notes taken in

K. Fredenhagen: On the renormalization group flow in perturbative algebraic quantum field theory , June 30, 2008, Max Planck institute for Math in Bonn, conference *The manifold geometry of quantum field theory*

a non-traditional approach to AQFT: perturbative field theory
Content:

1. introduction
2. generally covariant off shell formalism
3. extension of S to local interactions
4. algebraic adiabatic limit
5. cutoff, counter terms and flow equations
6. comparison of renormalization group

there are three main approaches to rigorous renormalized perturbative QFT:

1. BPHZ with Hopf algebraic structure [Connes-Kreimer]
2. renormalization group ("RNG") flow [Polchinsky,Kopper-Salmhofer]
3. Causal perturbation theory [Epstein-Glaser]

Among these, the **Epstein-Glaser approach** is that one which leads directly to a construction of algebras of observables and admits a generalization to generic Lorentzian spacetimes [Brunetti-Fredenhagen, Hollands-Wald, Dütsch-Fredenhagen]
Characteristic features:

- restriction to local interaction
- no cutoff needed
- easier on Lorentzian than on Riemannian spacetimes
- ultraviolet and infrared problems are completely disentangled

but the role of the renormalization group is not obvious in this approach [Hollands-Wald, Dütsch-Fredenhagen]
Questions:

- Where are the divergences in the Epstein-Glaser theory?
- Does Wilson's concept of theories at different scales apply?
- How to describe the RNG flow?

Different scales of theory invisible in Eppstein-Glaser approach.
Framework to be used in the following:

- M: a globally hyperbolic Lorentzian manifold;
- $C(M)$: space of smooth field configurations for a real scalar field;
- $F_0(M)$: space of smooth functionals on $C(M)$ whose derivatives are test functions with compact support;
- Δ_R, Δ_A: retarded and advanced propagator of the Klein-Gordon operator;
- $\Delta = \Delta_R - \Delta_A$: commutator function;
- $\Delta_D = \frac{1}{2}(\Delta_R + \Delta_A)$ Dirac propagator.

Warning: a generally covariant version of the Feynman propagator does not exist (no global concept of positive energy!)

Now consider the $\star$-product of functions defined by:

$$F \star G := \sum \frac{i^n \hbar^n}{2^n n!} \langle F^{(n)}, \Delta^{\otimes n} G^(n)) \rangle$$

The time ordered product of this is equivalent to the *pointwise* product:

$$F \cdot_T G := T(T^{-1} F \cdot T^{-1} G)$$

with the time-ordering operator

$$TF = \sum \frac{i^n \hbar^n}{2^n n!} \langle \Delta_D^{\otimes n}, F^{(2n)} \rangle$$

$\star$ and T, T^{-1} are defined on the space of formal power series in $\hbar$ with coefficients in $F_0(M)$

examples:

1. $$\phi(x) \star \phi(y) = \phi(x)\phi(y) + \frac{i\hbar}{2} \Delta(x, y)$$

2. $$T\phi(x)\phi(y) = \phi(x)\phi(y) + i\hbar \Delta_D(x, y)$$

[there was a longer list of examples...]

the formal S-matrix ($V \in F_0(M)$) (time-ordered exponential) is

$$S(V)(\phi) = T \exp T^{-} 1 V(\phi) =: \int d\mu_{i\Delta_D}(\phi) e^{:V(\phi - \varphi):}$$

this is to be thought of as the path integral where tadpoles are omitted

Warning: $S(V)$ is, in general, not unitary for imaginary V; unitarity cannot be defined for non-local functions.

The associated retarded interacting fields are [Bogoliubov]

$$R(V,F) = \left(\frac{d}{d\lambda}\right)_{\lambda=0} S(V)^{-1} \star S(V+\lambda F)$$

(where the inverse is taken with respect to $\star$-product)

Extension of the $\star$-product to local interaction V by continuity:

V is local if

$$V(\phi+\chi+\psi) = V(\phi+\chi) - V(\chi) + V(\chi+\psi)$$

provided $\mathrm{support}(\phi) \cap \mathrm{support}(\psi) = \emptyset$

As a consequence, the nth functional derivatives (if they exist) are supported on the thin diagonal $D_n \subset M^n$.

A local function is called *smooth* if all functional derivatives exist as distributions on cartesian powers of M with wave front sets in the co-normal bundle of the this diagonal.

Example:

$V(\phi) \int d\mathrm{vol} f(x)\phi(x)^n$

[again, there were more examples...]

Problem: the star product is ill defined on nonlinear local functionals. The traditional solution is: replace pointwise products of fields by Wick products.

This involves specification of a vacuum state.

Disadvantage: not compatible with general covariance.

May create infrared problems (e.g. for the massless scalar field in 2 dimensions).

Solution: choice of a Hadamard function.

This is a real valued, symmetric distribution H on M^2 such that $H + i\Delta$ satisfies the microlocal energy condition [Radzikowski].

H depends smoothly on the metric and on the other parameters of the free theory.

Define a linear isomorphism of $F_0(M)[[\hbar]]$ by

$$\alpha_H = \sum \frac{\hbar^n}{2^n n!} \langle H^{\otimes n}, F^{(2n)} \rangle$$

(e.g $\alpha_H \phi(x)\phi(y) = \phi(x)\phi(y) + \frac{\hbar}{2} H(x,y)$)

α_H transforms $\star$ to an equivalent product $\star_H$

$$F \star_H G = \alpha_H(\alpha_H^{-1} F \star \alpha_H^{-1} G)$$

which can be extended by continuity to the (sequential) completion $F(M)$ in a suitable topology. In particular [*missed an important statement here...*]

Removal of the H-dependence:

Equip $F_0(M)$ with the initial topology of α_H. This topology is independend of the choice of H. The sequential completion $A(M)$ is thus independent of H, and

$$\alpha_H : (A(M), \star) \to (F(M), \star_H)$$

is an isomorphism of algebras.

($\alpha_H^{-1}\phi(x)^n =: \phi(x)^n :_H$ corresponds to the normal ordered nth power with respect to H)

Dependence of the parameters $p := (g, m^2, \xi)$ of the free theory denoted by $A_p(M)$.

Introduce the bundle

$$B(M) = \sqcup_p A_p(M)\,.$$

Smooth section: $A = (A_p)_p$ is a smooth section of $B(M)$ if

$$\alpha_{H_p}(A_p)$$

is a smooth function of p.

$A(M)$ algebra (with respect to $\star$)

Scaling: scale transformations act in p by [Holland-Wald]

$$p(\lambda) = (\lambda^2 g, \lambda^{-1} m^2, \xi)$$

they induce an automorphism action of $A(M)$ by

$$\sigma_\lambda(A)_p = \sigma_\lambda(A_{p(\lambda)})\,.$$

Extension of S to localized interactions. So far this is known and established but pertains only to the free field.

$V \in A(M)$ is local, if $\alpha_H(V)$ is local.

causality: supp(A) temporally later than supp(B)

$$\Rightarrow: A \cdot_T B = A \star B$$

$$\Rightarrow S(A + B) = S(A) \star S(B)$$

star product is everywhere well defined, also on local functions

Construction of S as an alalytic function on the space of localized local interactions with values in $A(M)[[\hbar]]$ can be done by a recursive construction of the derivates of S with respect to V

causality leads to te following requirement on derivatives of S (as multilinear functionals on $A(M)$)

$$S^{(n)}(A^{\otimes k} \otimes B^{\otimes n-k}) = S^{(k)}(A^{\otimes k}) \star S^{(n-k)}(B^{\otimes (n-k)})$$

$$\Rightarrow$$

Causality fixes $S^{(n)}$ in terms of $S^{(k)}$, $k < n$ up to some n-linear functional $Z^{(n)}$ with values in local interactions

S exists and is unique up to composition with a map Z which maps local interactions into local interactions [...]

[*somehwhere around here I gave up taking notes... this was about half-way through the talk, the main point still to come*]