descent of the universal transition

Schreiber*

October 10, 2006

Abstract

To a given morphism $\mathcal{P}_2(Y) \to \mathcal{P}_2(X)$ of domains of 2-transport is associated the universal transition $\mathcal{P}_2(Y^\bullet)$. For sufficiently nice $Y$ we have

$$\mathcal{P}_2(Y^\bullet) \simeq \mathcal{P}_2(X).$$

*E-mail: urs.schreiber at math.uni-hamburg.de
Let \( p : \mathcal{P}_2(Y) \to \mathcal{P}_2(X) \) be a morphism of domains of 2-transport.

**Definition 1** We say that \( p : \mathcal{P}_2(Y) \to \mathcal{P}_2(X) \) contains all disks if each 2-morphism in \( \mathcal{P}_2(X) \) is the image of one in \( \mathcal{P}_2(Y) \).

For instance consider the union of a good covering of \( X \) by open sets with a collection consisting of one contractible open neighbourhood for every image of the standard disk in \( X \).

**Proposition 1** If \( Y \) contains all disks, then the universal transition \( \mathcal{P}_2(Y^\bullet) \) associated with \( Y \) is equivalent, as a 2-category, to \( \mathcal{P}_2(X) \),

\[
\mathcal{P}_2(Y^\bullet) \simeq \mathcal{P}_2(X) .
\]

This equivalence is established using a 2-functor

\[
s : \mathcal{P}_2(X) \to \mathcal{P}_2(Y^\bullet)
\]

with only weak respect for composition. Hence the proposition does not hold within the 3-category of strict 2-categories with strict 2-functors between them. We shall first construct a 2-functor

\[
\tilde{s} : \mathcal{P}_2(X) \to \mathcal{P}_2(p)
\]

and then show that it takes values only in the sub-2-category \( \mathcal{P}_2(Y^\bullet) \subset \mathcal{P}_2(X) \).

\( \tilde{s} \) is obtained from vertical composition with 2-morphisms of the kind \( t \) and \( \bar{t} \) in \( \mathcal{P}_2(p) \). In fact, we only need the combinations

\[
\begin{array}{c}
p(y_1) \xrightarrow{p(\gamma)} p(y_2) \\
\downarrow \quad \downarrow \\
y_1 \xrightarrow{\gamma} y_2
\end{array}
\]

and

\[
\begin{array}{c}
p(y_1) \xrightarrow{p(\gamma)} p(y_2) \\
\downarrow \quad \downarrow \\
y_1 \xrightarrow{\gamma} y_2
\end{array}
\]

which, by slight abuse of notation, we still call \( t \) and \( \bar{t} \).

It is crucial that these are one-sided inverses of each other:
Proposition 2

\[ p(y_1) \xrightarrow{\delta(y_1)} y_1 \xrightarrow{\gamma} y_2 \xrightarrow{\delta(y_2)} p(y_2) = p(y_1) \xrightarrow{\rho(\gamma)} p(y_2) \]

Proof. Use the fact that \( t \) and \( \bar{t} \) fit into a special ambidextrous adjunction. □

Definition 2 Let \( p \) be such that it contains all disks, and choose a lift for each morphism in \( P_2(X) \). The 2-functor

\[ \tilde{s} : P_2(X) \rightarrow P_2(p) \]

is defined by the assignment

where \((x, i)\) is the chosen lift of \( x \), \((\gamma, j)\) is the lift of \( \gamma \), and so on.

Proposition 3 \( \tilde{s} \) is indeed 2-functorial.

Proof. Respect for vertical composition is an immediate consequence of prop.
2. Horizontal composition is respected up to the compositor

Its coherence (associativity) is again a consequence of \( t \) being inverse to \( \bar{t} \).

\[ (x, r) \xrightarrow{t_r} (x, i) \xrightarrow{\bar{t}_i} (x, j) \xrightarrow{(\gamma_1, j)} (y, j) \xrightarrow{\bar{t}_j} (y, i) \xrightarrow{t_i} (x, r) \xrightarrow{\bar{t}_r} (z, r) \]

\[ (x, i) \xrightarrow{t_i} (x, y) \xrightarrow{\bar{t}_y} (x, l) \xrightarrow{t_l} (y, l) \xrightarrow{\bar{t}_l} (y, i) \xrightarrow{t_i} (x, r) \xrightarrow{\bar{t}_r} (z, r) \xrightarrow{\bar{t}_z} (z, o) \]

\[ (x, j) \xrightarrow{(\gamma_1, j)} (y, j) \xrightarrow{\bar{t}_j} (y, l) \xrightarrow{t_l} (y, i) \xrightarrow{\bar{t}_i} (y, l) \xrightarrow{t_l} (y, i) \xrightarrow{\bar{t}_i} (y, i) \xrightarrow{t_i} (x, r) \xrightarrow{\bar{t}_r} (z, r) \]

Proposition 4 \( \tilde{s} \) factors through \( \mathcal{P}_2(Y^\bullet) \).

\[ \tilde{s} : \xrightarrow{s} \mathcal{P}_2(X) \xrightarrow{-} \mathcal{P}_2(Y^\bullet) \xrightarrow{-} \mathcal{P}_2(p) \]

Proof. This is essentially trivial, since we know that \( \mathcal{P}_2(X) \) is the sub-2-category of \( \mathcal{P}_2(p) \) of all those 2-morphisms whose source and target object do not come from \( \mathcal{P}_2(X) \).
But more explicitly, we can write

\[
\begin{array}{c}
(x, j) \xrightarrow{(\gamma, j)} (y, j) \\
t_j(x) \quad \tilde{t}_j(y) \\
\downarrow \quad \downarrow \\
(x, i) \quad \tilde{x} \\
downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

\[
\begin{array}{c}
(x, j) \\
\downarrow \downarrow \\
(x, i) \quad \tilde{x} \\
\downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

\[
\begin{array}{c}
(x, i) \\
\downarrow \downarrow \\
(x, j) \xrightarrow{(\gamma, j)} (y, j) \\
t_j(x) \quad \tilde{t}_j(y) \\
\downarrow \quad \downarrow \\
(x, i) \quad \tilde{x} \\
downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

\[
\begin{array}{c}
(x, j) \\
\downarrow \downarrow \\
(x, i) \quad \tilde{x} \\
\downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

\[
\begin{array}{c}
(x, j) \xrightarrow{(\gamma, j)} (y, j) \\
t_j(x) \quad \tilde{t}_j(y) \\
\downarrow \quad \downarrow \\
(x, i) \quad \tilde{x} \\
downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

\[
\begin{array}{c}
(x, j) \\
\downarrow \downarrow \\
(x, i) \quad \tilde{x} \\
\downarrow \quad \downarrow \\
(x, k) \xrightarrow{(\gamma', k)} (y, k)
\end{array}
\]

making the 2-morphism in \( \mathcal{P}_2(Y^\bullet) \) manifest.

\[ \square \]

It remains to check that composing \( s \) with the canonical morphism

\[ \mathcal{P}_2(Y^\bullet) \to \mathcal{P}_2(X) \]

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is equivalent to the identity morphism. This is straightforward, but slightly tedious. There is an obvious choice for the pseudonatural isomorphisms one needs and all tin can equations involved, like

\[ (x, j) \xrightarrow{\gamma, j} (y, j) \]

and

\[ (x, j) \xrightarrow{\gamma, j} (y, j) = (x, i) \xrightarrow{\gamma, r} (x, r) \xrightarrow{\gamma, r} (y, r) \]

\[ (x, j) \xrightarrow{\gamma, j} (y, j) \]

\[ (x, i) \xrightarrow{\gamma, r} (x, r) \xrightarrow{\gamma, r} (y, r) \]
are seen to hold by repeatedly using the triangle and tetrahedron relation in $\mathcal{P}_2(Y^*)$. 