Structure of Lie n-Algebras

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Abstract

Higher order generalizations of Lie algebras have equivalently been conceived as Lie n-algebras, as L_∞-algebras, or, dually, as quasi-free differential graded commutative algebras.

Here we discuss morphisms and higher morphisms of Lie n-algebras, the construction of inner derivation Lie $(n+1)$-algebras, and the existence of short exact sequences of Lie $(2n+1)$-algebras for every transgressive Lie $(n+1)$-cocycle.

Contents

1 Introduction

Higher Lie algebras have been conceived as, equivalently, Lie n-algebras, L_∞-algebras, or, dually, quasi-free differential graded commutative algebras (quasi-"FDA"s, of "qfDGCA"s).

As Lie n-algebras, they arise through a process of categorification, as pioneered by Baez and his school. From their point of view, a Lie group is a Lie groupoid with a single object. Accordingly, a Lie n-group is a Lie n-groupoid with a single object.

Just as Lie groups have Lie algebras, Lie n-groups have Lie n-algebras, but in both cases, the algebra can be studied without recourse to the groups. Baez and Crans [2] have discussed how semistrict Lie n-algebras are the same as L_∞-algebras that are concentrated in the first n degrees.

An L_∞-algebra L can be described (see Definition 3) as a graded co-commutative coassociative coalgebra $S^c sL$ with a coderivation D of degree -1 that squares to 0.

Dually, on the space $\bigwedge^\bullet (sL)^*$, an L_∞-algebra L induces a differential graded commutative algebra which is free as a graded commutative algebras (we say "quasi-free DGCA" for short, but notice that in the physics literature these are known as "free differential algebras" or "FDA"s), whose derivation differential of degree 1 is given by

$$d\omega = -\omega(D(\cdot)) .$$

All these descriptions of higher Lie algebras have their advantages:

- the coalgebra picture is the most convenient one for many computations;
- the DCGA picture is most directly related to connections, curvatures and Bianchi identities with values in the given Lie n-algebra;
- the Lie n-algebra picture is conceptually the most powerful one.

We hope this work will be of interest to somewhat disparate readers: applied n-category theorists, homotopy theorists and cohomological physicists. Hopefully the table of contents will help each to find the parts most appealing to their individual tastes.

2 Main results

2.1 Higher morphisms of Lie n-Algebras

WARNING: what we currently actually do describe are higher morphisms on qfDGCAs which are free as differential algebras (inner derivation Lie n-algebras). The generalization to arbitary qfDGCAs requires certain transformations on the generators.

Lie n-algebras, being linear catgories, L, equipped with a bracket n-functor

$$[\cdot,\cdot] : L \times L \to L ,$$

are in particular monoidal n-categories. As such, they are naturally objects in an $(n+1)$-category.

The same should then hold for the equivalent L_∞-algebras and qfDGCAs concentrated in the first n-degree. But the right notion of higher morphisms for these objects has not been clear.

In principle, one would simply have to work out the god-given notion of morphism of Lie n-algebras in terms of the corresponding L_∞-data. While straightforward, this appears like a very tedious task in general. Baez and Crans [2] went through this for $n = 2$ and thus found the right notion of 2-morphisms of 2-term L_∞-algebras.

Here we propose that the right notion of higher morphisms of L_∞-algebas and quasi-free DGCAs are *derivation homotopies*, for which we give explicit formulas.

We prove that, for $n = 2$, algebra homomorphism chain maps and derivation homotopies of 2-term qfDGCAs are in bijection with the Baez-Crans notion of 1- and 2-morphisms of 2-term L_∞-algebras.

For $n = 2$ this proves that qfDGCAs, chain maps between these that respect the algebra structure, and derivation homotopies between those, form a 2-category.

We also give a general proof that qfDGCAs concentrated in the first n-degrees form an n-category

$$n\mathrm{Lie}$$

and that qfDGCAs without restriction on the degree form an $(\infty, 1)$-category

$$\omega\mathrm{Lie}$$

with derivation homotopies being the higher morphisms.

We also check in our concrete examples that derivation homotopies of qfDGCA morphisms from a Lie n-algebra to the deRham complex of some space induce the right notion of gauge transformations and higher gauge transformations for n-connections, see **??**.

2.2 The functor $\mathrm{inn}(\cdot) : n\mathrm{Lie} \to (n+1)\mathrm{Lie}$

Given any n-group $G_{(n)}$, we write $\Sigma G_{(n)}$ for the corresponding n-groupoid with a single object.

There is canonically associated an $(n + 1)$-group with any n-group $G_{(n)}$, the *automorphism* $(n + 1)$-group

$$\mathrm{AUT}(G_{(n)}) := \mathrm{Aut}_{n\mathrm{Cat}}(\Sigma G_{(n)})\,.$$

Its objects are (weakly) invertible n-functors $\Sigma G_{(n)} \to \Sigma G_{(n)}$, morphisms are (weakly) invertible transformations of these, and so on.

Inside $\mathrm{AUT}(G_{(n)})$ there is the sub $(n + 1)$-group

$$\mathrm{INN}(G_{(n)}) \subset \mathrm{AUT}(G_{(n)})$$

determined by restricting to those n-functors $\Sigma G_{(n)} \to \Sigma G_{(n)}$ which come from conjugation by morphisms in $\Sigma G_{(n)}$.

When everything is Lie, there should be Lie $(n + 1)$-algbras

$$\mathrm{DER}(\mathfrak{g}_{(n)}) := \mathrm{Lie}(\mathrm{AUT}(G_{(n)}))$$

and

$$\mathrm{inn}(\mathfrak{g}_{(n)}) := \mathrm{Lie}(\mathrm{INN}(G_{(n)}))\,,$$

where

$$\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})$$

is the Lie n-algebra corresponding to the Lie n-group $G_{(n)}$.

For reasons discussed in **??**, we shall be interested in particular in the inner derivation Lie $(n+1)$-algebra $\mathrm{inn}(\mathfrak{g}_{(n)})$ associated with a Lie n-algebra $\mathfrak{g}_{(n)}$.

While we fall short of deriving $\mathrm{inn}(\mathfrak{g}_{(n)})$ in general, we do construct a functor

$$\mathrm{inn}(\cdot) : n\mathrm{Lie} \to (n+1)\mathrm{Lie}$$

which we check to reproduce the above definition for Lie 1-algebra and strict Lie-2-algebras.

For example for the special case that $\mathfrak{g}_{(n)} = \mathfrak{g}_{(1)} = \mathfrak{g}$ is an ordinary Lie algebra, the Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ turns out to be an old friend: it is Cartan's conception of the Weil algebra (described in detail in **??**).

It turns out – and this is actually its *raison d'être* – that $\mathrm{inn}(\mathfrak{g}_n)$ is *trivializable* (but not trivial):

$$\begin{array}{ccc} \mathrm{inn}(\mathfrak{g}_{(n)}) & \xrightarrow{\mathrm{id}} & \mathrm{inn}(\mathfrak{g}_{(n)}) \\ & \Downarrow\sim & \\ \searrow & & \nearrow \\ & 0 & \end{array} \,.$$

Far from implying that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is uninteresting, this allows many interesting constructions.

In particular, we show that, generally, morphisms

$$\mathrm{Vect}(X) \to \mathfrak{g}_{(n)}$$

correspond to n-connection forms with values in $\mathfrak{g}_{(n)}$ which are *flat* in all degrees, while morphisms

$$\mathrm{Vect}(X) \to \mathrm{inn}(\mathfrak{g}_{(n)})$$

correspond to general n-connection forms with values in $\mathfrak{g}_{(n)}$.

One can see that this passage from $\mathfrak{g}_{(n)}$ to $\mathrm{inn}(\mathfrak{g}_{(n)})$ corresponds precisely to the procedure which is known as "softening of the Cartan Integrable System", or "passage to the soft group manifold" in parts of the physics literature [1, 12]. More on that in **??**.

2.3 Lie n-Algebras from Cocycles and Invariant Polynomials

The situation described in **??** turns out to be just a special case of a general phenomenon.

Baez-Crans had pointed out [2] that for every Lie algebra $(n+1)$-cocycle μ there is Lie n-algebra

$$\mathfrak{g}_\mu$$

which is concentrated in degree 1 and n. The above Lie 2-algebra $\mathfrak{g}_k$ is obatined by setting

$$\mu = k\langle \cdot, [\cdot,\cdot]\rangle \,,$$

which is a cocycle when $\mathfrak{g}$ is semisimple.

We observe that, moreover, for every invariant polynomial k of degree $(n+1)$ on $\mathfrak{g}$, there is a Lie $(2n+1)$-algebra

$$\mathrm{ch}_k(\mathfrak{g})$$

which is concentrated in degree 1,2, and $2n+1$.

We call this the *Chern* Lie $(2n+1)$-algebra since its connections have a curvature $(2n+2)$-form given by the Chern class defined by k:

$$\begin{array}{c} \mathrm{ch}_k(\mathfrak{g}) \\ {\scriptstyle (A,C)}\Big\uparrow{\scriptstyle dC = k((F_A)^{n+1})} \\ \mathrm{Vect}(X) \end{array} \,,$$

where

$$(A,C) \in \Omega^1(X,\mathfrak{g}) \times \Omega^{2n+2}(X) \,.$$

In some case the invariant polynomial k gives rise to an associated $(2n+1)$-cocycle μ_k, such that k is $d_{\mathrm{inn}(\mathfrak{g})} - exact$ (this is described in 5.3). If so, this is witnessed by the existence of a Lie $(2n+1)$-algebra

$$\mathrm{cs}_k(\mathfrak{g}) \,.$$

We call this the *Chern-Simons* Lie $(2n+1)$-algebra associated with k, since its connections explicitly know about the Chern-Simons potential $(2n+1)$-form $k((F_A)^{n+1}) = d\mathrm{CS}_k(A)$ of the Chern class of k:

$$\begin{array}{c} \mathrm{cs}_k(\mathfrak{g}) \\ {\scriptstyle (A,B,C)}\Big\uparrow{\scriptstyle C = dB + \mathrm{CS}_k(A)} \\ \mathrm{Vect}(X) \end{array} \,,$$

$$(A,B,C) \in \Omega^1(X,\mathfrak{g}) \times \Omega^2(X) \times \Omega^{2n+2}(X) \,.$$

Moreover, this Chern-Simons Lie $(2n+1)$-algebra is *isomorphic* to the inner derivations Lie $(2n+1)$-algebra coming from the Baez-Crans Lie $2n$-algebra which is defined by the cocycle μ_k defined by k:

$$\mathrm{inn}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{cs}_k(\mathfrak{g}) \,.$$

Since $\mathrm{inn}(\mathfrak{g}_{\mu_k})$ is trivializable (as described in 2.2), this implies, in particular, that also $\mathrm{cs}_k(\mathfrak{g})$ is *trivializable.*

In summary, the situation we find is

$$
\begin{array}{ccccccc}
 & & \text{Baez-Crans} & & \text{Chern-Simons} & & \text{Chern} \\
 & & & & & & \\
1 & & 2n & & 2n+1 & & 2n+1 \\
 & & & & & & \\
\mathfrak{g} & \twoheadleftarrow & \mathfrak{g}_{\mu_k} & \hookrightarrow & \mathrm{cs}_k(\mathfrak{g}) & \twoheadrightarrow & \mathrm{ch}_k(\mathfrak{g}) \\
\uparrow {\scriptstyle (A)} & & \uparrow {\scriptstyle (A,B)} & & \uparrow {\scriptstyle (A,B,C)} & & \uparrow {\scriptstyle (A,C)} \\
{\scriptstyle F_A=0} & & {\scriptstyle F_A=0,\; dB+\mathrm{CS}_k(A)=0} & & {\scriptstyle C=dB+\mathrm{CS}_k(A)} & & {\scriptstyle dC=k((F_A)^{n+1})} \\
\mathrm{Vect}(X) & & \mathrm{Vect}(X) & & \mathrm{Vect}(X) & & \mathrm{Vect}(X)
\end{array} .
$$

In fact, we have an exact sequence of Lie $(2n+1)$-algebras

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0$$

whenever a Lie algebra $(2n+1)$-cocycle μ_k is related by transgression to a characteristic class k.

3 Lie n-algebras in terms of graded differential algebra and coalgebra

An L_∞ algebra is essentially a free graded commutative coalgebra with nilpotent codifferential, $D^2 = 0$, of degree -1. The dual of that is a free graded commutative algebra with differential, $d^2 = 0$, of degree +1.

The (co)differential itself encodes the higher generalizations of the Lie bracket, while $d^2 = 0$ ($D^2 = 0$) encodes the higher i generalization of the Jacobi identity.

3.1 L_∞-algebras and their duals.

Definition 1 *Given a graded vector space V, the* tensor space $T^\bullet(V) := \bigoplus_{n=0} V^{\otimes n}$ *with V^0 being the ground field. We will denote by $T^a(V)$ the* tensor algebra *with the concatenation product on $T^\bullet(V)$:*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

and by $T^c(V)$ the tensor coalgebra *with the deconcatenation product on $T^\bullet(V)$:*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{p+q=n} x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n .$$

The graded symmetric algebra $S^a(V)$ *is the quotient of the tensor algebra $T^a(V)$ by the graded action of the symmetric groups $\mathbf{S}_n$ on the components $V^{\otimes n}$.*

The graded symmetric coalgebra $S^c(V)$ *is the sub-coalgebra of the tensor coalgebra* $T^c(V)$ *fixed by the graded action of the symmetric groups* $\mathbf{S}_n$ *on the components* $V^{\otimes n}$.

Remark. $S^c(V)$ is spanned by graded symmetric tensors

$$x_1 \vee x_2 \vee \cdots \vee x_p$$

for $x_i \in V$ and $p \geq 0$, where we use $\vee$ rather than $\wedge$ to emphasize the coalgebra aspect, e.g.

$$x \vee y = x \otimes y \pm y \otimes x.$$

Notice: no factor of 1/2 is needed.

In characteristic zero, the graded symmetric algebra can be identified with a sub-algebra of $T^a(V)$ but that is unnatural and we will try to avoid doing so.

The coproduct on $S^c(V)$ is given by

$$\Delta(x_1 \vee x_2 \cdots \vee x_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma)(x_{\sigma(1)} \vee x_{\sigma(2)} \cdots x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \cdots x_{\sigma(n)}) .$$

Here

- $\mathrm{Sh}(p,q)$ is the subset of all those bijections (the "unshuffles") of $\{1, 2, \cdots, p+q\}$ that have the property that $\sigma(i) < \sigma(i+1)$ whenever $i \neq p$;
- $\epsilon(\sigma)$, which is shorthand for $\epsilon(\sigma, x_1 \vee x_2, \cdots x_{p+q})$, the Koszul sign, defined by
$$x_1 \vee \cdots \vee x_n = \epsilon(\sigma) x_{\sigma(1)} \vee \cdots x_{\sigma(n)} .$$

For a graded vector space L, we denote by sL the suspended or shifted space: $(sL)_{i+1} = L_i$.

Definition 2 (L_∞-algebra) *An* L_∞-algebra *is a* non-negatively *graded vector space* L *with the associated structure of the free graded commutative coalgebra* $S^c sL$ *with a coderivation*

$$D : S^c sL \to S^c sL$$

of degree -1, *restricting to 0 on the ground field and such that*

$$D^2 = 0 .$$

Proposition 1 *Given a coderivation*

$$D : S^c sL \to S^c sL$$

of degree -1 and satisfying $D^2 = 0$, *there are linear maps ("higher brackets")*

$$l_n : L^{\otimes i} \to L$$

for $n = 1, 2 \cdots$ *such that with*

$$d_n : S^c sL \to S^c sL$$

given on $\vee^n(sL)$ *by*

$$d_n(x_1 \vee \cdots \vee x_n) = \tilde{\epsilon}(\sigma) l_n(x_1 \vee \cdots \vee x_n)$$

and extended as coderivations to $S^c(sL)$ *we have*

$$D = d_1 + d_2 + \cdots .$$

Here $\tilde{\epsilon}(\sigma)$, which is shorthand for $\tilde{\epsilon}(\sigma, x_1 \vee \cdots \vee x_i)$, is a sign given by the formula

$$\tilde{\epsilon}(\sigma, x_1 \vee \cdots \vee x_n) = \begin{cases} (-1)^{\sum_{k=1}^{n/2} |x_{2k-1}|} & n \text{ even} \\ (-1)^{\sum_{k=1}^{(n-1)/2} |x_{2k}|} & n \text{ odd} \end{cases} .$$

Remark. The collection of the higher brackets $\{l_i\}$ together with the "higher Jacobi relation" $D^2 = 0$ expressed in terms of these is what was historically first addressed as an L_∞-algebra or a strongly homotopy Lie algebra (sh-Lie algebra).

That the original definition of L_∞-algebra in terms of the $\{l_n\}$ produces a differential D as above was shown in [22]. That every such differential comes from a collection of $\{l_n\}$ this way is due to [21].

Remark. Notice that the extension of the d_n as coderivations means explicitly that

$$d_n(x_1 \vee \cdots x_r) = \sum_{\sigma \in \mathrm{Sh}(n,r-n)} \epsilon(\sigma) d_n(x_{\sigma(1)} \vee \cdots x_{\sigma(n)}) \vee x_{\sigma(n+1)} \vee \cdots x_{\sigma(r)} .$$

Remark. The full formulas obtained by expressing $D^2 = 0$ in terms of the $\{l_i\}$ are not very enlightening, but recognizing the first few relations may provide some insight.

First it follows that

$$l_1 l_1 = 0 ,$$

so l_1 is a differential. Then we have

$$l_1 l_2 - l_2(l_1 \otimes 1 + 1 \otimes l_1) = 0 ,$$

so l_2 is a chain map.

If we write $Jac_{[\ ,\ ]}(x, y, z) = 0$ for the standard Jacobi relation (with appropriate signs in the graded case), then Jac_{l_2} is not zero, but rather

$$Jac_{l_2} = l_1 l_3 + l_3(l_1 \otimes 1 \otimes 1 + 1 \otimes l_1 \otimes 1 + 1 \otimes 1 \otimes l_1).$$

Definition 3 (qfDGCA dual to L_∞-algebra) *For any* L_∞*-algebra* L *of finite dimension the dual space*

$$\bigwedge\nolimits^\bullet (sL)^*$$

naturally has the structure of a differential graded commutative algebra (DGCA). It is free as a graded commutative algebra, with the product given by

$$\omega_1 \wedge \omega_2(x_1 \vee x_2 \vee \cdots \vee x_n) := \omega_1 \otimes \omega_2(\Delta(x_1 \vee \cdots \vee x_n))$$

$$= \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma)\omega_1(x_{\sigma(1)} \vee \cdots x_{\sigma(p)})\omega_2(x_{\sigma(p+1)} \vee \cdots x_{\sigma(n)})$$

for all $\omega_1, \omega_2 \in (sL)^*$. Here we agree that $\omega(v) = 0$ unless $|\omega| = |v|$.

The differential

$$d : \bigwedge^\bullet (sL)^* \to \bigwedge^\bullet (sL)^*$$

is defined by

$$d\omega := -\omega(D(\cdot)) \tag{1}$$

for all $\omega \in (sL)^*$.

Remark. The sign in (1) is purely conventional. We include it here since, as demonstrated below, this way DGCA morphisms to the deRham complex reproduce common formulas for connections, curvatures and Bianchi identities in their natural form.

Remark. That $d^2 = 0$ follows directly from $D^2 = 0$. That d is a graded derivation of degree 1 follows similarly from the fact that D is a coderivation of degree -1:

$$\begin{aligned} d(\omega_1 \wedge \omega_2)(v) \quad &= -\omega_1 \wedge \omega_2(D(v)) \\ &= -\omega_1 \otimes \omega_2(\Delta(D(v))) \\ &= -\omega_1 \otimes \omega_2((D \otimes 1 + 1 \otimes D)\Delta(v))) \\ &= (d\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (d\omega_2) \end{aligned}$$

for all $\omega_1, \omega_2 \in (sL)^*$ of degree p and q, respectively. In the third line this uses the graded coderivation property. Notice that this involves

$$(1 \otimes D)(v \otimes w) := (-1)^{|v|} v \otimes Dw$$

for all $v, w \in sL$.

Remark. In concrete examples we make use of the fact that for checking $d^2 = 0$ it suffices to check this on generators. Dually, for checking $D^2 = 0$ it suffices to check this for terms $v \in S^c sL$ such that $D^2 v \in sL$.

Remark. When L is infinite dimensional the above construction has to be modified.

For our purposes here, we shall agree on the following terminology.

Definition 4 (semistrict Lie n-algebra) *A* semistrict Lie n-algebra *is an L_∞ algebra $(S^c(sL), D)$ where L is concentrated in the first n degrees.*

Equivalently, this is a quasi-free DGCA $(\bigwedge^\bullet (sL)^*, d)$ with L concentrated in the first n degrees, by convention starting with 0.

Remark. More precisely, a semistrict Lie n-algebra is an n-category L internal to vector spaces and equipped with a linear skew-symmetric n-functor $[\cdot,\cdot] : L \times L \to L$ which satisfies a Jacobi identity up to higher order coherent equivalences. Baez-Crans in [2] discuss how these are equivalent to n-term L_∞-algebras.

Here we find it convenient to calculate entirely in the world of L_∞-algebras and qfDGCAs, but to keep their interpretation as Lie n-algebras in mind for being able to naturally interpret the structures that we encounter.

3.1.1 Lie n-algebra Morphisms

Ordinary (1-)morphisms of L_∞-algebras and of DGCAs have an obvious definition.

Definition 5 (1-morphisms of Lie n-algebras) *A morphism*

$$f : L_1 \to L_2$$

between two L_∞-algebras is a coalgebra homomorphism (in particular of degree 0)

$$f : S^c sL_1 \to S^c sL_2$$

of the corresponding coalgebras, which commutes with the corresponding codifferentials

$$f \circ D_1 = D_2 \circ f .$$

Dually, it is an algebra morphism (in particular of degree 0)

$$f^* : {\bigwedge}^\bullet (sL_2)^* \to {\bigwedge}^\bullet (sL_1)^\bullet$$

that is at the same time a chain map

$$f \circ d_1 = d_1 \circ f .$$

The interpretation of L_∞-algebras and DGCA as Lie n-algebras amplifies the fact that n-term L_∞-algebras and DGCAs should live in an n-category instead of in a 1-category.

This means that there are 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, and so on, up to n-morphisms between $(n-1)$-morphisms.

It crucially matters what these higher morphisms are like, because their nature determines the notion of *equivalence* of Lie n-algebras.

For quasi-free DGCAs, the natural notion of higher morphisms are chain homotopies of the special kind known as derivation homotopies.

We show in proposition 4 that the Baez-Crans notion of 2-morphism of Lie 2-algebras is equivalent to derivation homotopies of the corresponding morphisms of the corresponding DGCAs.

3.1.2 Higher Morphisms

WARNING: THE FOLLOWING defintion of higher morphisms in terms of their action on products of generators so far works, as stated, only for DGCAs which are free as graded commutative algebras as well as as differential algebras. For the more general case one has to replace generators by appropiate combinations of generators.

Definition 6 (2-morphisms of Lie n-algebras) *Given quasi-free DG-CAs $(\bigwedge^\bullet(sV)^*, d)$ and $(\bigwedge^\bullet(sW)^*, d)$ and two DGCA morphisms*

$$f^*_{1,2} : (\bigwedge^\bullet(sW)^*) \to (\bigwedge^\bullet(sV)^*)$$

a derivation homotopy

$$\tau : f_1^* \to f_2^*$$

is a chain homotopy, hence in particular a degree -1 map

$$\tau : \bigwedge^\bullet(sW)^* \to \bigwedge^{\bullet-1}(sV)^* ,$$

whose action on products of n generators is determined by its action on single generators by

$$\tau : x_1 \wedge \cdots \wedge x_n \mapsto$$

$$\frac{1}{n!}\sum_\sigma \epsilon(\sigma) \sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} f_1(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge\tau(x_{\sigma(k)})\wedge f_2(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)}) ,$$

for all $x_i \in (sW)^*$.

Remark. Recall that, by our conventions, we have

$$V = V_0 \oplus \cdots \oplus V_{n-1} .$$

Remark. The graded symmetrization $\frac{1}{n!}\sum_\sigma \epsilon(\sigma)$ over all permutatations makes this well defined on $\bigwedge^\bullet(sW)^*$. Moreover, one checks that this does indeed satisfy the chain homotopy condition

$$f_2^* - f_1^* = [d, \tau] ,$$

where

$$[d, \tau] := d_V \circ \tau + \tau \circ d_W ,$$

by explicit computation:

$$\begin{aligned}
& [d,\tau](x_1 \wedge \cdots \wedge x_n) && (2)\\
= \; & \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n f_1(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)}) \wedge [d,\tau](x_{\sigma(k)}) \wedge f_2(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)}) && (3)\\
= \; & \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n f_1(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)}) \wedge (f_2^* - f_1^*)(x_{\sigma(k)}) \wedge f_2(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)}) && (4)\\
= \; & \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\Big(\sum_{k=1}^n f_1(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)}) \wedge f_2(x_{\sigma(k)}\wedge\cdots\wedge x_{\sigma(n)}) && (5)\\
& - \sum_{k=1}^n f_1(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k)}) \wedge f_2(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})\Big) && (6)\\
= \; & (f_2^* - f_1^*)(x_1 \wedge \ldots \wedge x_n) . && (7)
\end{aligned}$$

Example. For our examples of Lie n-algebras of low n here, we mainly need this formula only on products of two generators, where it becomes

$$\tau : a \wedge b \mapsto \frac{1}{2}\Big(\tau(a) \wedge (f_1^* + f_2^*)(b) + (-1)^{|a|}(f_1^* + f_2^*)(a) \wedge \tau(b)\Big) \qquad (8)$$

for all $a, b \in (sW)^*$. Proof. By applying the general formula to this special case, one gets

$$\begin{aligned}
\tau(a \wedge b) &= \frac{1}{2}\Big(\tau(a) \wedge f_2^*(b) + (-1)^{|a|} f_1^*(a) \wedge \tau(b)\Big) \\
&\quad + \frac{1}{2}(-1)^{|a||b|}\Big(\tau(b) \wedge f_2^*(a) + (-1)^{|b|} f_1^*(a) \wedge \tau(a)\Big) \\
&= \frac{1}{2}\Big(\tau(a) \wedge f_2^*(b) + (-1)^{|a|} f_1^*(a) \wedge \tau(b)\Big) \\
&\quad + \frac{1}{2}\Big((-1)^{|a|} f_2^*(a) \wedge \tau(b) + \tau(a) \wedge f_1^*(a)\Big)
\end{aligned}$$

□

Remark. Derivation homotopies $f_1^* \to f_2^*$ are very similar to the "f_1^*-Leibniz"-morphisms considered in [10]. The difference is that derivation homotopies do in fact constitute chain homotopies between algebra morphisms. For "f_1^*-Leibniz"-morphisms this is true only up to terms of higher order in the images of the generators. As a result, "f_1^*-Leibniz" morphisms yield for instance the right linearized gauge transformation formulas (discussed in **??** and **??**) when applied for low n and when evaluating everything only on generators. This was discussed in [24]. The symmetrization in f_1^* and f_2^* involved in derivation homotopies is crucial for the equivalence with Baez-Crans 2-morphisms of Lie 2-algebras (proposition 4).

Definition 7 (higher morphisms of Lie n-algebras) *A j-morphism of Lie n-algebras is a quasi-free DGCA homotopy*

$$h : \tau_1 \to \tau_2$$

between $(j-1)$-morphisms $\tau_{1,2}$ of Lie n-algebras as above, hence in particular a map of degree $-(j-1)$

$$h : \bigwedge\nolimits^{\bullet}(sW)^* \to \bigwedge\nolimits^{\bullet-(j-1)}(sV)^*$$

such that

$$\tau_2 - \tau_1 = [d, h]\,,$$

where

$$[d, h] := d_v \circ h + (-1)^j h \circ d_W\,,$$

which acts on products of generators as

$$h : x_1 \wedge \cdots \wedge x_n \mapsto$$

$$\frac{1}{n!}\sum_{\sigma}\epsilon(\sigma)\sum_{k=1}^{n}(-1)^{\sum_{i=1}^{k-1}(j-1)|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge h(x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})\,.$$

Remark. It is noteworthy that for the definition of (higher) derivation homotopies to make good sense, in fact only the source DGCA needs to be free as a graded commutative algebra. This means in particular that it makes sense to consider (higher) derivation homotopies of morphisms from some quasi-free DGCA to the deRham complex of any manifold X:

$$f^* : (\bigwedge^\bullet (sW)^*, d) \to (\Omega^\bullet(X), d)\,.$$

We discuss in several examples how such morphisms encode local connection data with values in the Lie n-algebra corresponding to $(\bigwedge^\bullet (sW)^*, d)$, and how (higher) derivation homotopies of these correspond to (higher) gauge transformation operations on these.

Proposition 2 *The sum of (the component maps of) two derivation homotopies*

$$\tau_1 : f_1^* \to f_2^*$$

and

$$\tau_2 : f_2^* \to f_3^*$$

is not in general itself (the component map of) a derivation homotopy, but is homotopic to the derivation homotopy

$$\tau_2 \circ \tau_1 : f_1^* \to f_3^*$$

which is defined on generators *by the sum of* τ_1 *and* τ_2 *and then uniquely extended as a derivation homotopy with respect to* f_1^* *and* f_3^*.

Proof. The socond order homotopy

$$\delta_{\tau_1,\tau_2} : \tau_2 \circ \tau_1 \to \tau_1 + \tau_2$$

is that coming from the degree -2 map

$$\delta_{\tau_1,\tau_2} : \bigwedge^\bullet (sW)^* \to \bigwedge^\bullet (sV)^*$$

defined by

$$\delta_{\tau_1,\tau} : (x_1 \wedge \cdots \wedge x_n) \mapsto$$

$$\frac{1}{n!}\sum_\sigma \epsilon(\sigma, x_1, \cdots, x_n) \sum_{1 \leq k_1 < k_2 \leq n} (-1)^{\sum_{i=k_1}^{k_2-1} |x_{\sigma(i)}|} f_1^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k_1-1)}) \wedge \tau_1(x_{\sigma(k_1)}) \wedge f_2^*(x_{\sigma(k_1+1)} \wedge \cdots \wedge x_{\sigma(k_2-1)}) \wedge \tau_2(x_{\sigma(k_2)}) \wedge f_3^*(x_{\sigma(k_2+1)} \wedge \cdots \wedge x_{\sigma(n)})\,.$$

This follows from direct computation, which makes use of (2):

$$\begin{aligned}
&(\tau_1 + \tau_2)(x_1 \wedge \cdots \wedge x_n) \\
&= \frac{1}{n!}\sum_\sigma \epsilon(\sigma) \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} f_1^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \wedge \tau_1(x_{\sigma(k)}) \wedge f_2^*(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) \\
&+ \frac{1}{n!}\sum_\sigma \epsilon(\sigma) \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} f_2^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \wedge \tau_2(x_{\sigma(k)}) \wedge f_3^*(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) \\
&= \frac{1}{n!}\sum_\sigma \epsilon(\sigma) \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} f_1^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \wedge (\tau_1 + \tau_2)(x_{\sigma(k)}) \wedge f_3^*(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)})
\end{aligned}$$

$$-\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge \tau_1(x_{\sigma(k)})\wedge [d,\tau_2](x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

$$+\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} [d,\tau_1](x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge \tau_2(x_{\sigma(k)})\wedge f_3^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

$$=\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge (\tau_1+\tau_2)(x_{\sigma(k)})\wedge f_3^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

$$-\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} \sum_{j=k+1}^n f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge \tau_1(x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(j-1)})[d,\tau_2](x_{\sigma(j)})\wedge f_3^*(x_{\sigma(j+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

$$+\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} \sum_{j=1}^{k-1} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(j-1)})\wedge [d,\tau_1](x_\sigma(j))\wedge f_2^*(x_{\sigma(j+1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge \tau_2(x_{\sigma(k)})\wedge f_3^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

$$=\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge (\tau_1+\tau_2)(x_{\sigma(k)})\wedge f_3^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$
$$+[d,\delta_{\tau_1,\tau_2}](x_1\wedge\cdots\wedge x_n)\,.$$

□

Remark. Notice that this in particular shows that the derivation homotopy

$$\tau_2\circ\tau_1$$

(defined as a sum on generators and then extended to a full derivation homotopy) is indeed a homotopy from f_1^* to f_3^*, since, by the above proof, we have

$$[d,\tau_2\circ\tau_1] = [d,\tau_1+\tau_2] - [d,[d,\delta_{\tau_1,\tau_2}]] = [d,\tau_1]+[d,\tau_2] = f_3^* - f_1^*\,.$$

3.1.3 The ∞-Category $\omega\mathrm{Lie}$

This fact motivates the following definition.

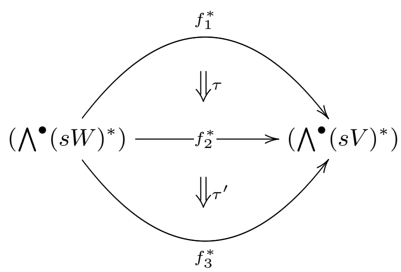
Definition 8 (composition of j-morphisms) *1-morphisms are composed with j-morphisms by pre- or postcomposing the component maps. All other compositions are by given by acting on generators with the sums of the respective maps and then uniquely extending to derivation homotopies.*

URS: Hope that's right. The computations are a little unwieldy.

Examples. Given derivations (1-morphisms)

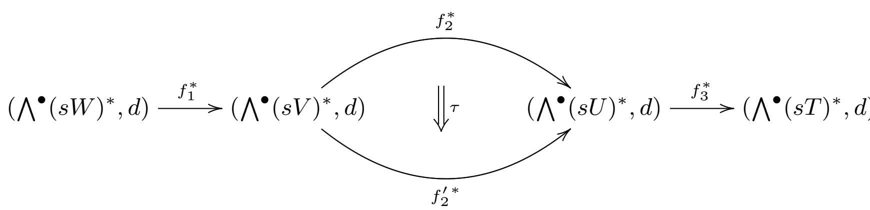
$$(\textstyle\bigwedge^\bullet(sV)^*,d)\xrightarrow{L_1}(\bigwedge^\bullet(sV)^*,d)\xrightarrow{L_2}(\bigwedge^\bullet(sV)^*,d)$$

the component map of their composite is $L_2 \circ f_1^*$. Given 2-morphisms



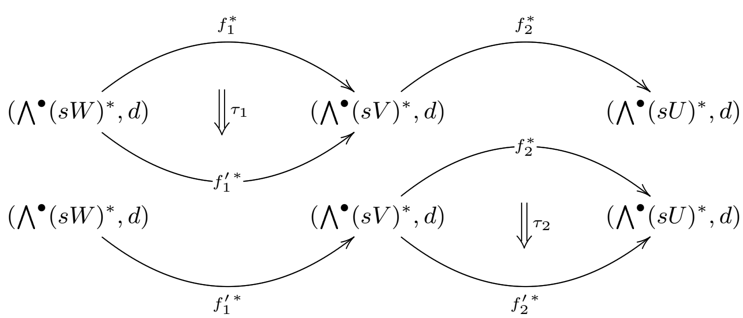
the component map of their composite restricted to generators is $\tau + \tau'$.

The component map of

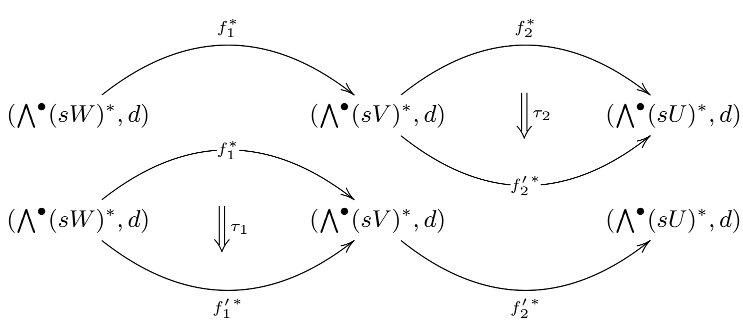


on generators is $f_3^* \circ \tau \circ f_1^*$.

The component map of the vertical composition of



on generators is $f_2^* \circ \tau_1 + \tau_2 \circ {f_1'}^*$. On the other hand, the component map of the vertical composition of



on generators is $\tau_2 \circ f_1^* + {f'_2}^* \circ \tau_1$. Notice that these two composites differ by an exact term

$$(\tau_2 \circ f_1^* + {f'_2}^* \circ \tau_1) - (f_2^* \circ \tau_1 + \tau_2 \circ {f'_1}^*) = [d, \tau_2] \circ \tau_1 - \tau_2 \circ [d, \tau_1] = [d, \tau_2 \circ \tau_1]\,,$$

which means that they are homotopic, which in turn means that the two ways to compose two 2-morphisms horizontally are connected by a 3-isomorphism, hence a 3-equivalence (in ωLie all higher morphisms are in fact already strictly invertible).

Definition 9 *Denote by*

$$n\mathrm{Lie}$$

the n-category whose objects are qfDGCAs concentrated in the lowesr n-degrees and whose morphisms are as above.

Denote by

$$\omega Lie$$

the ∞-category whose objects are $qfDGCAs$ concentrated in arbitrary degree and whose morphisms are as above.

URS: We need to spell this out in more detail and check if everything goes through as expected.

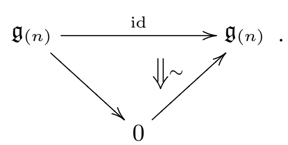
3.1.4 Trivializable Lie n-Algebras

Proposition 3 *Every qfDGCA which is free also as a differential algebra is equivalent, in* ωLie*, to the trivial qfDGCA.*

Proof. Let $\mathfrak{g}_{(n)}$ be any qfDGCA which is free also as a differential algebra. This means that there is a graded vector space $(sV)^*$ such that $\mathfrak{g}_{(n)}$ is the free differential graded algebra

$$\bigwedge^\bullet((sV)^* \oplus d(sV)^*)\,.$$

We need to show that there is a derivation homotopy



More concretely, this means that we need to find a derivation homotopy τ such that its component map of degree -1 satisfies

$$[d_{\mathfrak{g}_{(n)}}, \tau] = \mathrm{Id}_{\mathfrak{g}_{(n)}}\,.$$

By defining τ on generators by

$$\tau : a \mapsto 0$$

$$\tau : da \mapsto a$$

for all $a \in (sV)^*$ we get

$$[d_{\mathfrak{g}_{(n)}}, \tau] : a \mapsto a$$
$$[d_{\mathfrak{g}_{(n)}}, \tau] : da \mapsto da\,.$$

Then τ is uniquely extended as a derivation homotopy. □

Remark. This is a very important special case of the general result that if (W, d) and (W', d') are abelian DG Lie algebras, then if they are equivalent as DG Lie algebras, then the corresponding $\bigwedge^\bullet(sW)$ and $\bigwedge^\bullet(sW')$ are equivalent as DGCAs.

URS: How do I see that it's just a special case of that?

3.1.5 Equivalence with Baez-Crans definition of 2Lie

Lie 2-algebras, due to their category-theoretic nature, have a rather obvious notion of 1- and 2-morphisms between them, as explained by Baez-Crans. Here we show how the definition of higher morphisms of Lie n-algebras above reproduces the Baez-Crans definition for $n = 2$.

In order to set up the discussion of Baez-Crans 2-morphisms of 2-term L_∞-algebras, recall their notation for 1-morphisms of 2-term L_∞-algebras (which is of course just a special case of the general notion of 1-morphisms of L_∞-algebras).

Definition 10 *A morphism*

$$\varphi : V \to W$$

of 2-term L_∞-algebras V and W is a pair of maps

$$\phi_0 : V_0 \to W_0$$

$$\phi_1 : V_1 \to W_1$$

together with a skew-symmetric map

$$\phi_2 : V_0 \otimes V_0 \to W_1$$

satisfying

$$\phi_0(d(h)) = d(\phi_1(h))$$

as well as

$$\begin{aligned} d(\phi_2(x,y)) &= \phi_0(l_2(x,y)) - l_2(\phi_0(x), \phi_0(y)) \\ \phi_2(x, dh) &= \phi_1(l_2(x,h)) - l_2(\phi_0(x), \phi_1(h)) \end{aligned}$$

and finally

$$\begin{aligned} & l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x,y,z)) = \\ & \phi_2(x, l_2(y,z)) + \phi_2(y, l_2(z,x)) + \phi_2(z, l_2(x,y)) + \\ & l_2(\phi_0(x), \phi_2(y,z)) + l_2(\phi_0(y), \phi_2(z,x)) + l_2(\phi_0(z), \phi_2(x,y)) \,. \end{aligned}$$

for all $x, y, z \in V_0$ and $h \in V_1$.

For later reference, we spell out the dual DGCA formulation of this in the appendix **??**.

Definition 11 (Baez-Crans) *A 2-morphism*

$$\tau : \phi \Rightarrow \psi$$

of 1-morphisms of 2-term L_∞-algebras is a linear map

$$\tau : V_0 \to W_1$$

such that

$$\psi_0 - \phi_0 = t_W \circ \tau \tag{9}$$

$$\psi_1 - \phi_1 = \tau \circ t_v \tag{10}$$

and

$$\phi_2(x,y) - \psi_2(x,y) = l_2(\phi_0(x), \tau(y)) + l_2(\tau(x), \psi_0(y)) - \tau(l_2(x,y)) \tag{11}$$

Note that $[d, \tau] = d_W \tau + \tau d_V$ and that it restricts to $d_W \tau$ on V_0 and to τd_V on V_1.

Proposition 4 *The above conditions on 2-morphisms of L_∞-algebras are equivalent to that on derivation homotopy of the corresponding DGCAs morphisms.*

Proof. The straightforward computation is spelled out in the appendix 6.1.

3.2 Derivation Lie $(n+1)$-algebras $\mathrm{DER}(\mathfrak{g}_{(n)})$ and $\mathrm{inn}_k(\mathfrak{g}_{(n)})$

3.2.1 Lie n-group analog

Given any Lie n-group $G_{(n)}$, regarded as an n-groupoid $\Sigma G_{(n)}$ with a single object, we naturally obtain an $(n+1)$-group

$$\mathrm{AUT}(G_{(n)}) := \mathrm{Aut}_{n\mathrm{Cat}}(\Sigma G_{(n)})\,,$$

the *automorphism* $(n+1)$*-group*. For each $k \in \mathbb{N}$ this has a sub-$(n+1)$-group

$$\mathrm{INN}_k(G_{(n)}) \subset \mathrm{AUT}(G_{(n)})$$

obtained by restricting the 1-morphism to be those coming from *conjugation* of n-morphisms of $\Sigma G_{(n)}$ with j-morphisms of $\Sigma G_{(n)}$ for

$$1 \leq j \leq k\,.$$

See figure 1 for a description of

$$\mathrm{INN}_2(G_{(2)})$$

for $G_{(2)}$ a strict 2-group. We have a chain of canonical inclusions

$$\mathrm{INN}_0(G_{(n)}) \subset \mathrm{INN}_1(G_{(n)}) \subset \cdots \subset \mathrm{INN}_n(G_{(n)}) \subset \mathrm{AUT}(G_{(n)})\,.$$

We shall frequently write

$$\mathrm{INN}(G_{(n)}) := \mathrm{INN}_1(G_{(n)})\,.$$

This is motivated by the fact that only the 1-morphisms of $\mathrm{INN}(G_{(n)})$ are truly *inner* automorphisms with respect to the monoidal structure on the n-group $G_{(n)}$.

Of particular interest for us are the three parts

$$\mathrm{INN}(G_{(n)}) \subset \mathrm{INN}_n(G_{(n)}) \subset \mathrm{AUT}(G_{(n)})\,.$$

These (inner) automorphism $(n+1)$-groups ought to have an analogue at the level of Lie n-algebras, where (inner) automorphisms become (inner) derivations. By differentiating the above, we should obtain Lie $(n+1)$-algebras

$$\mathrm{inn}(\mathfrak{g}_{(n)}) \subset \mathrm{inn}_n(\mathfrak{g}_{(n)}) \subset \mathrm{DER}(\mathfrak{g}_{(n)})$$

for each Lie n-algebra $\mathfrak{g}_{(n)}$.

We discuss first general derivations of Lie n-algebras, then describe the action of a Lie n-algebra on itself by derivations concretely and then present the corresponding functor

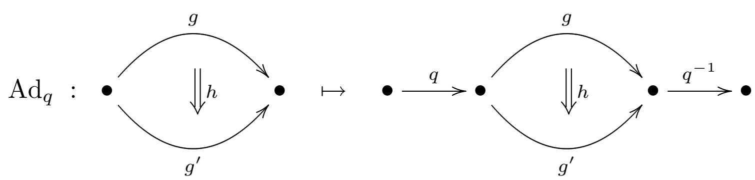
$$\mathrm{inn}(\cdot) : n\mathrm{Lie} \to (n+1)\mathrm{Lie}$$

in general.

- **horizontal conjugation** by any $q \in G$

$$\mathrm{Ad}_q \in \mathrm{Aut}_{2\mathrm{Cat}}(G_{(2)})$$

(true conjugation in the sense of the 2-group) acts as



- **vertical conjugation**

$$\mathrm{vAd}_f \in \mathrm{Aut}_{2\mathrm{Cat}}(G_{(2)})$$

by any map $f : G \to H$ which extends to a homomorphism

$$\mathrm{Id} \times f : G \to G \ltimes H \,,$$

acts as

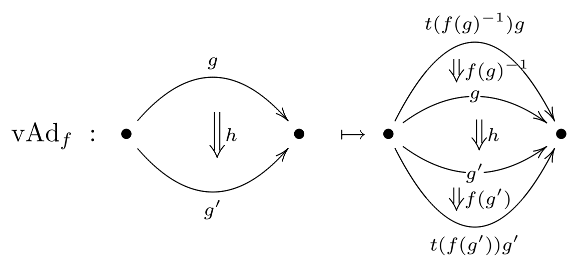


Figure 1: **The two notions of conjugation in a 2-group**, for the special case of a strict 2-group $G_{(2)}$, coming from a crossed module $(H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$ of groups.

3.2.2 General derivations

Remark. This discussion here is greatly motivated from discussion with Danny Stevenson [29].

Definition 12 (derivations of DGCAs) *A* derivation *of a differential graded commutative algebra* $(\bigwedge^\bullet (sV)^*, d)$ *is a linear degree* 0 *map*

$$L : \bigwedge^\bullet (sV)^* \to \bigwedge^\bullet (sV)^*$$

which is both a chain map as well as a derivation of the underlying graded commutative algebra.

Remark. A derivation τ of degree k is defined such that for any two homogeneous elements $\omega_1, \omega_2 \in sV^*$ we have

$$L(\omega_1 \wedge \omega_2) = (L\omega_1) \wedge \omega_2 + (-1)^{k|\omega_1|}\omega_1 \wedge (L\omega_2)\,.$$

Remark. One checks that for an ordinary Lie algebra derivations of the corresponding qfDGCA coincide with ordinary derivations of the Lie algebra.

Definition 13 *An* n-**morphism** *between derivations of Lie* n*-algebras is a higher chain homotopy whose component map is a derivation (necessarily of degree* $n-1$*).*

Remark. Derivations L naturally form a Lie algebra under the ordinary commutator

$$[L_1, L_2] := L_1 \circ L_2 - L_2 \circ L_1\,.$$

There should in fact be a Lie $(n+1)$-algebra structure on the space of all derivations and their higher morphisms on a given Lie n-algebra.

3.2.3 Action of a Lie n-algebra on itself

Definition 14 (interior product) *For any homogenous element*

$$v \in (sV)$$

of degree $k+1$ *we write*

$$\iota_v$$

for the derivation of degree $-(k+1)$ *of the graded commutative algebra* $\bigwedge^\bullet (sV)^*$ *which acts on generators* $\omega \in (sV)^*$ *as*

$$\iota_v : \omega \mapsto \omega(v)\,.$$

Remark. Notice that the interior product is, by definition, a derivation of the graded commutative algebra, but not in general a derivation of the *differential* algebra $(\bigwedge^\bullet (sV)^*, d)$, since it may not commute with the differential.

Proposition 5 *Under composition of maps $\iota_v : \bigwedge^\bullet (sV)^* \to \bigwedge^\bullet (sV)^*$, the interior product derivations ι_v for $v \in (sV)$ form a free graded commutative algebra isomorphic to $S^s(sV)$:*

$$\iota_v \circ \iota_w = (-1)^{|v||w|} \iota_w \circ \iota_v \,.$$

Definition 15 (Lie derivative and inner derivations) *For any derivation τ of degree $-(k+1)$ we call the derivation*

$$L_\tau : (\bigwedge^\bullet (sV)^*, d) \to (\bigwedge^\bullet (sV)^*, d)$$

of degree -1 given by

$$L_\tau := [d, \tau]$$

the Lie derivative *corresponding to τ.*

For any homogeneous element $v \in sV$ of degree $-(k+1)$, the Lie derivative

$$L_v := [d, \iota_v]$$

we call an inner derivation*.*

Remark. We have $[d, L_v] = 0$ due to $d^2 = 0$. Hence the Lie derivative is really a derivation on differential graded algebras.

Remark. It follows that a Lie n-algebra $\mathfrak{g}_{(n)} = (S^c(sV), D)$ most naturally acts on its *dual* qfDGCA: by Lie derivatives.

Remark. For any homogeneous $v \in sV$ of degree $k+1$, we may interpret the qfDGCA derivation

$$\iota_v : (\bigwedge^\bullet (sV)^*, d) \to (\bigwedge^\bullet (sV)^*, d)$$

of degree k as a $(k+1)$-morphism

$$\iota_v : [d, \iota_w] \to [d, \iota_{w+v}] \,,$$

for any $w \in sV$, due to the simple fact that

$$[d, \iota_{w+v}] - [d, \iota_v] = [d, \iota_v] \,.$$

It follows that every homogeneous element of $v \in sV$ plays a double role as

- an inner derivation ι_v of $(\bigwedge^\bullet (sV)^*, d)$
- a morphism between such inner derivations.

This phenomenon is responsible for the fact that the inner derivation Lie $(n+1)$-algebra $\mathrm{inn}(\mathfrak{g}_{(n)})$ defined below consists of two copies of the Lie n-algebra $\mathfrak{g}_{(n)}$, with one of them in the original degree and the other shifted by one.

Definition 16 *To any Lie n-algebra* $\mathfrak{g}_{(n)} = (S^c(sV), D)$ *is canonically associated the Lie (1-)algebra*

$$L(\mathfrak{g}_{(n)})$$

on the space of all degree -1 *derivations* τ *on* $(\bigwedge^\bullet (sV)^*, d)$ *with bracket defined by*

$$[\tau, \tau'] := [L_\tau, \tau'] .$$

This is indeed a Lie bracket, due to the standard computation

$$L_{[\tau,\tau']} = [d, [[d,\tau], \tau']] = [[d,\tau],[d,\tau']] = [L_\tau, L_{\tau'}] .$$

3.2.4 The generalized inner derivation Lie $(n+1)$-algebra $\mathrm{inn}_k(\mathfrak{g}_{(n)})$

Definition 17 *For* $\mathfrak{g}_{(n)}$ *any Lie n-algebra and for* $k \in \mathbb{N}$*, the Lie* $(n+1)$*-algebra*

$$\mathrm{inn}_k(\mathfrak{g}_{(n)}) \subset \mathrm{DER}(\mathfrak{g}_{(n)})$$

is obtained by restricting all derivations L *to be of the form*

$$L = [d, \iota] ,$$

where ι *is a degree -1 derivation which restricts to 0 on elements* $\omega \in sV^*$ *of degree larger than* k*:*

$$|\omega| > k \;\Rightarrow \iota(\omega) = 0 .$$

3.2.5 The inner derivation Lie $(n+1)$-algebra $\mathrm{inn}(\mathfrak{g}_{(n)})$

Here we give a description of $\mathrm{inn}(\mathfrak{g}_{(n)})$ in terms of qfDGCA. WARNING: At the moment we do not yet strictly derive this description from definition 17.

Definition 18 *Let* V *be the graded vector space underlying* $\mathfrak{g}_{(n)}$ *and let*

$$(\bigwedge^\bullet (sV)^*, d)$$

be the corresponding qfDGCA defined on it. Denote by

$$\sigma : \; (sV)^* \xrightarrow{\sim} (ssV)^*$$

the canonical isomorphism of degree +1 and let

$$\Sigma : \bigwedge^\bullet((sV)^* \oplus (ssV)^*) \to \bigwedge^\bullet((sV)^* \oplus (ssV)^*)$$

be the graded differential of degree +1 which restricts to σ *on* $(sV)^*$ *and to zero on* $(ssV)^*$*.*

Then define the graded differential of degree +1

$$d' : \bigwedge^\bullet((sV)^* \oplus (ssV)^*) \to \bigwedge^\bullet((sV)^* \oplus (ssV)^*)$$

by demanding that it restricts to $d' := d + \sigma$ *on* $(sV)^*$ *and to* $-\Sigma \circ d \circ \sigma^{-1} = -d' \circ d \circ \sigma^{-1}$ *on* $(ssV)^*$*. Since* $d \circ \sigma^{-1} : (ssV)^* \to \bigwedge^\bullet (sV)^*$ *this makes sense.*

Proposition 6

$$d'^2 = 0\,.$$

Proof. For any $a \in (sV)^*$ we have

$$d'd'a = d'(da + \sigma(a)) = \Sigma(da) - \Sigma da = 0\,.$$

Hence $d'^2 = 0$ when restricted to $\bigwedge^\bullet (sV)^*$. Next, for any $b \in (ssV)^*$, we have

$$d'd'b = -d'd'd\sigma^{-1}(b)\,.$$

But $d\sigma^{-1}(b) \in \bigwedge^\bullet (sV)^*$, hence $d'd'b = 0$. □

Definition 19 (inner derivation Lie $(n+1)$-algebra) *For $\mathfrak{g}_{(n)}$ any Lie n-algebra given by the qfDGCA $(\bigwedge^\bullet (sV)^*, d)$, we say that*

$$\mathrm{inn}(\mathfrak{g}_{(n)})$$

given by the qfDGCA

$$(\bigwedge\nolimits^\bullet ((sV)^* \otimes (ssV)^*), d')$$

with d' as in definition 18 is the inner derivation Lie $(n+1)$-algebra *of $\mathfrak{g}_{(n)}$.*

Proposition 7 ($\mathrm{inn}(\mathfrak{g}_{(n)})$ is free and therefore trivializable) *For any $\mathfrak{g}_{(n)}$ coming from the qfDGCA $(\bigwedge^\bullet (sV)^*, d_{\mathfrak{g}_{(n)}})$, the qfDGCA of $\mathrm{inn}(\mathfrak{g}_{(n)})$ is isomorphic to the free differential graded commutative algebra on $(sV)^*$ and therefore trivializable.*

Proof. Write $F(V)$ for the free differential graded commutative algebra over $(sV)^*$. Define a morphism

$$f : F(V) \to (\mathrm{inn}(\mathfrak{g}_{(n)}))^*$$

by setting

$$f : a \mapsto a$$

$$f : d_{F(V)}a \mapsto d_{\mathrm{inn}(\mathfrak{g}_{(n)})}a$$

for all $a \in (sV)^*$. This clearly satisfies the morphism property. One checks that its inverse is given by

$$f^{-1} : a \mapsto a$$

$$f^{-1} : \sigma a \mapsto d_{F(V)}a - d_{\mathfrak{g}_{(n)}}a\,.$$

□

Since $\mathrm{inn}(\mathfrak{g}_{(n)})$ is isomorphic to a free qfDGCA according to prop. 7, and since every free qfDGCA is trivializable according to proposition 3, it follows that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is trivializable.

Remark. That $\mathrm{inn}(\mathfrak{g}_{(n)})$ is *trivializable* (as opposed to *trivial*) does not mean that there is no useful information contained in it. We will see various examples in which $\mathrm{inn}(\mathfrak{g}_{(n)})$ is useful.

Proposition 8 (inclusion of a Lie n-algebra into its inner derivations) *Given any Lie n-algebra $\mathfrak{g}_{(n)}$, we canonically get an inclusion (a monomorphism)*

$$\mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g}_{(n)})\,.$$

Proof. Let $\mathfrak{g}_{(n)}$ be defined on a graded vector space V, i.e. such that it corresponds to a qfDGCA defined on $\bigwedge^\bullet(sV)^*$. Then define a morphism

$$f : \mathfrak{g}_{(n)} \to \mathrm{inn}(\mathfrak{g}_{(n)})$$

by its dual f^* being the identity on $(sV)^*$ and zero on $(ssV)^*$.

This is clearly a morphism of the corresponding qfDGCAs, since for all $a \in (sV)^*$ we have

$$f^*(d_{\mathrm{inn}(\mathfrak{g}_{(n)})}a) = f^*((d_{\mathfrak{g}_{(n)}} + \Sigma)(a)) = f^*(d_{\mathfrak{g}_{(n)}}a) = d_{\mathfrak{g}_{(n)}} f^*(a)$$

and

$$f^*(d_{\mathrm{inn}(\mathfrak{g}_{(n)})}\Sigma a) = f^*(\Sigma da) = 0 = d_{\mathfrak{g}_{(n)}}(f^*\Sigma a)\,.$$

Moreover, it is clear that f^* is an epimorphism. Hence f is a monomorphism. $\square$

Definition 20 *For any 1-morphism*

$$f^* \;:\; (\bigwedge^\bullet(sW)^*, d) \to (\bigwedge^\bullet(sV)^*, d)$$

of qfDGCAs, let

$$\mathrm{inn}(f^*) \;:\; \mathrm{inn}(\bigwedge^\bullet(sW)^*, d) \to \mathrm{inn}(\bigwedge^\bullet(sV)^*, d)$$

be the 1-morphism whose component map is given by

$$\mathrm{inn}(f^*) : a \mapsto f^*(a)$$

and

$$\mathrm{inn}(f^*) : \sigma_W a \mapsto \sigma_V(f^*(a))$$

for all $a \in (sW)^*$.

Proposition 9 *The map $\mathrm{inn}(f^*)$ does indeed satisfy the morphism property.*

Proof. For all $a \in (sW)^*$ we have

$$f^*(d'_W a) = f^*(d_W + \Sigma_W)(a) = (d_V + \Sigma_V) f^*(a) = d' f^*(a)$$

and

$$f^*(d'_W \sigma_W a) = f^*(\Sigma_W d_W a) = \Sigma_V d_V f^*(a) = \Sigma_V d_V \sigma_V^{-1}(\sigma_V f^*(a)) = d'_V f^*(\sigma_W a)\,.$$

$\square$

Definition 21 *For any $(j > 1)$-morphism of qfDGCAs*

$$h : \bigwedge^\bullet (sW)^* \to \bigwedge^{\bullet-(j-1)}(sV)^*$$

let

$$\mathrm{inn}(h) : \bigwedge^\bullet ((sW)^* \otimes (ssW)^*) \to \bigwedge^{\bullet-(j-1)}((sV)^* \otimes (ssV)^*)$$

be the j-morphism of the corresponding inner derivation qfDGCAs whose component map is given by

$$\mathrm{inn}(h) \; : \; a \mapsto h(a)$$

and

$$\mathrm{inn}(h) \; : \; \sigma_W a \mapsto (-1)^{j-1}\Sigma_V h(a)\,.$$

To prove that this indeed satisfies the required morphism properties, first consider the following lemma:

Lemma 1

$$[\Sigma, \mathrm{inn}(h)]|_{\bigwedge^\bullet (sW)^*} = 0\,.$$

Proof. Both operations are derivations, hence it suffices to compute the graded commutator on generators. There it vanishes by definition of inn(h). □

Proposition 10 *Definition 21 is compatible with the required morphism properties.*

Proof. For all $a \in (sW)^*$ we have

$$[d', \mathrm{inn}(h)](a) = (d_W + \Sigma_W)h(a) + (-1)^j \mathrm{inn}(h)(d_W + \Sigma_W)(a) = [d,h](a)$$

and

$$\begin{aligned} [d', \mathrm{inn}(h)](\sigma_W a) &= (-1)^{j-1} d'_V \Sigma_V h(a) + (-1)^j \mathrm{inn}(h)\Sigma_V da \\ &= (-1)^j \Sigma_V dh(a) - \Sigma_V h(da) \\ &= (-1)^{j-2}\Sigma_V [d,h](a)\,, \end{aligned}$$

where the second step uses lemma 1. □

This means that inn$(\cdot)$ respects the composition of j-morphisms with themselves. It also respects all other compositions.

Lemma 2 *The operation*

$$\mathrm{inn}(\cdot) : \mathrm{Mor}(\omega Lie) \to \mathrm{Mor}(\omega Lie)$$

is compatible with all compositions.

Proof. Most compositions correspond to adding the component functions and inn$(\cdot)$ is clearly compatible with that. One checks that it is also compatible with compositions involving 1-morphisms. □

URS: this needs to be checked again once we have to definition of ωLie completely spelled out.

In summary, we find the following:

Corollary 1 *Forming inner derivations is an ω-functor*

$$\mathrm{inn}(\cdot) : \omega\mathrm{Lie} \to \omega\mathrm{Lie}\,.$$

4 Characteristic classes in terms of $\mathrm{inn}(\mathfrak{g})^*$ cohomology

Lie algebra cohomology, invariant polynomials and Chern-Simons elements can all be conveniently conceived in terms of the quasi-free differential graded algebra corresponding to the Lie 2-algebra

$$\mathrm{inn}(\mathfrak{g})$$

of inner derivations of the Lie algebra $\mathfrak{g}$.

The relation to the more common formulation of these phenomena in terms of the cohomology of the universal G-bundle comes from the fact that this universal bundle is the realization of the nerve of $\mathrm{INN}(G)$.

4.1 Formulation in terms of the cohomlogy of EG

Let G be a compact, simply connected simple Lie group.

The classical formulation of

- Lie algebra cocycles
- invariant polynomials
- transgression induced by Chern-Simons elements

is the following.

Consider the fibration corresponding to the universal principal G-bundle:

$$G \longrightarrow EG \xrightarrow{p} BG \;.$$

- A Lie algebra $(2n+1)$-cocycle μ (with values in a trivial module) is an element
$$\mu \in H^{2n+1}(\mathfrak{g}, \mathbb{R})\,.$$
By compactness of G, this is the same as an element in de Rham cohomology of G:
$$\mu \in H^{2n+1}(G, \mathbb{R})\,.$$
- An invariant polynomial k of degree $n+1$ represents an element in
$$k \in H^{2n+2}(BG, \mathbb{R})\,.$$
- A transgression form mediating between μ and k is a cochain $cs \in \Omega^{2n+1}(EG)$ such that
$$cs|_G = \mu$$
and
$$d\,cs = p^* k\,.$$

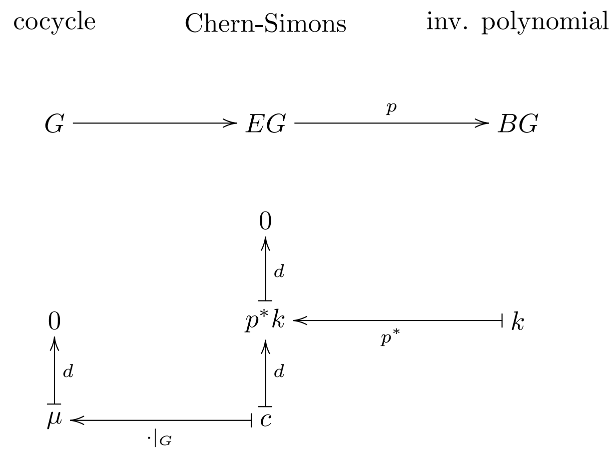


Figure 2: **Lie algebra cocycles, invariant polynomials and transgression forms** in terms of cohomology of the universal G-bundle.

4.2 Formulation in terms of cohomology of $\mathrm{inn}(\mathfrak{g})^*$

The universal G-bundle may be obtained from the sequence of groupoids

$$\mathrm{Disc}(G) \to \mathrm{INN}(G) \to \Sigma G$$

by taking geometric realizations of nerves:

$$\begin{array}{ccccc} \mathrm{Disc}(G) & \longrightarrow & \mathrm{INN}(G) & \longrightarrow & \Sigma G \\ \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} \\ G & \longrightarrow & EG & \longrightarrow & BG \end{array} .$$

$\mathrm{Disc}(G)$ and $\mathrm{INN}(G)$ are strict 2-groups, coming from the crossed modules

$$\mathrm{Disc}(G) = (1 \to G)$$

and

$$\mathrm{INN}(G) = (\mathrm{Id} : G \to G) .$$

On the other hand, ΣG is a 2-group only if G is abelian.

4.2.1 Cocycles, invariant polynomials and Chern-Simons elements

Differentially, this corresponds to the sequence

$$\begin{array}{ccccc} \mathrm{Disc}(G) & \longrightarrow & \mathrm{INN}(G) & \xrightarrow{p} & \Sigma G \\ \downarrow{\scriptstyle \mathrm{Lie}} & & \downarrow{\scriptstyle \mathrm{Lie}} & & \downarrow \\ \bigwedge^\bullet s\mathfrak{g}^* & \longleftarrow & \bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*) & \xleftarrow{p^*} & \bigwedge^\bullet (ss\mathfrak{g}^*) \end{array} .$$

In terms of this, we have

- A Lie algebra $(2n+1)$-cocycle μ (with values in a trivial module) is an element
$$\mu \in \bigwedge^{(2n+1)}(s\mathfrak{g}^*)$$
$$d_{\mathfrak{g}}\mu = 0\,.$$
- An invariant polynomial k of degree $n+1$ is an element
$$k \in \bigwedge^{n+1}(ss\mathfrak{g}^*)$$
$$d_{\mathrm{inn}(\mathfrak{g})}k = 0\,.$$
- A transgression form cs inducing a transgession between a $(2n+1)$-cocycle μ and a degree $(n+1)$-invariant polynomial is a degree $(2n+1)$-element
$$cs \in \bigwedge(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$
such that
$$cs|_{\bigwedge^\bullet(s\mathfrak{g}^*)} = \mu$$
and
$$d_{\mathrm{inn}(\mathfrak{g})}cs = p^*k\,.$$

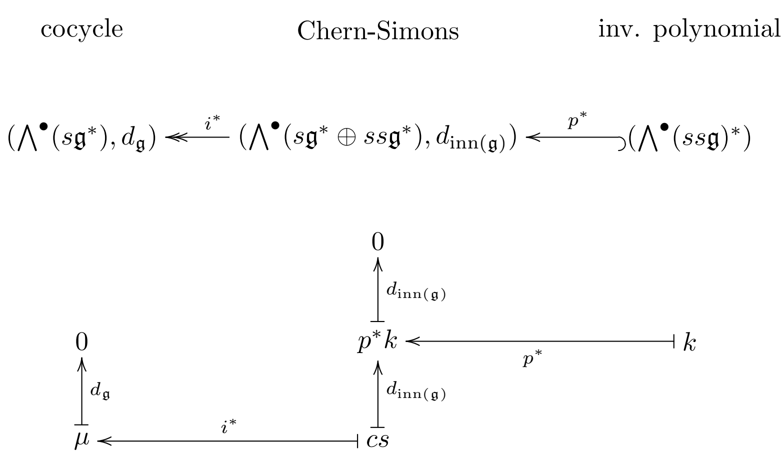


Figure 3: **Lie algebra cocycles, invariant polynomials and transgression elements** in terms of cohomology of $\mathrm{inn}(\mathfrak{g})$.

4.2.2 Transgression and the trivializability of $\mathrm{inn}(\mathfrak{g})$

It is important that

EG is contractible
$\Leftrightarrow$ $\mathrm{INN}(G)$ is trivializable
$\Leftrightarrow$ the cohomology of $\mathrm{inn}(\mathfrak{g})^* = (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})})$ is trivial
$\Leftrightarrow$ there is a homotopy $\tau : 0 \to \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}$, i.e. $[d_{\mathrm{inn}(\mathfrak{g})}, \tau] = \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}$.

This implies that if

$$cs$$

is to be a transgression element mediating between μ and k, then we have

$$cs = \tau(p^* k) + d_{\mathrm{inn}(\mathfrak{g})} q \,.$$

So for every invariant polynomial k

$$d_{\mathrm{inn}(\mathfrak{g})} k = 0$$

a “potential” c does exist. The nontrivial condition is then that cs restricted to $\mathfrak{g}$ is a cocyle.

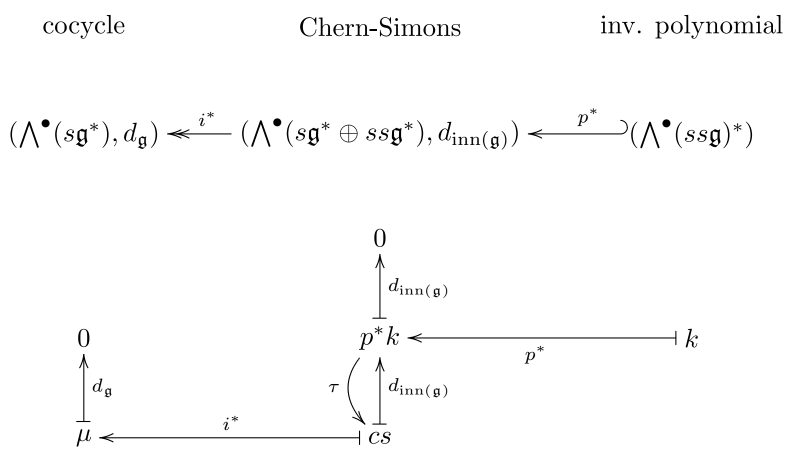


Figure 4: **The homotopy operator** τ exists due to the trivializability of $\mathrm{inn}(\mathfrak{g})$.

4.3 Formulation in terms of components

From the $\mathrm{inn}(\mathfrak{g})$-description it is easy to read off the properties of cocycles and invariant polynomials in terms of components:

Fix a Lie algebra $\mathfrak{g}$ and a basis $\{X_a\}$ with dual basis $\{t^a\}$, regarded as a basis of $s\mathfrak{g}^*$ and $\{r^a\}$, regarded as a basis of $ss\mathfrak{g}^*$.

- A Lie $(2n+1)$-cocylce is a completely antisymmetric tensor

$$\mu = \mu(t) = \mu_{a_1 \cdots a_{2n+1}} t^{a_1} \wedge \cdots t^{a_{2n+1}}$$

  such that

$$\sum_{i=1^{2n+1}} (-1)^i \mu_{[a_1 \cdots a_i \cdots a_{2n+1}} C^{a_i}{}_{bc]} = 0 \,.$$

- A degree $n+1$ symmetric invariant polynomial is a completely symmetric tensor

$$k = k(r) = k_{a_1 \cdots a_{n+1}} r^{a_1} \wedge \cdots \wedge r^{a_{n+1}}$$

such that

$$\sum_{i=1^{2n+1}} k_{a_1 \cdots a_i \cdots a_{n+1}} C^{a_i}{}_{bc} = 0\,.$$

By explicitly computing the homotopy operator τ (compare Chern and Simons [**?**]), using the theory of derivation homotopies, we find that the restriction

$$\tau(k(r))|_{\bigwedge^\bullet(s\mathfrak{g}^*)}$$

has components proportional to

$$k_{a_1 a_2 \cdots a_{n+1}} t^{a_1} \wedge (d_{\mathfrak{g}} t^{a_1}) \wedge \cdots (d_{\mathfrak{g}} t^{a_{n+1}})\,.$$

5 Lie n-algebras from cocycles and from invariant polynomials

We shall Lie n-algebras which come from

- Lie algebra cocycles μ – these are the Lie n-algebras $\mathfrak{g}_\mu$ studied by Baez-Crans [2]
- invariant polynomials $k$on $\mathfrak{g}$ – these we identify as Chern Lie $(2n+1)$-algebras $\mathrm{ch}_k(\mathfrak{g})$
- invariant polynomials k of degree $(n+1)$ and an associated cocycle μ_k of degree $(2n+1)$ – these we identify as Chern-Simons Lie $(2n+1)$-algebras.

5.1 Baez-Crans Lie n-algebras $\mathfrak{g}_\mu$ from $(n+1)$-cocycles μ

Definition 22 *Let μ be an $(n+1)$-cocycle on $\mathfrak{g}$ as in proposition* **??**. *Then the Lie n-algebra*

$$\mathfrak{g}_\mu$$

is defined by the qfDGCA

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (s^n\mathbb{R})^*)$$

with the differential

$$dt^a = -\frac{1}{2} C^a{}_{bc} t^b t^c$$

$$db = -\mu(t)\,.$$

We may reformulate this equivalently, in fact slightly more generally, in L_∞-language:

Baez-Crans showed [2] that Lie n-algebras which are concentrated in top and bottom degree are all equivalent to Lie n-algebras of the following form.

Definition 23 *For $\mathfrak{g}$ any Lie algebra, m any $\mathfrak{g}$-module and*

$$h \in H^{n+1}(\mathfrak{g}, m)$$

a Lie algebra $(n+1)$-cocycle for $\mathfrak{g}$ with values in m, the semistrict Lie n-algebra

$$\mathfrak{h}^n(\mathfrak{g}, m)$$

is defined to be the L_∞-algebra on

$$S^c(s\mathfrak{g} \oplus s^n m)$$

with codifferential

$$D = d_1 + d_2 + d_{n+1}$$

defined by

$$\begin{aligned} d_2(sX \vee sY) &= s[X,Y] \\ d_1(sX \vee s^n B) &= s^n X(B) \\ d_{n+1}(sX_1 \vee \cdots \vee sX_{n+1}) &= s^n h(X_1, \cdots, X_{n+1})\,, \end{aligned}$$

for all $X, Y, X_i \in \mathfrak{g}$ and all $B \in m$.

We find that $D^2(sX \vee sY) = 0$ is the Jacobi identity on $\mathfrak{g}$, as before, and $D^2(sX \vee sY \vee B) = 0$ is the Lie module property of m. Finally

$$\begin{aligned} D^2(sX_1 \vee \cdots \vee sX_{n+2}) &= D\Bigg(\sum_{\sigma \in \mathrm{Sh}(1,n+1)} \epsilon(\sigma) sX_{\sigma(1)} \vee s^n h(X_{\sigma(2)}, \cdots, X_{\sigma(n+2)}) \\ &\quad + \sum_{\sigma \in \mathrm{Sh}(2,n)} \epsilon(\sigma) s[X_{\sigma(1)}, X_{\sigma(2)}] \vee sX_{\sigma(3)} \vee \cdots \vee sX_{\sigma(n+2)}\Bigg) \\ &= s^n \sum_{\sigma \in \mathrm{Sh}(1,n+1)} \epsilon(\sigma) X_{\sigma(1)} \left(h(X_{\sigma(2)}, \cdots, X_{\sigma(n+2)})\right) \\ &\quad + s^n \sum_{\sigma \in \mathrm{Sh}(2,n)} \epsilon(\sigma) h([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma(3)}, \cdots X_{\sigma(n+2)}) \\ &= 0 \end{aligned}$$

is precisely the Lie cocycle property of h.

URS: I think I have the signs right here, but should be checked again.

Remark. The Lie 2-algebra $\mathfrak{g}_k$ from **??** is a special case of this for $n = 2$, $m = \mathbb{R}$ the trivial $\mathfrak{g}$-module and $h(X, Y, Z) = \langle X, [Y, Z]\rangle$.

5.2 Chern Lie $(2n+1)$-algebra $\mathrm{cs}_k(\mathfrak{g})$ from invariant polynomials k.

Definition 24 *Let k be an invariant polynomial of degree $(n+1)$ on $\mathfrak{g}$ as in proposition* **??***. Then the Lie $(2n+1)$-algebra*

$$\mathrm{ch}_k(\mathfrak{g})$$

is defined in terms of the qfDGCA on

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}\mathbb{R})^*)$$

by

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c + r^a$$
$$dr^a = -C^a{}_{bc}t^b r^c$$
$$dc = k(r)\,.$$

Here $\{t^a\}$, $\{r^a\}$ is our basis choice for $\mathrm{inn}(\mathfrak{g})^*$ as usual and $\{b\}$ is the canonical basis of $(s^{2n+1}\mathbb{R})^*$.

That $d^2 = 0$ follows directly from the defining property $d_{\mathrm{inn}(\mathfrak{g})}k(r)$ of k.

5.3 Chern-Simons Lie $(2n+1)$-algebras $\mathrm{cs}_k(\mathfrak{g})$ from invariant polynomials k

Definition 25 *Let k be an invariant polynomial of degree $n+1$ such that it has a Chern-Simons potential $\tau(k(r))$ as in* **??** *Then the Lie $(2n+1)$-algebra*

$$\mathrm{cs}_k(\mathfrak{g})$$

is defined on

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n}\mathbb{R}) \oplus (s^{2n+1}\mathbb{R})^*)$$

by

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b t^c + r^a$$
$$dr^a = -C^a{}_{bc}t^b r^c$$
$$db = -(\mu_k(t) + Q(t,r)) + c$$
$$dc = k(r)\,,$$

*where $Q(t,r)$ is as in (**??**).*

That $d^2 = 0$ here follows directly from the defining properties of $k(r)$.

5.4 Higher abelian Chern-Simons Lie $(2n+1)$-algebras $\mathrm{cs}_k(\Sigma^n\mathfrak{u}(1))$

There is another notion of higher Chern-Simons Lie $(2n+1)$-algebras, coming from just the abelian Lie algebra $\mathfrak{u}(1)$ but involving higher differential forms.

Definition 26 *For odd $n \in \mathbb{N}$ and any $k \in \mathbb{R}$, define the Lie $(2n+1)$-algebra*

$$\mathrm{cs}_k(\Sigma^n\mathfrak{u}(1))$$

by the qfDGCA which is defined on

$$\bigwedge^\bullet((s^n\mathfrak{u}(1))^* \oplus (s^{n+1}\mathfrak{u}(1))^* \oplus (s^{2n}\mathfrak{u}(1))^*) \oplus (s^{2n+1}\mathfrak{u}(1))^*)$$

by

$$da = r$$
$$dr = 0$$
$$db = -k\, a \wedge r + c$$
$$dc = k\, r \wedge r\,,$$

for $\{a\}$ *a choice of basis of* $(s^n \mathfrak{u}(1))^*$, $\{r\}$ *a choice of basis of* $(s^{n+1}\mathfrak{u}(1))^*$ *and* $\{c\}$ *a choice of basis of* $(s^{2n+1}\mathfrak{u}(1))^*$ *.*

That $d^2 = 0$ is immediate.

Remark. From the point of view of n-connections, the objects studied in [15] can be regarded as $(2n+1)$-connections with values in $\mathrm{cs}_k(\Sigma^n \mathfrak{u}(1))$. See 5.5.1.

5.5 Morphisms

5.5.1 Higher Chern-Simons forms

Proposition 11 (Chern forms) $(2n+1)$-*Connections with values in the Chern Lie* $(2n+1)$-*algebras* $\mathrm{ch}_k(\mathfrak{g})$ *are in bijective correspondence with tuples*

$$(A, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n+1}(X)$$

such that

$$dC = dk(F_A \wedge \cdots \wedge F_A)\,.$$

Proposition 12 (Chern-Simons forms) $(2n+1)$-*Connections with values in the Chern-Simons Lie* $(2n+1)$-*algebras* $\mathrm{cs}_k(\mathfrak{g})$ *are in bijective correspondence with tuples*

$$(A, B, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n}(X) \times \Omega^{2n+1}(X)$$

such that

$$C = dB + k\mathrm{CS}_k(A)\,.$$

Here $\mathrm{CS}_k(A)$ is the k-Chern-Simons form, such that

$$dC = k(F_A \wedge \cdots \wedge F_A)\,.$$

Proposition 13 $(2n+1)$-*Connections with values in the Chern-Simons Lie* $(2n+1)$-*algebras* $\mathrm{cs}_k(\Sigma^n \mathfrak{u}(1))$ *from 5.4, i.e. qfDGCA-morphisms*

$$f^* : (\mathrm{cs}_k(\Sigma^n \mathfrak{u}(1)))^* \to \Omega^\bullet(X)\,,$$

are in bijective correspondence with p-forms

$$(A, B, C) \in \Omega^n(X) \times \Omega^{2n}(X) \times \Omega^{(2n+1)}(X)$$

such that

$$C = dB + k\, A \wedge dA\,.$$

5.5.2 The isomorphism $\mathrm{inn}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{cs}_k(\mathfrak{g})$

Proposition 14 *We have an equivalence (even an isomorphism)*

$$\mathrm{inn}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{cs}_k(\mathfrak{g})$$

whenever the latter exists.

Proof. One checks that the assignments

$$\begin{aligned} t^a &\mapsto t^a \\ r^a &\mapsto r^a \\ b &\mapsto b \\ c &\mapsto c \pm Q \end{aligned}$$

define morphisms between the two Lie $(2n+1)$-algebras. These are clearly strict inverses of each other. □

Remark. Together with proposition 7 this implies that $\mathrm{cs}_k(\mathfrak{g})$ is in fact trivializable.

5.5.3 The exact sequence $0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0$

Suppose that the degree $n+1$ invariant polynomial $k(r)$ admits a Chern-Simons potential, i.e. such that all three Lie $(2n+1)$-algebras

- $\mathfrak{g}_{\mu_k}$ – 5.1
- $\mathrm{cs}_k(\mathfrak{g})$ – 5.3
- $\mathrm{ch}_k(\mathfrak{g})$ – 5.2 .

Then we have the following morphisms between these.

Proposition 15 *We have a canonical surjection*

$$i : \ \mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g}) \ .$$

Proof. One checks that the canonical inclusion of vector spaces

$$\bigwedge\nolimits^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}\mathbb{R})^*) \hookrightarrow \bigwedge\nolimits^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n}\mathbb{R})^* \oplus (s^{2n+1}\mathbb{R})^*)$$

gives a monomorphic qfDGCA-morphism

$$(\mathrm{ch}_k(\mathfrak{g}))^* \to (\mathrm{cs}_k(\mathfrak{g}))^*$$

hence defines an epimorphic dual morphism. □

Proposition 16 *We have a canonical injection*

$$i : \ \mathfrak{g}_{\mu_k} \hookrightarrow \mathrm{cs}_k(\mathfrak{g}) \ .$$

Proof. One checks that the canonical surjection of vector spaces

$$\bigwedge\nolimits^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n}\mathbb{R})^*) \hookrightarrow \bigwedge\nolimits^\bullet((s\mathfrak{g})^* \oplus (s^{2n+1}\mathbb{R})^*)$$

gives an epimorphic qfDGCA-morphism

$$(\mathrm{cs}_k(\mathfrak{g}))^* \to (\mathfrak{g}_{\mu_k})^*$$

hence defines a monomorphic dual morphism. □

Remark. Notice that it is the algebra property of this map which crucially depends on the fact (**??**) that in

$$k(r) = d_{\mathrm{inn}(\mathfrak{g})}(\mu(t) + Q(t,r))$$

the $Q(t,r)$ vanishes when restricted to $\bigwedge^\bullet(s\mathfrak{g})^*$.

Proposition 17 *The composite morphism*

$$\mathfrak{g}_{\mu_k} \hookrightarrow \mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g})$$

is homotopic to the zero-morphism.

Proof. By the above, the dual morphism is the identity on the generators of $(s\mathfrak{g})^*$

$$f^* : t^a \mapsto t^a$$

and sends everything else to zero. This is reproduced by the derivation homotopy which acts as

$$\tau : t^a \mapsto 0$$
$$\tau : r^a \mapsto t^a$$
$$\tau : c \mapsto 0\,.$$

On t^a this is immediate, on r^a this depends crucuially on the prefactors obtained by extending to a derivation homotopy

$$\begin{aligned}[d,\tau](r^a) &= d_{\mathfrak{g}_{\mu_k}} t^a + \tau(-C^a{}_{bc}t^b r^c)\\ &= -\frac{1}{2}C^a{}_{bc}t^b t^c + \frac{1}{2}C^a{}_{bc}t^b t^c\\ &= 0\,.\end{aligned}$$

□

In summary this gives

Corollary 2 *Whenever the* $(2n+1)$*-cocycle* μ_k *on* $\mathfrak{g}$ *and the invariant degree* $(n+1)$*-polynomial* k *are related by transgression, we have an exact sequence of Lie* $(2n+1)$*-algebras*

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0\,.$$

6 Remaining Proofs

6.1 Relation of Baez-Crans 2-morphisms to derivation homotopies

Proof of proposition 4:

Let $(\bigwedge^\bullet(V_0^* \oplus V_1^*), d_V)$ and $(\bigwedge^\bullet(W_0^* \oplus W_1^*), d_W)$ be DGCAs and

$$f_1^*, f_2^* : (\bigwedge^\bullet(W_0^* \oplus W_1^*), d_W) \to (\bigwedge^\bullet(V_0^* \oplus V_1^*), d_V)$$

be two morphisms as in the proof of proposition **??**.

A morphism

$$\tau : q' \to q$$

between these (a derivation homotopy) is a map $\tau : \bigwedge^\bullet(W_0^* \oplus W_1^*) \to \bigwedge^\bullet(V_0^* \oplus V_1^*)$ of degree -1 which acts on the generators as

$$\tau : b^i \mapsto \tau^i{}_a a'^a$$

and

$$\tau : a^a \mapsto 0\,.$$

We have

$$[d,\tau] : a^a \mapsto -t^a{}_i \tau^i{}_b a'^b$$

and

$$[d,\tau] : b^i \mapsto -\frac{1}{2}\tau^i{}_a C'^a{}_{bc} a'^b a'^c - \tau^i{}_a t'^a{}_j b'^j + \alpha^i{}_{aj}(q + \frac{1}{2}[d,\tau])^a{}_b \tau^j{}_c a'^b a'^c\,. \quad (12)$$

Then

$$q - q' = [d,\tau]$$

is equivalent to

$$(q^a{}_b - q'^a{}_b) a'^b = -t^a{}_i \tau^i{}_b a'^b$$

and

$$(q^i{}_j - q'^i{}_j) b'^j = -\tau^i{}_a t'^a{}_j b'^j$$

and

$$\frac{1}{2}(q^i{}_{ab} - q'^i{}_{ab}) a'^a a'^b = -\frac{1}{2}\tau^i{}_a C'^a{}_{bc} a'^b a'^c + \alpha^i{}_{aj}(q + \frac{1}{2}[d,\tau])^a{}_b \tau^j{}_c a'^b a'^c\,,$$

where we have used the property (8) of a derivation homotopy.

The first two equation express the fact that τ is a chain homotopy with respect to t and t'. The last equation is equivalent to

$$\begin{aligned} q_2(x,y) - q_2'(x,y) &= -\tau([x,y]) + [q(x) + \frac{1}{2}t(\tau(x)), \tau(y)] - [q'(y) - \frac{1}{2}t(\tau(y)), \tau(x)] \\ &= -\tau([x,y]) + [q(x), \tau(y)] + [\tau(x), q'(y)] \end{aligned}$$

This is exactly the Baez-Crans condition on a 2-morphism. □

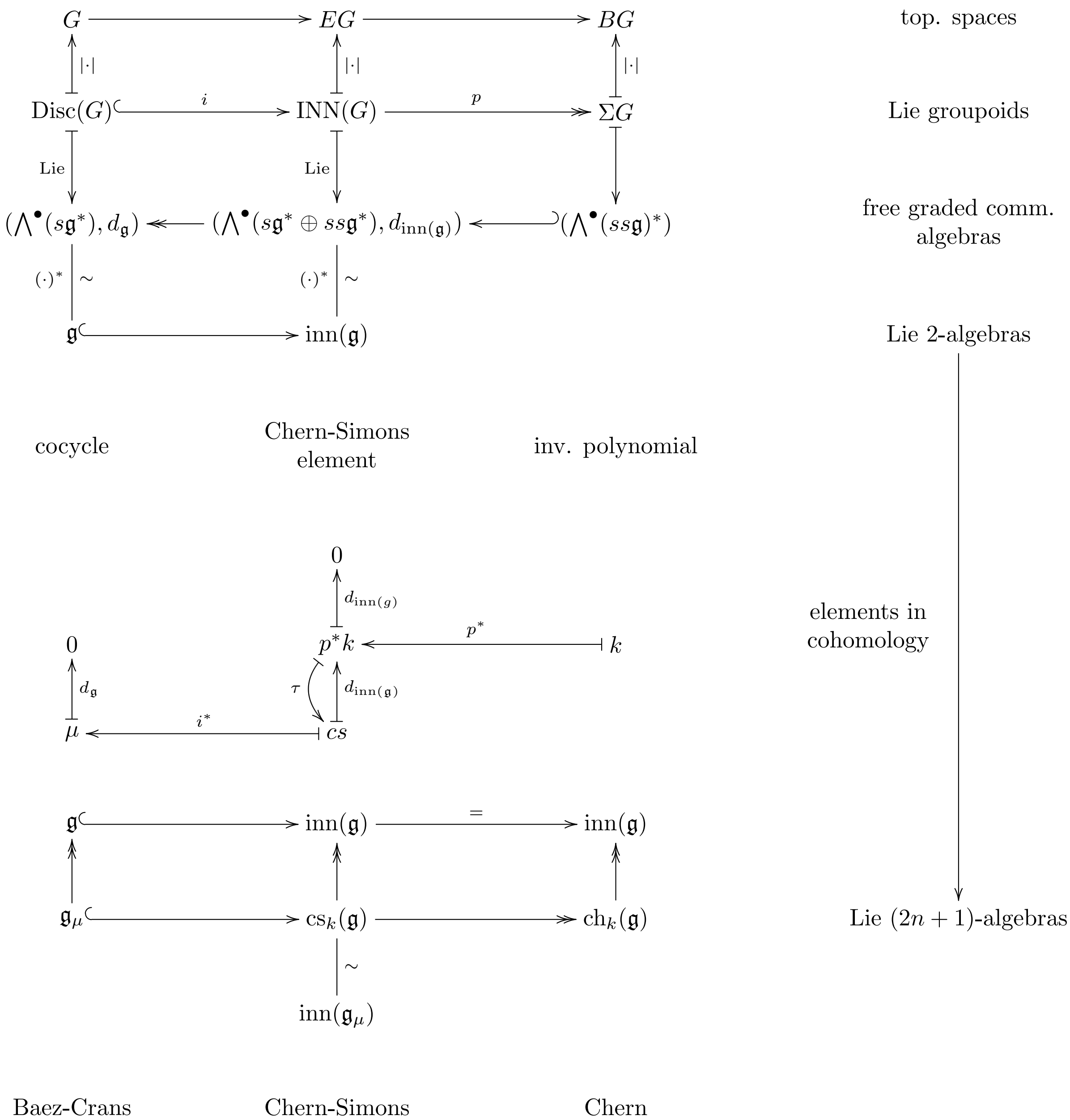


Figure 5: **Chern Lie** $(2n+1)$**-algebras**: for each Lie algebra $(n+1)$ cocycle μ which is related by transgression to an invariant polynomial k we obtain an exact sequence of Lie $(2n+1)$-algebras.

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