Tangent Categories

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September 5, 2007

Abstract

For any n-category C we consider the sub-n-category $TC \subset C^2$ of squares in C with pinned left boundary. This resolves the space of objects in C in a natural way. We describe various properties of TC and indicate why it deserves to be addressed as the tangent n-category of C.

Contents

1 Introduction

URS: Instead of a real introduction, at the moment I offer only the following reflection.

Tangent categories play two different important roles, which a priori seem to be rather unrelated:

- The tangent n-category TC is a puffed up version of the space of objects C_0. For C an n-groupoid, the canonical inclusion

$$C_0 \to TC$$

is an equivalence. The canonical sequence

$$\mathrm{Mor}(C) \to TC \to C$$

is the n-groupoid incarnation of the universal C-bundle [2].

- At the same time, TC does know about the tangency relations on C_0 induced by $\mathrm{Mor}(C)$: for C an n-groupoid, G-flows

$$\Gamma_G(TC) := \{G \to \Gamma(TC) \subset T_{\mathrm{Id}_C}(\mathrm{End}(C))\}$$

do provide a generalization of the concept of vector fields on C_0 in that for $G = \mathbb{R}$ and with everything taken to be smooth we have that sections

$$\Gamma_{\mathbb{R}}(TC) \simeq \Gamma(\mathrm{Lie}(C))$$

do coincide with the sections of the Lie n-algebroid associated with C.

The apparent dichotomy – universal C spaces on one hand, differentials on C on the other – is resolved by noticing that TC is actually to be regarded as the *universal C-bundle equipped with the universal C-connection* [3].

2 Tangent Categories

2.1 Definition

We write

$$\mathrm{pt} := \{\bullet\}$$

for the terminal category and

$$2 := \{\bullet \to \circ\}$$

for the category with two objects and one nontrivial morphism, going between them. Then for C any category, we have the category

$$C^2 := \mathrm{Hom}_{\mathrm{Cat}}(2, C)$$

of commuting squares in C, with composition being the vertical pasting of squares. C^2 has two obvious projections onto C

$$C^2 \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} C$$

which may be thought of as arising from pullback along the two injections

$$\mathrm{pt} \underset{\circ}{\overset{\bullet}{\rightrightarrows}} 2$$

in that

$$\mathrm{dom} \;:\; \left(2 \underset{a_2}{\overset{a_1}{\underset{\Downarrow f}{\rightrightarrows}}} C \right) \mapsto \left(\mathrm{pt} \xrightarrow{\bullet} 2 \underset{a_2}{\overset{a_1}{\underset{\Downarrow f}{\rightrightarrows}}} C \right).$$

Definition 1 (tangent category) *For C any category, its tangent category TC is defined to be the strict pullback*

$$\begin{array}{ccc} TC & \longrightarrow & C^2 \\ \downarrow & & \downarrow^{\mathrm{dom}} \\ C_0 & \xrightarrow{i_C} & C \end{array}$$

in Cat*.*

Here $C_0 := \mathrm{Obj}(C)$ is regarded as a discrete category and $i_C : C_0 \to C$ sends objects to identity endomorphisms.

Hence TC is the co-slice category

$$TC = \bigoplus_{a \in \mathrm{Obj}(C)} (a \downarrow C)\,.$$

Objects of TC are morphisms $f : a \to b$ in C, and morphisms $f \xrightarrow{h} f'$ in TC are commuting triangles

$$\begin{array}{ccc} & & b \\ & \overset{f}{\nearrow} & \\ a & \| & \downarrow^{h} \\ & \underset{f'}{\searrow} & \\ & & b' \end{array}$$

in C.

Remark. The definition of the tangent category is an exact analogue of the sum of path spaces, construed as a pullback in Top:

$$\begin{array}{ccc} PX & \longrightarrow & X^I \\ \downarrow & & \downarrow^{\mathrm{eval}_0} \\ |X| & \longrightarrow & X \end{array}.$$

Here $I = [0,1]$ is the interval, eval_0 is evaluation at the left boundary of the interval and $|X|$ is the set underlying the topological space X, equipped with the discrete topology.

That and how TC is still usefully thought of as a *tangent* bundle is discussed in 4.

Definition 2 (tangent category functor) *We write*

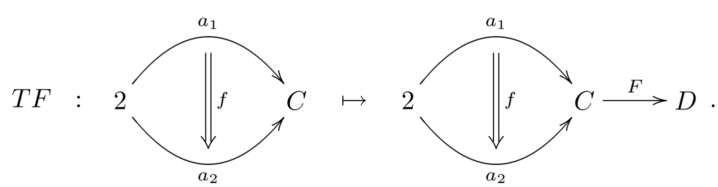
$$T : \mathrm{Cat} \to \mathrm{Cat}$$

for the corresponding functor.

Hence for $F : C \to D$ any functor, the functor

$$TF : TC \to TD$$

acts by postcomposition with F, in that



2.2 Inner automorphisms

For C any category, the categorical tangent space

$$T_{\mathrm{Id}_C}(\mathrm{End}(C))$$

in $\mathrm{End}(C)$ at the identity endomorphism plays a special role. It makes good sense to address these endomorphisms connected to the identity as *inner* endomorphisms.

Definition 3 *For C any groupoid, we address*

- $$\mathrm{AUT}(C) := \mathrm{Aut}_{\mathrm{Cat}}(C)$$
  as the automorphism 2-group of C;

- $$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{End}(C))$$
  as the inner automorphisms 2-group of C

- $$Z(C) := \Sigma\mathrm{End}_{\mathrm{Id}_C}$$
  as the center of C

- $$\mathrm{OUT}(C) := \mathrm{coker}(\mathrm{INN}(C) \hookrightarrow \mathrm{AUT}(C))$$

  as the outer automorphism 2-group of C.

URS: I need to think about how to define $\mathrm{OUT}(C)$ properly.
These fit into an exact sequence of 2-groups

$$Z(C) \to \mathrm{INN}(C) \to \mathrm{AUT}(C) \to \mathrm{OUT}(C)\,.$$

2.3 Properties

Proposition 1 *The discrete category over the space* $\mathrm{Mor}(C)$ *arises as the pullback*

$$\begin{array}{ccccc} \mathrm{Mor}(C) & & \xrightarrow{\mathrm{cod}} & & C_0 \\ \downarrow & & & & \downarrow{\scriptstyle i_C} \\ TC & \longrightarrow & C^2 & \xrightarrow{\mathrm{cod}} & C \end{array} \quad .$$

We may read that as a "short exact sequence"

$$\mathrm{Mor}(C) \to TC \to C\,.$$

Proposition 2 *When C is a groupoid, then*

$$TC \simeq C_0$$

and in fact the projection

$$TC \to C_0$$

is weakly inverse to the canonical section

$$C_0 \to TC\,.$$

Definition 4 *We write*

$$\Gamma(TC)$$

for the category of sections

$$e : C_0 \to TC$$

of $TC \to C_0$.

Proposition 3 *When C is a groupoid, then we have a canonical equivalence (isomorphism, even)*

$$\Gamma(TC) \simeq T_{\mathrm{Id}_C}(\mathrm{End}(C))\,.$$

Remark. Recall from 2.2 that this equips $\Gamma(TC)$ with a monoidal structure, even the structure of a 2-group

$$\mathrm{INN}(C) := T_{\mathrm{Id}}(\mathrm{End}(C)) \,.$$

This monoidal structure is crucial in 4 and for relating tangent categories to ordinary notions of tangent spaces.

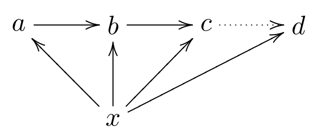
Notice that for $n > 1$ one finds that

$$\Gamma(TC) \to T_{\mathrm{Id}_C}(\mathrm{End}(C))$$

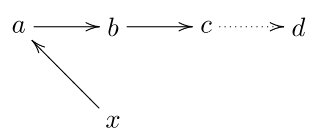
is no longer an equivalence but becomes a proper inclusion.

2.4 Simplicial aspects

Notice that a sequence of k composable morphisms



in TC is, since all triangles commute, the same as a sequence



of $k + 1$ composable morphisms in C.

This means that when passing to nerves, the tangent functor T becomes what is known as *décalage* in the simplicial context:

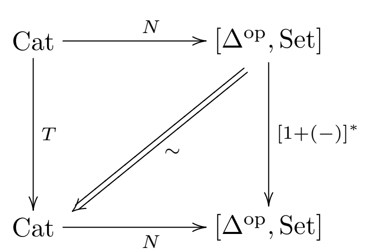
Definition 5 *Denote by*

$$[1 + (-)] : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$$

the obvious functor which acts on objects as

$$[n] \mapsto [n + 1] \,.$$

Proposition 4 *We have a weakly commuting square*



Proof. Notice that n-simplices in TC are commuting squares

$$\begin{array}{ccc}
[n] & \longrightarrow & 2 \times [n] \\
\downarrow & & \downarrow \\
[0] & \longrightarrow & C
\end{array}$$

but the pushout of this co-cone is $[n+1]$

$$\begin{array}{ccc}
[n] & \longrightarrow & 2 \times [n] \\
\downarrow & & \downarrow{\scriptstyle f} \\
[0] & \longrightarrow & [n+1]
\end{array} \;,$$

where

$$f \;:\; \begin{array}{ccccccc}
0 & \to & 1 & \to & 2 & \dashrightarrow & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
x_0 & \to & x_1 & \to & x_2 & \dashrightarrow & x_k
\end{array} \;\mapsto\; \begin{array}{ccccccc}
0 & \to & 1 & \to & 2 & \dashrightarrow & k \\
& \nwarrow & \uparrow & \nearrow & & \nearrow & \\
& & x_0 & & & &
\end{array} \;\mapsto\; \begin{array}{ccccccc}
a & \to & b & \to & c & \dashrightarrow & d \\
& \nwarrow & & & & & \\
& & x & & & &
\end{array} \;.$$

Hence we functorially assign $(n+1)$-simplices in C to n-simplices in TC

$$\begin{array}{ccccc}
[n] & & \longrightarrow & & 2 \times [n] \\
\downarrow & & & \swarrow{\scriptstyle f} & \downarrow \\
& & [n+1] & & \\
\downarrow & \nearrow & & \searrow{\scriptstyle !} & \downarrow \\
[0] & & \longrightarrow & & C
\end{array} \;.$$

□

3 Tangent n-Categories

3.1 Strict tangent 2-Categories

4 Arrow-fields and Flows on n-Categories

Isham [1] coined the term *arrow field* on a category C for what we conceive as a section

$$e \in \Gamma(TC)\,,$$

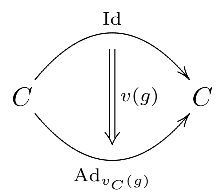
thinking of it as a model for a tangent vector field on C_0. On the other hand, such an e is far from being "infinitesimal" in any sense. We shall now make use of the monoidal structure on $\Gamma(TC)$ – also noticed by Isham – to obtain a sensible notion of categorical vector fields. We exhibit the special cases in which this reproduces ordinary sections of ordinary vector bundles.

4.1 G-flows

Definition 6 *For C any groupoid and G any group, we address a group homomorphism*

$$v : G \to \mathrm{INN}(C)$$

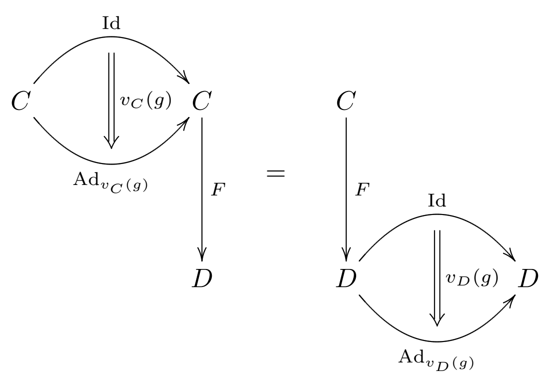
as a G-flow on C. For each $g \in G$ we write



for the corresponding element in INN(G)*. A morphism between two categories C and D equipped with G-flows is a functor*

$$F : C \to D$$

which respects the flows in that



for all $g \in G$.

We write

$$\Gamma_G(TC) := \mathrm{Hom}(G, \mathrm{INN}(C))$$

for the collection of G-flows on C.

Example (ordinary vector fields). Let X be a smooth manifold and $\Pi_1(X)$ its fundamental groupoid. Smooth $\mathbb{R}$-flows on $\Pi_1(X)$ are in canonical bijection with ordinary vector fields on X

$$\Gamma(TX) \simeq \Gamma_{\mathbb{R}}(T\Pi_1(X))\,.$$

(Here on the left TX denotes the ordinary tangent bundle of X.)

Example (odd vector fields). Let sVect be the category of super vector spaces. The parity shift operator

$$\Pi : \mathrm{sVect} \to \mathrm{sVect}$$

characterized by the fact that

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \Pi V & \xrightarrow{\Pi f} & \Pi W \end{array}$$

for all morphisms $V \xrightarrow{f} W$ with $V \xrightarrow{\sim} \Pi V$ the canonical isomorphism manifestly is a $\mathbb{Z}_2$-flow on sVect, hence an element

$$\Pi \in \Gamma_{\mathbb{Z}_2}(T\mathrm{sVect})\,.$$

It might be useful to think of Π as the "flow of an odd vector field" in supergeometry.

Example (Lie algebroids). For C any Lie groupoid with Lie algebroid $\mathrm{Lie}(C)$, we have

$$\Gamma_{\mathbb{R}}(TC) \simeq \Gamma(\mathrm{Lie}(C))\,.$$

The canonical morphism

$$C \to \mathrm{codisc}(C_0)$$

induces the anchor map

$$\Gamma_{\mathbb{R}}(TC) \to \Gamma_{\mathbb{R}}(T\Pi_1(C_0))\,.$$

The Lie bracket on sections is obtained from the group commutator in $\mathrm{INN}(C)$ in the usual way.

5 Universal n-Bundles

5.1 Principal 1-Bundles with connection

For the following, let

$$C := \Sigma G$$

be the one-object groupoid given by a group G.

The sequence

$$\mathrm{Mor}(C) \to TC \to C$$

which each tangent category sits in then becomes

$$G \to T\Sigma G \to \Sigma G\,,$$

which we also frequently denote

$$G \to \mathrm{INN}(G) \to \Sigma G\,.$$

This sequence is the universal G-bundle in the world of groupoids.

Proposition 5 (Segal) *The geometric realization of the nerve of*

$$G \to T\Sigma G \to \Sigma G$$

is a model for the universal G-bundle

$$\begin{array}{ccccc} G & \longrightarrow & T\Sigma G & \longrightarrow & \Sigma G \\ \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} \\ G & \longrightarrow & EG & \longrightarrow & BG \end{array}\,.$$

But here we shall find it useful not to pass to spaces by realizing nerves. The entire discussion can usefully be done entirely within the world of groupoids.

For X some space, choose a good cover

$$\pi : Y \to X$$

and denote by

$$Y^{[2]} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y$$

the corresponding groupoid. Noticing that $|Y^{[2]}| \simeq X$ we may take this as a groupoid model of X.

Proposition 6 *Equivalence classes of principal G-bundles on X are in bijection with equivalence classes*

$$f \in [Y^{[2]}, \Sigma G]/_\sim$$

of functors.

Proof. By unwrapping the relevant definitions one finds that functors $Y^{[2]} \to \Sigma G$ are precisely G-cocycles on X, while transformations of these functors are precisely isomorphisms of G-cocycles. $\square$

We may hence regard $f : Y^{[2]} \to \Sigma G$ as a classifying map. By pulling this back along the groupoid version of the universal G-bundle

$$\begin{array}{ccc} & & G \\ & & \downarrow \\ & & \mathrm{INN}(G) \\ & & \downarrow \\ Y^{[2]} & \xrightarrow{f} & \Sigma G \end{array}$$

we obtain the groupoid version of the total space

$$\begin{array}{ccc} Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) & \longrightarrow & \mathrm{INN}(G) \\ \downarrow & & \downarrow \\ Y^{[2]} & \xrightarrow{f} & \Sigma G \end{array}$$

of the G-bundle classified by f

But there is more. Since for $C = \Sigma G$ we have that TC is again itself a 2-group, we may iterate the tangent category construction to obtain

$$\begin{array}{ccccc} \mathrm{Mor}(C) & \longrightarrow & TC & \longrightarrow & C \\ \downarrow & & \downarrow & & \\ TC & \longrightarrow & T\Sigma TC & \longrightarrow & \Sigma TC \\ \downarrow & & \downarrow & & \\ C & & \Sigma TC & & \end{array} .$$

This does not close strictly, but up to pseudonatural transformation

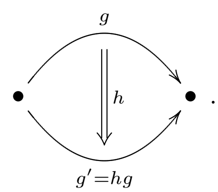
$$\begin{array}{ccccc} \mathrm{Mor}(C) & \longrightarrow & TC & \longrightarrow & C \\ \downarrow & & \downarrow & \swarrow\!\!\!\swarrow \simeq & \downarrow \\ TC & \longrightarrow & T\Sigma TC & \longrightarrow & \Sigma TC \\ \downarrow & \swarrow\!\!\!\swarrow \simeq & \downarrow & & \\ C & \longrightarrow & \Sigma TC & & \end{array} .$$

Here the sequence in the middle is the universal $\mathrm{INN}(G)$-2-bundle.

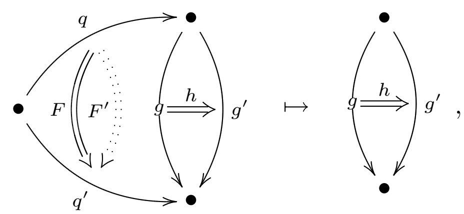
We may regard

- ΣTC as the fundamental 2-groupoid of the universal G-bundle;
- $T\Sigma TC$ as the pair groupoid $EG \times EG$ of the fundamental G-bundle, pulled back to the fundamental 2-groupoid, such that after diving out G it becomes the Atiyah groupoid $\mathrm{At}(EG) := EG \times_G EG$ pulled back to the fundamental 2-groupoid.

A 2-morphism in ΣTC looks like



A 2-morphism in $T\Sigma TC$ covering this looks like



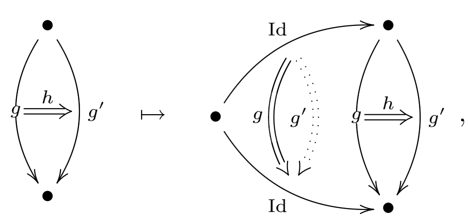
where all labels are elements in G. Here one should think of q and q' as elements in the fiber of EG over the chosen base point $\bullet$, and of F and F' as two choices of fiber isomorphisms over the paths g and g', respectively.

Since $TC = \mathrm{INN}(G)$ is equivalent to the trivial 2-group, its universal 2-bundle $T\Sigma TC$ is trivializable, and we have a canonical section

$$\Sigma TC \to T\Sigma TC\,.$$

But this section of the universal $\mathrm{INN}(G)$ 2-bundle we can regard as a choice of connection on the 1-bundle, namely as a splitting of the Atiyah groupoid projection

$$T\Sigma TC \to \Sigma TC\,.$$



But this is essentially nothing but the identity 2-functor

$$\Sigma TC \to \Sigma TC\,.$$

We should think of this as the universal connection on the universal G-bundle:

$$\mathrm{curv}_{EG} := \mathrm{Id} : \Sigma TC \to \Sigma TC\,.$$

Write $\Pi_2(X)$ for the fundamental 2-groupoid of the space X (objects are the points in X, morphisms are thin-homotopy classes of paths in X, 2-morphisms are homotopy classes of surfaces in X).

The weak pushout $\mathcal{C}_2(Y)$

$$\begin{array}{ccc} \Pi_2(Y^{[2]}) & \longrightarrow & \Pi_2(Y) \\ \downarrow & \swArrow \simeq & \downarrow \\ \Pi_1(Y) & \longrightarrow & \mathcal{C}_2(Y) \end{array} \quad ,$$

addressed as the *path pushout* in [4] is the 2-groupoid modelling the fundamental 2-groupoid of X with respect to the covering Y. It is the 2-groupoid generated from $\Pi_2(Y^{[2]})$ and from $Y^{[2]}$, modulo the relations

$$\begin{array}{ccc} \pi_1(x) & \xrightarrow{\pi_1(\gamma)} & \pi_1(y) \\ \downarrow & & \downarrow \\ \pi_2(x) & \xrightarrow{\pi_2(\gamma)} & \pi_2(y) \end{array}$$

for all $x \xrightarrow{\gamma} y$ in $\mathrm{Mor}_1(\Pi_2(Y^{[2]}))$.

We find that extending our classifying map

$$\begin{array}{ccc} Y^{[2]} & & \\ \downarrow f & & \\ C & \longrightarrow & \Sigma TC \end{array}$$

from points to paths

$$\begin{array}{ccc} Y^{[2]} & \longrightarrow & \mathcal{C}_2(Y) \\ \downarrow g & & \downarrow (\mathrm{curv}, f) \\ C & \longrightarrow & \Sigma TC \end{array}$$

is the same as choosing a G-connection on the bundle classified by f, by [4]. Locally, i.e. on generators of $\mathcal{C}_2(Y)$ coming from $\Pi_2(Y^{[2]})$ this 2-functor curv is precisely the curvature 2-functor of a parallel transport

$$\mathrm{tra} : \mathcal{P}_1(Y) \to \Sigma G\,.$$

Notice that we may think of the connection on our bundle

$$(\mathrm{curv}, f) : \mathcal{C}_2(Y) \to \Sigma TC$$

as the pullback of the universal connection

$$\mathrm{curv}_{EG} := \mathrm{Id} : \Sigma TC \to \Sigma TC$$

along a refinement of our classifying map f, simply as

$$(\mathrm{curv}, f) : \ \mathcal{C}_2(Y) \longrightarrow \Sigma TC \xrightarrow{\mathrm{curv}_{EG}} \Sigma TC \ .$$

References

[1] C. J, Isham, *Quantising on a general category*, Adv.Theor.Math.Phys. 7 (2003) 331-367, available as `arXiv:gr-qc/0303060`

[2] D. M. Roberts, U. Schreiber, *The inner automorphism 3-group of a strict 2-group*, available as `arXiv:0708.1741`

[3] U. Schreiber, J. Stasheff, *Connections with values in Lie n-algebras*, in preparation

[4] U. Schreiber, K. Waldorf, *Parallel transport and functors*, available as `arXiv:0705.0452`

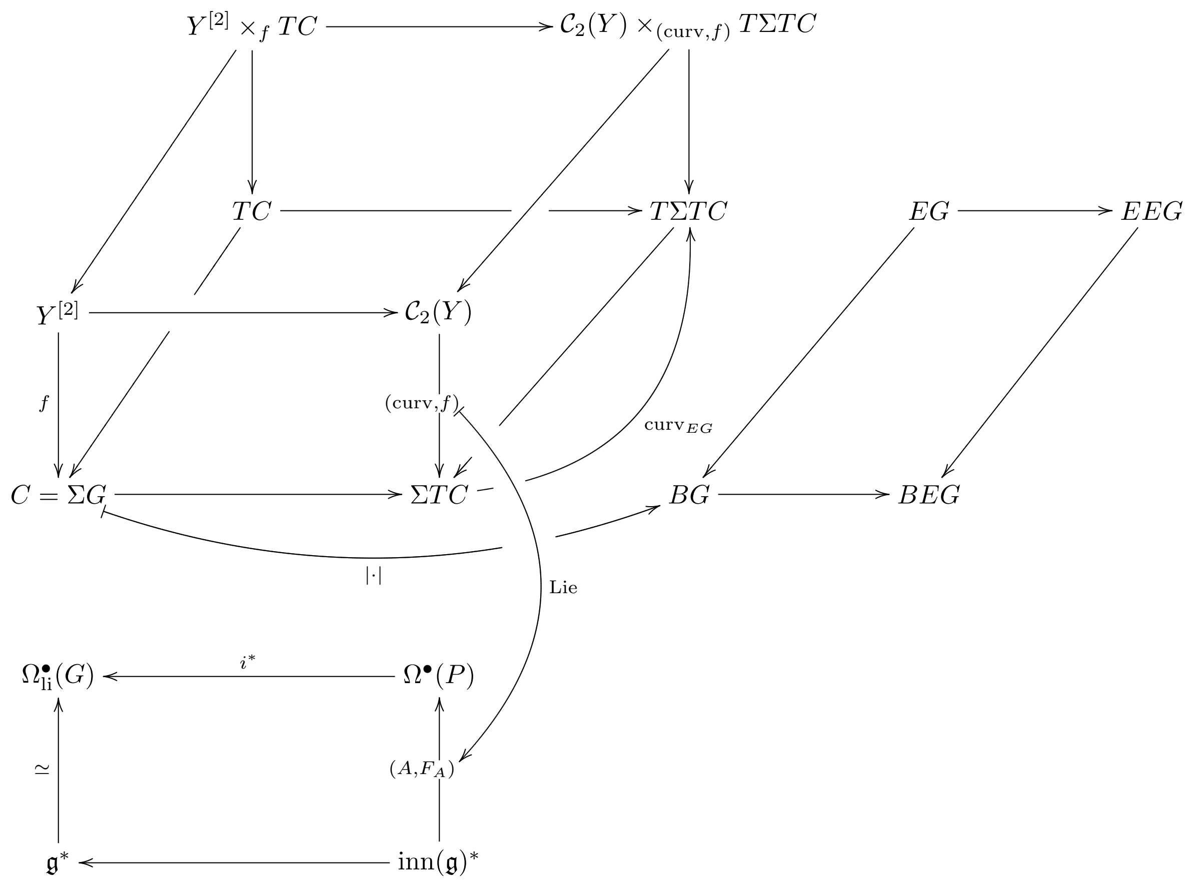


Figure 1: **The universal G-bundle** and its pullbacks in the world of groupoids and Lie algebroids. $C = \Sigma G$ is the one-object groupoid corresponding to G, $TC = \mathrm{INN}(G)$ its inner automorphism 2-group, whose underlying groupoid is the total space of the universal G-bundle. $Y \to X$ is a good cover of base space and $\mathcal{C}_2(Y)$ the fundamental 2-groupoid of X relative to this cover. Assuming $Y = P$ to be the total space of a G-bundle itself, differentiation takes us to the world of Lie algebroids, here presented in terms of their Koszul dual qDGCAs, as indicated. The geometric realization $|\cdot|$ is indicated only for orientation purposes. Notice that $|\mathrm{INN}(G)| \simeq EG$ implies that EG has the structure of a topological group.