Arrow-theoretic differential theory

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Abstract

We propose and study a notion of a tangent $(n+1)$-bundle to an arbitrary n-category. Despite its simplicity, this notion turns out to be useful, as we shall indicate.

Contents

1 Introduction

Various applications of (n-)categories in quantum field theory indicate that (n-)categories play an important role over and above their more traditional role as mere organizing principles of the mathematical structures used to describe the world: they appear instead themselves as the very models of this world.

For instance there are various indications that thinking of configuration spaces and of physical processes taking place in these as categories, with the configurations forming the objects and the processes the morphisms, is a step of considerably deeper relevance than the tautological construction it arises from seems to indicate.

While evidence for this is visible for the attentive eye in various modern mathematical approaches to aspects of quantum field theory – for instance [1], [2] but also [3] – the development of this observation is clearly impeded by the lack of understanding of its formal underpinnings.

If we ought to think of configuration spaces as categories, what does that imply for our formulation of physics involving these configuration spaces? In particular: how do the morphisms, which we introduce when refining traditional spaces from 0-categories to 1-categories, relate to existing concepts that must surely secretly encode the information contained in these morphisms. Like tangent spaces for instance.

Possibly one of the first places where this question was at all realized as such is Isham's [4]. That this is a piece of work which certainly most physicists currently won't recognize as physics, while mathematicians might not recognize it as interesting mathematics, we take as further indication for the need of a refined formal analysis of the problem at hand.

Several of the things we shall have to say here may be regarded as an attempt to strictly think the approach indicated in Isham's work to its end. Our particular goal here is to indicate how we may indeed naturally, generally and usefully relate morphisms in a category to the wider concept of tangency.

For instance his "arrow fields" on categories we identify as categorical tangents to identity functors on categories and find their relation to ordinary vector fields as well as to Lie derivatives, thereby, by the nature of arrow-theory, generalizing the latter concepts to essentially arbitrary categorical contexts.

While there is, for reasons mentioned, no real body of literature yet, which we could point the reader to, on the concrete question we are aiming at, the reader can find information on the way of thinking involved here most notably in the work of John Baez, the *spiritus rector* of the idea of extracting the appearance of n-categories as the right model for the notion of state and process in physics. In particular the text [7] as well as the lecture notes [8] should serve as good background reading.

On the other hand, there are indications that possibly large parts of the n-categorical picture of n-dimensional quantum field theory which we have in mind already secretly exist in disguise: some of the most sophisticated approached to quantum field theory known, like BV-quantization methods, heavily rely on L_∞-algebras as well as, dually, quasi-free differential graded algebras. But, as discussed for instance in [5], L_∞-algebras concentrated in the first n degrees are canonically equivalent to n-categorical analogues of ordinary Lie algebras – called Lie n-algebras. While equivalent, in many cases realizing the n-categorical structure behind n-term L_∞-algebras is necessary to obtain a coherent conceptual picture of what is really going on.

Here we shall demonstrate this by describing the theory of n-connections and their $(n+1)$-curvature in two parallel ways: once in the integral picture of tangent n-groupoid and once in the corresponding differential picture [6] of Lie n-algebra and Lie n-algebroids.

The work that our particular developments here have grown out is described in [9]. Our discussion of the Bianchi identity for n-functors should be compared with the similar but different constructions in the world of n-fold categories given in [10] and in the context of posets in [16].

I thank Bruce Bartlett, David Roberts, Jim Stasheff, Sean Tilson for general discussion of the notion of tangent categories described here. Special thanks go to Roberto Conti for pointing out the work by Roberts and Ruzzi to me, part of which we shall reproduce below, and to Calin Lazaroiu for pointing out his work [17] to me, which happened to have secretly some overlap with other parts of our discussion here.

2 Main results

Our working model for all concrete computations in the following is 2Cat, the Gray category whose objects are strict 2-categories, whose morphisms are strict 2-functors, whose 2-morphisms are pseudonatural transformations and whose 3-morphisms are modifications of these. It is clear that all our statements ought to have analogs for weaker, more general and higher n versions of n-categories. But with a good general theory of higher n-categories still being somewhat elusive, we won't bother to try to go beyond our model 2Cat.

So we shall now set $n = 2$ once and for all and take the liberty of using n instead of 2 in our statements, to make them look more suggestive of the general picture which ought to exist.

2.1 Tangent $(n+1)$-bundle

We define for any n-category C an n-category TC which is an $(n+1)$-bundle $p : TC \to \mathrm{Obj}(C)$ over the space of objects of C. This we address as the tangent bundle of C.

The definition of this tangent bundle is morally similar to but in detail somewhat different form the way tangent bundles are defined in synthetic differential geometry and in supergeometry:

we consider the category

$$\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \}$$

as an arrow-theoretic model for the "infinitesimal interval" or the "superpoint" in that it is a puffed-up version of the mere point

$$\mathrm{pt} := \{\, \bullet \,\}$$

to which it is equivalent, by way of the injection

$$\mathrm{pt} \hookrightarrow \mathbf{pt}\ ,$$

but not isomorphic. This suble difference, rooted deeply in the very notion of category theory, we claim usefully models the notion of tangency as "extension which hardly differs from no extension". Concretely, we consider

$$TC \subset \mathrm{Hom}_{n\mathrm{Cat}}(\mathbf{pt}, C)$$

to be that subcategory of morphisms from the fat point into C which collapses to a 0-category after pulled back to the point pt.

The characteristic property of the tangent $(n+1)$-bundle is that it sits inside the short exact sequence

$$\mathrm{Mor}(C) \to TC \to C\,.$$

For later use notice that dual to its realization as a projection

$$TC \to \mathrm{Obj}(C)$$

the tangent bundle may be thought of as an n-functor

$$TC : C^{\mathrm{op}} \to n\mathrm{Cat}$$

which sends objects a to the tangent categories T_aC over them and sends morphisms the the pullback of these along them

$$TC \ : \ (\, a \xleftarrow{f} b\,) \ \mapsto \ (\, T_aC \xrightarrow{T_fC} T_bC\,)\,.$$

2.2 G-Flows on Categories

It is important that the notion of tangent category does not *necessarily* involve the concept of the *infinitesimal*. We find that one strength of the notion of tangent categories is that it captures ordinary vector fields on manifolds just as well as exotic generalizations of these, like tangent stacks or odd vector fields as they appear in supergeometry.

The crucial property of the n-category of sections

$$\Gamma(TC)$$

of the tangent n-category is that it inherits a monoidal structure, in fact the structure of an $(n+1)$-group, through a canonical inlcusion

$$\Gamma(TC) \hookrightarrow T_{\mathrm{Id}_C}(\mathrm{Aut}(C))\,.$$

We speak of the right hand here as the *inner automorphism* $(n+1)-group$

$$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{Aut}(C))$$

of C. In this context the image of the above inclusion is denoted

$$\Gamma(TC) := \mathrm{INN}_0(C)\,.$$

We find that different notions of vector fields and their generalizations correspond to different kinds of group homomorphisms

$$v : G \to \Gamma(TC)$$

of this group of sections of the tangent category.

Ordinary vector fields correspond to smooth $\mathbb{R}$-flows on path groupoids. Tangent stacks of orbifolds correspond to $\mathbb{R}$-flows of the corresponding Lie groupoids. Images of $\mathbb{Z}_2$ and other abelian groups in the category of sections of the tangent category appear in the context of supergeometry (playing the role of certain "odd vector fields").

2.2.1 Vector fields and Lie derivatives

Let X be a smooth manifold and let $\mathcal{P}_1(X)$ be the groupoid of thin homotopy classes of paths in X.

Then ordinary vector fields $v \in \Gamma(TX)$ on X are in canonical bijection with smooth 1-parameter families of categorical tangent vectors to the identity map on $\mathcal{P}_1(X)$:

$$\Gamma(TX) \stackrel{\simeq}{\to} \left\{ \mathbb{R} \to T_{\mathrm{Id}_{\mathcal{P}_1(X)}}(\mathrm{Aut}(\mathcal{P}_1(X))) \right\} .$$

Relation to Chris Isham's work On a general category C, it may be useful to consider generalizations of this where $\mathbb{R}$ is replaced by some other group G.

The "arrow fields" on a category C, considered by Isham in [4], are $\mathbb{Z}$-flows

$$\{\mathbb{Z} \to T_{\mathrm{Id}_C}(\mathrm{End}(C))\}$$

on C.

Lie derivatives There is a differential analog of all the structures we are discussing here, with Lie n-groupoids replaced by Lie n-algebroids.

Possibly a helpful way, in this context, to appreciate the above conception of a vector field as a smooth group homomorphism

$$t \mapsto \left(\mathcal{P}_1(X) \underset{\mathrm{Ad}_{\exp(v)(t)}}{\overset{\mathrm{Id}}{\Downarrow \exp(v)(t)}} \mathcal{P}_1(X) \right)$$

integrally	$t \;\mapsto\; \mathcal{P}_1(X) \underset{\mathrm{Ad}_{\exp(v)(t)}}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathcal{P}_1(X)$ with $\Downarrow \exp(v)(t)$	$\exp(v) : \mathbb{R} \to \mathrm{INN}(\mathcal{P}_1(X))$
differentially	$\Omega^\bullet(X) \underset{L_v=[d,\iota_v]}{\overset{0}{\rightrightarrows}} \Omega^\bullet(X)$ with $\sim \Downarrow \iota_v$	$d\exp(v) \in \mathrm{inn}(\Omega^\bullet(X))$

Figure 1: **That families of categorical inner automorphisms can usefully be thought of as vector fields** is made particularly plausible by noticing that, in the differential picture corresponding to a vector field flow by inner automorphisms on the path groupoid, the Lie derivative on differential forms is a derivation connected, by Cartan's famous formula, to the 0-derivation.

is to notice that an ordinary *Lie derivative* on differential forms is itself a derivation of the graded-commutative algebra of differential forms which is, by Cartan's famous formula, connected by a chain homotopoy (given by contraction with the vector field) to the 0-derivation:

$$\Omega^\bullet(X) \underset{L_v=[d,\iota_v]}{\overset{0}{\rightrightarrows}} \Omega^\bullet(X) \quad (\sim \Downarrow \iota_v) \,.$$

More on that in [6].

2.2.2 Inner automorphism n-groups

Of particular importance are the tangent bundles, in our sense, to n-categories which are 1-object $(n-1)$-groupoids $\Sigma G_{(n)}$, hence n-groups $G_{(n)}$. In our context these $(n-1)$-groupoids must be thought of as 1-point orbifolds. Accordingly, they have just a single "tangent space" (tangent n-category)

$$T_\bullet \Sigma G_{(n)} \,.$$

This turns out to have interesting properties [15]:

- For G an ordinary group, one has that

$$T_\bullet \Sigma G \simeq T_{\mathrm{Id}_{\Sigma G}}(\mathrm{Aut}(\Sigma G))$$

is a 2-group, which we call $\mathrm{INN}(G)$. It sits inside the exact sequence

$$Z(G) \longrightarrow \mathrm{INN}(G) \longrightarrow \mathrm{AUT}(G) \longrightarrow \mathrm{OUT}(G)$$

of 1-groupoids. Here $Z(G)$ is the categorical center of ΣG (which coincides with the ordinary center of G), regarded as a 1-object groupoid. This identifies $\mathrm{INN}(G)$ as the 2-group of inner automorphisms of G.

But $\mathrm{INN}(G)$ also sits inside the exact sequence

$$G \longrightarrow \mathrm{INN}(G) \longrightarrow \Sigma G \ .$$

Moreover, it is equivalent to the trivial 2-group, hence "contractible". This identifies $\mathrm{INN}(G)$ as the categorical version of the universal G-bundle.

- For $G_{(2)}$ a strict 2-group, one finds that

$$T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Aut}(\Sigma G_{(2)}))$$

is a 3-group, which we call $\mathrm{INN}(G_{(2)})$. It sits inside the exact sequence

$$Z(G_{(2)}) \longrightarrow \mathrm{INN}(G_{(2)}) \longrightarrow \mathrm{AUT}(G_{(2)}) \longrightarrow \mathrm{OUT}(G_{(2)})$$

of 2-groupoids. Here $Z(G_{(2)})$ is the 2-categorical center of ΣG, regarded as a 1-object 2-groupoid.

Inside $\mathrm{INN}(G_{(2)})$ we have $\mathrm{INN}_0(G_{(2)})$, the image of the inclusion

$$T_\bullet \Sigma G_{(2)} \subset T_{\mathrm{Id}_{\Sigma G_{(2)}}}(2\mathrm{Cat}) .$$

This sits inside the exact sequence

$$G_{(2)} \longrightarrow \mathrm{INN}_0(G_{(2)}) \longrightarrow \Sigma G_{(2)} \ .$$

Moreover, it is equivalent to the trivial 3-group, hence "contractible". This identifies $\mathrm{INN}_0(G_{(2)})$ as the categorical version of the universal $G_{(2)}$-2-bundle.

2.2.3 Supercategories

In the context of supergeometry one encounters categories which exhibit actions by certain abelian groups, usually extensions $G \twoheadrightarrow \mathbb{Z}_2$ of $\mathbb{Z}_2$. But often more is true: there are specified graded isomorphisms relating any object with its shifted copies.

The basic example for this is the category $\mathrm{Vect}[\mathbb{Z}_2]_s$ of ordinary super-vector spaces i.e. the symmetric braided monoidal category of Z_2-graded vector spaces equipped with the unique nontrivial symmetric braiding:

$\mathbb{Z}_2$ acts on this category by changing the degree of all vector spaces $V \mapsto sV$. The existence of the canonical isomorphisms $s_V : V \to sV$ distinguishes $\mathrm{Vect}[\mathbb{Z}_2]_s$ from the generic category with a $\mathbb{Z}_2$-action.

In our arrow-theoretic differential language, we can express this by the fact that on $\mathrm{Vect}[\mathbb{Z}_2]_s$ there is an "odd vector field" in that we have a $\mathbb{Z}_2$-flow.

$$s : \mathbb{Z}_2 \to \Gamma(T\mathrm{Vect}[\mathbb{Z}_2]_s)\,.$$

One finds that the 2-category of categories with G-flow is canonically isomorphic to the 2-category of "categories with G-shifts" defined in [17] and found useful there for the discussion of categories of graded D-branes.

2.2.4 Categorical flows and quotients

The existence of a G-flow on a category C is essentially distinguished by that of a mere G-action by the fact that there exists morphisms in C (the "G-flow lines") which connect the objects that are related by the G-action.

For C any 1-category and

$$\rho : G \to \mathrm{Aut}(C)$$

an action of the group G on C by automorphisms, the *weak* coequalizer of this action

$$C \underset{\rho(g)}{\overset{\mathrm{Id}}{\rightrightarrows}} C \longrightarrow C/G$$

is the category *generated* from C and from a collection of new morphisms

$$a \xrightarrow{s_a g} \rho(g)(a)$$

for all $a \in \mathrm{Obj}(C)$ and all $g \in G$, subject to the *relations*

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle s_a(g)}\downarrow & & \downarrow{\scriptstyle s_b(g)} \\ \rho(g)(a) & \xrightarrow{\rho(g)(f)} & \rho(g)(b) \end{array}$$

and

$$\begin{array}{ccccc} & & \rho(g)(a) & & \\ & {\scriptstyle s_a(g)}\nearrow & & \searrow{\scriptstyle g'} & \\ a & & \xrightarrow[s_a(g'g)]{} & & \rho(g'g)(a) \end{array}$$

for all $g, g' \in G$ and all $f \in \mathrm{Mor}(C)$.

This is essentially the descent through the G-action as discussed in [11].

The quotient category C/G obtained this way is special in that, while it still canonically admits a G-action, now this G-action is *inner*, in that we have a G-flow

$$G \to \mathrm{INN}(C/G)\,.$$

In [17] C/G is called the "skew category" associated with the category C with G action.

2.3 Curvature and Bianchi Identity for functors

We give a purely arrow-theoretic definition of the differential of an arbitrary n-functor, show that this notion satisfies some of the general properties one would wish such a differential to satisfy and demonstrate that in the suitable special case it does indeed reproduce the exterior derivative on differential forms, including its nonabelian generalizations.

2.3.1 General functors

Using the functorial incarnation $TC : C^{\mathrm{op}} \to n\mathrm{Cat}$ of the tangent bundle, we may push forward any n-functor

$$F : C \to D$$

to a connection on the tangent bundle of C, simply by postcomposing

$$\delta F : \; C \xrightarrow{F} D \xrightarrow{\quad TD \quad} n\mathrm{Cat}\,.$$

The crucial point of this construction is that it extends *uniquely* (up to equivalence) to an $(n+1)$-functor

$$\delta F : C_{(n+1)} \to n\mathrm{Cat}$$

on the $(n+1)$-category

$$C_{(n+1)} := \mathrm{Codisc}(C)$$

which is obtained from C by replacing all Hom-$(n-1)$-categories by the corresponding codiscrete n-groupoids over them.

By introducing the terminology

- δF is the curvature of F
- F is flat if δF is degenerate (sends all $(n+1)$-morphisms to identities)

we obtain the technically easy but conceptually important generalization of the *Bianchi identity*: for any functor F

- δF is flat

or equivalently

- $\delta\delta F$ is degenerate .

2.3.2 Curvature and inner automorphisms

By combining our defintion of curvature of an n-functor using the arrow-theoretic differential with the discussion in 2.2.2 we arrive at the statement that the curvature of an n-functor

$$\mathrm{tra} : C \to \Sigma G_{(n)}$$

taking values in an n-group $G_{(n)}$ is an $(n+1)$-functor

$$\mathrm{curv} = \delta\mathrm{tra} : C \to \Sigma\mathrm{INN}_0(G_{(n)})$$

taking values in the inner automorphism $(n+1)$-group of $G_{(n)}$.

Accordingly, the curvature of the curvature, hence the Bianchi identity, takes values in $\mathrm{INN}_0\mathrm{INN}_0(G_{(n)})$.

Comparison with Roberts-Ruzzi. For $n = 1$ almost this statement had been proposed in [16] as the right framework for discussing curvature and Bianchi identities.

The 1-, 2- and 3-groupoids $1G$, $2G$ and $3G$ which they use are the quotients of our inner automorphism $(n+1)$-froups by the respective categorical centers

$$1G = \Sigma G \quad (1)$$

$$2G = \Sigma(\mathrm{INN}(G)/Z(G)) \quad (2)$$

$$3G = \Sigma(\mathrm{INN}_0(2G)/Z(2G)) \,. \quad (3)$$

We shall indicate in 2.4 that by forming these quotients here one loses some crucial properties.

We now describe how our abstract arrow-theoretic notion of curvature does reproduce the familiar one on differential forms in suitable smooth context.

2.3.3 Parallel transport functors and differential forms

When $F : C \to D$ is the smooth parallel transport functor [11] in an n-bundle with connection [12, 13, 14], the arrow-theoretic notion of curvature described above does reproduce the theory of curvature forms of connection forms. The general Bianchi identity we have discussed then reduces to the ordinary Bianchi identity familiar from differential geometry.

More precisely, let $C := \mathcal{P}_3(X)$ be the strict 3-groupoid of thin homotopy classes of k-paths in a smooth manifold X. And let $G_{(2)}$ be a strict Lie 2-group coming from the Lie crossed module $H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$.

Then, according to [13, 14, 15, 18] we have the following bijections of smooth n-functors with differential forms

- $\{\text{smooth 1-functors } \mathcal{P}_1(X) \to \Sigma G\ \} \xrightarrow{\simeq} \{A \in \Omega^1(X, \mathrm{Lie}(G))\}$
- $\{\text{smooth 2-functors } \mathcal{P}_2(X) \to \Sigma G_{(2)}\ \} \xrightarrow{\simeq} \left\{ \begin{array}{l} (A, B) \in \Omega^1(X, \mathrm{Lie}(G)) \times \Omega^2(X, \mathrm{Lie}(H)) \\ F_A + t_* \circ B = 0 \end{array} \right\}$

- $\left\{\text{smooth 3-functors } \mathcal{P}_3(X) \to \Sigma\mathrm{INN}(G_{(2)})\right\} \xrightarrow{\sim} \left\{ \begin{array}{ll} (A,B,C) \in & \Omega^1(X, \mathrm{Lie}(G)) \\ & \times\Omega^2(X,\mathrm{Lie}(H)) \\ & \times\Omega^3(X,\mathrm{Lie}(H)) \\ C = d_A B & \end{array} \right\}$

Now let

$$\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma G$$

be a smooth 1-functor with values in the Lie group G. Then, under these bijections, we find that its curvatures correspond to the following differential forms at top level

$$\begin{array}{ccc} \mathrm{tra} & \mapsto & A \\ \mathrm{curv} = \delta\mathrm{tra} & \mapsto & F_A := dA + A \wedge A \\ \delta\mathrm{curv} = \delta\delta\mathrm{tra} & \mapsto & d_A F_A = 0 \end{array} .$$

This way the ordinary Bianchi identity for the curvature 2-form F_A of A is reproduced. Notice that for this result come out the way it does, just by turning our abstract crank for differential arrow-theory, the result of 2.2.2 is crucial, which says that the curvature $(n+1)$-functor of a $G_{(n)}$-transport is itself an $\mathrm{INN}(G_{(n)})$-transport.

2.4 Sections and covariant derivatives

The curvature curv $= d$tra of a parallel transport n-functor is typically trivializable, in that it admits morphisms

$$e : I \xrightarrow{\sim} \mathrm{curv}$$

for I some "trivial" $(n+1)$-transport. As with the inner automorphism $(n+1)$-groups $\mathrm{INN}(G_{(n)})$, this trivializability, far from making these objects uninteresting, turns out to control the entire theory.

(Compare this to the contractibility of the universal G-bundle: while equivalent to a point, it is far from being an uninteresting object, due to the morphisms which go into and out of it. According to 2.2.2, this comparison is far more than an mere analogy.)

A basic fact of n-category theory has major implications here:
recall that for F and G n-functors, a transformation

$$G \to F$$

is given in components itself by an $(n-1)$-functor. Now if G is trivial in some sense to be made precise, and if the transformation is an equivalence

$$I \xrightarrow[\sim]{f} F ,$$

then this implies that the n-functor F is entirely encoded in the $(n-1)$-functor f.

We show that for $F = \mathrm{curv} = \delta\mathrm{tra}$ the curvature $(n+1)$-functor of a transport n-functor tra, the latter essentially encodes the component map of the transformation

$$I \xrightarrow[\sim]{\mathrm{tra}} \mathrm{curv} \; .$$

In components this is nothing but a generalization of Stokes' law

$$\int_X d\omega = \int_{\partial X} \omega \, .$$

Moreover, it turns out that there may be other trivializations of curv, not by isomorphisms but by mere equivalences. On objects, the component functions of these correspond to sections of the original bundle. On morphisms it corresponds, under the identification of smooth functors and differential forms mentioned in **??**, to the covariant derivative of these sections.

In [9] it is indicated how all these statements have a quantum analogue as we push our n-functors forward. There it is indicated how the fact that transport n-functors have sections which are themselves transport $(n-1)$-functors translates in the context of extended functorial quantum field theory to essentially what is known in physics as the holographic principle. This needs to be discussed elsewhere, clearly.

3 Differential arrow theory

3.1 Tangent categories

Definition 1 (the point)**.** *The point is the n-category*

$$\mathrm{pt} := \{\bullet\}$$

with a single object and no nontrivial morphisms. We shall carefully distinguish this from the n-category

$$\mathbf{pt} := \{\, \bullet \xrightarrow{\sim} \circ \,\}\,,$$

consisting of two objects connected by a 1-isomorphism.

The category **pt** might be called the "fat point". It is of course equivalent to the point – but not isomorphic. We fix one injection

$$i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$$

$$i : \bullet \mapsto \bullet$$

once and for all.

It is useful to think of morphisms

$$\mathbf{f} : \mathbf{pt} \to C$$

from the fat point to some codomain C as labeled by the corresponding image of the ordinary point

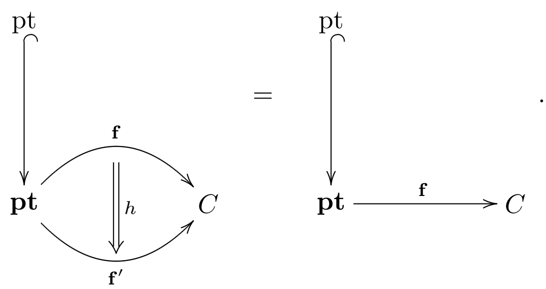
$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{f} & C \\ \big\downarrow & & \big\downarrow {\scriptstyle =} \\ \mathbf{pt} & \xrightarrow{\mathbf{f}} & \mathrm{C} \end{array} .$$

3.1.1 The tangent $(n+1)$-bundle as a projection

Definition 2 (tangent $(n+1)$-bundle)**.** *Given any n-category C, we define its tangent $(n+1)$-bundle*

$$TC \subset \mathrm{Hom}_{n\mathrm{Cat}}(\mathbf{pt}, C)$$

to be that sub n-category of morphisms from the fat point into C which collapses to a 0-category when pulled back along the fixed inclusion $i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$: the *morphisms h in $T\,C$ are all those for which*



The tangent $(n+1)$-bundle is a disjoint union

$$TC = \bigoplus_{x \in \mathrm{Obj}(C)} T_x C$$

of tangent n-categories at each object x of C. In this way it is an $(n+1)$-bundle

$$p : TC \longrightarrow \mathrm{Disc}(C)$$

over the space of objects of C.

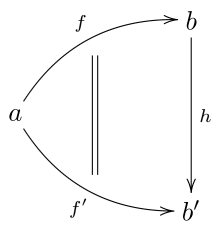
Example (slice categories). For C any 1-groupoid, i.e. a strict 2-groupoid with only identity 2-morphisms, its tangent 1-category is the comma category

$$TC = ((\mathrm{Disc}(C) \hookrightarrow C) \downarrow \mathrm{Id}_C)\,.$$

This is the disjoint union of all co-over categories on all objects of C

$$TC = \bigoplus_{a \in \mathrm{Obj}(C)} (a \downarrow C)$$

Objects of TC are morphisms $f : a \to b$ in C, and morphisms $f \xrightarrow{h} f'$ in TC are commuting triangles

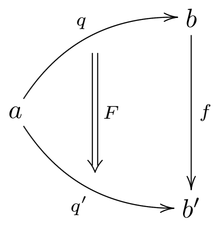


in C.

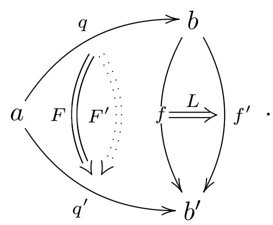
Example (strict tangent 2-groupoids). The example which we are mainly interested in is that where C is a strict 2-groupoid. For a any object in C, an object of T_aC is a morphism

$$a \xrightarrow{q} b\,.$$

A 1-morphism in T_aC is a filled triangle



in C. Finally, a 2-morphism in T_aC looks like



The composition of these 2-morphisms is the obvious one.

Proposition 1. *The tangent $(n+1)$-bundle of C fits into the short exact sequence*

$$\mathrm{Mor}(C) \to TC \to C\,.$$

Proposition 2. *The tangent $(n+1)$-bundle TC is a "deformation retract" of the underlyin space of objects in that it is equivalent to the discrete n-category over the set of objects of C*

$$TC \simeq \mathrm{Obj}(C)\,.$$

This equivalence goes through also if everything is internal to smooth spaces.

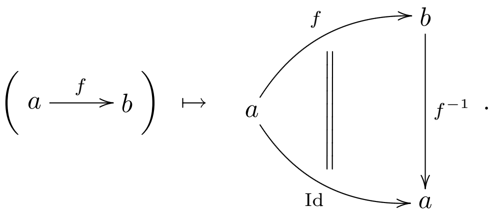
Proof. This is to say that each tangent n-groupoid T_aC, for all $a \in \mathrm{Obj}(C)$, is equivalent to the trivial n-category

$$T_aC \simeq \mathrm{pt}\,.$$

To establish this, it is sufficient to exhibit an invertible transformation from the identity on T_aC to the functor which sends everything to the identity morphism on a. Now making explicit use of our assumption that we are working with strict 2-categories, we can take the components of this on objects

$$a \xrightarrow{f} b$$

to be simply the morphism



There is then a unique choice for the component of the transformation on morphisms. □

Remark. This result shows in which sense one should think of our tangent n-categories: the right intuition is to think of the space of objects of C as the space in question, and of the morphisms and higher morphisms of C as encoding "tangency relations" among these objects. This means that the way we are to think here of *(n-)categories as spaces* is the way in which in the theory of stacks orbifolds are regarded as groupoids, rather than, say, the identification of categories with spaces induced by the nerve construction.

3.1.2 The tangent $(n+1)$-bundle as a fiber-assigning functor

The tangent $(n+1)$-bundle of a category comes equipped with a canonical parallel transport over the opposite of the category:

Definition 3. *For any n-category C, let*

$$TC : C^{\mathrm{op}} \to n\mathrm{Cat}$$

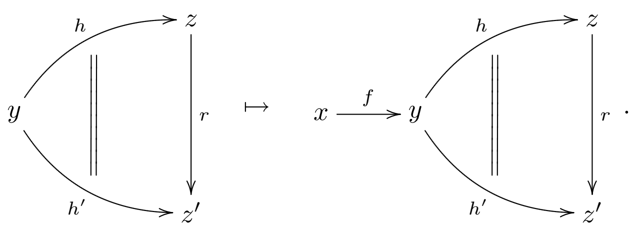
be the n-functor which sends

$$x \mapsto T_x C$$

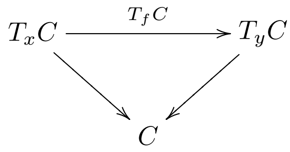
for each $x \in \mathrm{Obj}(C)$ and which sends a morphism $x \xrightarrow{f} y$ to the n-functor

$$T_f C : T_y C \to T_x C$$

given by



Proposition 3. *This functor respects the morphism $TC \to C$ from proposition 1 in that*



commutes for all morphisms f in C.

This way we even get an n-functor $T : \mathrm{Cat} \to T^*_{\mathrm{Cat}}(n\mathrm{Cat})$ as follows.

Definition 4. *Define an n-functor*

$$T : \mathrm{Cat} \to T^{\mathrm{op}}_{\mathrm{Cat}}(n\mathrm{Cat})$$

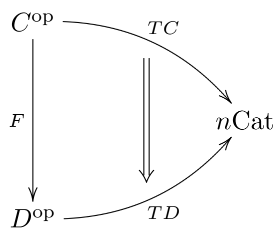
as follows. It sends any n-category C to the morphism

$$TC : C^{\mathrm{op}} \to n\mathrm{Cat}\,.$$

It sends any n-functor

$$F : C \to D$$

to the triangle



,

with the only obvious transformation filling this triangle.

The fact mentioned in proposition 2, that

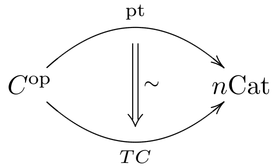
$$TC \simeq \mathrm{Obj}(C)$$

reads in terms of the functor $TC : C^{\mathrm{op}} \to \mathrm{Cat}$ as follows.

Proposition 4. *For any n-groupoid C, let*

$$\mathrm{pt} : C^{\mathrm{op}} \to n\mathrm{Cat}$$

be the n-functor which sends everything to the identity on the n-category pt*. Then there is an equivalence*



.

3.2 Differentials of functors

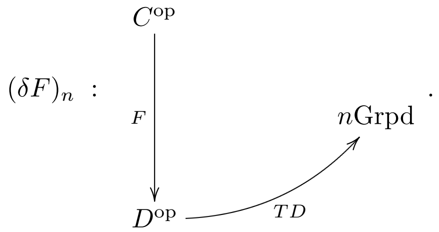
For any n-functor

$$C \xrightarrow{F} D$$

with D an n-groupoid, let

$$C_{(n+1)} := \mathrm{Codisc}(C)$$

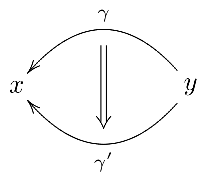
be the $(n+1)$-category whose Hom-categories are the codiscrete n-groupoids over the Hom-$(n-1)$-categories of C (i.e. with a unique n-morphism for every pair of parallel $(n-1)$-morphisms) and set



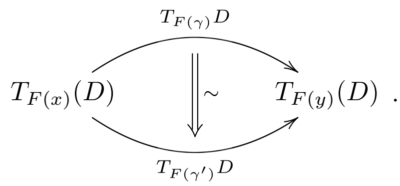
Proposition 5. *$(\delta F)_n$ extends essentially uniquely to an $(n+1)$-functor*

$$\delta F : C_{(n+1)} \to n\mathrm{Grpd}\,.$$

Proof. As a warmup, to see the idea, consider $n = 1$. Then for any two parallel morphisms



in C we need a transformation



By writing out what this must be like, and using the fact that D is assumed to be an n-groupoid, one finds that there is, up to equivalence, only one possible choice.

The case $n = 2$ is entirely analogous. □

Notice that δ commutes with pullbacks: given an n-functor $p : C' \to C$ and a n-functor $F : C \to D$ we write

$$p^*F : \; C' \xrightarrow{p} C \xrightarrow{F} D$$

and call this the pullback of F along p. Then we trivially have the following important statement:

Proposition 6. *The operation δ commutes with pullbacks.*

Hence

$$p^*(\delta F)_n = \delta(p^*F)_n$$

for all p and all F.

Definition 5 (curvature)**.** *We say that*

- *δF is the curvature of F.*
- *An n-functor is degenerate if it sends all n-morphisms to identity n-morphisms.*
- *An n-functor F is flat if its curvature $(n+1)$-functor δF is degenerate.*

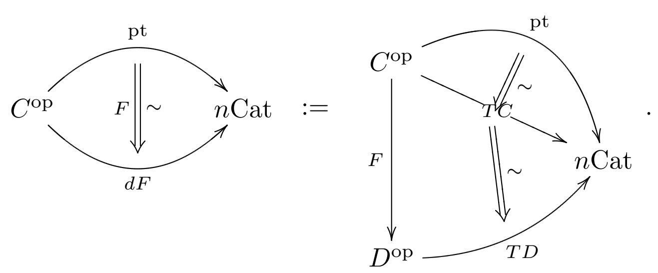
Proposition 7 (Bianchi identity)**.** *Let $F : C \to D$ be a 1-functor with values in a groupoid D. Then*

- *δF is flat*

or equivalently

- *$\delta\delta F$ is degenerate.*

Definition 6 (Stokes' theorem)**.** *The curvature $(n+1)$-functor is canonically trivialized by the functor it comes from, by combination of the transformations from definition 4 and proposition 4:*



3.3 Sections of functors

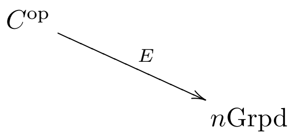
In our context, we may think of an n-functor

$$F : C \to D$$

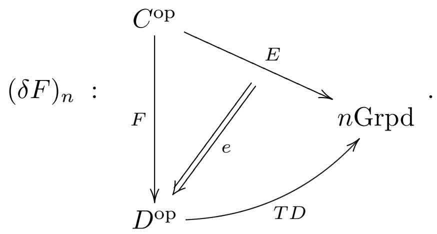
as encoding [11, 13, 14] an n-bundle with connection on $\mathrm{Obj}(C)$. From this point of view one is interested in determining the *space of n-sections* of this n-bundle, as well as the covariant derivative of these sections with respect to the given n-connection.

Definition 7 (sections)**.** *A section of F is a morphism into δF .*

Hence a section of an n-functor is a pair, consisting of an $n+1$-functor $F : C \to D$



together with a transformation



4 Parallel transport functors and their curvature

4.1 Principal parallel transport

4.1.1 Trivial G-bundles with connection

Let S be a category playing the role of the path groupoid of some space.

For us, a trivial G-bundle with connection on S is a functor

$$\mathrm{tra} : S \to \Sigma G$$

which we interpret as sending each morphism in S to the parallel G-transport along it.

The differential of this functor we may think of as the curvature of the parallel transport

$$\mathrm{curv} = d\mathrm{tra}$$

in that the component map of the transformation which it assigns to a 2-morphism Σ is essentially the parallel transport $\mathrm{tra}(\partial\Sigma)$ around the boundary of that 2-morphism.

Notice that curv sends every point to a fiber of the form

$$T_\bullet \Sigma G = \mathrm{INN}(G)$$

over it. Moreover, by proposition 3 this assignment respects the morphism $\mathrm{INN}(G) \to \Sigma G$

$$\begin{array}{ccc} \mathrm{INN}(G) & \xrightarrow{\mathrm{curv}(\gamma)} & \mathrm{INN}(G) \\ & \searrow \quad \swarrow & \\ & \Sigma G & \end{array} .$$

It turns out that curv is trivializable in various ways. There is interesting information in the morphisms that trivialize it.

To see this, consider two special bundles with connection over S, the trivial point bundle

$$J : S \longrightarrow \{\bullet\} .$$

and the trivial G-bundle with trivial connection

$$I : S \longrightarrow \{\bullet\} \longrightarrow \Sigma G .$$

Proposition 8. *There is an isomorphism (not just an equivalence)*

$$S_2 \underset{\mathrm{curv}}{\overset{dI}{\Downarrow \sim}} \mathrm{Cat}$$

whose component map is essentially a map $\mathrm{Mor}(S) \to G$ *and as such coincdes with* tra*.*

Hence we write, with conventient and suggestive abuse of notation,

$$dI \xrightarrow[\sim]{\mathrm{tra}} \mathrm{curv} .$$

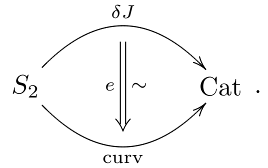
Gauge equivalent transport functors

$$g : \mathrm{tra} \to \mathrm{tra}'$$

correspond to different but equivalent trivializations of the same curvature 2-functor

$$dI \underset{\mathrm{tra}'}{\overset{\mathrm{tra}}{g \Downarrow \sim}} \mathrm{curv} .$$

There may also be trivialization of curv which correspond to sections of the original bundle. These come from equivalences of curv with the trivial point bundle J



Their component maps pick out an element of G over each object of S.

5 The last section

in which I admit that I am running out of steam for the moment.

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