Globally Smooth Parallel Transport

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1 Introduction.

A *Chen-smooth space* (or *diffeological space*) is a set X together with a collection pl_X of maps into X, called *plots*, whose domains are some ordinary notion of smooth spaces, and which satisfy some obvious compatibility conditions.

These conditions ensure that pl_X is in fact a sheaf, or a stack. Hence Chen-smooth spaces form a subcategory of all sheaves, or stacks, on the chosen domain. They are special among all such stacks in that they are *concrete*ly coming from maps into a given set X.

(The precise technical formulation of this is, apparently, that Chen-smooth spaces form a *concrete quasi-topos*.)

Parallel transport in a smooth bundle, principal or associated, is a functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

or

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

which is smooth in some sense.

Elsewhere we have discussed how to capture this smoothness property without equipping Vect or GTor with a (Chen-)smooth structure, but rather by forming a local model

$$i : \Sigma G \xhookrightarrow{\sim} G\mathrm{Tor}$$

or

$$i : \cup_n \Sigma U(n) \overset{\sim}{\hookrightarrow} \mathrm{Vect}$$

of these large categories, which is smooth in the ordinary sense.

Functors tra as above come from parallel transport in a smooth bundle precisely if they admit what we call a *smooth local i-trivialization*. This is then guaranteed to be unique, up to isomorphism.

One motivating advantage of this concept is that it allows, due to the uniqueness result, to handle smooth such functors globally, like any other functors, while at the same time allowing for direct access to the local transition data of the functor.

But we might just as well want to equip the target categories $G\mathrm{Tor}$ or Vect with Chen-smooth structures themselves, and consider Chen-smooth functors to these.

Here I consider one way how to do this, and try to indicate how this connects the concept of smoothness in terms of existence of smooth local i-trivialization with the concept of smoothness of transport functors as used by Stolz, Teichner and Dumitrescu.

2 Chen-smooth spaces.

Let

$$\mathrm{Dom}$$

be some site, like that of open (covex, maybe) subsets of $\mathbb{R}^n$s, or like the site of smooth manifolds.

Remark. There are various flavors of Chen-smooth spaces in use, that differ by the precise choice of Dom. Originally Chen chose Dom to be open subsets of $\mathbb{R}^n$s, but later switched to *convex* subsets. Iglesias-Zemmour (who says "diffeological" instead of "Chen-smooth") in his work uses open subsets.

2.1 As stacks.

Definition 1 *A* **Chen-smooth space** (X, pl_X) *is a set X together with an assignment*

$$\mathrm{pl}_X : U \mapsto \mathrm{pl}_X(U) \subset \mathrm{Hom}_{\mathrm{Set}}(U, X)$$

of maps, called **plots***, to every $U \in \mathrm{Dom}$ such that $\mathrm{pl}_X(U)$ contains all constant maps and such that pl_X is a sheaf on Dom, with the restriction map given by pullback.*

We may trivially regard a smooth space as giving a stack on Dom.

Definition 2 *A* **morphism of Chen-smooth spaces**

$$f : (X, \mathrm{pl}_X) \to (Y, \mathrm{pl}_Y)$$

is a map of sets

$$F : X \to Y$$

that induces a morphism

$$F_*(U) : \mathrm{pl}_X(U) \to \mathrm{pl}_Y(U)$$

by

$$\phi \mapsto F \circ \phi$$

on the corresponding sheaves of plots.

Notice that this is really a morphism of sheaves. The naturality condition for any $f : V \to U$, which reads

$$\begin{array}{ccc} \mathrm{pl}_X(U) & \xrightarrow{F_*(U)} & \mathrm{pl}_Y(U) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{pl}_X(V) & \xrightarrow{F_*(V)} & \mathrm{pl}_Y(V) \end{array} ,$$

is nothing but the associativity of the composition

$$V \xrightarrow{f} U \xrightarrow{\phi} X \xrightarrow{F} Y .$$

Write

$$C^\infty$$

for the category of smooth manifolds and

$$S^\infty$$

for the category of Chen-smooth spaces.

2.2 As fibered categories.

We may equivalenty think of a Chen-smooth space (X, pl_X) as a category pl_X (by abuse of notation) with a surjection to Dom

$$X : \mathrm{pl}_X \to \mathrm{Dom}^{\mathrm{op}} .$$

The objects of pl_X are all plots $\phi : U \to X$ of X, for all $U \in \mathrm{Dom}$. Morphisms are pullbacks along morphisms $(f : V \to U)$ in Dom

$$\phi \xrightarrow{f_*} \phi \circ f .$$

The functor to Dom is the obvious projection that remembers only the domain of the plots.

Then a morphism

$$f : (X, \mathrm{pl}_X) \to (Y, \mathrm{pl}_Y)$$

yields a morphism of fibered categories, namely a functor (again abusing notation)

$$f : \mathrm{pl}_X \to \mathrm{pl}_Y$$

that respects the projection to $\mathrm{Dom}^{\mathrm{op}}$:

$$\begin{CD} \mathrm{pl}_X @>{f}>> \mathrm{pl}_Y \\ @V{X}VV @VV{Y}V \\ \mathrm{Dom}^{\mathrm{op}} @= \mathrm{Dom}^{\mathrm{op}} \end{CD} \quad .$$

This way, Chen-smooth spaces may be regarded as a subcategory of the category of all categories over $\mathrm{Dom}^{\mathrm{op}}$.

2.3 Examples.

Example 1

The category S^∞ is closed. The smooth structure on the set

$$\mathrm{Hom}_{S^\infty}(X, Y)$$

for (X, pl_X) and (Y, pl_Y) smooth spaces is that which assigns to $U \in \mathrm{Dom}$ all those maps

$$\phi : U \to \mathrm{Hom}_{S^\infty}(X, Y)$$

such that for any plot $\tilde{\phi} : V \to X$ of X the map

$$U \times X \xrightarrow{\tilde{\phi} \times \mathrm{Id}} \mathrm{Hom}_{S^\infty}(X, Y) \times X \xrightarrow{\mathrm{ev}} Y$$

is a morphism in S^∞.

Example 2

For any manifold $X \in C^\infty$, the path groupoid

$$\mathcal{P}_1(X)$$

is internal to S^∞. Its morphisms are thin homotopy classes of smooth paths $\gamma : [0, 1] \to X$ in X, equipped with the smooth structure inherited from $\mathrm{Hom}_{S^\infty}([0, 1], X)$.

Example 3

We may turn the category Vect of vector spaces into a smooth category Vect by declaring that a map

$$\phi : U \to \mathrm{Mor}(\mathrm{Vect})$$

from any open subset $U \subset \mathbb{R}^n$ is a plot if and only if it is the component map of a smooth morphism

$$f : E \to E'$$

between two smooth vector bundles $p : E \to U$ and $p' : E' \to U$ over U that fixes the base, i.e. which is such that

$$\phi(u) = f|_{\{u\}} : E_u \to E'_u$$

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & U & \end{array} .$$

A completely analogous construction works for GTor. Here we take a map $U \to \mathrm{Mor}(G\mathrm{Tor})$ to be a plot if it is the component map of a morphisms of smooth principal G-bundles on U.

(Notice, by the way, that our category GTor is taken to contain only those G-torsors which at other places would be called G-torsors *over a point*: an object here is a G-space that is isomorphic to G as a G-space.)

3 Globally Smooth Parallel Transport.

Florin Dumitrescu, in his thesis, mentions the following fact (reformulated here in terms of the above Chen-smooth structure on Vect).

Proposition 1 *Parallel transport in a smooth vector bundle $E \to X$ with connection is a smooth functor*

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}\,.$$

Proof. For any plot

$$\tilde{\phi} : U \to PX$$

of path space we pull back E to U along the two endpoint evaluations

$$s: \; U \xrightarrow{\phi} PX \xrightarrow{\sim} PX \times \{0\} \xrightarrow{\mathrm{ev}} X$$

and

$$t: \; U \xrightarrow{\phi} PX \xrightarrow{\sim} PX \times \{1\} \xrightarrow{\mathrm{ev}} X \; .$$

The parallel transport then provides a smooth bundle isomorphism

$$s^*E \xrightarrow{\phi^*\mathrm{tra}} t^*E \; .$$

One can check this explicitly by choosing around every point of U a neighbourhood over which both s^*E as well as t^*E trivialize. $\square$

Using the technology of smooth π-local i-trivializations, I think we can also show that, for Vect the category of hermitean vector bundles (for convenience):

Proposition 2 *Every functor*

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

that is smooth with respect to the above smooth structure on Vect *is also smoothly locally i-trivializable over a contractible cover, with*

$$i : \bigcup_n \Sigma U(n) \xhookrightarrow{\sim} \mathrm{Vect} \;.$$

Proof. By restricting to plots to paths with fixed endpoints, the smoothness condition on tra translates into the property that tra has smooth i-Wilson lines. This implies that tra is locally smoothly i-trivializable. $\square$

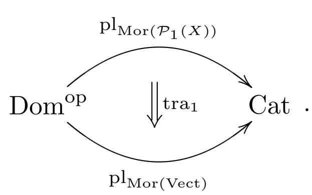
Theorem 1 *A functor*

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

is smooth with respect to the above smooth structure on Vect *if and only it it is smoothly locally i-trivializable.*

Proof. I think I can prove this. But for the moment I won't. $\square$

Remark. By definition, the smooth functor $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ is in particular a morphism of stacks



4 Chen-smooth Bordism Categories.

Let

$$1\mathrm{Cob}_{\mathrm{Riem}}$$

be the category whose objects are disjoint unions of oriented points, and whose morphisms are 1-dimensional Riemannian manifolds cobounding these points and equipped with a suitable notion of collars, such that composition can sensibly be defined by gluing of cobordisms.

We now equip the category $1\mathrm{Cob}_{\mathrm{Riem}}$ and the category Vect with smooth structures in two at least superficially different ways: first by equipping them with the obvious sheaf of plots, then by equipping them with the obvious sheaf of smooth bundles.

Definition 3 (smooth structure as sheaf of plots) *Equip this category with a Chen-smooth structure as follows. A map*

$$\phi : U \to \mathrm{Mor}(1\mathrm{Cob}_{\mathrm{Riem}})$$

is a plot if and only if there is a smooth bundle

$$B \to U$$

over U, with a fiberwise Riemannian structure, such that

$$s : u \mapsto B_u \,.$$

It then makes good sense to consider Chen-smooth functors

$$F : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Vect}$$

with respect to the above given Chen-smooth structure on $1\mathrm{Cob}_{\mathrm{Riem}}$ and Vect
These take plots

$$\phi : U \to \mathrm{Mor}(1\mathrm{Cob}_{\mathrm{Riem}})$$

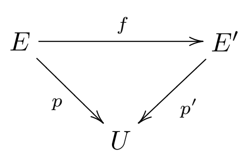
to plots

$$F_1 \circ \phi : U \to \mathrm{Mor}(\mathrm{Vect})$$

and hence, by definition, are such that for the bundle $B \to U$ of Riemannian cobordisms whose existence is guaranteed by the existence of ϕ,

$$\phi(u) = B_u \,,$$

there is a bundle morphism



of vector bundles over U such that

$$F_1 \circ \phi : u \mapsto f_u \,.$$

Definition 4 (smooth structure as sheaf of smooth bundles) *Alternatively, it makes good sense to turn $1\mathrm{Cob}_{\mathrm{Riem}}$ into a category internal to categories fibred over $\mathrm{Dom}^{\mathrm{op}}$ associating with any $U \in \mathrm{Obj}(\mathrm{Dom})$ the collection of smooth bundles over U whose fibers are 1-dimensional Riemannian cobordisms.*

Analogously, we may consider Vect *as internal to categories fibered over* $\mathrm{Dom}^{\mathrm{op}}$ *by associating with any $U \in \mathrm{Obj}(\mathrm{Dom})$ the collection of smooth vector bundles over U.*

Remark. I am deliberately glossing over some slight technical subtleties which I believe are not essential for the argument to follow. One is that we are here trying to fomulate a setup in the language of sheaves which really wants to live in stacks.

Question. *Are functors*

$$F : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Vect}$$

that are smooth in the sense of Def. 3 in bijection with functors

$$F : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Vect}$$

that are smooth in the sense of Def. 4?

At first sight the answer seems to be obviously yes. But there is possibly a subtle subtlety related to the difference between the guarantee of the existence of a bundle, and the bundle itself. I haven't made up my mind yet.