0.1 Equivalence of Anafunctors and Local id-Trivializations.

We have already seen that every smooth π-local id-trivialization gives rise to an anafunctor

$$\begin{array}{ccc} \mathcal{C}_1(\pi) & \xrightarrow{\mathrm{R}(g,\mathrm{triv})} & T' \\ \downarrow{\scriptstyle \pi_{\mathcal{C}}} & & \\ \mathcal{P}_1(X) & & \end{array}$$

Conversely, we want to find a condition which guarantees that a smooth anafunctor

$$F : \mathcal{P}_1(X) \to T'$$

is of this form, for some surjective submersion $\pi : Y \to X$.

Proposition 1 *If*

$$p : |F| \to \mathcal{P}_1(X)$$

is a smooth surjective equivalence, whose component maps are surjective submersions, then there exists a surjective submersion

$$\pi : Y \to X$$

such that

$$|F| = \mathcal{C}_1(\pi)\,.$$

Proof. We simply define

$$Y := \mathrm{Obj}(|F|)\,.$$

Then we need to show that indeed $\mathcal{C}_1(\pi) = |F|$, for $\pi = p_0$.

In order to do so, we repeatedly make use of the fact that, since p is a surjective equivalence, there is, for every morphism in $\mathcal{P}_1(X)$ and every lift of its endpoints to $\mathrm{Obj}(|F|)$, a *unique* lift of the entire morphism.

This immediately implies that we have pullback squares of the form

$$\begin{array}{ccc} \mathrm{Mor}(\mathcal{P}_1(Y)) & \hookrightarrow & \mathrm{Mor}(|F|) \\ \| & & \downarrow{\scriptstyle p_1} \\ \mathrm{Mor}(\mathcal{P}_1(Y)) & \hookrightarrow & \mathrm{Mor}(\mathcal{P}_1(X)) \end{array}$$

and

$$\begin{array}{ccc} Y^{[2]} & \hookrightarrow & \mathrm{Mor}(|F|) \\ \| & & \downarrow p_1 \\ Y^{[2]} & \xrightarrow{r} & \mathrm{Mor}(\mathcal{P}_1(X)) \end{array} \quad ,$$

which define the inclusions

$$\mathcal{P}_1(Y) \hookrightarrow \mathrm{Mor}(|F|)$$

and

$$Y^{[2]} \hookrightarrow \mathrm{Mor}(|F|) \; .$$

Here r sends (x, y) to $\mathrm{Id}_{\pi(x)}$ $(= \mathrm{Id}_{\pi(y)})$.

The fact that these generators satisfy the relations that hold in $\mathcal{C}_1(\pi)$ again follows from uniqueness of lifts. Therefore we even have an inclusion

$$\mathcal{C}_1(\pi) \hookrightarrow |F| \; .$$

Finally, by lifting any path in X piecewise to morphisms in $\mathcal{P}_1(Y)$ and in $Y^{[2]}$ we obtain a lift for each choice of lift of the endpoints. By the uniqueness of lifts, this means that $\mathcal{C}_1(\pi)$ already coincides with $|F|$. □

Theorem 1 *Let $F : \mathcal{P}_1(X) \to T'$ be a smooth anafunctor such that the component maps of*

$$p : |F| \to \mathcal{P}_1(X)$$

are surjective submersions. Then there is a smoothly locally id*-trivializable transport functor*

$$\mathrm{tra}_F : \mathcal{P}_1(X) \to T'$$

with transition data (triv, g) *such that*

$$\tilde{F} : |F| \to T'$$

equals

$$R_{(\mathrm{triv},g)} : \mathcal{C}_1(\pi) \to T' \; .$$

Proof. According to prop. 1 there is a surjective submersion $\pi : Y \to X$ such that $|F| = \mathcal{C}_1(\pi)$, so that

$$\tilde{F} : \mathcal{C}_1(\pi) \to T' \; .$$

But using the equivalence of such functors with transition data, it follows that there is $(\mathrm{triv}, g) \in \mathrm{TD}^{\infty}_{\pi}(i)$ such that $\tilde{F} = R_{(\mathrm{triv},g)}$. Finally, by applying Ex_π we get the corresponding transport functor $\mathrm{Ex}_\pi(\mathrm{triv}, g)$. □