The principle of general tovariance

Klaas Landsman

Radboud Universiteit Nijmegen
Universität Hamburg (1993-1994)

Motto

Our civilization is characterized by the word "progress". Progress is its form rather than making progress one of its features. Typically it constructs. It is occupied with building an ever more complicated structure. And even clarity is only sought as a means to this end, not as an end in itself. For me on the contrary clarity, perspicuity are valuable in themselves. I am not interested in constructing a building, so much as in having a perspicuous view of the foundations of typical buildings. **Ludwig Wittgenstein**

Einstein's principle of general covariance:

> **Laws of physics must be definable in any coordinate system and are preserved under arbitrary coordinate transformations**

Originally intended to express the general relativity of motion, it actually says that general relativity uses differential geometry

Principle of general tovariance:

> **Any mathematical structure appearing in the laws of physics must be definable in an arbitrary topos and must be preserved under canonical ("geometric") morphisms between topoi**

Like general covariance, general tovariance has no physical content; it identifies the mathematical language of physics and allows the formulation of an equivalence principle

Contexts of topos theory:

- **Categorical methods in physics**
- **Generalized notions of logic**
- **Generalized notions of space**

Categorical methods in physics

- **AQFT:** Doplicher–Roberts **(1972–), BF (1982), FRS (1992)**
- **Conformal field theory:** Segal **(1986–)**
- **Topological quantum field theory:** Atiyah **(1989–)**
- **Locally covariant quantum field theory:**
  Brunetti–Fredenhagen–Verch **(2003)**
- **Functoriality of quantization:** L. **(2001–)**
- Topos theory **in foundations of physics**
  - **Kochen–Specker Theorem:** Butterfield–Isham **(1998)**
  - **Framework for all of physics:** Döring–Isham **(2007)**
  - **Algebraic quantum theory:** Heunen–L.–Spitters **(2007)**
    using "internal" C^*-algebras: Banaschewski–Mulvey **(2006)**

"The natural language is provided by category theory. This need not be a deterrent for a theoretical physicist of our days"

Generalized notions of logic

Propositional logic of classical physics (à la von Neumann):

- **Propositions** $a \in U$ **are defined in terms of functions** $a : M \to \mathbb{R}$ **and ranges** $U \subset \mathbb{R}$ **(all measurable)**
- **These fall into equivalence classes** $[a \in U] = a^{-1}(U) \subset M$
- **Which form the "Lindenbaum algebra" of the logic (i.e. propositions modulo provable equivalence), which is Boolean:**

  $X \vee Y = X \cup Y$, $X \wedge Y = X \cap Y$, $\neg X = X^c \equiv M \backslash X$, $X \to Y = V^c \cup W$, $\perp = \emptyset$, $\top = M$, **and** $X \leq Y \equiv X \Rightarrow Y = X \subset Y$
- **Truth values** $\{0, 1\}$ **are assigned by pure states** $\psi \in M$**: Proposition** $a \in U$ **is true if** $\psi \in a^{-1}(U)$ **and false if** $\psi \notin a^{-1}(U)$
- **Physicists: state-observable pairing** $\langle \psi, a \in U \rangle = 1$ **or 0**
- **Generalization to mixed states:** $\langle \mu, a \in U \rangle \in [0, 1]$

This generalization is innocent: mixed states can be decomposed into pure ones with truth values 0 or 1 and probabilities admit ignorance interpretation ('human frailty')

Propositional logic of quantum physics (à la von Neumann):

- **Propositions** $a \in U$ are defined in terms of self-adjoint operators $a : D(a) \subset H \to H$ and ranges $U \subset \mathbb{R}$
- These fall into equivalence classes $[a \in U] = a^{-1}(U) \in P(H)$
- Which projections form a **nondistributive orthomodular lattice** under $p \le q$ iff $pH \subset qH$ and hence a new kind of "quantum" logic (Birkhoff–von Neumann): closed linear subspaces of H replace (measurable) subsets of phase space
- Even pure states $\Psi \in H$ assign **probabilistic truth values** under pairing $\langle \psi, a \in U \rangle = (\Psi, [a \in U]\Psi) \in [0, 1]$ (Born rule)

This "quantum" logic seems dubious:

1. No ignorance interpretation of probabilities
2. No logical structure at all in any recognizable sense
3. No generalization to "quantum" predicate logic
4. Piron's reconstruction program has failed

Categorical logic of classical physics (à la Döring–Isham):

- Classical physics is formulated in category (topos) ***Sets***
- **Phase space** M is object in ***Sets***
- **Pure state** is arrow $1 \xrightarrow{\psi} M$ (i.e. **point** of M)
- **Proposition** $a \in U$ defines **subobject** $a^{-1}(U) \hookrightarrow M$ with **classifying arrow** $M \xrightarrow{\chi_{a^{-1}(U)}} \Omega = \{0, 1\} \equiv \{\text{false}, \text{true}\}$
- **Pairing** $1 \xrightarrow{\langle\psi, a\in U\rangle} \Omega := 1 \xrightarrow{\psi} M \xrightarrow{\chi_{a^{-1}(U)}} \Omega$ given by composition: $1 \xrightarrow{\langle\psi, a\in U\rangle} \Omega = \text{true}$ if $a(\psi) \in U$ and $= \text{false}$ if not

Categorical logic of quantum physics:

- Formulate quantum theory in appropriate topos containing:
- **"Quantum" phase space** as an object
- **States** and **propositions** as arrows
- **Pairing** $\langle\psi, a \in U\rangle$ that is not *a priori* probabilistic but takes values in a canonical multi-valued logical object Ω

Slogan: Truth is prior to probability - *derive* **Born rule**

Generalized notions of space

1. GELFAND–NAIMARK: Topological space $X \to$ commutative C^*-algebra $C_0(X) \to$ noncommutative C^*-algebra
   - CONNES: noncommutative geometry
   - DOPLICHER–FREDENHAGEN–ROBERTS (1995): noncommutative space-time at Planck scale
2. "Pointless topology": Space $X \to$ open sets $\mathcal{O}(X)$ as lattice ($U \leq V$ if $U \subseteq V$) $\to$ locale i.e. sup-complete distributive lattice such that $x \wedge \vee_\lambda y_\lambda = \vee_\lambda x \wedge y_\lambda$; N.B. many locales $\not\cong \mathcal{O}(X)$
3. GROTHENDIECK, LAWVERE: Set $\to$ Category of sets $\to$ Topos; "Generalized set" is object in topos

Combination of 1 and 3: commutative C^*-algebra in topos

Combination of 2 and 3: (completely regular cpt) locale in topos

Relationship: Gelfand theory for commutative C^*-algebras in topoi

Copenhagen

These uncertainties are simply a consequence of the fact that we describe the experiment in terms of classical physics (1958)

However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms (1949)

The “topic” equivalence principle

- **Bohr’s “doctrine of classical concepts”:** quantum theory is empirically accessible only through classical physics

  Mathematical translation: the physics of a noncommutative C^*-algebra lies in its family of commutative subalgebra’s

- **Einstein’s equivalence principle:** free fall in gravitational field is ‘infinitesimally’ indistinguishable from rest (or uniform motion) in Minkowski space-time without gravitational forces

  Einstein’s own mathematical translation: Geodesic motion is special choice of space-time coordinates which ‘transforms gravitational force away’ and hence restores special relativity

Perhaps in quantum theory one might be able to ‘transform noncommutativity away’ by moving to a suitable topos?

- **Topic equivalence principle:** The physics of a noncommutative C^*-algebra $\mathfrak{A}$ of observables (defined in *Sets*) is contained in a commutative C^*-algebra (defined in a suitable topos $T(\mathfrak{A})$)

What is a topos?

'A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry and on the other hand, logic and set theory.'

S. Mac Lane & I. Moerdijk, Sheaves in geometry and logic: A first introduction to topos theory. Springer, 1994.

Recapitulation: topos theory provides a setting for

- Generalized notions of space (Grothendieck)
- Generalized notions of logic (Lawvere)
- Categorical approach to physics (Isham)

Topoi generalize category *Sets* so that "everything" you can do with sets can still be done except - surprise - classical logic:

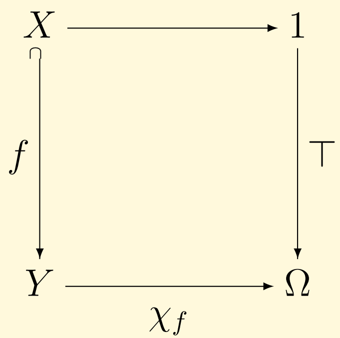
Topoi provide semantics for intuitionistic predicate logic

Definition: A topos is a category with

1. **Terminal object:** 1 such that $X \xrightarrow{\exists !} 1$ for each object X
2. **Exponentials:** object Y^X generalizing $\{X \xrightarrow{f} Y\}$ in ***Sets***
3. **Pullbacks:** fibered product of $B \xrightarrow{f} A$ and $C \xrightarrow{g} A$
4. **Subobject classifier:** generalization of $\Omega = \{0, 1\} = \{\text{false}, \text{true}\}$ and $B \xrightarrow{\chi_A} \Omega$ (carries logical structure of classical mathematics)

Arrow $X \overset{f}{\hookrightarrow} Y$ defines **subobject** $X \subset Y$ if f is "injective" (monic)

There exists an object Ω and an arrow $1 \xrightarrow{\top} \Omega$, such that for every monic f there is a *unique* arrow χ_f defining a pullback



The Nijmegen topos

Starting point: noncommutative C^*-algebra $\mathfrak{A}$ (in topos ***Sets***)

Note: C^*-algebras can be defined "internally" in any topos

Goal: implement "topic equivalence principle" - physics of $\mathfrak{A}$ is contained in commutative C^*-algebra A in suitable topos $T(\mathfrak{A})$

1. $\mathsf{C}(\mathfrak{A})$:= set of all commutative unital C^*-subalgebras of $\mathfrak{A}$
2. This is a poset and hence a category under $\leq := \subset$
3. $T(\mathfrak{A}) := \mathbf{Sets}^{\mathsf{C}(\mathfrak{A})} = \{\mathbf{functors}: \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}\}$
4. "Tautological" functor $A: \mathsf{C}(\mathfrak{A}) \to$ Sets defined by $A(C) = C$

A is a commutative C^*-algebra in $T(\mathfrak{A})$ under natural operations

"Quantum" phase space $\Sigma \in T(\mathfrak{A})$ is the **Gelfand spectrum** of A:

$\Sigma(C) = \mathcal{O}(P(C))$ is internal locale s.t. $A \cong C(\Sigma, \mathcal{O}(\mathbb{C}))$ in $T(\mathfrak{A})$

States, propositions, pairing

Exercise in classical physics: replace space M by locale $\mathcal{O}(M)$

- **Pure state** $\psi \in M$ yields **subobject** $S_\psi := \{U \in \mathcal{O}(M) \mid \delta_\psi(U) = 1\}$ of $\mathcal{O}(M)$ with classifying arrow $\mathcal{O}(M) \xrightarrow{\chi_\psi} \Omega = \{0, 1\}$
- **Proposition** $a \in U$ yields **point** $1 \xrightarrow{a^{-1}(U)} \mathcal{O}(M)$ (i.e. open in M)
- **Pairing** $1 \xrightarrow{\langle \psi, a \in U \rangle} \Omega := 1 \xrightarrow{a^{-1}(U)} \mathcal{O}(M) \xrightarrow{\chi_\psi} \Omega$ (same result as before)

Analogous construction in quantum physics in topos $T(\mathfrak{A})$:

- **State** ψ on $\mathfrak{A}$ defines **subobject** $S_\psi \overset{\psi}{\hookrightarrow} \Sigma$ with $S_\psi : \mathsf{C}(\mathfrak{A}) \to$ Sets

$$S_\psi(C) = \{U \in \mathcal{O}(P(C)) \mid \mu_\psi^C(U) = 1\}$$

  (μ_ψ^C is probability measure on $P(C)$ induced by state ψ on $\mathfrak{A}$)

- **Proposition** $a \in U$ defines **point** $1 \overset{a^{-1}(U)}{\hookrightarrow} \Sigma$ via "Daseinization"
- **Pairing** $1 \xrightarrow{\langle \psi, a \in U \rangle} \Omega := 1 \xrightarrow{a^{-1}(U)} \Sigma \xrightarrow{\chi_\psi} \Omega$ yields truth value in $\Omega_{T(\mathfrak{A})}$

Outlook

Topos theory provides an attractive setting for:

- **Generalized spaces**, e.g. internal locales as "quantum" phase spaces, occurring as Gelfand spectra of commutative C^*-algebras
- **Generalized logics** (replacing e.g. quantum logic)
  N.B. logics are intuitionistic (no middle third, no Choice)
- **States, propositions, and their pairing:** truth attribution to propositions is no longer probabilistic but Ω-valued

Topos theory may soften the noncommutativity of a C^*-algebra, which in a suitable topos "becomes commutative"
(perhaps this is useful for quantum gravity?)

Topoi carry the hope of deriving the probabilisitic structure of quantum mechanics from its logical structure (von Neumann)

N.B. Almost nothing has been achieved yet!