the 1-dimensional 3-vector space

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November 23, 2006

Abstract

We explain how, for any braided abelian monoidal category $C$, the 3-category $\Sigma(\text{Bim}(C))$ plays the role of the 3-category of canonical 1-dimensional 3-vector spaces. We make some comments on the resulting concept of line-3-bundles with connection and show how the 3-category of twisted bimodules arises from morphisms of almost-trivial line-3-bundles with connection.

Let $C$ be a braided abelian monoidal category.

You may want to think of the examples $C = \text{Vect}_k$ for some field $k$, or $C = \text{Mod}_R$, for some commutative ring $R$. But for the applications we have in mind, we will have a nontrivial braiding. In particular, $C$ might be a modular tensor category.

I denote the 2-category whose objects are algebras internal to $C$, whose morphisms are bimodules and whose 2-morphisms are bimodule homomorphisms by $\text{Bim}(C)$.

We can think of this as a 2-category of 2-vector spaces, due to the canonical inclusion

$$\text{Bim}(C) \hookrightarrow \text{Mod}_C.$$

Remarkably, since $C$ is assumed to be braided, we get that $\text{Bim}(C)$ is a monoidal 2-category.

For $A$ and $A'$ two algebras, their tensor product $A \otimes A'$ is the algebra which is $A \otimes A'$ as an object in $C$ and equipped with the product obtained by using the braiding to exchange $A$ with $A'$:

$$A \otimes A' \quad \text{and} \quad A' \otimes A.$$

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Accordingly, the left $A$-module $N$ and the left $A'$-module $N'$ are tensored to form the $A \otimes A'$-module $N \otimes N'$ with the action given by using the braiding:

\[
\begin{array}{c}
A \quad A' \\
N \quad N'
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
N \quad N'
\end{array}
\]

Similarly, if $N$ is a right $B$-module and $N'$ is a right $A'$-module, the right action of $B \otimes B'$ on $N \otimes N'$ is

\[
\begin{array}{c}
N \quad N' \\
\uparrow \quad \uparrow \\
B \quad B'
\end{array}
\]

A simple special case of this turns out to be interesting in applications. The tensor unit $1$ of $\mathcal{C}$ with the trivial algebra structure on it is always an algebra internal to $\mathcal{C}$. Any object of $\mathcal{C}$ is a $1$-$1$ bimodule. This yields a canonical inclusion

\[
\Sigma(\mathcal{C}) \rightarrow \text{Bim}(\mathcal{C})
\]

This means that for any $A$-$B$ bimodule $N$, and any object $U$ in $\mathcal{C}$, we may consider $N \otimes U$ as another $A$-$B$ bimodule, with the obvious left action and with the right action given by

\[
\begin{array}{c}
N \quad U \\
\downarrow \quad \downarrow \\
B
\end{array}
\]

Similarly, for $V$ any object of $\mathcal{C}$, we obtain the $A$-$B$ bimodule $V \otimes N$ with the obvious right action and the left action given by

\[
\begin{array}{c}
A \quad V \quad N \\
\downarrow \quad \downarrow \\
V \quad N
\end{array}
\]

Quite literally, we can think of the tensor structure on $\text{Bim}(\mathcal{C})$ as obtained from arranging bimodules in front of each other.

The formal expression of this geometric intuition is that from the monoidal 2-category $\text{Bim}(\mathcal{C})$ we can form the suspension, $\Sigma(\text{Bim}(\mathcal{C}))$, which is the 3-category with a single object $\bullet$, such that $\text{End}(\bullet) = \text{Bim}(\mathcal{C})$, and such that composition across that single object is the tensor product on $\text{Bim}(\mathcal{C})$. 

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If

\[
\begin{array}{c}
A \\
\downarrow^\rho \\
B
\end{array}
\]

is a 2-morphism in $\text{Bim}(\mathcal{C})$, we draw the corresponding 3-morphism in $\Sigma(\text{Bim}(\mathcal{C}))$ as

\[
\begin{array}{c}
A \\
\downarrow^\rho \\
B
\end{array}
\]

Since $\mathcal{C}$ is braided, by assumption, it can itself be regarded as a 3-category with a single object and a single morphism. This is the double suspension $\Sigma(\Sigma(\mathcal{C}))$ of $\mathcal{C}$. As before, we have a canonical inclusion

\[
\Sigma(\Sigma(\mathcal{C})) \subset \Sigma(\text{Bim}(\mathcal{C})).
\]

This inclusion should be thought of as analogous to the canonical inclusion

\[
\Sigma(\mathcal{C}) \subset \text{Vect}_\mathcal{C}.
\]

Notice that we may think of $\Sigma(\text{Bim}(\mathcal{C}))$ as the 3-category obtained by acting with $\text{Bim}(\mathcal{C})$ on itself. The single object then corresponds to $\text{Bim}(\mathcal{C})$ itself, a morphism colored by an algebra $A$ then corresponds to the 2-functor

\[
A \otimes \cdot : \text{Bim}(\mathcal{C}) \to \text{Bim}(\mathcal{C}),
\]

and so on.

Therefore we have a canonical embedding

\[
\Sigma(\text{Bim}(\mathcal{C})) \subset \text{Mod}_{\text{Bim}(\mathcal{C})}.
\]

I suspect that under suitable conditions the similar inclusion $\text{Bim}(\mathcal{C}) \subset \text{Mod}_\mathcal{C}$ is in fact an equivalence. It seems that Ostrik has at least shown that for well behaved $\mathcal{C}$ this inclusion is at least essentially surjective on objects.

We might even be tempted to define the well-behaved part of $\text{Mod}_\mathcal{C}$ to be that in the image of this inclusion.

Just suppose for the moment this were so. Then

\[
\text{Mod}_{\text{Bim}(\mathcal{C})} \simeq \text{Mod}_{\text{Mod}_\mathcal{C}}.
\]
and
\[ \Sigma(\text{Bim}(C)) \xrightarrow{\subset} \text{Mod}_{\text{Mod}_C}. \]
But here the right-hand side is rightly addressed as the 3-category of 3-vector spaces.

For that reason, just like we may address \( C \) itself as the canonical 1-dimensional \( C \)-module category, it seems right to address \( \text{Bim}(C) \) as the canonical 1-dimensional \( \text{Mod}_C \)-module 2-category. Or, more suggestively, as the canonical 1-dimensional 3-vector space.

Adopting this point of view, we make the following definitions, all with respect to a fixed choice of braided abelian monoidal category \( C \).

**Definition 1** A **3-vector-bundle** with connection is a transport 3-functor
\[ \mathcal{P} \rightarrow \text{Mod}_{\text{Mod}_C}. \]

Recall that we have talked about this chain of inclusions:
\[ \Sigma(\Sigma(\Sigma(C))) \xrightarrow{j} \Sigma(\text{Bim}(C)) \xrightarrow{i} \text{Mod}_{\text{Mod}_C}. \]

If \( C \) is itself already a category of modules, for instance if \( C = \text{Vect}_C = \text{Mod}_C \), we get yet another inclusion:
\[ \Sigma(\Sigma(\Sigma(C))) \xrightarrow{k} \Sigma(\text{Vect}_C) \xrightarrow{j} \Sigma(\text{Bim}(\text{Vect}_C)) \xrightarrow{i} \text{Mod}_{\text{Mod}_{\text{Vect}_C}}. \]

For each such inclusion, we get a notion of trivial, or locally trivial, 3-vector bundle.

**Definition 2** An \( i \)-trivial 3-vector bundle with connection, called a **line-3-bundle with connection**, is a transport 3-functor
\[ \mathcal{P} \rightarrow \Sigma(\text{Bim}(C)). \]

The \( i \circ j \circ k \)-trivial \( n \)-vector bundle shall be denoted by \( 1 \). It plays a role for defining the spaces of (flat) sections of a 3-vector bundle. In general, we say

**Definition 3** The 3-functor
\[ 1 : \mathcal{P} \rightarrow \Sigma(\text{Bim}) \]
is that which sends everything to the identity.

**Proposition 1** Let the domain \( \mathcal{P} \) be a 2-category, i.e. a 3-category with only identity 3-morphisms. Endomorphisms of the trivial 3-vector bundle \( 1 \) on \( \mathcal{P} \) are the same as 2-functors to \( \text{Bim}(C) \).
\[ \text{End}(1) \simeq [\mathcal{P}, \text{Bim}(C)]. \]
Proof. This will become clear, shortly. □

A degenerate but interesting example in between general line 3-bundles and the completely trivial bundle 1 are those that are \( i \circ j \)-trivial.

We shall be interested in those especially for the case where the domain \( P \) is what we call the (open, disklike) 2-particle.

**Definition 4** The 3-particle is, for the present purpose, the 2-category

\[
\text{par} = \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 k \\
 s \\
 b
\end{array}
\end{array}
\end{array}
\end{array}\right\}
\]

that consists of two objects, two nontrivial 1-morphisms and one nontrivial 2-morphism, as shown.

**Example 1** (morphisms of \((i \circ j)\)-trivial line 3-bundles over the open 3-particle)

A general line-3-bundle on par is nothing but any bimodule.

An \((i \circ j)\)-trivial line-3-bundle with connection on par is nothing but any \( \mathbb{1} \)-\( \mathbb{1} \)-bimodule, hence nothing but any object of \( \mathcal{C} \).

Let’s write

\[
1_U : \text{par} \to \Sigma(\text{Bim}(\mathcal{C}))
\]

for the \((i \circ j)\)-trivial 3-bundle with connection that assigns \( U \in \text{Obj}(\mathcal{C}) \) to \( S \):

\[
1_U : \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 k \\
 s \\
 b
\end{array}
\end{array}
\end{array}\right) \mapsto \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 1 \\
 U \\
 1
\end{array}
\end{array}
\end{array}\right) .
\end{array}\right)
\]
A morphism $\rho 1_U \to 1_V$ is a filled tin can 3-morphism

in $\Sigma(\text{Bim}(\mathcal{C}))$.

Cutting this open, this is a 3-morphism $\rho$ from

In other words, $\rho$ is a morphism from the $A \otimes \mathbb{1}-B \otimes \mathbb{1}$-bimodule $N \otimes U$ to the
$\mathbb{1} \otimes A \otimes \mathbb{1} \otimes B$-bimodule $V \otimes N'$:

$$
\begin{array}{c}
N \quad U \\
\downarrow \rho \\
V \quad N'
\end{array}
$$

All tin cans $\rho$ in $\Sigma(\text{Bim}(C))$ of this kind, with top and bottom a $\mathbb{1} \otimes \mathbb{1}$ bimodule, form a 2-category in the obvious way. We will address this as

**Definition 5** The 2-category $\text{TwBim}(C)$ of **twisted bimodules** is the 2-category of tin cans in $\Sigma(\text{Bim}(C))$ whose top and bottom are $\mathbb{1} \otimes \mathbb{1}$-bimodules,

$$
\text{TwBim}(C) \equiv \left\{ \begin{array}{c}
A \\
\downarrow \rho^U \\
N' \\
\downarrow N
\end{array} \right\}.
$$

Here

Sometimes it is useful to think of $\text{TwBim}$ as a 3-category, too. The 3-morphisms then come from composing 3-morphisms in $\Sigma(\text{Bim}(C))$ at the top and bottom of those tin cans.