,B)\xrightarrow{\left(e^{-\frac{s}{2}\lambda},\frac{s}{2}a\right)}(A'',B'')\xrightarrow{\left(e^{-\frac{s}{2}\lambda},\alpha_{\exp(\frac{s}{2}\lambda)}(\frac{s}{2}a)\right)}(A',B')$$

at $s=1$. Computing (A'',B'') from this in the two different ways yields

$$g^{-\frac{s}{2}\lambda}Ag^{\frac{s}{2}\lambda}+g^{-\frac{s}{2}\lambda}dg^{\frac{s}{2}\lambda}-t\circ(\frac{s}{2}a) = A'' = g^{\frac{s}{2}\lambda}A'g^{-\frac{s}{2}\lambda}+g^{\frac{s}{2}\lambda}dg^{-\frac{s}{2}\lambda}+t\circ(\alpha_{\exp(s\lambda)}(\frac{s}{2}a))$$

and

$$\begin{aligned}\alpha_{\exp(-\frac{s}{2}\lambda)}(B) + d_A(\frac{s}{2}a) + (\frac{s}{2}a)\wedge(\frac{s}{2}a) = B''\\ = \alpha_{\exp(\frac{s}{2}\lambda)}(B') - d_{A'}(\alpha_{\exp(s\lambda)}\frac{s}{2}a) + \alpha_{\exp(\frac{s}{2}\lambda)}((\frac{s}{2}a)\wedge(\frac{s}{2}a))\,.\end{aligned}$$

Definition 6 *We say that a linearized gauge transformation*

$$(\lambda,a):(A,B)\to(A',B')$$

between two 2-connections as in proposition 7 is a $\mathfrak{g}$-valued 0-form $\lambda\in\Omega^0(X,\mathfrak{g})$ and an $\mathfrak{h}$-valued 1-form $a\in\Omega^1(X,\mathfrak{h})$ such that the above equations hold when differentiated with respect to s and evaluated at $s=0$:

$$A' - \frac{1}{2}[A',\lambda] = A + \frac{1}{2}[A,\lambda] + d\lambda - t\circ a$$

and

$$B' - \frac{1}{2}[B',\lambda] = B + \frac{1}{2}[B,\lambda] + \frac{1}{2}d_{A+A'}a\,.$$

Proposition 8 *There is a bijective correspondence between linearized gauge transformations*

$$(\lambda, a) : (A,B) \to (A',B')$$

and derivation homotopies

$$\tau : f^*_{(A,b)} \to f^*_{(A',B')}$$

of the corresponding morphisms $f^*_{(A,B)} : \bigwedge^\bullet(t : \mathfrak{h} \to \mathfrak{g})^* \to \Omega^\bullet(X)$ of DGCAs.

Proof. This is again a straightforward computation. λ is the component of the homotopy on $(s\mathfrak{g})^*$ and a is the component on $(ss\mathfrak{h})^*$. $\square$

5.2.4 Gauge transformations of gauge transformations

Let (λ, a) and (λ', a') be linearized gauge transformations of 2-connections as in definition 6.

We define a transformation

$$f : (\lambda, a) \to (\lambda', a')$$

between these linearized transformations to be an $\mathfrak{h}$-valued 0-form

$$f \in \Omega^0(X, \mathfrak{h})$$

such that

$$\lambda' - \lambda = t \circ f$$

and

$$a' - a = \frac{1}{2}d_{A+A'}f\,.$$

Proposition 9 *There is a bijective correspondence between such transformations of linearized gauge transformations of 2-connections and second order derivation homotopies (definition* **??***) between the corresponding derivation homotopies from proposition 8.*

Proof. This is a straightforward computation; f is the component of the second order homotopy of DGCA morphisms on $(ss\mathfrak{h})^*$. $\square$

Remark. We have concentrated above on working out the transformation properties of 2-connections with values in strict Lie 2-algebras, showing that they reproduce the linearized transformations known from connections on principal 2-bundles and nonabelian gerbes for given strict Lie 2-group $G_{(2)}$.

With only slightly more computational effort, the entire discussion above may be carried through also for the most general semistrict Lie 2-algebras from 5.1.5. This is left to the reader.

5.3 Lie algebras of derivations

5.3.1 Derivation Lie algebra of $(\mathfrak{h} \to \mathfrak{g})$

Proposition 10 *The derivation Lie algebra of the crossed module Lie 2-algebra*

$$(t : \mathfrak{h} \to \mathfrak{g})$$

is the semidirect sum Lie algebra

$$\mathfrak{g} \oplus (\mathfrak{g}^* \otimes \mathfrak{h}) \,,$$

where $\mathfrak{g}$ *acts on* $(\mathfrak{g}^* \otimes \mathfrak{h})$ *by*

$$x(\alpha, h) = ((\mathrm{ad}_x)^* \alpha, x(h)) \,,$$

with $h \mapsto x(h) := \alpha(x, h)$ *the given action of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *by derivations. The bracket on* $(\mathfrak{g}^* \otimes \mathfrak{h})$ *itself is given by*

$$[(\alpha_1, h_1), (\alpha_2, h_2)] = \alpha_2(t(h_1))(\alpha_1, h_2) - \alpha_1(t(h_2))(\alpha_2, h_1)$$

Proof. The proof is given in B.1. $\square$

This should be the Lie algebra of the conjugation Lie algebra of the strict 2-group coming from a crossed module of groups $(t : H \to G)$ which integrates the above differential crossed module. See figure 1.

5.3.2 Derivation Lie algebra of $\mathfrak{g}_\mu$

Let $\mathfrak{g}$ be semisimple and $\mu := \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle. Recall the Baez-Crans Lie 2-algebra $\mathfrak{g}_\mu$ from 5.1.6.

Proposition 11 *The derivation Lie algebra of* $\mathfrak{g}_\mu$ *is isomorphic to*

$$\mathfrak{g} \oplus \mathbb{R}^{\dim(\mathfrak{g})} \,.$$

Proof. The proof is given in B.2. $\square$

- **horizontal conjugation** by any $q \in G$

$$\mathrm{Ad}_q \in \mathrm{Aut}_{2\mathrm{Cat}}(G_{(2)})$$

(true conjugation in the sense of the 2-group) acts as

$$\begin{tikzcd}
\mathrm{Ad}_q \;:\; \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet & \mapsto & \bullet \arrow[r, "q"] & \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "q^{-1}"] & \bullet
\end{tikzcd}$$

- **vertical conjugation**

$$\mathrm{vAd}_f \in \mathrm{Aut}_{2\mathrm{Cat}}(G_{(2)})$$

by any map $f : G \to H$ which extends to a homomorphism

$$\mathrm{Id} \times f : G \to G \ltimes H \,,$$

acts as

$$\begin{tikzcd}
\mathrm{vAd}_f \;:\; \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet & \mapsto & \bullet \arrow[r, bend left=80, "t(f(g)^{-1})g"] \arrow[r, bend left=25, "g"] \arrow[r, bend right=25, "g'"'] \arrow[r, bend right=80, "t(f(g'))g'"'] \arrow[r, phantom, "\Downarrow h"] \arrow[r, phantom, shift left=14, "\Downarrow f(g)^{-1}"] \arrow[r, phantom, shift right=14, "\Downarrow f(g')"] & \bullet
\end{tikzcd}$$

Figure 1: **The two notions of conjugation in a 2-group**, for the special case of a strict 2-group $G_{(2)}$, coming from a crossed module $(H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$ of groups.

6 Lie 3-Algebras

6.1 Examples

6.1.1 Inner derivation Lie 3-algebra of $(\mathfrak{h} \to \mathfrak{g})$

We wish to define the Lie 3-algebra which we call $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$, using just data contained in a strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$. The idea is that, given a strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$ as in example 5.1.1, we may find a strict Lie 2-group $G_{(2)} = (H \to G)$ integrating it. Since a Lie 2-group can be regarded as a 1-object 2-groupoid, we may naturally form its automorphism 3-group $\mathrm{AUT}(G_{(2)})$. This has a sub-3-group

$$\mathrm{INN}(G_{(2)}) \subset \mathrm{AUT}(G_{(2)})$$

coming from restricting to all *inner* automorphisms. This Lie 3-group, in turn, may be differentiated to a Lie 3-algebra

$$\mathrm{Lie}(\mathrm{INN}(G_{(2)}))\,.$$

For brevity we will write

$$\mathrm{inn}(\mathfrak{h} \to \mathfrak{g}) := \mathrm{Lie}(\mathrm{INN}(H \to G))$$

and address this as the *Lie 3-algebra of inner derivations* of the strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$.

Here we shall not go into the details of this derivation. Instead, the following example simply defines the Lie 3-algebra which we call $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$, using just data contained in a strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$. For the present purposes, the reader may just as well read "$\mathrm{inn}(\cdot)$" as a mere shorthand for this definition. However, it may be helpful to keep in mind the Lie 2-algebra of inner derivations of a Lie 1-algebra, discussed in example 5.1.3.

Definition 7 (Coalgebra version of $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$) *Given $(\mathfrak{h} \to \mathfrak{g})$ as in 5.1.1, the codifferentiual coalgebra*

$$\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$$

is the free coalgebra

$$S^c(s\mathfrak{g} \oplus (ss\mathfrak{g} \oplus ss\mathfrak{h}) \oplus sss\mathfrak{h})$$

equipped with the codifferential D defined as follows:

$$d_1(ssX) = sX$$

$$d_1(ssB) = stB$$
$$d_1(sssB) = sstB$$
$$d_2(sX \vee sY) = s[X,Y]$$
$$d_2(sX \vee ssY) = ss[X,Y]$$
$$d_2(sX \vee ssB) = ss[X,B]$$
$$d_2(sX \vee sssB) = sss[X,B]$$

and

$$d_2(ssX \vee ssB) = sss[X,B]\,.$$

Proposition 12 $D^2 = 0$.

Proof. One checks that the above definition is indeed a special case of **??**. $\square$

Dually, consider the vector space

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus ((ss\mathfrak{g})^* \oplus (ss\mathfrak{h})^*) \oplus (sss\mathfrak{h})^*) .$$

Denote a chosen basis of $(s\mathfrak{g})^*$ by $\{q^a\}$, a basis of $(ss\mathfrak{g})^*$ by $\{r^a\}$, a basis of $(ss\mathfrak{h})^*$ by $\{s^i\}$ and, finally, a basis of $(sss\mathfrak{h})^*$ in degree 3 by $\{t^i\}$. Let $C^a{}_{bc}$, $\alpha^i{}_{aj}$ and $t^i{}_a$ be the tensors characterizing the crossed module $(\mathfrak{h} \to \mathfrak{g})$ as in example 5.1.1.

The differential, induced by the codifferential D above using $d\omega = -\omega(D(\cdot))$, acts on these basis elements as

$$\begin{aligned}
dq^a &= -\frac{1}{2}C^a{}_{bc}q^b q^c - t^a{}_i s^i - r^a \\
dr^a &= -C^a{}_{bc}q^b r^c - t^a{}_i t^i \\
ds^i &= -\alpha^i{}_{aj}q^a s^j + t^i \\
dt^i &= -\alpha^i{}_{aj}q^a t^j - \alpha^i{}_{aj}r^a s^j .
\end{aligned}$$

Remark. Except for the constants, this is the only differential on

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \otimes ((ss\mathfrak{g})^* \otimes (ss\mathfrak{h})^*) \otimes (sss\mathfrak{h})^*)$$

which can be written down using just the data of the crossed module $(\mathfrak{h} \to \mathfrak{g})$.

Remark. As in the examples before, while the expression of the differential in a basis looks awkward, this already essentially makes the nature of 3-connections with values in $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$ – example 6.2.1 below – manifest.

6.1.2 Chern-Simons Lie 3-algebra $\mathrm{cs}_k(\mathfrak{g})$

For any semisimple Lie algebra $\mathfrak{g}$ and every $k \in \mathbb{R}$ define a Lie 3-algebra

$$\mathrm{cs}_k(\mathfrak{g}) ,$$

called the *Chern-Simons Lie 3-algebra* of $\mathfrak{g}$ at level k as follows.

The underlying coalgebra is

$$S^c(s\mathfrak{g} \oplus (ss\mathfrak{g} \oplus ss\mathbb{R}) \oplus sss\mathbb{R}) .$$

The coderivation $D = d_1 + d_2 + d_3$ on this is given by

$$d_1(ssX) = sX$$

$$d_1(sssC) = ssC$$

$$d_2(sX \vee sY) = s[X,Y]$$

$$d_2(sX \vee ssY) = ss[X,Y] + ssk\langle X,Y\rangle$$
$$d_2(ssX \vee ssY) = -sssk\langle X,Y\rangle$$
$$d_3(sX \vee sY \vee sZ) = -ssk\,,$$

for all $X, Y \in \mathfrak{g}$ and $C \in \mathbb{R}$.

The corresponding dual qfDGCA is defined on the vector space

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus ((ss\mathfrak{g})^* \oplus (ss\mathbb{R})^*) \oplus (sss\mathbb{R})^*)\,.$$

To express the differential, choose a basis $\{t^a\}$ of $(s\mathfrak{g})^*$, a basis $\{r^a\}$ of $(ss\mathfrak{g})^*$, a basis $\{b\}$ spanning $(ss\mathbb{R})^*$ and a basis $\{c\}$ for $(sss\mathbb{R})^*$.

The codifferential D from above induces the differential d that acts on these basis elements as

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c + r^a$$
$$dr^a = C^a{}_{bc}t^b r^c$$
$$db = k\left(\frac{1}{6}C_{abc}t^a t^b t^c - k_{ab}t^a r^b\right) + c$$
$$dc = k(k_{ab}r^a r^b)\,.$$

6.1.3 The Lie 3-algebra $\mathrm{inn}(\mathfrak{g}_k)$

Notice that the Chern-Simons Lie 3-algebra from 6.1.2 is similar to, but different from the inner derivation Lie 3-algebra of the Baez-Crans Lie 2-algebra $\mathfrak{g}_k$ from 5.1.6.

Using definition **??**, one finds that the qfDGCA corresponding to

$$\mathrm{inn}(\mathfrak{g}_k)\,,$$

which is defined on the same free graded commutative algebra as $\mathrm{cs}_k(\mathfrak{g})$, has a differential which reads, in the basis chosen in 6.1.2:

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c + r^a$$
$$dr^a = C^a{}_{bc}t^b r^c$$
$$db = k\frac{1}{6}C_{abc}t^a t^b t^c + c$$
$$dc = -k\frac{1}{2}C_{abc}t^a t^b r^c\,.$$

While $\mathrm{inn}(\mathfrak{g}_k)$ is not equal to $\mathrm{cs}_k(\mathfrak{g})$, we show in 6.2.7 that both are *equivalent* in 3Lie.

6.1.4 Supergravity Lie 3-algebra $\mathrm{sugra}(10,1)$

The theory of supergravity is the study of generalizations of the Einstein-Hilbert functional on Riemannian manifolds to the world of super-Riemannian manifolds, hence to functionals on manifolds equipped with a Riemannian metric and certain further extra structure. One notable phenomenon in this context is that this extra structure involves – beyond the spinorial structures that one expects – in particular differential forms of various degree on these manifolds.

We shall see that these differential forms may in fact naturally be conceived as components of connections with values in higher Lie algebras and that at least some supergravity functionals may be regarded as functionals on a space of n-connections.

In 1982, D'Auria and Fré [1] demonstrated that a useful tool for dealing with the intricate structures appearing in these studies is a formalism that they call a *Cartan integrable system*.

A Cartan integrable system is defined by these authors as a collection of graded generators $\{\Theta^A\}$, with ^{A} running over some index set, and structure constants $C^A_{B_1,\cdots,B_n}$ defining a differential

$$d\Theta^A = -\sum \frac{1}{n} C^A_{B_1\cdots B_n}\Theta^{B_1}\wedge\cdots\wedge\Theta^{B_n}\,.$$

of degree $+1$ such that

$$d^2 = 0\,.$$

In other words, a "Cartan integrable system" is precisely a qfDGCA, hence an L_∞-algebra, hence a Lie n-algebra.

Definition 8 (D'Auria-Fré) *Consider the qfDGCA of the super Poincaré Lie algebra from 3.1.3. Extend this by adding one generator in degree 3*

$$\bigwedge{}^\bullet(\mathrm{siso}^s(10,1)\oplus sss\mathbb{R})^*\,.$$

For $c \in (sss\mathbb{R})^*$ a choice of basis, extend the qfDGCA-differential of $\mathfrak{iso}^s(10,1)$ to this space by setting

$$dc = \frac{1}{2}(\Gamma^{ab})^\alpha{}_\beta\bar{\psi}_\alpha\wedge\psi^\beta\wedge v^a\wedge v^b\,.$$

This defines the super Lie 3-algebra which we call

$$\mathrm{sugra}(10,1)\,.$$

As discussed in [1], one shows that with this definition $d^2=0$ follows from Fierz identities on spinors in eleven dimensions.

Remark. With the contraction of the spinor indices understood implicitly, we have, in summary, a qfDGCA defined on generators $\{v^a,\omega^{ab},\psi^\alpha,c\}$ by

$$\begin{aligned} d\omega^{ab} &= \omega^{ac}\omega^{cb} \\ dv^a &= \omega^{ab}v^b + \frac{i}{2}\bar{\psi}\Gamma^a\psi \\ d\psi &= \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi \end{aligned}$$

$$dc = \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi v^a v^b\,.$$

Accordingly, qfDGCA morphisms

$$f^* : \mathrm{sugra}(10,1)^* \to \Omega^\bullet(X)$$

are in bijective correspondence with p-forms

$$(V,\Omega,\Psi,C) \in \Omega^1(X,\mathbb{R}^{11}) \times \Omega^1(X,\mathfrak{so}(10,1)) \times \Omega^1(X,\mathbb{C}^{32}) \times \Omega^3(X)$$

satisfying

$$d\Omega + [\Omega \wedge \Omega] = 0$$

$$dV + \Omega \cdot V + \frac{i}{2}\bar{\psi} \wedge \Gamma\Psi = 0$$

$$d\Psi + \Omega \cdot \Psi = 0$$

$$dC = \bar{\psi} \wedge \Gamma^{ab}\psi \wedge V^a \wedge V^b\,.$$

URS: THIS is for the moment modulo signs etc.

Remark. A useful brief summary of the concepts and terminology used in the relevant supergravity part of the physics literature can be found at the beginning of of [12]. Notice for instance that the notion of gauge transformation considered in this context – usually conceived as a Lie derivative as in (2.16) of [12] – is essentially nothing but a homotopy of qfDGCA maps

$$\begin{tikzcd} \mathrm{sugra}(10,1)^* \arrow[rr, bend left, "{(V,\Omega,\Psi,C)}"] \arrow[rr, bend right, "{(V',\Omega',\Psi',C')}"'] \arrow[rr, phantom, "\epsilon\Downarrow"] & & \Omega^\bullet(X) \end{tikzcd}\,.$$

This relation is also considered in [10]. However, recall from our discussion in **??** that derivation homotopies in general differ from the transformations considered in [10] by terms of higher order in the "gauge transformation parameter". But in contexts such as [12], these higher terms would be dropped anyway.

6.2 Morphisms

6.2.1 Non-flat 2-connections as flat $\mathrm{inn}(\mathfrak{h} \to \mathfrak{g})$-connections

Proposition 13 *DGCA morphisms from the inner derivation Lie 3-algebra of example 6.1.1 to the deRham complex of some manifold X*

$$f^* : \mathrm{inn}(\mathfrak{h} \to \mathfrak{g})^* \to \Omega^\bullet(X)$$

are in bijective correspondence with pairs consisting of a $\mathfrak{g}$-valued 1-form A, an $\mathfrak{h}$-valued 2-form B such that with

$$\beta := F_A + \delta(B)$$

and

$$H = d_A B$$

we have

$$d_A\beta = t \circ H$$

and

$$d_A H + \beta \wedge B = 0\,.$$

Proof. This is a straightforward computation. A is the component of f^* on $(s\mathfrak{g})^*$, B is the component on $(ss\mathfrak{h})^*$, β is the component on $(ss\mathfrak{g})^*$ and H is the component on $(sss\mathfrak{h})$. The given relations between these forms are equivalent to the chain map condition on f^*. $\square$

Remark. Setting $\beta = 0$ and $H = 0$ leads back to example 5.2.1.

Remark. In the case that the crossed module $(\mathfrak{h} \to \mathfrak{g})$ is $\mathrm{DER}(\mathfrak{g})$, discussed in 5.1.3, the above differential form data is exactly that which Breen and Messing [9] give for the connection on a trivial nonabelain G-gerbe. In particular, the "fake curvature" β – which in [4] was found to vanish for strict parallel transport 2-functors from 2-paths to $\mathrm{AUT}(G)$ – here is arbitrary. In the light of 5.2.1 and the above, it is now clear what is going on:

the parallel n-transport which integrates the Breen-Messing connection data should in fact not take values in the 2-group $\mathrm{AUT}(G)$, but in the 3-group $\mathrm{INN}(\mathrm{AUT}(G))$. This is indeed the case, as will be discussed in [25].

6.2.2 Gauge transformations

6.2.3 Gauge transformations of gauge transformations

6.2.4 Gauge transformations of gauge transformations of gauge transformations

6.2.5 Chern-Simons 3-connection

Proposition 14 *Let* $(\mathrm{cs}_k(\mathfrak{g}))^*$ *be the DGCA describing the Lie 3-algebra from 6.1.2. Then DGCA morphisms*

$$f^* : (\mathrm{cs}_k(\mathfrak{g}))^* \to \Omega^\bullet(X)$$

are in bijective correspondence with pairs consisting of a $\mathfrak{g}$*-Chern-Simons 3-form*

$$\mathrm{CS}(A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle$$

for $A \in \Omega^1(X, \mathfrak{g})$ *and a 2-form*

$$B \in \Omega^2(X)\,.$$

Proof. The 1-form A is the value of f^* on $(s\mathfrak{g})^*$, the 2-form is the value on $(ss\mathbb{R})^*$. The value on $(ss\mathfrak{g})^*$ is constrained to be the curvature of A and the value on $(sss\mathbb{R})^*$ is constrained to be the 3-form

$$dB + k\mathrm{CS}(A)\,.$$

$\square$

6.2.6 10D supergravity 2-connection

Just as we could regard non-flat 2-connections with values in a strict Lie 2-algebra as flat Lie 3-connections (their curvature) in 6.2.1, it makes sense to interpret a morphism

$$f^* : (\mathrm{cs}_k(\mathfrak{g}))^* \to \Omega^\bullet(X)$$

as in 6.2.5 as the 3-curvature of a non-flat 2-connection of sorts.

Regarded from this point of view, the 2-connection corresponding to 6.2.5 is given by a pair of forms

$$(A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X)$$

whose curvature 3-form is

$$H = dB + k\mathrm{CS}(A)$$

In string theoretic application one would address the 2-form B as the Kalb-Ramond field.

The modified field strength H which appears this way plays a crucial role in the Green-Schwarz anomaly cancellation mechanism [18] for the heterotic string background theory.

6.2.7 The equivalence $\mathrm{inn}(\mathfrak{g}_k) \simeq \mathrm{cs}_k(\mathfrak{g})$

The Chern-Simons Lie 3-algebra is in fact equivalent (even isomorphic) to the inner derivation Lie 3-algebra of $\mathfrak{g}_k$. We prove this by explicitly constructing the isomorphism

Proposition 15 *The map*

$$f^* : \mathrm{inn}(\mathfrak{g}_k)^* \to \mathrm{cs}_k(\mathfrak{g})^*$$

defined by

$$\begin{aligned} f^* &: t^a \mapsto t^a \\ f^* &: r^a \mapsto r^a \\ f^* &: b \mapsto b \\ f^* &: c \mapsto c - kk_{ab}t^a r^b \end{aligned}$$

is a morphism of Lie 3-algebras.

Proof. We check $[d, f^*] = 0$ on all generators. On t^a and r^a this is trivial. On b we find

$$\begin{aligned} f^*(d_{\mathrm{inn}(\mathfrak{g}_k)}b) &= f^*(k\frac{1}{6}C_{abc}t^a t^b t^c + c) \\ &= k\frac{1}{6}C_{abc}t^a t^b t^c - kk_{ab}t^a r^b + c\,, \end{aligned}$$

which is indeed equal to $d_{\mathrm{cs}_k(\mathfrak{g})}(f^*b)$. On c we find

$$f^*(d_{\mathrm{inn}(\mathfrak{g}_k)}c) = f^*(-k\frac{1}{2}C_{abc}t^a t^b r^c) = -k\frac{1}{2}C_{abc}t^a t^b r^c$$

and

$$
\begin{aligned}
d_{\mathrm{cs}_k(\mathfrak{g})}(f^*(c)) &= d_{\mathrm{cs}_k(\mathfrak{g})}(c - kk_{ab}t^a r^b) \\
&= kk_{ab}r^a r^b - kk_{ab}r^a r^b + \frac{1}{2}kC_{abc}t^a t^b r^c - kC_{abc}t^a t^b r^c \,.
\end{aligned}
$$

□

Proposition 16 *The map*

$$
f^* : \mathrm{cs}_k(\mathfrak{g})^* \to \mathrm{inn}(\mathfrak{g}_k)^*
$$

defined by

$$
\begin{aligned}
f^* &: t^a \mapsto t^a \\
f^* &: r^a \mapsto r^a \\
f^* &: b \mapsto b \\
f^* &: c \mapsto c + kk_{ab}t^a r^b
\end{aligned}
$$

is a morphism of Lie 3-algebras.

Proof. We check $[d, f^*] = 0$ on all generators. On t^a and r^a this is trivial. On b we find

$$
\begin{aligned}
f^*(d_{\mathrm{cs}_k(\mathfrak{g})}b) &= f^*(k\frac{1}{6}C_{abc}t^a t^b t^c - kk_{ab}t^a r^b + c) \\
&= k\frac{1}{6}C_{abc}t^a t^b t^c + c \,,
\end{aligned}
$$

which is indeed equal to $d_{\mathrm{inn}(\mathfrak{g}_k)}(f^* b)$. On c we find

$$
f^*(d_{\mathrm{cs}_k(\mathfrak{g})}c) = f^*(k\frac{1}{2}k_{ab}r^a r^b) = k\frac{1}{2}k_{ab}r^a r^b
$$

and

$$
\begin{aligned}
d_{\mathrm{inn}(\mathfrak{g}_k)}(f^*(c)) &= d_{\mathrm{inn}(\mathfrak{g}_k)}(c + kk_{ab}t^a r^b) \\
&= -k\frac{1}{2}C_{abc}t^a t^b r^c + kk_{ab}r^a r^b - k\frac{1}{2}C_{abc}t^a t^b r^c + kC_{abc}t^a t^b r^c \,.
\end{aligned}
$$

□

Proposition 17 *We have an equivalence (in fact an isomorphism)*

$$
\mathrm{inn}(\mathfrak{g}_k) \simeq \mathrm{cs}_k(\mathfrak{g}) \,.
$$

Proof. It is immediate that the two morphisms above compose to the identity, either way. □

7 Lie 4-Algebras

7.1 Supergravity field strength Lie 4-algebra $\mathrm{inn}(\mathrm{sugra}(10,1))$

Definition 9 (Supergravity field strength Lie 4-algebra) *On*

$$\bigwedge^\bullet((s\mathfrak{iso}^s(10,1))^* \oplus (ss\mathfrak{iso}^s(10,1))^* \oplus (sss\mathbb{R})^* \oplus (ssss\mathbb{R})^*)$$

with a basis for $(s\mathfrak{iso}^s(10,1))^* \oplus (sss\mathbb{R})^*$ chosen as in 6.1.4 and another basis $\{r^{ab}, r^a, \psi\}$ chosen for $(ss\mathfrak{iso}^s(10,1))^*$ and a basis $\{g\}$ chosen for $(sss\mathbb{R})^*$ define a differential as follows

$$\begin{aligned}
d\omega^{ab} &= \omega^{ac}\omega^{cb} + r^{ab} \\
dv^a &= \omega^{ab}v^b + \frac{i}{2}\bar{\psi}\Gamma^a\psi + r^a \\
d\psi &= \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi + \rho \\
dc &= \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi v^a v^b + g \\
dr^{ab} &= -d(\omega^{ac}\omega^{cb}) \\
dr^a &= -d(\omega^{ab}v^b + \frac{i}{2}\bar{\psi}\Gamma^a\psi) \\
d\rho &= -d(\frac{1}{4}\omega^{ab}\Gamma^{ab}\psi) \\
dg &= -d(\frac{1}{2}\bar{\psi}\Gamma^{ab}\psi v^a v^b)\,.
\end{aligned}$$

This qfDGCA defines the Lie 4-algebra

$$\mathrm{inn}(\mathrm{sugra}(10,1))\,.$$

URS: HERE AS before, no guarantee yet on signs etc.

claim: the "rheonomy constraints" [1] on the curvatures in 11-dimensional supergravity express nothing but the qfDGCA-morphism property of

$$d\mathrm{curv} : \mathrm{Vect}(X) \to \mathrm{inn}(\mathrm{sugra}(10,1))\,.$$

8 Open problems

What is the structure unifying $\mathrm{cs}_k(\mathfrak{g})$ and $\mathrm{sugra}(10,1)$? From string-theoretical considerations [14] we might expect that there is a Lie n-algebra $\mathfrak{m}$ which naturally unifies the Chern-Simons Lie 3-algebra of $\mathfrak{e}_8$ with that of $\mathfrak{spin}(10,1)$ and with the supergravity Lie 3-algebra:

$$\begin{tikzcd}
 & \mathrm{sugra}(10,1) \arrow[d, hook] & \\
 & \mathfrak{m} & \\
\mathrm{cs}_k(\mathfrak{e}_8) \arrow[ur, hook] & & \mathrm{cs}_k(\mathfrak{spin}(10,1)) \arrow[ul, hook']
\end{tikzcd}\,.$$

A direct sum construction would do, but it is conceivable that this really points to an interesting indecomposable Lie n-algebra $\mathfrak{m}$.

It seems that we should regard sugra$(10,1)$ as a super Lie n-algebra of Baez-Crans type **??**. That would suggest to extend it to a corresponding super Chern and super Chern-Simons Lie n-algebra.

A Useful component formulas

It so happens that computations in a DGCA are often practical only after choosing a basis. Since some of our proofs rely on such computations in a chosen basis, we here list some useful formulas concerning our examples.

A.1 DGCA for strict Lie 2-algebra

Consider the DGCA with its chosen basis as in example 5.1.1.

The fact that α is an action

$$\alpha(a_a)(\alpha(a_b)(b_i)) = \alpha([a_a, a_b])(b_i) + \alpha(a_b)(\alpha(a_a)(b_i))$$

reads in components

$$2\alpha^i{}_{[a|j|}\alpha^j{}_{b]k} = \alpha^i{}_{ck}C^c{}_{ab}\,. \tag{5}$$

One of the conditions on the differential crossed module is

$$t(\alpha(a_a)(b_i)) = [a_a, t(b_i)]\,.$$

In components this reads

$$t^b{}_j\alpha^j{}_{ai} = C^b{}_{ac}t^c{}_i\,. \tag{6}$$

The other condition is

$$\alpha(t(b_i))(b_j) = [b_i, b_j]\,.$$

This says that

$$\alpha^i{}_{ak}t^a{}_j = \tilde{C}^i{}_{jk} \tag{7}$$

are the structure constants of $\mathfrak{h}$. In particular, this implies that the expression is antisymmetric in the two lower indices.

Nilpotency of d follows from

$$\begin{aligned} d^2a^a &= d(-\frac{1}{2}C^a{}_{bc}a^ba^c - t^a{}_ib^i) \\ &= \frac{1}{2}C^a{}_{bc}C^b{}_{de}a^ca^da^e - C^a{}_{bc}a^bt^c{}_ib^i + t^a{}_i\alpha^i{}_{bj}a^bb^j \\ &= 0\,, \end{aligned}$$

where the first term vanishes again by the Jacobi identity on $\mathfrak{g}$ and the second two terms by the first of the two crossed module conditions. Also

$$\begin{aligned} d^2b^i &= d(-\alpha^i{}_{aj}a^ab^j) \\ &= -\alpha^i{}_{aj}(-\frac{1}{2}C^a{}_{bc}a^ba^c - t^a{}_kb^k)b^j + \alpha^i{}_{aj}a^a(-\alpha^j{}_{bk}a^bb^k) \\ &= (\alpha^i{}_{aj}\frac{1}{2}C^a{}_{bc} - \alpha^i{}_{bk}\alpha^k{}_{cj})a^ba^cb^j + \alpha^i{}_{a(j}t^a{}_{k)}b^kb^j \\ &= 0\,. \end{aligned}$$

The first term vanishes by (5). The second vanishes by the antisymmetry of (7) combined with the symmetry of b^kb^j, due to b being of even degree.

A.2 Lie 2-algebra morphisms as DGCA morphisms

The following is guaranteed by the general relation between L_∞-morphisms and the morphisms of the dual DGCA. We spell out the proof because the details will be helpful for our computations in section 6.

Proposition 18 *The morphisms of 2-term L_∞-algebras from Def.* **??** *are in bijective corespondence with morphisms of the corresponding DGCAs (which are algebra homomorphisms that are at the same time chain maps).*

Proof. Let $\{a^a\}$ and $\{b^i\}$ as above be a basis of generators for $(\Lambda^\bullet(W_0^* \oplus W_1^*)$ and let $\{a'^a\}$ and $\{b'^i\}$ be a basis of generators for another Lie 2-algebra $(\Lambda^\bullet(V_0^* \oplus V_1^*)$.

Then a morphism

$$q : (\Lambda^\bullet(W_0^* \oplus W_1^*), d_W) \to (\Lambda^\bullet(V_0^* \oplus V_1^*), d_V)$$

reads in terms of these bases

$$q : a^a \mapsto q^a{}_b a'^b$$

and

$$q : b^i \mapsto q^i{}_j b'^j + \frac{1}{2} q^i{}_{ab} a'^a a'^b .$$

The chain map condition demands that the coefficients satisfy

$$-\frac{1}{2} C^a{}_{bc} q^b{}_d q^c{}_e a'^d a'^e - \frac{1}{2} t^a{}_i q^i{}_j b'^j - t^a{}_i q^i{}_{bc} a'^b a'^c = -q^a{}_d \frac{1}{2} C'^d{}_{bc} a'^b a'^c - q^a{}_d t'^d{}_i b'^i$$

and

$$\begin{aligned} & -\alpha^i{}_{aj} q^a{}_b q^j{}_k a'^b b'^k - \frac{1}{2}\alpha^i{}_{aj} q^a{}_b q^j{}_{cd} a'^b a'^c a'^d - \frac{1}{6} r^i{}_{abc} q^a{}_d q^b{}_e q^c{}_f a'^d a'^e a'^f \\ = \; & d'(q^i{}_j b'^j + \frac{1}{2} q^i{}_{ab} a'^a a'^b) \\ = \; & -q^i{}_j \alpha'^j{}_{ak} a'^a b'^k - \frac{1}{6} q^i{}_j r^j{}_{abc} a'^a a'^b a'^c + \frac{1}{2} q^i{}_{ab} C'^b{}_{cd} a'^a a'^c a'^d + q^i{}_{ab} t'^b{}_j a'^a b'^j . \end{aligned}$$

Hence

$$C^a{}_{de} q^d{}_b q^e{}_c + t^a{}_i q^i{}_{bc} = q^a{}_d C'^d{}_{bc}$$

and

$$t^a{}_i q^i{}_j = q^q{}_d t'^d{}_j$$

and

$$\alpha^i{}_{aj} q^q{}_b q^j{}_k = q^i{}_j \alpha'^j{}_{ak} - q^i{}_{ab} t^b{}_j$$

and

$$\alpha^i{}_{dj} q^d{}_{[a} q^j{}_{bc]} + \frac{1}{6} r^i{}_{def} q^d{}_{[a} q^e{}_b q^f{}_{c]} = \frac{1}{6} q^i{}_j r^j{}_{[abc]} - \frac{1}{2} q^i{}_{[a|b|} C'^b{}_{cd]} .$$

Here the square bracket of indices means antisymmetrization over all indices included in the bracket (except for those exempted by being included in $|\cdot|$).

One can check that these are indeed the equations defining a morphism of Lie-2-algebras. $\square$

B Remaining proofs

B.1 Derivation Lie algebra of $(\mathfrak{h} \to \mathfrak{g})$

We prove proposition 10.

Proof. Recall that the corresponding qfDGCA defined on $\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{h})^*)$ has a differential defined by

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c - t^a{}_i b^i$$

and

$$db^i = -\alpha^i{}_{aj}t^a b^j ,$$

in terms of the basis chosen in 5.1.1.

The derivation of degree -1 are

$$\{\iota_{X_a}\}$$

and $\{\tau_{ai}\}$, defined on generators by

$$\tau_{ai} : t^a \mapsto 0$$

and

$$\tau_{ai} : b^j \mapsto \delta_i^j t^a .$$

This yields Lie derivatives acting as

$$L_{X_a}(t^b) = C^b{}_{ca}t^c$$

$$L_{X_a}(b^i) = -\alpha^i{}_{aj}b^j$$

and

$$L_{\tau^a{}_i}(t^b) = t^b{}_i t^a$$

$$L_{\tau^a{}_i}(b^j) = \alpha^j{}_{ci}t^c t^a .$$

From this one finds

$$[L_{X_a}, L_{X_b}] = C^c{}_{ab}L_{X_c}$$

and

$$\begin{aligned}
[L_{X_a}, L_{\tau^b{}_j}](t^c) &= L_{X_a}(t^c{}_j t^b) - L_{\tau^b_j}(C^c{}_{da}t^d) \\
&= t^c{}_j C^b{}_{da}t^d - C^c{}_{da}t^d{}_j t^b \\
&= (C^b{}_{da}L_{\tau^d{}_j} + \alpha^i{}_{aj}L_{\tau^b{}_i})(t^c)
\end{aligned}$$

In the last step we used the property (6) satisfied by a differential crossed module.

Finally

$$\begin{aligned}
[L_{\tau^a{}_i}, L_{\tau^b{}_j}](t^c) &= L_{\tau^a{}_i}(t^c{}_j t^b) - L_{\tau^b{}_j}(t^c{}_i t^a) \\
&= t^c{}_j t^b{}_i t^a - t^c{}_i t^a{}_j t^b \\
&= (t^b{}_i L_{\tau^a{}_j} - t^a{}_j L_{\tau^b{}_i})(t^c)
\end{aligned}$$

and

$$\begin{aligned}
[L_{\tau^a{}_i}, L_{\tau^b{}_j}](b^k) &= L_{\tau^a{}_i}(\alpha^k{}_{cj}t^c t^b) - L_{\tau^b{}_j}(\alpha^k{}_{ci}t^c t^a) \\
&= (t^b{}_i L_{\tau^a{}_j} - t^a{}_j L_{\tau^b{}_i})(b^k)
\end{aligned}$$

□

B.2 Derivation Lie algebra of $\mathfrak{g}_\mu$

We prove proposition 11.

Proof. Recall that the corresponding qfDGCA is defined on the vector space $\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathbb{R})^*)$ by a differential acting as

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c$$

$$db = -\frac{1}{6}C_{abc}t^a t^b t^c\,,$$

where $\{X_a\}$ is a chosen basis of $s\mathfrak{g}$ and $\{t^a\}$ the dual basis.

For each basis vector X_a let τ^a be the derivation of degree -1 which acts on generators as

$$\tau^a : t^b \mapsto 0$$

$$\tau^a : b \mapsto -2t^a\,.$$

(All indices are raised and lowered with the Killing form corresponding to μ.)

One computes

$$[[d, \iota_{X^a}], [d, \iota_{X^b}]]\,(b) = C_{abc}C^c{}_{de}t^d t^e$$

and then uses the Jacobi identity to find that the Lie derivatives

$$\{L_{X^a} := [d, \iota_{X^a}]\}$$

and

$$\{L_{\tau_{X^a}} := [d, \tau_{X^a}]\}$$

generate a Lie algebra $\mathfrak{g}$ whose nonvanishing brackets are

$$[L_{X^a}, L_{X^b}] = C^c{}_{ab}(L_{X^c} + L_{\tau_{X^c}})\,.$$

This is ismomorphic to the direct sum Lie algebra

$$\mathfrak{g} \oplus \mathbb{R}^{\dim(\mathfrak{g})}\,.$$

□

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