3-category of twisted bimodules

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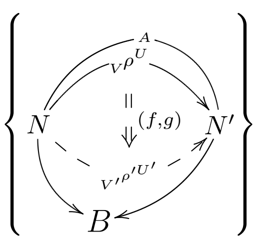
Abstract

For $\mathcal{C}$ an abelian monoidal category, functors into $\mathrm{Bim}(\mathcal{C})$ are lax functors into $\Sigma(\mathcal{C})$. On the other hand, lax functors into $\mathrm{End}(\Sigma(\mathcal{C}))$, for $\mathcal{C}$ braided, yield bimodules, the action on which is twisted by over- or under-braiding past objects of $\mathcal{C}$. Here we want to describe aspects of the resulting 3-category of twisted bimodules.

1 Introduction

For $\mathcal{C}$ an abelian and braided monoidal category, there should be a weak 3-category

$$\mathrm{TwBim}(\mathcal{C}) =$$



whose objects are special Frobenius algebra objects internal to $\mathcal{C}$, whose 1-morphisms are bimodules for these, whose 2-morphisms are triples $(N \xrightarrow{\rho} N', U, V)$, where ρ is a homomorphism of bimodules from $N \otimes U$ to $V \otimes N'$, and where 3-morphisms are pairs (f, g) of morphisms going between the twisting objects U and V.

Here the right module structure on $N \otimes U$ is understood to be by overbraiding U, and the left module structure on $V \otimes N'$ by underbraiding V.

This structure can be motivated by looking at lax functors into $\Sigma(\mathrm{End}(\mathcal{C}))$.

Below we give the precise definition of the morphisms and their composition and check some of the required properties.

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2 The definition of $\mathrm{TwBim}(\mathcal{C})$

Write

$$A \xrightarrow{N} B$$

for an object N of $\mathcal{C}$ with the structure of an A-B bimodule, for A and B algebra objects internal to $\mathcal{C}$.

If we assume that all algebras are special Frobenius, then the relations we need for checking the exchange law below are easily obtained.

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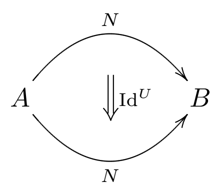
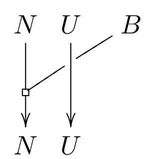


for the **twisted** or **induced** bimodule object $N \otimes U$ whose action is that of N combined with braiding under U



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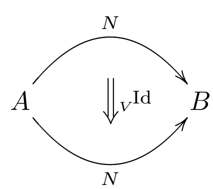
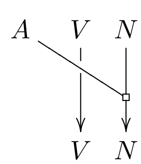
Similarly, write



for the bimodule obtained by braiding over V



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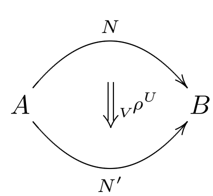
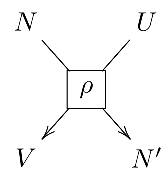
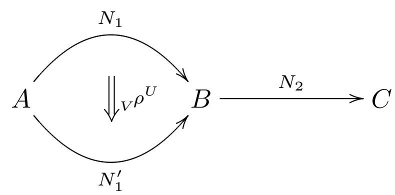


for a morphism



of such induced bimodules.

Write



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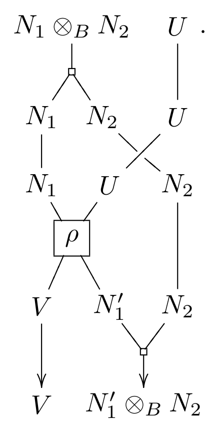

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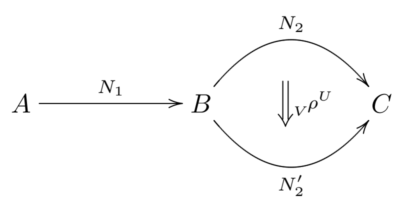

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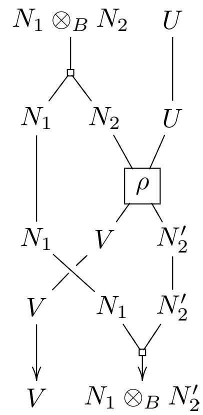

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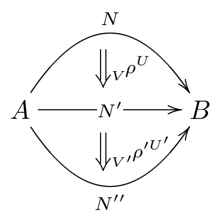

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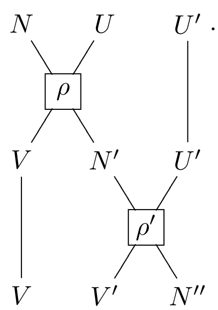

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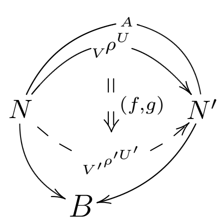

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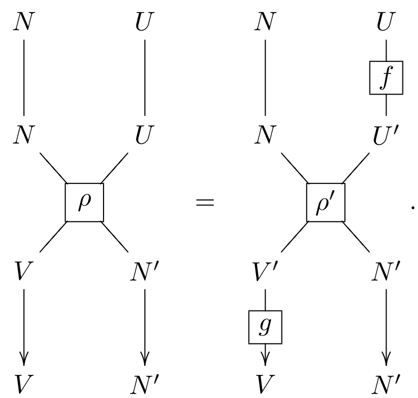
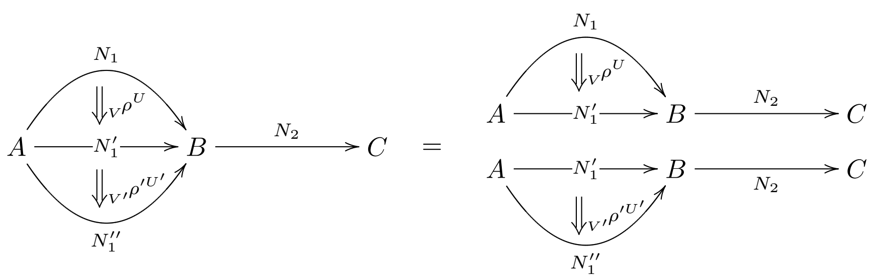
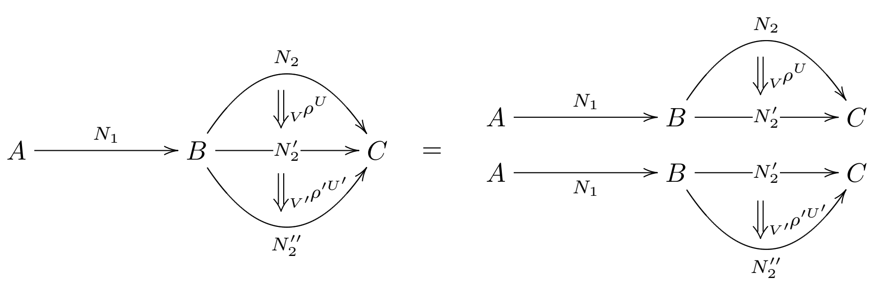


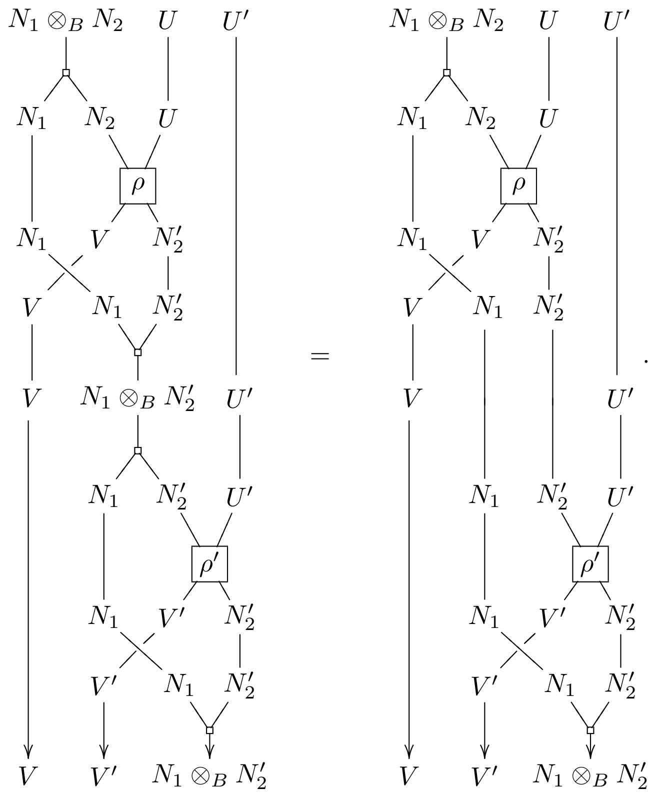
Proposition 1 *From the above definitions it follows that horizontal composition with identity 2-morphisms satisfies the exchange law strictly:*



and

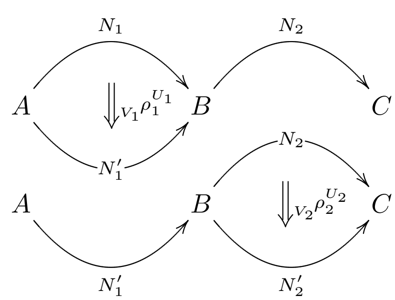


Proof.

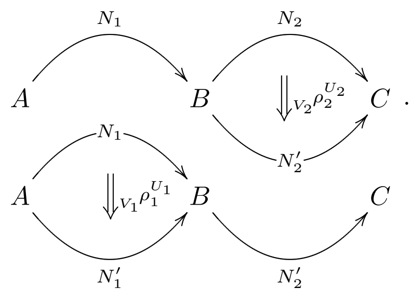


□

Proposition 2 *The composition*

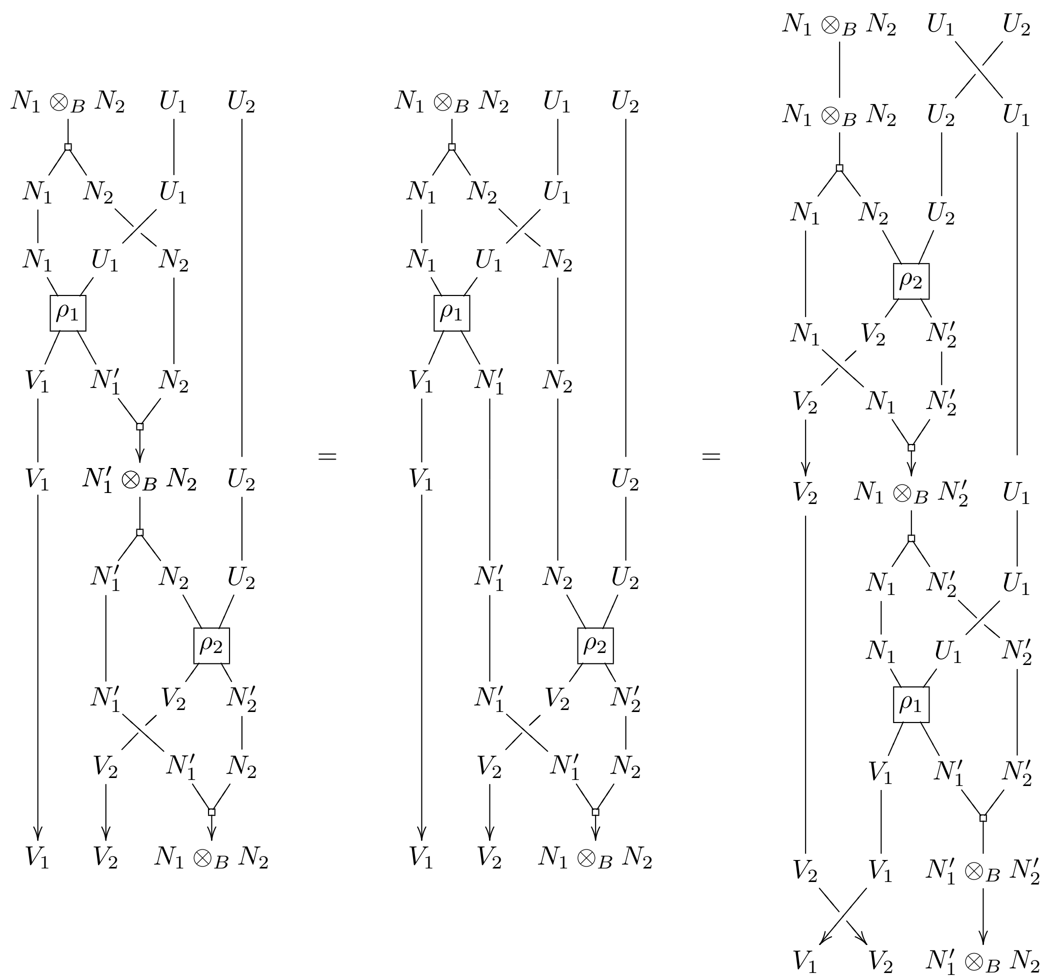


is isomorphic to



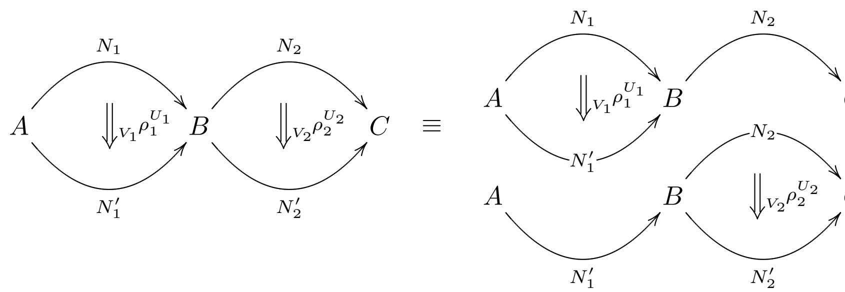
The isomorphism is is given by the braiding on $U_1 \otimes U_2$ and $V_1 \otimes V_2$, respectively. It is unique.

Proof.



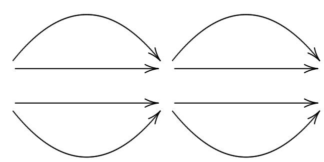
□

Definition 1 *Given the above, there are different but isomorphic ways to define the horizontal composition of 2-morphisms. For definiteness, we set*



Proposition 3 *Composition of 2-morphisms satisfies the exchange law up to isomorphism.*

Proof.



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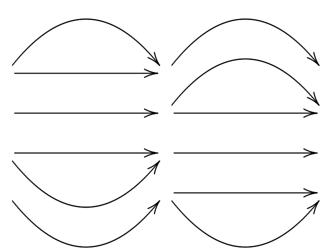

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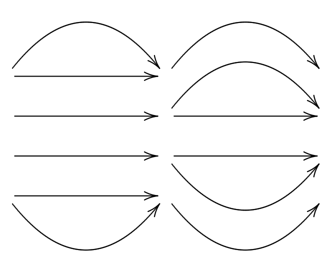

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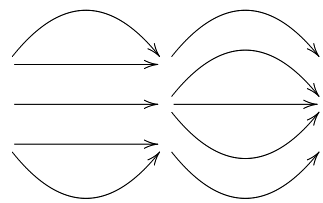

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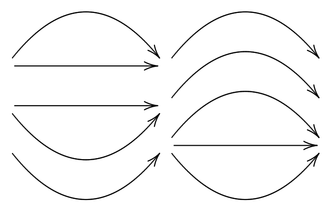

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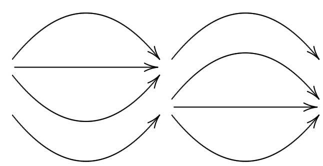

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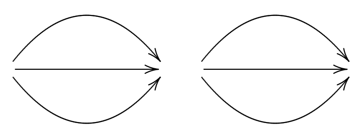

$\square$

Remark. There is an obvious associator in TwBim$(\mathcal{C})$ induced from the associator in Bim$(\mathcal{C})$. The twisting of bimodules does affect neither its existence nor its coherence.

Since the isomorphism replacing the exchange law is unique, it should automatically be coherent.

Hence TwBim$(\mathcal{C})$ should be weak 3-category.