Some Stuff about the String Gerbe

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1 Introduction

Let G be a compact, simple and simply connected Lie group and let $\pi\colon P \to M$ be a principal G bundle over a smooth manifold M. Let ν denote the universally transgressive generator of $H^3(G;\mathbb{Z}) = \mathbb{Z}$ and let $c \in H^4(BG;\mathbb{Z}) = H^3(G;\mathbb{Z})$ be the transgression of ν. By regarding $H^4(BG;\mathbb{Z})$ as a lattice in $(S^2\mathfrak{g}^*)^G$ we will allow ourselves to confuse c and the **basic** inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Recall [23] that $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{g}$, normalised so that the longest root θ has length $\sqrt{2}$. In the physics literature, M is said to be **string**, or admit a **string structure**, if a certain characteristic class in $H^3(LM;\mathbb{Z})$ vanishes (here LM denotes the free loop space of M). This characteristic class is the obstruction to lifting the structure group of the principal LG-bundle $LP \to LM$ to $\widehat{LG}$ — the Kac-Moody group. As has been observed by several authors [12, 13, 19] the obstruction in $H^3(LM;\mathbb{Z})$ is closely related to the characteristic class $c \in H^4(M;\mathbb{Z})$: if M is 2-connected a lift of the structure group to $\widehat{LG}$ exists precisely when the map $c\colon M \to K(\mathbb{Z},4)$ is null-homotopic. As is well known, if $G = \mathrm{Spin}(n)$ then $2c = p_1$. This obstruction problem on LM can be phrased in the language of homotopy theory down on M. Recall [25] that G fits into a short exact sequence of topological groups

$$1 \to K(\mathbb{Z},2) \to \hat{G} \to G \to 1$$

where $\hat{G}$ is the 3-connected cover of G. $\hat{G}$ is a topological group which can be defined in a homotopy theoretic manner as the homotopy fibre of the canonical map $G \to K(\mathbb{Z}, 3)$ classifying ν. When $G = \mathrm{Spin}(n)$ the group $\hat{G}$ is called $\mathrm{String}(n)$. $\hat{G}$ has vanishing third homotopy group and therefore cannot have the homotopy type of any Lie group. The obstruction problem on LM of finding a lift of the structure group of LP to $\widehat{LG}$ translates into the problem down on M of finding a lift of the structure group of P from G to $\hat{G}$ (from this perspective it is slightly easier to see that if M is 2-connected a lift exists precisely when the characteristic class c on M vanishes).

Suppose that M is string, and a lift $\hat{P}$ of P to a principal $\hat{G}$-bundle exists. In the work [25] of Stolz and Teichner on elliptic objects it is important to make sense of the geometry of the bundle $\hat{P}$. Since $\hat{G}$ is only a topological group and not a Lie group, connections and curvature cannot be understood in the conventional sense. To get around this problem, Stolz and Teichner introduce the notion of a *string connection*, which appears to be closely related to the notion of higher dimensional parallel transport studied in [4, 9]. In this note we want to promote the point of view that one can make sense of the geometry of the bundle $\hat{P}$ by replacing the topological group $\hat{G}$, by the Fréchet Lie 2-group $\hat{\mathbb{G}}$ considered in [2], and study the geometry of what one might well call a 'principal $\hat{\mathbb{G}}$-bundle' $\mathbb{P}$ with $\hat{\mathbb{G}}$ as its 'structure 2-group'. The notion of a principal bundle for a 2-group is introduced in Definition 9. It is essentially a groupoid version of the notion of $\mathcal{G}$-*torsor* for a gr-stack introduced by Breen in [7]. We recall [7] that isomorphism classes of $\mathcal{G}$-torsors parametrise the degree non-abelian cohomology set $H^1(M; \mathcal{G})$ in the same way that isomorphism classes of principal G-bundles parametrise $H^1(M; G)$ for G a topological group. If $\mathcal{G}$ is the gr-stack associated to a crossed module $t\colon H \to G$, then $\mathcal{G}$-torsors correspond bijectively to (G, H)-gerbes. Our main result is

Theorem 15. *Suppose that M is 2-connected and that the class $c \in H^4(M; \mathbb{Z})$ vanishes. Then there is a principal $\hat{\mathbb{G}}$-bundle $\mathbb{P}$ on $\mathbb{M}$, together with a morphism of principal bundles*

$$\begin{array}{ccc} \mathbb{P} & \longrightarrow & P \\ {\scriptstyle \hat{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{M} & \longrightarrow & M \end{array}$$

$\mathbb{P}$ corresponds to a $(\widehat{\Omega G}, P_0 G)$-gerbe on M which we call the **string gerbe** *on M.*

Here $t\colon \widehat{\Omega G} \to P_0G$ is the crossed module defining the 2-group $\hat{\mathbb{G}}$. The point of doing this is that one can then make sense of the geometry of this torsor through the theory of connections on non-abelian gerbes as developed by several authors [1, 4, 9] but most notably Breen and Messing. It would be very interesting to see how such an approach is related to Stolz and Teichner's notion of a string connection.

The subject of gerbes by now barely needs an introduction. Suffice it to say that $U(1)$-gerbes and their higher dimensional analogues, $U(1)$-n-gerbes, provide geometric realisations of $H^3(M;\mathbb{Z})$ and $H^{n+2}(M;\mathbb{Z})$ respectively, generalising the correspondence between $H^2(M;\mathbb{Z})$ and isomorphism classes of line bundles provided by the Chern class. In this sense $U(1)$-gerbes and their higher analogues should be thought of as 'higher line bundles'. Classically, one thinks of $H^{n+2}(M;\mathbb{Z})$ as isomorphism classes of principal $K(\mathbb{Z},n)$-bundles. From the point of view of differential geometry however such a description is not quite what one would hope for, as for increasing n it is progressively more difficult if not impossible to realise $K(\mathbb{Z},n)$ as a Lie group in any conventional sense. What one would of course like is a higher dimensional analogue of the classical Weil-Kostant theory of line bundles with connection; one would like to realise the characteristic class in $H^{n+2}(M;\mathbb{Z})$ associated to a principal $K(\mathbb{Z},n)$-bundle P as the 'curvature $(n+2)$-form' of some 'connection' on P, and for this one needs $K(\mathbb{Z},n)$ to be a Lie group. As pointed out to me by John Baez, it is profitable to think of $K(\mathbb{Z},n)$ as the n-group $U(1)[n-1]$, i.e the group object in $(n-2)-\mathbf{Gpd}(\mathbf{Man})$ ...

- explain this

In this picture, a $K(\mathbb{Z},n)$-bundle corresponds to a torsor in $(n-2)-\mathbf{Gpd}(\mathbf{Man})$ over M for the group object $U(1)[n-1]$. According to one's taste and the applications at hand, one may consider weak or strict $(n-2)$-groupoids. The classical picture in terms of principal $K(\mathbb{Z},n)$-bundles may be recovered by taking geometric realisations of the nerves of these objects. The point of this approach is that the n-group $U(1)[n-1]$ is a *Lie* n-group and one can thus hope to adapt methods of differential geometry to this setting. This approach has been carried out, at least for $n=3$ and 4, beginning with the work of Brylinski [10] and Murray [21] and continuing with the work of Brylinski and McLaughlin [11] (see also [24] for a continuation of the ideas in [21] to $n=4$). To our mind this theory of higher $U(1)$-gerbes is just a convenient language for interpreting the geometry of higher degree cohomology classes, and there certainly exist other languages which do the same task, sometimes more efficiently, for instance the Hopkins-Singer language of differential functions [16].

Of course, just as one should not restrict attention to principal $U(1)$-bundles alone but instead consider also principal G-bundles for other groups G, one should also replace the 2-group $U(1)[1]$ and its higher analogues by other 2-groups G, and consider torsors in $\mathbf{Gpd}(\mathbf{Man})$ for G. This leads one into the study of non-abelian gerbes. In fact, for gerbes to be anything other than such a convenient language, the theory of non-abelian gerbes should be developed in analogy with the theory of principal bundles. Such a development has begun to be undertaken by the authors cited above, particularly Larry Breen. However, it is to the detriment of the subject, we feel, that hitherto examples of non-abelian gerbes of interest in mathematical physics and differential geometry have not been forthcoming. The traditional example of a non-abelian gerbe described in [7] is the Schreier gerbe associated to an extension of groups $1 \to G \to H \to K \to 1$, but we are not aware of any other examples significantly different from this one. One of the purposes of this note is to present a novel example of a non-abelian gerbe — the so-called 'string gerbe' — which we feel is worthy of further study.

The main theme of this paper is *internalisation*. Recall that if C is a category, then one can consider the notion of a *category in* C or a category *internal* to C. A category E is said to be a category in C if the objects and morphisms E_0 and E_1 respectively of E are objects of C. The structural maps of E are also required to be morphisms in C. In this paper we will focus on the cases where C is the category of groups and the category of torsors. In §2 we review the notion of a group object in a category and the notion of a 2-group, i.e a category internal to the category of groups. In §3 we review various notions of torsors and corresponding gr-stacks following the discussion in [6]. In §4 we review the definition of the non -abelian cohomology set and describe various geometric objects giving rise to classes in this set. The geometric objects we focus on here can be thought of as torsor objects or as categories internal to the category of torsors. We relate these objects to the torsors of [6] and the crossed module bundle gerbes of [1]. §3 and §4 owe an obvious debt to the paper [6] of Breen. To our mind this is the classic work on a geometric description of non-abelian cohomology but it dosen't seem to have been as widely quoted as it should. In §5 we recall the construction of the *string* 2-group $\hat{\mathbb{G}}$ from [2]. We show how this 2-group is related to the group $\hat{G}$ of [25]. Finally in §6 we describe the construction of the string gerbe.

2 Internal Groups

2-groups are an important example of internal categories. Rather than giving a hackneyed exposition of the theory we will just recall the main points and refer the interested reader to [3]. Let $\mathcal{E}$ be a category with finite products and a terminal object, for instance $\mathcal{E}$ could be a topos T. Recall that a **group object** in $\mathcal{E}$ is an object G of $\mathcal{E}$ together with morphisms in $\mathcal{E}$

$$m\colon G \times G \to G, \qquad \mathrm{e}\colon 1 \to G, \qquad i\colon G \to G$$

where 1 is the terminal object in $\mathcal{E}$ such that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{1\times m} & G \\ {\scriptstyle m\times 1}\downarrow & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(1,i)} & G\times G \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} \\ & & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(1,e)} & G\times G \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} \\ & & G \end{array}$$

Write $\mathbf{Grp}(\mathcal{E})$ for the category of group objects in $\mathcal{E}$.

Definition 1. A (**strict**) 2-**group** in $\mathcal{E}$ is category internal to $\mathbf{Grp}(\mathcal{E})$, the category of group objects in $\mathcal{E}$. Therefore a 2-group is a category $\mathcal{G}$ with

- a **group object** of **objects** $\mathcal{G}_0$
- a **group object** of **morphisms** $\mathcal{G}_1$

together with

- **source** and **target homomorphisms** of group objects $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$
- a **composition homomorphism** $\circ\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$
- an **identity assigning homomorphism** $i\colon \mathcal{G}_0 \to \mathcal{G}_1$

making the usual diagrams commute.

For example $\mathcal{E}$ could be the category **Set** of sets, in which case a 2-group in $\mathcal{E}$ would be more usually called a 2-group. Or $\mathcal{E}$ could be the category **Man** of manifolds, then a 2-group in $\mathcal{E}$ would more usually be called a **Lie** 2-**group**. It is straightforward to see that for any of these categories $\mathcal{E}$, the 2-groups in $\mathcal{E}$ are precisely the group objects in $\mathbf{Cat}(\mathcal{E})$. The notion of a group object in **Cat** is the starting point for various weakenings of the notion of 2-group described in [3]. One can for instance require that the diagrams above do not commute on the nose, but instead commute up to some coherent natural isomorphism — this leads one to the notion of a *weak* 2-group (see [3]). 2-groups can also be described in terms of *crossed modules*.

- give definition of crossed module

Recall that a **crossed module** of groups consists of a pair of groups H and G together with a homomorphism $t\colon H \to G$, and a left action of G on H described by a homomorphism $\alpha\colon G \to \mathrm{Aut}(H)$ which satisfy the two conditions

$$t(\alpha(g)(h)) = gt(h)g^{-1}$$
$$\alpha(t(h))(h') = hh'h^{-1}$$

A prime example of a crossed module of groups is the crossed module $i\colon G \to \mathrm{Aut}(G)$ associated to any group G. Here the homomorphism α is just the identity.

3 Torsors and Gr-stacks

Let E be a site, for definiteness, suppose that $E = \mathbf{Top}$, with the local section topology. The notion of a *torsor* under a group object makes sense in any topos, in particular the Grothendieck topos $Sh(E)$ of sheaves on E. In [15] Giraud defines a *torsor in* E under a group object G in E to be an object P representing a torsor in $Sh(E)$. This is equivalent to the following requirements:

Definition 2. A G-**torsor in** E, for G a group object in E, consists of an object P of E together with

- an epimorphism $P \twoheadrightarrow *$, where $*$ is the final object of E, (in other words there is a cover $U_i \to *$ such that the sets $\mathrm{Hom}(U_i, P)$ are non-empty),
- a (right) action $P \times G \to P$ of G on P,

such that

- the natural map $P \times G \to P \times P$ is an isomorphism.

If M is an object of E then a G-**torsor in** E **over** M is a torsor in E/M.

We have chosen to use the language of sites and Grothendieck topologies even when it is clear that we are just interested in topological spaces, because we want to illustrate how, through the process of internalisation, the later Definition 9 is just a special instance of the present definition. If $E = \mathbf{Top}$ then a torsor in E over M is the same thing as a topological principal bundle over M. The G-torsors in E together with the morphisms between them form a groupoid G-TORS. Similarly, the G-torsors over objects in E together with the morphisms between them form a fibred category $G - \mathrm{TORS}_{/E} \to E$. If $E = \mathbf{Top}$ then this fibred category is just the usual classifying stack BG of topological principal G-bundles.

Suppose that $\mathbb{G}$ is a groupoid in E with objects G_0 and morphisms G_1. A $\mathbb{G}$-**torsor** in E is a non-empty object P over G_0 in E (i.e there is an epimorphism $P \twoheadrightarrow 1$ in E/G_0) together with a free and transitive action

$$P \times_{G_0} G_1 \to P.$$

If $\mathbb{G}$ is a groupoid in $Sh(E)$ then the $\mathbb{G}$-torsors in $Sh(E)$ provide a realisation of the stack associated to $\mathbb{G}$. If $\mathbb{G}$ is the 2-group associated to a crossed module of groups $t\colon H \to G$ in E, torsors for the groupoid $\mathbb{G}$ are the same as (G, H)-*torsors* in the sense of the following definition of Breen.

Definition 3. Suppose that $t\colon H \to G$ is a crossed module of groups in E. We say that (G, H)-**torsor** in E is a (right) H-torsor P together with a morphism $\phi\colon P \to G$ which satisfies $\phi(uh) = t(h)^{-1}\phi(u)$ on sections. Similarly, if M is an object of E, then we say that a (G, H)-**torsor over** M is a (G, H)-torsor in E/M.

Again, in the case where $E = \mathbf{Top}$, a (G, H)-torsor over M in E is the same thing as a (G, H)-*principal bundle* over M in the sense of the following definition:

Definition 4. Let $t\colon H \to G$ be a crossed module of topological groups. We say that a (G, H)-**principal bundle** on a topological space M consists of a local section admitting surjection $\pi\colon P \to M$ together with

- a right action $P \times H \to P$ of H on P
- a map $\phi\colon P \to G$

such that

- the H-action preserves the fibres of π in the sense that the diagram

$$\begin{array}{ccc} P \times H & \longrightarrow & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

  commutes

- the H-action is free and transitive in the sense that the canonical map

$$P \times H \to P \times_M P$$

  is a homeomorphism

- ϕ satisfies the equivariance property $\phi(uh) = t(h)^{-1}\phi(u)$ for all $u \in P$ and $h \in H$.

Clearly (G, H)-principal bundles can be understood as principal H-bundles P equipped with a trivialisation of the principal G-bundle $P \times^H G$ associated to P via the homomorphism $t\colon H \to G$. It is then clear that isomorphism classes of principal (G, H)-bundles on X correspond bijectively to homotopy classes of maps from X into the homotopy fibre of the map $BH \to BG$. In [18] Jurčo shows that this homotopy fibre is homotopy equivalent to the space $EH \times^H G$ which in turn identifies with $|\mathbb{G}|$, the geometric realisation of the nerve of the 2-group $\mathbb{G}$ associated to the crossed module $t\colon H \to G$. Here the quotient $EH \times^H G$ is formed with respect to the right action of H on $EH \times G$ in the usual way by

$$(x, g) \cdot h = (xh, t(h)^{-1}g) \tag{1}$$

Notice that the projection $EH \times G \to EH \times^H G$ is a (G, H)-torsor, where the equivariant map $\phi\colon EH \times G \to G$ is just projection onto the second factor in $EH \times G$. This is in fact the *universal* (G, H)-torsor.

In the special case where $t\colon H \to G$ is the crossed module $i\colon G \to \mathrm{Aut}(G)$ associated to a group G in E an $(\mathrm{Aut}(G), G)$-torsor is the same thing as a *G-bitorsor*. Recall that a G-**bitorsor** is a (right) G-torsor P in E equipped with a left action $G \times P \to P$ of G, commuting with the right G-action and is such that P is a left G-torsor for this action. Given a G-bitorsor P, we define an equivariant map $\phi\colon P \to \mathrm{Aut}(G)$ making P into a $(\mathrm{Aut}(G), G)$-torsor as follows: if $g \in G$ and $u \in P$ then $gu = u\phi(u)(g)$. Conversely, given a right G-torsor P together with such an equivariant map $\phi\colon P \to \mathrm{Aut}(G)$ we define a left G-action on P by the same formula. It is easy to check that this left G-action is free and transitive and also commutes with the right G-action. Given two G-bitorsors P and Q, we can form their **product** $P \overset{G}{\wedge} Q$ which is the G-bitorsor defined by

$$P \overset{G}{\wedge} Q = \frac{P \times Q}{G}$$

where G acts on the product $P \times Q$ by $(u, v)g = (ug, g^{-1}v)$. We see therefore that the collection of all G-bitorsors in E, together with the morphisms between them, forms a (weak) 2-group $G-\mathrm{BITORS}$. Similarly, the collection of G-bitorsors over objects in E togther with the morphisms between them, forms a fibred category $G - \mathrm{BITORS}_{/E} \to E$, which in fact is a gr-stack.

Suppose now that $t\colon H \to G$ is a crossed module and (P, ϕ) is a (G, H)-torsor. Composing ϕ with the homomorphism $\alpha\colon G \to \mathrm{Aut}(H)$ defines an equivariant map $P \to \mathrm{Aut}(H)$ making P into an H-bitorsor. We can use this fact to define the product $P \overset{H}{\wedge} Q$ of two (G, H)-torsors (P, ϕ) and (Q, ψ). As a right H-torsor $P \overset{H}{\wedge} Q$ is the product of the H-bitorsors P and Q in the above sense. It is easy to check that $\phi \times \psi$ descends to a map

$$\phi \overset{H}{\wedge} \psi\colon P \overset{H}{\wedge} Q \to G$$

satisfying the equivariance property of Definition 3. Again the (G, H)-torsors form the objects of a weak 2-group $(G, H) - \mathrm{TORS}$ and similarly the (G, H)-torsors over objects in $\mathcal{E}$ form the objects of a fibred category $(G, H) - \mathrm{TORS}_{/\mathcal{E}} \to \mathcal{E}$ which is in fact a gr-stack.

The group structure on $|N\mathcal{G}|$ arising from the simplicial group structure on $\mathcal{G}$ can be understood as follows (see [18]). First observe that the action of G on H by automorphisms extends to an action of G on EH by automorphisms. We can therefore define a 'semi-direct product' structure on $EH \times G$ by

$$(x_1, g_1) \cdot (x_2, g_2) = (x_1\alpha(g_1)(x_2), g_1g_2)$$

It is then easy to check that this product descends to a product on $EH \times^H G$

4 Non-abelian Cohomology and Higher Torsors

Let X be a topological space and let $T = Sh(X)$ be the Grothendieck topos of sheaves on X. Suppose that $\mathcal{G}$ is a category internal to $\mathbf{Grp}(T)$ corresponding to a crossed module $t\colon H \to G$ of groups in T. In [6] it is explained how to define the non-abelian cohomology $H^1(T;\mathcal{G})$ of T with coefficients in $\mathcal{G}$. Recall that one first regards $\mathcal{G}$ as a 2-category with one object and one then forms the nerve $B\mathcal{G}$ of $\mathcal{G}$. This is a simplicial object in T. Illusie [17] defines a quasi-isomorphism in $Simp(T)$ to be a morphism $f\colon X \to Y$ in $Simp(T)$ which induces isomorphisms on homotopy groups. He defines $D(T)$ to be the localisation of the category $Simp(T)$ at the quasi-isomorphisms. Breen [6] defines $H^1(T;\mathcal{G})$ as

$$H^1(T;\mathcal{G}) = \mathrm{Hom}_{D(T)}(e, B\mathcal{G})$$

where e is the final object of $D(T)$. We usually write $H^1(X;\mathcal{G})$ for $H^1(T;\mathcal{G})$. This definition simplifies substantially in the case where the sheaves of groups H and G are representable by group objects. In this case we have

$$H^1(X,\mathcal{G}) = \varinjlim_{V_\bullet \to X} [V_\bullet, B\mathcal{G}]$$

the directed limit of the set of homotopy classes of simplicial maps from hypercoverings $V_\bullet$ of X into $B\mathcal{G}$. If X is paracompact then we can compute the limit using ordinary coverings instead of hypercoverings.

Theorem 5 (Jurčo [18]). *Let X be a paracompact topological space and suppose that $t\colon H \to G$ is a crossed module of topological groups. Then there is an isomorphism*

$$H^1(X;\mathcal{G}) = [X, |B\mathcal{G}|]$$

where the right hand side denotes homotopy classes of maps from X into the geometric realisation $|B\mathcal{G}|$ of the simplicial topological space $B\mathcal{G}$.

Having given a combinatorial description of $H^1(\mathcal{G})$ we would now like a geometric description of the elements in this set. Just as there is an interpretation of $H^1(G)$ where G is a group in T as the set of isomorphism classes of G-torsors in T, there is a similar interpretation of $H^1(\mathcal{G})$ in terms of $\mathcal{G}$-torsors in T due to Breen in [6].

Definition 6 (Breen [6]). Let $\mathcal{G}$ be a gr-stack in T. A **torsor** under $\mathcal{G}$ is a stack $\mathcal{P}$ together with

- an epimorphism $\mathcal{P} \twoheadrightarrow 1$
- a morphism of stacks
$$m\colon \mathcal{P} \times \mathcal{G} \to \mathcal{P}$$
- a natural transformation μ between the two morphisms in the diagram:

$$\begin{array}{ccc} \mathcal{P} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{m \times 1} & \mathcal{P} \times \mathcal{G} \\ {\scriptstyle 1\times m}\downarrow & \not\swarrow \mu & \downarrow{\scriptstyle m} \\ \mathcal{P} \times \mathcal{G} & \xrightarrow[m]{} & \mathcal{P} \end{array}$$

which is required to be compatible with the associativity natural isomorphism for $\mathcal{G}$ in the sense described by Breen in (6.1.3) of [6]. We also require that the morphism of stacks $m\colon \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ induces an equivalence of stacks $(m, p_2)\colon \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times \mathcal{P}$. There is a further constraint on unit objects which we will not bother to write down.

The quintessential example of this kind of structure is the stack $\mathrm{TORS}(G)$ of G-torsors in T: this is a torsor under the gr-stack $\mathrm{BITORS}(G)$ associated to the crossed module $G \to \mathrm{Aut}(G)$. The action $\mathrm{TORS}(G) \times \mathrm{BITORS}(G) \to \mathrm{TORS}(G)$ is

$$(P, Q) \mapsto P \overset{G}{\wedge} Q$$

where P is a G-torsor in $\mathcal{E}$ and Q is a G-bitorsor in $\mathcal{E}$. The importance of the notion of $\mathcal{G}$-torsor comes from the following Theorem of Breen [6], showing that $\mathcal{G}$-torsors provide a geometric realisation of $H^1(\mathcal{G})$.

Theorem 7 (Breen [6] Proposition 6.2). *Let $\mathcal{G}$ be a 2-group in T. Then there is a bijective correspondence between the pointed set $H^1(\mathcal{G})$ and equivalence classes of torsors under the gr-stack $(G, H)-\mathrm{TORS}$ associated to $\mathcal{G}$.*

As part of his proof of this theorem, Breen shows that every $(G, H)-\mathrm{TORS}$-torsor $\mathcal{P}$ has a 'cocyclic description' as follows. Let $S \to *$ be a covering of the final object $*$ of $Sh(X)$. Then there exist (G, H)-torsors $P \to S \times_* S$ over $S^{[2]} = S \times_* S$ together with isomorphisms

$$d_0^* P \overset{H}{\wedge} d_2^* P \to d_1^* P$$

over $S^{[3]} = S \times_* S \times_* S$ which satisfy the obvious coherency condition over $S^{[4]}$. The 'crossed module bundle gerbes' considered by [1] are clearly special instances of this cocyclic description.

Definition 8 ([1]). Let $t\colon H \to G$ be a crossed module of topological groups. A **crossed module bundle gerbe** on X, or a (G, H)-**bundle gerbe** consists of the following data:

- a local section admitting surjection $\pi\colon Y \to X$
- a (G, H)-principal bundle $Q \to Y^{[2]}$ (see Definition 4)

together with

- an isomorphism
$$d_0^* Q \overset{H}{\wedge} d_2^* Q \to d_1^* Q$$
of (G, H)-principal bundles over $Y^{[3]}$ satisfying the obvious coherency condition over $Y^{[4]}$. When we want to make the dependence on $\pi\colon Y \to X$ clear we sometimes say that Q is a crossed module bundle gerbe **over** $Y \to X$.

There are of course many examples of these objects when $t\colon H \to G$ is the crossed module $U(1) \to 1$, in which case a crossed module bundle gerbe is just a $U(1)$-bundle gerbe in the sense of Murray [21]. We want to recast the definition of a crossed module bundle gerbe in a slightly different form, one in which we feel is more conceptually closer to the definition of a principal bundle. To this end, suppose that $\mathbb{M}$ is an object of **Gpd**(**Top**), i.e a topological groupoid. For instance $\mathbb{M}$ could be the groupoid

$$\mathbb{M}\colon Y^{[2]} \rightrightarrows Y \tag{2}$$

associated to a local section admitting surjection $Y \to X$. We make the following definition.

Definition 9. A **principal bundle object** in **Gpd**(**Top**) over $\mathbb{M}$ for a 2-group $\mathbb{G}$ in **Top**, or simply a $\mathbb{G}$-**principal bundle over** $\mathbb{M}$, is an object $\mathbb{P}$ in **Gpd**(**Top**) together with

- a local surjection admitting surjection $\pi\colon \mathbb{P} \to \mathbb{M}$
- an action of $\mathbb{G}$ on $\mathbb{P}$ in the sense described above

such that

- the diagram $\begin{array}{ccc} \mathbb{P}\times\mathbb{G} & \longrightarrow & \mathbb{P} \\ & \searrow \quad \swarrow & \\ & \mathbb{M} & \end{array}$ commutes,
- the natural map
$$\mathbb{P}\times\mathbb{G}\to\mathbb{P}\times_{\mathbb{M}}\mathbb{P}$$
is an isomorphism.

We could of course think of this definition as just a special instance of Definition 2 for, if (E, J) is a site, then there is an induced topology on the category $\mathbf{Gpd}(E)$, where we say that an E-functor $F\colon G\to H$ between groupoids G and H belongs to J iff the morphisms $F_0\colon G_0\to H_0$ and $F_1\colon G_1\to H_1$ belong to J. In our case we take **Top** equipped with the local section topology J and say that a **Top**-functor admits local sections if it belongs to the induced topology J on $\mathbf{Gpd}(\mathbf{Top})$. Then a torsor object in $\mathbf{Gpd}(\mathbf{Top})$ over $\mathbb{M}$ for the group object $\mathbb{G}$ in $\mathbf{Gpd}(\mathbf{Top})$ is exactly a principal bundle object in $\mathbf{Gpd}(\mathbf{Top})$ over $\mathbb{M}$ in the sense of the above definition. Let TORS(**Top**) denote the category of torsors in **Top** for groups in **Top**. We have the following straightforward proposition.

Proposition 10. *There is an isomorphism of* 2*-categories between the* 2*-category of torsor objects in* $\mathbf{Gpd}(\mathbf{Top})$ *and the* 2*-category of groupoids internal to* TORS(**Top**).

We would like to relate $\mathbb{G}$-principal bundles for 2-groups $\mathbb{G}$ associated to crossed modules $t\colon H\to G$ to the $\mathcal{G}$-torsors of Breen (Definition 6). We first need to remind the reader of the relevant part of the theory of stacks. If $\mathbb{X}$ is a topological groupoid, it gives rise to a stack $\underline{\mathbb{X}}$ on **Top** in the following manner. $\mathbb{X}$ represents a pre-sheaf of groupoids on **Top**, which in turn gives rise to a fibration over **Top**. We let $\underline{\mathbb{X}}$ denote the associated stack. This construction defines a functor

$$St\colon \mathbf{Gpd}(\mathbf{Top})\to\mathbf{Stack}_{/\mathbf{Top}}$$

Another way of viewing this functor is that it sends $\mathbb{X}$ to the pesudo-functor $\underline{\mathbb{X}}\colon \mathbf{Top}^{\mathrm{op}}\to\mathbf{Gpd}$ defined by setting $\underline{\mathbb{X}}(M)$ equal to the groupoid of $\mathbb{X}$-torsors over $\mathbb{M}$. This pseudo-functor is easily seen to define a stack on **Top**. Note that the image of the 2-group $\mathbb{G}$ under this functor is the gr-stack

$(G,H) - \mathrm{TORS}$ on **Top**. Applying this functor to the principal $\mathbb{G}$-bundle $\mathbb{P} \to \mathbb{M}$ yields a $(G,H) - \mathrm{TORS}$ torsor $\underline{\mathbb{P}}$ over $\underline{\mathbb{M}}$ in the sense of Definition 6. An obvious direction in which to generalise this notion is to consider 'weak' principal bundle objects in which the diagrams ?? above are only required to commute up to coherent natural isomorphism. This is the approach taken in [5] which is in turn closely related to Definition 6. Note that the 2-bundles considered in [5] have a strong local triviality condition which is not required in the framework considered here.

Notice that any $U(1)$-bundle gerbe $P \to Y^{[2]}$ over X can be thought of as a $U(1)[1]$-principal bundle. More generally, if $t\colon H \to G$ is a crossed module, then a crossed module bundle gerbe Q on X for $t\colon H \to G$ in the sense of Definition 8 gives rise to a $\mathbb{G}$-principal bundle on $\mathbb{M}$, for the 2-group $\mathbb{G}$ associated to the crossed module $t\colon H \to G$, as follows (here $\mathbb{M}$ denotes the groupoid (2) above). We define a topological groupoid $\mathbb{P}$ with objects P_0 and P_1 where P_0 is the trivial G_0-bundle $Y \times G_0$ on Y. The principal G_1-bundle P_1 over $Y^{[2]}$ is defined by setting $P_1 = Q \times G$, where the right G_1-action is defined by

$$(u,g)\cdot(g_1,h_1) = (u\alpha(g)(h_1), gg_1)$$

for $u \in Q$, $g \in G$ and $(g_1,h_1) \in G_1 = G \ltimes H$. It is immediate that this G_1-action is free and transitive on the fibres of P_1, making P_1 into a principal G_1-bundle. Define source and target maps $s\colon P_1 \to P_0$ and $t\colon P_1 \to P_0$ respectively by

$$s(u,g) = (s(\pi(u)), g)$$
$$t(u,g) = (t(\pi(u)), \phi(u)^{-1}g)$$

where $\pi\colon Q \to Y^{[2]}$ denotes the projection (so that $\pi_1\colon P_1 \to Y^{[2]}$ is defined by $\pi_1(u,g) = \pi(u)$). It is easy to check that these maps are equivariant for the source and target homomorphisms in $\mathbb{G}$ respectively. To define composition, suppose we have composable morphisms $(u,g)\colon (y_1,g) \to (y_2,\phi(u)^{-1}g)$ and $(u',\phi(u)^{-1}g)\colon (y_2,\phi(u)^{-1}g) \to (y_3,\phi(u')^{-1}\phi(u)^{-1}g)$; then the composite morphism $(u',\phi(u)g)\circ(u,g)$ is given by

$$(u',\phi(u)^{-1}g)\circ(u,g) = (m(u' \overset{H}{\wedge} u), g).$$

One can check that this map is equivariant for the composition homomorphism $\circ$ in $\mathbb{G}$. Finally one defines identity morphisms for $\mathbb{P}$ using the identity section e of Q. So we have seen that every crossed module bundle gerbe on

X gives rise to a principal bundle object over the groupoid $\mathbb{M}$ associated to $\pi\colon Y \to X$. Every principal bundle object $\mathbb{P} \to \mathbb{M}$ for a 2-group $\mathbb{G}$ in **Top**, gives rise to a $\mathcal{G}$-torsor $\mathcal{P}$ over X where $\mathcal{P}$ and $\mathcal{G}$ are the stacks associated to $\mathbb{P}$ and $\mathbb{G}$ respectively. Therefore one may expect that $\mathbb{P}$ is the principal bundle object associated to some crossed module bundle over $\pi\colon Y \to X$. However, this is not quite true, in general the covering $Y \to X$ may not be fine enough to construct Breen's cocyclic description (Theorem 7) of the $\mathcal{G}$-torsor $\mathcal{P}$. Clearly, a necessary condition for $\mathbb{P}$ to arise from a (G, H)-bundle gerbe in the above fashion is that the principal G_0-bundle $P_0 \to Y$ should be trivialised. That this is also a sufficient condition is the subject of the following Lemma, whose proof is left to the reader.

Lemma 11. *Let $t\colon H \to G$ be a crossed module of topological groups with associated 2-group $\mathbb{G}$. Then there is a bijective correspondence between principal bundle objects $\mathbb{P}$ on $\mathbb{M}$ for the 2-group $\mathbb{G}$ such that the G_0-bundle $P_0 \to Y$ is equipped with a trivialisation and (G, H)-bundle gerbes over $Y \to X$.*

5 The String 2-group

Let G be a compact, simple and simply connected Lie group. We first explain our conventions regarding loop groups. For us, LG will denote the group of free loops in G, i.e. the piece-wise smooth maps from the circle S^1 into G under pointwise multiplication. The based loop group ΩG will mean for us something slightly different than usual. ΩG will denote the group of smooth maps f from the interval $[0, 2\pi]$ into G such that $f(0) = f(2\pi) = 1$. The group P_0G of based paths will denote the smooth maps $f\colon [0, 2\pi] \to G$ with $f(0) = 1$ under pointwise multiplication.

In [2] the authors construct a Fréchet Lie 2-group $\hat{\mathbb{G}}$ arising from a crossed module $\widehat{\Omega G} \to P_0G$ associated to an action[1] of P_0G on the Kac-Moody group $\widehat{\Omega G}$ covering the action of P_0G on ΩG by conjugation. It is easily seen that there is a short exact sequence of 2-groups

$$1 \to \mathbb{K}(\mathbb{Z}, 2) \to \hat{\mathbb{G}} \to G \to 1 \qquad (3)$$

where $\mathbb{K}(\mathbb{Z}, 2)$ is the 2-group associated to the crossed module $\widehat{\Omega G} \to \Omega G$. Here the action of ΩG on $\widehat{\Omega G}$ lifts the adjoint action of ΩG on itself (this

[1] Although I would be amazed if this action were not well-known to experts, I have been unable to find a reference in the literature where it is described.

is well defined because $1 \to \mathbb{T} \to \widehat{\Omega G} \to \Omega G \to 1$ is a *central* extension). Notice that the 2-group $\mathbb{K}(\mathbb{Z},2)$ is Morita equivalent to the 2-group $U(1)[1]$: using the homomorphism $p\colon \widehat{\Omega G} \to \Omega G$ we can pullback the 2-group $\mathbb{K}(\mathbb{Z},2)$ to obtain a 2-group with objects $\widehat{\Omega G}$ and morphisms $\widehat{\Omega G} \times \widehat{\Omega G} \times U(1)$. We therefore have a homomorphism of 2-groups

$$\begin{array}{ccc} \widehat{\Omega G} \times \widehat{\Omega G} \times U(1) & \rightrightarrows & \widehat{\Omega G} \\ \downarrow & & \downarrow \\ \widehat{\Omega G} \ltimes \Omega G & \rightrightarrows & \Omega G \end{array}$$

which is clearly a Morita equivalence. Similarly we have a homomorphism of 2-groups

$$\begin{array}{ccc} \widehat{\Omega G} \times \widehat{\Omega G} \times U(1) & \rightrightarrows & \widehat{\Omega G} \\ \downarrow & & \downarrow \\ U(1) & \rightrightarrows & * \end{array}$$

which is also clearly a Morita equivalence. Therefore the 2-groups $\mathbb{K}(\mathbb{Z},2)$ and $U(1)[1]$ are Morita equivalent as 2-groups (this accounts for our choice of name $\mathbb{K}(\mathbb{Z},2)$). The relation of $\hat{\mathbb{G}}$ to the group $\hat{G}$ is described in the following theorem from [2].

Theorem 12 ([2]). *Let $|\hat{\mathbb{G}}|$ denote the geometric realisation of the nerve of the 2-group $\hat{\mathbb{G}}$. $|\hat{\mathbb{G}}|$ fits into a short exact sequence of topological groups*

$$1 \to K(\mathbb{Z},2) \to |\hat{\mathbb{G}}| \to G \to 1$$

Moreover $|\hat{\mathbb{G}}| \simeq \hat{G}$, the 3-connected cover of G.

To fix notation, we briefly recall the construction of the 2-group $\hat{\mathbb{G}}$ from [2]. We first construct the Kac-Moody group, following [22] (we remind the reader that Jouko Mickelsson in [20] gave the first construction of $\widehat{\Omega G}$, the construction given later in [22] closely parallels his). Equip $P_0\Omega G \times \mathbb{T}$ with the product

$$(f,z)\cdot(g,w) = (fg, c(f,g)zw)$$

for $z, w \in \mathbb{T}$. Here we write $f = f(t,\theta)$ and $g = g(t,\theta)$ for based paths in ΩG (so that t refers to the path variable and θ refers to the loop variable, we sometimes suppress the dependence on t or θ). Here $c(f,g)$ is the 2-cocycle

$$c(f,g) = \exp\left(\int_0^{2\pi}\int_0^{2\pi}\langle f(t)^{-1}f'(t), g'(\theta)g(\theta)^{-1}\rangle d\theta dt\right).$$

It is easy to check that the subset N of $P_0\Omega G \times \mathbb{T}$ consisting of all pairs (γ, z) where $\gamma\colon [0,2\pi] \to \Omega G$ is a loop based at 1 in ΩG and

$$z^{-1} = \exp\left(\int_{D_\gamma}\omega\right)$$

is a normal subgroup of $P_0\Omega G \times \mathbb{T}$, for the product structure defined above. Here $\omega\colon \Omega\mathfrak{g} \times \Omega\mathfrak{g} \to i\mathbb{R}$ is the Kac-Moody 2-cocycle

$$\omega(f,g) =$$

thought of as a left invariant 2-form on ΩG. $\widehat{\Omega G}$ is defined as a quotient as in the following diagram

$$\begin{array}{ccccl} P_0\Omega G \times \mathbb{T} & \longrightarrow & \widehat{\Omega G} & = & (P_0\Omega G \times \mathbb{T})/N \\ \downarrow & & \downarrow & & \\ P_0\Omega G & \longrightarrow & \Omega G & = & P_0\Omega G/\Omega\Omega G \end{array}$$

The action of P_0G on $\widehat{\Omega G}$ is defined by first defining an action of P_0G on $P_0\Omega G \times \mathbb{T}$ by

$$\alpha(p)(f,z) = (pfp^{-1}, z\exp(i\int_0^{2\pi}\int_0^{2\pi}\langle f(t)^{-1}f'(t), p(\theta)^{-1}p'(\theta)\rangle d\theta dt)$$

and then observing that this action preserves the normal subgroup N and hence descends to an action on the quotient $\widehat{\Omega G} = (P_0\Omega G \times \mathbb{T})/N$. This action of P_0G on $\widehat{\Omega G}$ defines a crossed module $t\colon \widehat{\Omega G} \to P_0G$, where t is the projection $\widehat{\Omega G} \to \Omega G$ onto the base, followed by the inclusion of the based loops ΩG into the based paths P_0G.

In [25] Stolz and Teichner give a construction of the group $\hat{G}$ in terms of von Neumann algebras. The main purpose of this section is to compare their construction with the 2-group $\hat{\mathbb{G}}$ (Theorem 14). We start by reviewing Stolz and Teichner's construction. Choose first of all a positive energy representation $\tilde{\rho}\colon \mathbb{T} \ltimes \widetilde{LG} \to U(\mathcal{H})$ of the free loop group LG on some complex Hilbert space $\mathcal{H}$ at the 'basic level' c, i.e. corresponding to the basic inner product $\langle\,,\,\rangle$. We identify the interval $I = [0, 2\pi]$ with the upper semi-circle of S^1 consisting of all $z \in S^1$ with non-negative imaginary part. We can regard ΩG as contained in the subgroup of LG consisting of those smooth loops in LG with support in I. $\widetilde{\Omega G}_{pw}$ is then the pre-image in $\widetilde{LG}$ of ΩG_{pw}, where the subscript pw here denotes piece-wise smooth loops. Here $\mathbb{T}$ acts on $\widetilde{LG}$ by rotating loops. Stolz and Teichner define

$$A_c = \tilde{\rho}(\widetilde{\Omega G}_{pw})'' \subset B(\mathcal{H})$$

to be the von Neumann algebra generated by the operators $\tilde{\rho}(\hat{\gamma})$ with $\tilde{\gamma} \in \widetilde{\Omega G}_{pw}$. $\hat{\rho}$ induces a $U(1)$-equivariant homomorphism $\tilde{\rho}\colon \widehat{\Omega G} \to U(A_c)$, the unitary group of the von Neumann algebra A_c, and thus descends to a homomorphism $\rho\colon \Omega G \to PU(A_c)$. Stolz and Teichner make the observation that the action of the group of based paths P_0G on ΩG by conjugation extends to an action of P_0G on $PU(A_c)$. This action is constructed as follows: given a path $\delta \in P_0G$, extend δ to a piece-wise smooth loop $\gamma \in LG$ and choose a lift $\tilde{\gamma} \in \widetilde{LG}$ of γ. An action of δ on $PU(A_c)$ is then defined via the formula $[a] \mapsto [\tilde{\rho}(\tilde{\gamma})a\tilde{\rho}(\tilde{\gamma}^{-1})]$. It is shown in [25] that this well-defined. Observe that this in fact gives an action $\alpha\colon P_0G \to \mathrm{Aut}(U(A_c))$ of P_0G on $U(A_c)$. It is not hard to see at this stage that this action α just described extends the action of P_0G on $\widehat{\Omega G}$ described earlier, in the sense that we have a commutative diagram

$$\begin{array}{ccc} \widehat{\Omega G} \times P_0G & \xrightarrow{\tilde{\rho}\times 1} & U(A_c) \times P_0G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \widehat{\Omega G} & \xrightarrow[\tilde{\rho}]{} & U(A_c). \end{array}$$

Observe that for any $p \in P_0G$ we can define a function $f_p\colon \Omega G \to U(1)$ by

$$\tilde{\rho}(\alpha(p)\hat{\gamma}) = \alpha(p)\tilde{\rho}(\hat{\gamma})f_p(\gamma)$$

where $\hat{\gamma}$ is a lift of γ to $\widehat{\Omega G}$. Clearly this is independent of the choice of lift $\hat{\gamma}$. It is easy to see that $f_p\colon \Omega G \to U(1)$ is a homomorphism. However, ΩG is a perfect group ([23] Proposition 3.4.1) and hence $f_p = 1$. Returning to the construction of $\hat{G}$ in [25], the authors show in Lemma 5.4.7 that one can regard ΩG as a normal subgroup of the semi-direct product $P_0G \ltimes PU(A_c)$, where the semi-direct product structure is given by the action of P_0G on

$PU(A_c)$ described above. Finally, Stolz and Teichner define

$$\hat{G} = (P_0G \ltimes PU(A_c))/\Omega G, \tag{4}$$

the quotient by this normal subgroup. It is clear, however, that $\hat{G}$ could also be obtained as a quotient of $(P_0G \ltimes U(A_c))/\widehat{\Omega G}$. We state this as a Lemma.

Lemma 13. *There is an isomorphism of topological groups*

$$(P_0G \ltimes PU(A_c))/\Omega G = (P_0G \ltimes U(A_c))/\widehat{\Omega G}$$

To be completely honest here, the group $\hat{G}$ defined above is slightly smaller than the group defined by Stolz and Teichner; they use piecewise smooth loops and paths throughout, so that $\hat{G}$ is defined as $((P_0G)_{pw} \ltimes PU(A_c))\Omega G_{pw}$. The group defined in (4) above is homotopy equivalent to the group $\hat{G}$ defined in [25]. Another perspective on the relation of $\hat{\mathbb{G}}$ to $\hat{G}$ can be obtained by thinking of the 2-group $\hat{\mathbb{G}}$ as a groupoid presentation of a gr-stack. In fact, from the point of view of gr-stacks, $\hat{\mathbb{G}}$ and $\hat{G}$ are equivalent, as the next theorem shows (note incidentally that this theorem allows for a very easy proof of the preceding one).

Theorem 14. $\hat{\mathbb{G}}$ *is Morita equivalent as a 2-group to the discrete 2-group* $\hat{G}$.

Proof. First of all, since the group $\hat{G}$ is constructed as a quotient $\hat{G} = (P_0G \ltimes U(A_c))/\widehat{\Omega G}$, it is Morita equivalent (when considered as a 2-group with only identity morphisms) to the 2-group

$$(P_0G \ltimes U(A_c)) \ltimes \widehat{\Omega G} \rightrightarrows P_0G \ltimes U(A_c)$$

It is clear that this 2-group is Morita equivalent to the 2-group $\mathcal{P}G$: we have a forgetful morphism of 2-groups

$$\begin{array}{ccc} (P_0G \ltimes U(A_c)) \ltimes \widehat{\Omega G} & \longrightarrow & P_0G \ltimes \widehat{\Omega G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ P_0G \ltimes U(A_c) & \longrightarrow & P_0G \end{array}$$

which is obviously a Morita morphism. This establishes the Theorem. □

6 The String Gerbe

Suppose that P is principal G-bundle where G is a compact, simple and simply connected Lie group. By choosing a connection on P with connection 1-form A we can represent the image of characteristic class $c \in H^4(M;\mathbb{Z})$ in de Rham cohomology by the Chern-Weil 4-form

$$c = \frac{1}{8\pi^2}\langle F_A, F_A\rangle$$

where $\langle\ ,\ \rangle$ is the basic inner product on $\mathfrak{g}$ and F_A is the curvature 2-form of A. Recall that the pullback of c to P is exact: $\pi^* c = dCS(A)$, where $CS(A)$ is the Chern-Simons form

$$CS(A) = \frac{1}{8\pi^2}\langle A, F_A\rangle - \frac{1}{48\pi^2}\langle A, [A,A]\rangle. \tag{5}$$

Note that $CS(A)$ restricts to the basic 3-form $\nu = -1/48\pi^2\langle\theta_L, [\theta_L,\theta_L]\rangle$ on the fibres of P so that c is the transgression of ν. Let $\mathbb{M}$ denote the topological groupoid (2) associated to the submersion $\pi\colon P_0M \to M$ where P_0M is the space of based paths in M and π is the map which evaluates a path at its endpoint. We have the following Theorem.

Theorem 15. *Suppose that M is 2-connected and that the class $c \in H^4(M;\mathbb{Z})$ vanishes. Then there is a principal $\hat{\mathbb{G}}$-bundle $\mathbb{P}$ on $\mathbb{M}$, together with a morphism of principal bundles*

$$\begin{array}{ccc} \mathbb{P} & \longrightarrow & P \\ {\scriptstyle \hat{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{M} & \longrightarrow & M \end{array}$$

$\mathbb{P}$ corresponds to a $(\widehat{\Omega G}, P_0G)$-gerbe on M which we call the **string gerbe** *on M.*

We construct $\mathbb{P}$ as follows. Choose a basepoint m_0 in M and a compatible base point p_0 in P so that the projection $\pi\colon P \to M$ is a based map. Form the based path fibration $p\colon P_0P \to P$, where P_0P is the space of all smooth maps $f\colon [0,2\pi] \to P$ such that $f(0) = p_0$. P_0P is a Fréchet manifold. Note that P_0P is a Fréchet principal P_0G-bundle over P_0M where P_0M is the space of smooth based paths in M. Notice that the connection on P provides a natural trivialisation of this P_0G-bundle: if f is a smooth path in M beginning at m_0 then we can uniquely lift it to a horizontal path $\hat{f}$ in P beginning at p_0. This defines a trivialisation of the principal P_0G-bundle $P_0P \to P_0M$.

Now we would like to construct a principal G_1-bundle $P_1 \to P_0M^{[2]}$ in such a way that P_1 forms the space of morphisms of a topological groupoid $\mathbb{P}$ with objects $P_0 = P_0P$. We construct P_1 by first constructing a $U(1)$-bundle gerbe $Q \to P_0P \times_P P_0P$. The bundle gerbe product on Q will provide the composition law for the groupoid $\mathbb{P}$.

So, we first construct a principal $U(1)$-bundle $Q \to P_0P^{[2]}$ where $P_0P^{[2]} = P_0P \times_P P_0P$ is the space of pairs of based paths (f_1, f_2) with $f_1(2\pi) = f_2(2\pi)$. $P_0P^{[2]}$ is a Fréchet submanifold of $P_0P \times P_0P$ since the map $P_0P \to P$ which evaluates a path at its endpoint is a submersion. The principal $U(1)$-bundle Q will have a product, in other words a morphism of $U(1)$-bundles which on the fibres looks like

$$Q_{(f_2,f_3)} \otimes Q_{(f_1,f_2)} \to Q_{(f_1,f_3)}$$

where $(f_1, f_2, f_3) \in P_0P^{[3]}$. In other words this is a morphism $\pi_1^*Q \otimes \pi_3^*Q \to \pi_2^*Q$ of principal $U(1)$-bundles where $\pi_i\colon P_0P^{[3]} \to P_0P^{[2]}$ are the maps which omit the i-th factor in the triple fibre product $P_0P^{[3]} = P_0P \times_P P_0P \times_P P_0P$. Here $\otimes$ stands for the *contracted product* of $U(1)$-bundles. Thus $Q \to P_0P^{[2]}$ is a $U(1)$-bundle gerbe on P. Q is obtained from the tautological construction described by Murray in section 10 of [21]. Note that it is important that P is 2-connected, this follows from our assumptions on M and G. Under our hypotheses we can write $c = d\mu$ for some 3-form on M. As already noted, the pullback π^*c of c to P is always exact; $\pi^*c = dCS(A)$ where $CS(A)$ is the Chern-Simons form (5) associated to A. Therefore on P we have the closed 3-form $\omega = CS(A) - \pi^*\mu$. Notice that the class in $H^3(P)$ represented by the 3-form $\omega = CS(A) - \pi^*\mu$ is unchanged if we change μ by a *closed* 3-form. This observation allows us to suppose that ω is an *integral* 3-form (this is precisely the same argument used by the authors in [12] — in fact this whole construction owes a great deal to that paper).

Q is defined as follows. Let D_0P be the space of maps $\phi\colon [0,1] \times [0, 2\pi] \to P$ from the rectangle $[0, 1] \times [0, 2\pi]$ into P such that $\phi(t, 0) = p_0$ for all $0 \le t \le 1$. D_0P is a Fréchet manifold. We have a smooth map $D_0P \to P_0P^2$ which just evaluates such a map ϕ at either $t = 0$ or $t = 1$. Let D_0Q denote the restriction of D_0P to the submanifold $P_0P^{[2]} \subset P_0P^2$. Thus the fibre of D_0Q over a point $(f_1, f_2) \in P_0P^{[2]}$ is the set of all smooth maps $\phi\colon [0,1] \times [0, 2\pi] \to P$ such that $\phi(0, \theta) = f_1(\theta)$, $\phi(1, \theta) = f_2(\theta)$ and $\phi(t, 0) =$

p_0 for all $\theta \in [0, 2\pi]$ and all $t \in [0, 1]$. D_0Q is also a Fréchet manifold. Q is constructed as a quotient

$$Q = (D_0Q \times U(1))/\sim$$

by a certain equivalence relation $\sim$. Two pairs (ϕ_1, z_1) and (ϕ_2, z_2), where $\phi_1, \phi_2 \in D_0Q$ and $z_1, z_2 \in U(1)$, are related under $\sim$ if

$$z_2 = \exp(2\pi i \int_B \psi^*\omega) z_1$$

where B denotes the ball in $\mathbb{R}^3$ with boundary S^2 and $\psi \colon B \to P$ is an extension of the map $S^2 \to P$ formed from ϕ_1 and ϕ_2. It is easy to see that $\sim$ is an equivalence relation. The product $\pi_1^*Q \otimes \pi_3^*Q \to \pi_2^*Q$ described above is constructed as follows. If $[\phi_1, z_1] \in Q_{(f_1,f_2)}$ and $[\phi_2, z_2] \in Q_{(f_2,f_3)}$ then their product $[\phi_1, z_1] \cdot [\phi_2, z_2] \in Q_{(f_1,f_3)}$ is defined to be

$$[\phi_1, z_1] \cdot [\phi_2, z_2] = [\phi_1 \circ \phi_2, z_1 z_2]$$

where $\phi_1 \circ \phi_2$ is the map from the rectangle $[0, 1] \times [0, 2\pi]$ into P obtained by gluing ϕ_1 and ϕ_2 along their common boundary. In other words,

$$(\phi_1 \circ \phi_2)t, \theta) = \begin{cases} \phi_1(t, 2\theta) & \text{if } 0 \leq \theta \leq \pi \\ \phi_2(t, 2\theta - 2\pi) & \text{if } \pi < \theta \leq 2\pi \end{cases}$$

The condition on the angular derivatives ensure that this is still a smooth map from $[0, 1] \times [0, 2\pi]$ into P. It follows from the discussion in [21] that this is well defined and that we have a groupoid $\mathcal{P}$ with objects $\mathcal{P}_0 = P_0P$ and morphisms $\mathcal{P}_1 = Q$ where source and target are defined by the projection $Q \to P_0P^{[2]}$ and where composition is defined via the product in Q.

Define an action of $P_0G \ltimes (P_0\Omega G \times U(1))$ on $D_0P \times U(1)$ by

$$(\phi, z)\cdot(p, (g, w)) = (\phi g p, z w \exp(\frac{i}{4\pi} \int_D \langle \phi^* A, g^*\theta_R + \mathrm{ad}(g) p^*\theta_R \rangle + \langle g^*\theta_L, p^*\theta_R \rangle))$$

The product $\phi g p$ here requires a little explanation. As remarked above, we can think of ϕ as a map from the square $[0, 1] \times [0, 2\pi]$ into P which sends the top edge to p_0 and which is constant along the bottom edge. We can form the product ϕg by regarding ϕ in this fashion and then re-interpreting the resulting map ϕg from $[0, 1] \times [0, 2\pi]$ into P as a map from the unit disk

D into P. If ϕ is an element of D_0P and $p \in P_0G$ then ϕp is just the product $(\phi p)(r,\theta) = \phi(r,\theta)p(\theta)$. It is a long and tedious calculation to show that this is an action.[2] We have the following Lemma (which is just an extension of the properties 1–4 on page 447 of [12]).

Lemma 16. *The following statements are true*

1. *The action of $P_0G \ltimes (P_0\Omega G \times U(1))$ respects the equivalence relation $\sim$ and hence descends to an action on Q.*
2. *The isotropy subgroup of $P_0G \ltimes (P_0\Omega G \times U(1))$ at any point of Q is the normal subgroup $1 \times N$. Therefore $P_0G \ltimes \widehat{\Omega G}$ acts on Q.*
3. *The orbits of the free action of $P_0G \ltimes \widehat{\Omega G}$ on Q are the fibres of the projection $Q \to P_0M^{[2]}$.*
4. *The action of $P_0G \ltimes \widehat{\Omega G}$ on Q covers the action of $P_0G \ltimes \Omega G$ on $P_0P^{[2]}$*
5. *Suppose that $u \in Q_{(f_1,f_2)}$, $u' \in Q_{(f_2,f_3)}$, $(p,\hat\gamma) \in P_0G \ltimes \widehat{\Omega G}$ and $(pt(\hat\gamma),\hat\gamma') \in P_0G \ltimes \widehat{\Omega G}$ so that we can form the composites $u' \circ u$ and $(pt(\hat\gamma),\hat\gamma') \circ (p,\hat\gamma)$. Then*

$$u'(pt(\hat\gamma),\hat\gamma') \circ u(p,\hat\gamma) = (u' \circ u)(p,\hat\gamma\hat\gamma')$$

Proof. We will prove 1 and 5, the other statements are easy and are left to the reader. To prove 1 we must show that if $(\phi,z) \sim (\phi',z')$ then $(\phi,z)\cdot(p,(g,w)) \sim (\phi',z')\cdot(p,(g,w))$ for all $(p,(g,w)) \in P_0G \ltimes (P_0\Omega G \times U(1))$. Since $(\phi,z) \sim (\phi',z')$ we have

$$z' = z\exp(2\pi i \int_B \psi^*\omega)$$

for any extension $\psi\colon B \to P$ of the map $S^2 \to P$ formed from ϕ and ϕ'. By the integrality of the differential forms involved, it is enough to establish the identity

$$\int_B \psi^*\omega + \frac{i}{4\pi}\int_D \langle(\phi')^*A, g^*\theta_R + \mathrm{ad}(g)p^*\theta_R\rangle = \frac{i}{4\pi}\int_D \langle\phi^*A, g^*\theta_R + \mathrm{ad}(g)p^*\theta_R\rangle + 2\pi i\int_B (\psi gp)^*\omega$$

[2]We respect the tradition of leaving such calculations to the reader.

This follows immediately from the identity

$$(R_g)^*CS(A) = CS(A) + \frac{1}{8\pi^2}d\langle A, g^*\theta_R\rangle - \frac{1}{48\pi^2}g^*\langle\theta_L, [\theta_L, \theta_L]\rangle$$

and Stokes' Theorem. □

7 Conclusion

We have presented here a definition of a notion of a 'higher-dimensional principal bundle' and discussed how these are related to familiar objects such as gerbes. In particular we have given an example of such an object which we believe is worthy of further study. In work in progress we have developed a parallel theory of higher dimensional vector bundles, using the notion of internalisation. In particular we believe that this theory provides a very useful language with which to discuss higher dimensional notions of connection and curvature [1, 4, 9] (we point out that in this note we have made no attempt to study the geometry of the $\hat{\mathbb{G}}$-bundle $\hat{\mathbb{P}}$). In particular, to make contact with Stolz and Teichner's notion of a string connection it would be necessary to consider some sort of associated vector bundle to $\hat{\mathbb{P}}$; this would involve a consideration of the representation theory of $\hat{\mathbb{G}}$.

Acknowledgments

This paper grew out of some notes written for Urs Schreiber and the present form has been very influenced by to his comments throughout our email exchanges. I feel that there is nothing really new in this paper, which is why I have hesitated to make it available, but I hope that the point of view presented here may be of some interest.

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