Twisted L_∞-connections

February 29, 2008

Abstract

The concept of a vector bundle twisted by a 2-bundle ($\simeq$ gerbe) is, by now, familiar. It leads, notably, to twisted K-theory. In the context of bundle gerbes, such twisted bundles are usually addressed as *gerbe modules*. Twisted 2-bundles, i.e. modules for bundle 2-gerbes, have been defined analogously.

Here we describe the notion of n-bundles twisted by an $(n+1)$-bundle in the context of our notion of L_∞-algebra connections. As one application, we interpret the Green-Schwarz mechanism in heterotic String theory as saying that the Kalb-Ramond field (a 2-bundle with connection) is twisted, in this sense, by the supergravity C-field restricted to the end-of-the-world 9-brane.

Contents

1 Plan and Setting

1.1 Twisted bundles and gerbe modules

Twisted bundles in the guise of bundle gerbe module were introduced in [2], where it was also shown that twisted K-theory is the K-theory of such twisted bundles.

In the language of Cheeger-Simons differential characters abelian twisted n-bundles for all n are described in [3], the discussion of which is further developed in [4].

1.2 n-Bundle twists in physics: Kalb-Ramond and Green-Schwarz

A gerbe-theoretic review of the relation between twisted bundles on D-branes and the Kalb-Ramond field followed by an ambitious 2-gerbe-theoretic description of the analogous situation for twisted gerbes on M5-branes and the supergravity C field is given in [1].

Here we want to shift perspective from M5-branes to end-of-the-world 9-branes (I think ;-)

In [1] twisted bundles and twisted gerbes are conceived in terms of local transition data, using a nonabelian variant of Deligne-cohomology notation. Twisted bundles appear on the second half of p. 9, while twisted gerbes are described in section 4. It is not hard to see that their equation in between equations (55) and (56) expresses the idea which we emphasize here: that twisted n-bundles are potentially failed lifts through $b^{n-1}\mathfrak{u}(1)$-extensions

1.3 Twisted L_∞-connections

In [5] we had discussed that the obstruction to lifting a $\mathfrak{g}$-connection

$$\begin{array}{ccc}
\Omega^\bullet_{\mathrm{vert}}(Y) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(Y) & \xleftarrow{(A,F_A)} & \mathrm{W}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(X) & \xleftarrow{\{P_i\}} & \mathrm{inv}(\mathfrak{g})
\end{array}$$

through a String-like central extension

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

is the $b^n\mathfrak{u}(1)$-connection obtained by canonically completing this diagram to the right as shown in figure 1.

The construction crucially involves first forming the lift of the $\mathfrak{g}$-connection to a $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection, where $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ is the "weak cokernel" or "homotopy quotient" of the injection of $b^{n-1}\mathfrak{u}(1)$ into $\mathfrak{g}_\mu$. This lift through the homotopy quotient always exists, since the homotopy quotient is in fact equivalent to just $\mathfrak{g}$. But performing the lift to the homotopy quotient also extracts the failure of the underlying attempted lift to $\mathfrak{g}_\mu$ proper. This failure may be projected out under

$$(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \twoheadrightarrow b^n\mathfrak{u}(1)$$

to yield the $b^n\mathfrak{u}(1)$-connection which obstructs the lift. It is the morphism denoted f^{-1} in 1 which picks up the information about the twist/obstruction. This was constructed in proposition 40 of [5].

However, the $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection itself deserves to be considered in its own right: this is just the L_∞-connection version of "twisted bundles" or "gerbe modules".

In particular, the obstruction problem can also be read the other way round:

given a $b^n\mathfrak{u}(1)$-bundle, we may ask for which $\mathfrak{g}$-bundles it is the obstruction to lifting these to a $\mathfrak{g}_\mu$-bundle. In string theory, this is actually usually the more natural point of of view:

- given the Kalb-Ramond background field (a $b\mathfrak{u}(1)$-connection) pulled back to the worldvolume of a D-brane, the "twisted $U(H)$-bundles" corresponding to it are the "Chan-Paton bundles" supported on that D-brane;
- given the supergravity 3-form field (a $b^2\mathfrak{u}(1)$-connection) pulled back to the end-of-the-world 9-branes, the "twisted $BU(1)$-2-bundle" corresponding to it is the Kalb-Ramond field, with the twist giving the failure of its 3-form curvature to close

$$dH_3 = G_4 .$$

	ordinary $\mathfrak{g}$-connection	attempted lift to $\mathfrak{g}_\mu$-connection	obstructing $b^n\mathfrak{u}(1)$-connection	**obstruction interpretation**
	ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	twisting $b^n\mathfrak{u}(1)$-connection	**twisting interpretation**
	ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	magnetic charge	**charge interpretation**

$$
\begin{array}{ccccccc}
\Omega^\bullet_{\mathrm{vert}}(Y) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) & \longleftarrow & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) & \longleftarrow & \mathrm{CE}(b^n\mathfrak{u}(1)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^\bullet(Y) & \xleftarrow{(A,F_A)} & \mathrm{W}(\mathfrak{g}) & \xleftarrow{f^{-1}} & \mathrm{W}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) & \longleftarrow & \mathrm{CE}(b^n\mathfrak{u}(1)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^\bullet(X) & \xleftarrow{\{P_i\}} & \mathrm{inv}(\mathfrak{g}) & \longleftarrow & \mathrm{inv}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) & \longleftarrow & \mathrm{inv}(b^n\mathfrak{u}(1))
\end{array}
$$

Figure 1: **Obstructing $b^n\mathfrak{u}(1)$ $(n+1)$bundles and "twisted" $\mathfrak{g}_\mu$ n-bundles** are two aspects of the same mechanism: the $(n+1)$-bundle is the obstruction to "untwisting" the n-bundle. The n-bundle is "twisted by" the $(n+1)$-bundle. There may be many non-equivalent twisted n-bundles corresponding to the same twisting $(n+1)$-bundle. We can understand these as forming a collection of *n-sections* of the $(n+1)$-bundle.

2 Twisted L_∞-connections

2.1 Ordinary twisted bundles in terms of L_∞-connections

Let $\mathfrak{g}$ be a Lie algebra with 2-cocycle $\mu \in \mathrm{CE}(\mathfrak{g})$ which induces a central extension

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{u}(1) & \longrightarrow & \hat{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
& & & & \Big\| {\scriptstyle =} & & & & \\
& & & & \mathfrak{g}_\mu & & & &
\end{array}
$$

which we can think of as a special case of our "string-like" central extensions, according to the first example in section 6.4.1 of [5].

Then the weak cokernel Lie 2-algebra

$$(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$$

is in fact a special case of a strict Lie 2-algebra as in the third example of 6.1.1 in [5]. Accordingly, the following discussion is really a special case of the kind of computations shown in section 6.3.1 of [5]. But it deserves to be spelled out for the present case in detail here.

By inspection, one finds that forms on Y with values in $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$ have the following characterization, as displayed in figure 2.1:

$$
\begin{array}{ccc}
\mathrm{CE}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}}) & \twoheadleftarrow & \mathrm{W}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}}) \\
\Bigg| & & \Big| \\
\begin{array}{l} A \in \Omega^1(Y, \hat{\mathfrak{g}}) \\ B \in \Omega^2(Y) \\ \\ (F_A)^a = 0 \\ (F_A)^0 = B \\ dB = 0 \end{array} & & \begin{array}{l} A \in \Omega^1(Y, \hat{\mathfrak{g}}) \\ \beta \in \Omega^1(Y, \hat{\mathfrak{g}}) \\ B \in \Omega^2(Y) \\ C \in \Omega^3(Y) \\ \\ (F_A)^a = \beta^a \\ (F_A)^0 - B = \beta^0 \\ (d_A\beta)^0 = C \end{array} \\
\Big\downarrow & & \downarrow \\
\Omega^\bullet(Y) & \overset{=}{\rule{6em}{0.4pt}} & \Omega^\bullet(Y)
\end{array}
$$

let, as usual $\{t^a\}$ be a chosen basis of $\mathfrak{g}^*$ and let t^0 denote the canonical basis of the central part of $\hat{\mathfrak{g}}$. Then a $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$-descent object

$$
\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})
$$

is a $\mathfrak{g}$-descent object whose failure to be a $\hat{\mathfrak{g}}$-descent object is measured by a closed vertical 2-form B.

Analogously, a $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$-connection descent object has a $\hat{\mathfrak{g}}$-valued curvature 2-form β whose $\mathfrak{g}$-valued part satisfies the ordinary $\mathfrak{g}$-Bianchi identity, $(d_A\beta)^a = 0$, but whose central part satisfies $(d_A\beta)^0 = C$, for C the curvature 3-form of the twisting 2-bundle.

This phenomenon is sometimes addressed as a *failure of the Bianchi identity*, but of course it is just the Bianchi identity of a 1-connection which fails, while what we see is actually the Bianchi identity of a 2-connection.

2.2 Twisted 2-bundles

Now for μ a 3-cocycle on $\mathfrak{g}$, repeating the above for the String-extension

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{u}(1) & \longrightarrow & \hat{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & & & \Big\| {\scriptstyle =} & & & & \\
 & & & & \mathfrak{g}_\mu & & & &
\end{array}
$$

yields a “twisted 2-bundle” whose 3-form crvature H_3 suffers an similar “failure of the Bianchi identity”

$$
dH_3 = G\,.
$$

References

[1] P. Aschieri and B. Jurco, *Gerbes, M5-brane anomalies and E_8 gauge theory* J. High Energy Phys. **0410** (2004) 068, [`arXiv:hep-th/0409200v1`].

[2] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, [`arXiv:hep-th/0106194v2`]

[3] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, [`arXiv:hep-th/0011220`]

[4] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology and M-theory*, [`arXiv:math/0211216`]

[5] H. Sati, U. Schreiber and J. Stasheff, *L_∞-connections and applications to String- and Chern-Simons n-transport*, [`arXiv:0801.3480`]