On weak cokernels

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Abstract

We show that the weak cokernels of morphisms of 2-groups studied in [CarrascoGarzónVitale:2006] are isomorphic to the corresponding construction in [RobertsSchreiber:2007] which can be thought of in terms of mapping cones. Motivated by this we adopt the discussion of weak cokernels to Lie n-algebras, for arbitrary n, following [SchreiberStasheff].

Contents

1 Introduction

Given two 2-groups $G_{(2)}$ and $H_{(2)}$ and a strictly injective morphism of 2-groups

$$t : H_{(2)} \to G_{(2)}$$

[CarrascoGarzónVitale:2006] showed how to construct a weak cokernel $\mathrm{wcoker}(t)$

$$H_{(2)} \xrightarrow{t} G_{(2)} \longrightarrow \mathrm{wcoker}(t)$$

which is a Gray 3-group. We demonstrate that one may think of this as the mapping cone of t in a generalization of the construction considered in [RobertsSchreiber:2007] and write

$$\mathrm{wcoker}(t) := (H_{(2)} \xrightarrow{t} G_{(2)})\,.$$

It follows that for any given short exact sequence of strict 2-groups

$$K_{(2)} \xrightarrow{t} G_{(2)} \longrightarrow B_{(2)}$$

one obtains the setup

$$\begin{tikzcd} K_{(2)} \arrow[r, "t"] & G_{(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow \simeq"] & (H_{(2)} \xrightarrow{t} G_{(2)}) \arrow[dl, "f"] \\ & B_{(2)} & \end{tikzcd} .$$

We want to eventually understand the obstruction to lifting a $\Sigma B_{(2)}$-valued 2-functor

$$\mathcal{P} \longrightarrow \Sigma B_{(2)}$$

through the exact sequence

$$\begin{tikzcd} \Sigma K_{(2)} \arrow[r, "\Sigma t"] & \Sigma G_{(2)} \arrow[d] \\ & \Sigma B_{(2)} \end{tikzcd}$$

i.e. to construct

$$\begin{tikzcd} \Sigma K_{(2)} \arrow[r] & \Sigma G_{(2)} \arrow[d] \\ P \arrow[r] \arrow[ur, dotted] & \Sigma B_{(2)} \end{tikzcd}$$

The obstruction to this should be the composite denoted obst in

$$\begin{tikzcd} \Sigma K_{(2)} \arrow[r, "\Sigma t"] & \Sigma G_{(2)} \arrow[r, "i"] \arrow[d] & \Sigma(K_{(2)} \xrightarrow{t} G_{(2)}) \arrow[r] & \mathrm{coker}(i) \\ P \arrow[r] \arrow[ur, dotted] \arrow[urrr, bend right, "\mathrm{obst}"'] & \Sigma B_{(2)} \arrow[ur, "f^{-1}"'] & & \end{tikzcd}$$

with f^{-1} some suitable "local inverse" to f. This should exist in the context of ana-2-functors.

While we do not try to make this more precise at the level of 2-groups, we can study the analogous situation in the context of (semistrict) Lie n-algebras.

Following [StasheffSchreiber] these we can conceives as quasi-free differential graded commutative algebras living in the obious 2-category of algebra chain maps and homotopies. This allows us to work with arbitrary n.

We reproduce the construction analogous to the above one for sequences

$$\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)} \longleftarrow \mathfrak{b}^*_{(n)}$$

with t^* assumed to be particularly well behaved. (A generalization away from this assumption is certainly expected to exists, but not studied here.)

Thinking of the weak cokernel of 2-groups as a mapping cone proves to be useful for the generalization to Lie n-algebras:

we define the mapping cone Lie $(n+1)$-algebra

$$(\mathfrak{k}_{(n)} \xrightarrow{t} \mathfrak{g}_{(n)})$$

and show that it does fit into

$$\begin{tikzcd} \mathfrak{k}^*_{(n)} & \mathfrak{g}^*_{(n)} \arrow[l, "t^*"'] & (\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}) \arrow[l] \\ & \mathfrak{b}^*_{(n)} \arrow[u] \arrow[ur, dashed, "f"'] & \end{tikzcd} .$$

Moreover, we show that in this context now the map f does have a weak inverse

$$f^{-1} : (\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}) \to \mathfrak{b}^*_{(n)} .$$

2 Mapping cones and weak cokernels of Lie n-algebras

We conceive semistrict Lie n-algebras dually as differential graded commutative algebras which are freely generated, as graded commutative algebras, in degree $1 \leq d \leq n$. We refer to them as quasi-free differential graded commutative algebras (qDGCAs).

These we take here to live in the 2-category whose morphisms are chain maps that are at the same time algebra homomorphisms, and whose 2-morphisms are chain homotopies.

Definition 1 *(mapping cone of qDGCAs) Let*

$$\mathfrak{h}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}$$

be a morphism of qDGCAs such that t^* restricts to a surjective morphism on the underlying vector spaces, hence that it surjectively maps generators to generators.

The mapping cone of t^* is the qDGCA whose underlying graded algebra is

$$\bigwedge^\bullet (s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{f}^*_{(n)})$$

and whose differential d_{t^*} is such that it acts on generators schematically as

$$d_{t^*} = \begin{pmatrix} d_{\mathfrak{g}_{(n)}} & 0 \\ t^* & d_{\mathfrak{f}_{(n)}} \end{pmatrix} .$$

More in detail, d_{t^*} is defined as follows.

We write σt^* for the degree $+1$ derivation on $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{f}^*_{(n)})$ which acts on $s\mathfrak{g}^*_{(n)}$ as t^* followed by a shift in degree and which acts on $s\mathfrak{f}^*_{(n)}$ as 0.

Then, for any $a \in s\mathfrak{g}^*_{(n)}$ we have

$$d_{t^*}a := d_{\mathfrak{g}_{(n)}}a + \sigma t^*(a)\,.$$

and

$$d_{t^*}\sigma t^*(a) := -\sigma t^*(d_{\mathfrak{g}_{(n)}}a) = -d_{t^*}d_{\mathfrak{g}_{(n)}}a\,.$$

Proposition 1 *The differential d_{t^*} defined this way indeed satisfies $(d_{t^*})^2 = 0$.*

Proof. For $a \in s\mathfrak{g}^*_{(n)}$ we have

$$d_{t^*}d_{t^*}a = d_{t^*}(d_{\mathfrak{g}_{(n)}} + \sigma t^*(a)) = \sigma t^*(d_{\mathfrak{g}_{(n)}}a) - \sigma t^*(d_{\mathfrak{g}_{(n)}}a) = 0\,.$$

Hence $(d_{t^*})^2$ vanishes on $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)})$. Since

$$d_{t^*}d_{t^*}\sigma t^*(a) = -d_{t^*}d_{t^*}d_{\mathfrak{g}_{(n)}}a$$

and since $d_{\mathfrak{g}_{(n)}}a \in \bigwedge^\bullet(s\mathfrak{g}^*_{(n)})$ this implies $(d_{t^*})^2 = 0$. □

We write

$$(\ \mathfrak{h}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}\ ) := \left(\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{f}_{(n)}*), d_{t^*}\right)$$

for the resulting qDGCA and

$$(\mathfrak{h}_{(n)} \xrightarrow{t} \mathfrak{g}_{(n)})$$

for the corresponding Lie $(n+1)$-algebra.

Proposition 2 *There is a canonical morphism*

$$\mathfrak{g}^*_{(n)} \longleftarrow (\mathfrak{h}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)})$$

with the property that

$$\begin{tikzcd} \mathfrak{h}^*_{(n)} & \mathfrak{g}^*_{(n)} \arrow[l, "t^*"'] \arrow[d, phantom, "\Downarrow\tau"] & (\mathfrak{h}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}) \arrow[l] \arrow[ll, bend left, "0"] \\ & {} & \end{tikzcd}\ .$$

Proof. On components, this morphisms is the identity on $s\mathfrak{g}^*_{(n)}$ and 0 on $ss\mathfrak{h}^*_{(n)}$. One checks that this respects the differentials. The homotopy to the 0-morphism sends

$$\tau : \sigma t^*(a) \mapsto t^*(a)\,.$$

□

Proposition 3 *Let*

$$\mathfrak{h}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)} \longleftarrow \mathfrak{f}^*_{(n)}$$

be a sequence of qDGCAs with t^* as above and with the property that $\mathfrak{g}^*_{(n)} \longleftarrow \mathfrak{f}^*_{(n)}$ *restricts, on the underlying vector spaces of generators, to the kernel of the linear map underlying t^*.*

Then there is a unique morphism $f : \mathfrak{f}^*_{(n)} \to (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)})$ such that

$$\begin{tikzcd} \mathfrak{h}^*_{(n)} & \mathfrak{g}^*_{(n)} \arrow[l, "t^*"'] & (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) \arrow[l] \\ & \mathfrak{f}^*_{(n)} \arrow[u] \arrow[ur, dashed, "f"'] & \end{tikzcd} .$$

Proof. The morphism f has to be in components the same as $\mathfrak{g}^*_{(n)} \leftarrow \mathfrak{f}^*_{(n)}$. By the assumption that this is in the kernel of t^*, the differentials are respected. □

Remark. It must be possible to relax the assumptions on $\mathfrak{g}^*_{(n)} \leftarrow \mathfrak{f}^*_{(n)}$ while retaining a unique $\mathfrak{f}^*_{(n)} \to (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)})$ up to isomorphism. This would then show that

$$(\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) = \mathrm{coker}(t)^*$$

is the weak kernel of t^*.

Proposition 4 *With the assumptions as before, the morphism $\mathfrak{f}^*_{(n)} \to (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)})$ has a – noncanonical – weak inverse*

$$f^{-1} : (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) \to \mathfrak{f}^*_{(n)} .$$

Proof. We first construct a morphism f^{-1} and then show that it is weakly inverse to f. Choose a splitting of the vector space V underlying $\mathfrak{g}^*_{(n)}$ as

$$V = \ker(t^*) \oplus V_1 .$$

Take the component map of f^{-1} to be the identity on $\ker(t^*)$ and 0 on V_1. Moreover, for $a \in V_1$ set

$$f^{-1} : \sigma t^*(a) \mapsto -d_{\mathfrak{g}_{(n)}} a|_{\bigwedge^\bullet \ker(t^*)} .$$

For $a \in \ker(t^*)$ we have

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto, "f^{-1}"'] & d_{\mathfrak{g}_{(n)}} a \arrow[d, mapsto, "f^{-1}"'] \\ a \arrow[r, mapsto] & d_{\mathfrak{g}_{(n)}} a \end{tikzcd} .$$

For $a \in V_1$ we have

$$\begin{tikzcd} a \arrow[rr, mapsto] \arrow[d, mapsto, "f^{-1}"'] & & d_{\mathfrak{g}_{(n)}} a + \sigma t^*(a) \arrow[d, mapsto, "f^{-1}"'] \\ 0 \arrow[r, mapsto] & d_{\mathfrak{g}_{(n)}} a|_{\bigwedge^\bullet \mathrm{ker}(t^*)} - d_{\mathfrak{g}_{(n)}} a|_{\bigwedge^\bullet \mathrm{ker}(t^*)} \end{tikzcd} .$$

and

$$\begin{tikzcd} \sigma t^*(a) \arrow[r, mapsto] \arrow[d, mapsto, "f^{-1}"'] & -\sigma t^*(d_{\mathfrak{g}_{(n)}} a) \arrow[d, mapsto, "f^{-1}"'] \\ d_{\mathfrak{g}_{(n)}} a|_{\bigwedge^\bullet \mathrm{ker}(t^*)} \arrow[r, mapsto] & d_{\mathfrak{g}_{(n)}} (d_{\mathfrak{g}_{(n)}} a|_{\bigwedge^\bullet \mathrm{ker}(t^*)}) \end{tikzcd} .$$

Hence this is indeed a morphism of qDGCAs.

Next we check that f^{-1} is a weak inverse of f. Clearly

$$\mathfrak{f}^*_{(n)} \longleftarrow (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) \longleftarrow \mathfrak{f}^*_{(n)}$$

is the identity on $\mathfrak{f}^*_{(n)}$. What remains is to construct a homotopy

$$\begin{tikzcd} (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) & \mathfrak{f}^*_{(n)} \arrow[l] & (\mathfrak{h}^*_{(n)} \stackrel{t^*}{\leftarrow} \mathfrak{g}^*_{(n)}) \arrow[l] \arrow[ll, bend left, "\mathrm{Id}"] \arrow[l, phantom, shift left=4, "\Downarrow\tau"] \end{tikzcd} .$$

One checks that this is accomplished by taking τ to act on σV_1 as $\tau : \sigma V_1 \stackrel{\simeq}{\to} V_1$ and extend suitably. $\square$

Example. For μ an $(n+1)$-cocycle on $\mathfrak{g}$ and $\mathfrak{g}_\mu$ the corresponding Baez-Crans type Lie n-algebra, the qDGCA of $(\Sigma^{n-1}\mathfrak{u}(1)^* \leftarrow \mathfrak{g}_\mu^*)$ is based on the vector space $(\bigwedge^\bullet(s\mathfrak{g}^* \oplus s^n\mathbb{R}^* \oplus s^{n+1}\mathbb{R}^*))$ with the differential on $s\mathfrak{g}^*$ being the Chevalley-Eilenberg differential. For $\{b\}$ the canonical basis of $s^n\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^{n+1}\mathbb{R}^*$ the differential on these generators is

$$db = -\mu + c$$

and

$$dc = 0 \,.$$

Then the morphism

$$f^{-1} : (\Sigma^{n-1}\mathfrak{u}(1)^* \leftarrow \mathfrak{g}_\mu^*) \to \mathfrak{g}^*$$

is the identity on $s\mathfrak{g}^*$, vanishes on b and sends

$$f^{-1} : c \mapsto \mu \,.$$

3 Mapping cones and weak cokernels of 2-groups

3.1 Mapping cone of the identity

In [RobertsSchreiber:2007] the mapping cone of the identity morphism on a strict 2-group was studied.

Definition 2 *The Gray groupoid which we denote either*

$$T\Sigma G_{(2)}$$

and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$*, or*

$$\mathrm{INN}_0(G_{(2)})$$

and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ *or simply*

$$(\ G_{(2)} \xrightarrow{\mathrm{Id}} G_{(2)}\ )$$

and address it as the mapping cone of $\mathrm{Id}_{G_{(2)}}$ *or as the 2-crossed module induced by* $\mathrm{Id}_{G_{(2)}}$.

This 2-groupoid $T\Sigma G_{(2)}$ *is defined to be the the strict pullback*

$$\begin{tikzcd}
T\Sigma G_{(2)} \arrow[r] \arrow[d] & (\Sigma G_{(2)})^2 \arrow[d, "\mathrm{dom}"] \\
\{\bullet\} \arrow[r] & \Sigma G_{(2)}
\end{tikzcd}$$

This means the following. An object of $T\Sigma G_{(2)}$ is a morphism

$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

A 1-morphism in $T\Sigma G_{(2)}$ is a filled triangle

$$\begin{tikzcd}
 & \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\
 & \bullet
\end{tikzcd}$$

in $\Sigma G_{(2)}$. Finally, a 2-morphism in $T^t\Sigma G_{(2)}$ looks like

$$\begin{tikzcd}
 & \bullet \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "f'"] \arrow[dd, phantom, "\Rightarrow L"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "F \Downarrow \; \Downarrow F'"] & \\
 & \bullet
\end{tikzcd} .$$

The monoidal structure on $T\Sigma G_{(2)}$ is that induced from the embedding

$$T\Sigma G_{(2)} := \mathrm{INN}_0(\Sigma G_{(2)}) \hookrightarrow \mathrm{AUT}(G_{(2)})$$

discussion in [RobertsSchreiber:2007].

Recall for later use that this canonically sits in the sequence

$$G_{(2)} \hookrightarrow T\Sigma G_{(2)} \twoheadrightarrow \Sigma G_{(2)} \ .$$

3.2 Mapping cone of a faithful morphism

This has an obvious generalization to non-identity but faithful morphisms:

Let $G_{(2)}$ and $H_{(2)}$ be strict 2-groups and write $\Sigma G_{(2)}$ and $\Sigma H_{(2)}$ be the corresponding strict one object 2-groupoids.

Let

$$t : H_{(2)} \hookrightarrow G_{(2)}$$

be a morphism of strict 2-groups, faithful as a functor of the underlying 1-groupoids. This means we have a strict 2-functor

$$\Sigma t : \Sigma H_{(2)} \hookrightarrow \Sigma G_{(2)} \,.$$

Definition 3 *The morphism t defines a strict 2-groupoid with a weak monoidal structure that makes it a Gray groupoid, which we denote either*

$$T^t \Sigma G_{(2)}$$

and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$ relative to t, or

$$\mathrm{INN}_0^t(G_{(2)})$$

and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ relative to t or simply

$$(\ H_{(2)} \xrightarrow{t} G_{(2)}\ )$$

and address it as the mapping cone of t or as the 2-crossed module induced by t.

This 2-groupoid $T^t\Sigma G_{(2)}$ is defined to be the the strict pullback

$$\begin{tikzcd}
 & & T^t\Sigma G_{(2)} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{\bullet\} \arrow[dr] & & (\Sigma G_{(2)})^2 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{codom}"] & & \Sigma H_{(2)} \arrow[dl, "\Sigma t"'] \\
 & \Sigma G_{(2)} & & \Sigma G_{(2)} &
\end{tikzcd}$$

where

$$2 := \{\ \bullet \xrightarrow{\simeq} \circ\ \}$$

is the fat point.

Equivalently this means that $T^t\Sigma G_{(2)}$ is the strict pullback

$$\begin{tikzcd}
T^t\Sigma G_{(2)} \arrow[r] \arrow[d] & \Sigma G_{(2)} \arrow[d, "="] \\
\Sigma H_{(2)} \arrow[r, "\Sigma t"] & \Sigma G_{(2)}
\end{tikzcd} .$$

An object of $T^t\Sigma G_{(2)}$ is a morphism

$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

A 1-morphism in $T^t\Sigma G_{(2)}$ is a filled triangle

$$\begin{tikzcd}
 & \bullet \arrow[dd, "t(f)"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\
 & \bullet
\end{tikzcd}$$

in $\Sigma G_{(2)}$, with f a morphism in $\Sigma H_{(2)}$, hence an object of $H_{(2)}$. Finally, a 2-morphism in $T^t\Sigma G_{(2)}$ looks like

$$\begin{tikzcd}
 & \bullet \arrow[dd, bend right, "t(f)"'] \arrow[dd, bend left, "t(f')"] \arrow[dd, phantom, "\overset{t(L)}{\Rightarrow}"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "F \Downarrow \; \Downarrow F'"] & \\
 & \bullet
\end{tikzcd}$$

with

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow L"] & \bullet
\end{tikzcd}$$

a 2-morphism in $\Sigma H_{(2)}$, hence a morphism in $H_{(2)}$.

The monoidal structure on $T^t\Sigma G_{(2)}$ is that induced from the embedding

$$T^t\Sigma G_{(2)} \hookrightarrow T\Sigma G_{(2)} .$$

Proposition 5 *The 2-groupoid $T^t\Sigma G_{(2)}$ is codiscrete at top level. Therefore it is equivalent to its quotient by its 2-morphisms*

$$T^t\Sigma G_{(2)} \simeq \pi_1(T^t\Sigma G_{(2)}) .$$

This quotient is isomorphic to what in [CarrascoGarzónVitale:2006] is called (p. 595) the quotient pointed groupoid*:* $G_{(2)}/\langle H_{(2)}, t\rangle$*:*

$$\pi_1(T^t\Sigma G_{(2)}) \simeq G_{(2)}/\langle H_{(2)}, t\rangle \,.$$

Proof. This is a matter of matching the items of the componentwise definition on the top of p. 595 in [CarrascoGarzónVitale:2006] to the above definition. □

[CarrascoGarzónVitale:2006] prove that $G_{(2)}/\langle H_{(2)}, t\rangle$ is indeed the cokernel of t. See the last paragraph on p. 595 and item 2 on p. 596.