Integration over supermanifolds

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1 Introduction

When generalizing some structure, it usually matters which of various equivalent definitions of the ordinary structure one takes as the starting point.

Integration over ordinary manifolds X is often regarded as being governed by non-degenerate "top degree" differential forms $\omega \in \Omega^{\dim(X)}(X)$.

When saying this in a more general context, such as that of supermanifolds, one finds that the notion of "top degree" becomes secondary, and even ill defined, while what matters is that we have something "non-degenerate" of the dimension of X:

a "non-degenerate top-form" is locally a wedge product of a basis *of 1-forms.*

	unsuitable definition	**suitable definition**
integration is against...	a nondegenerate top-degree form	locally a wedge product of a basis of 1-forms

If this is taken as the starting point for the ordinary definition of integration, that definition goes through seamlessly for supermanifolds, too.

2 Integration over supermanifolds

We define integration over $\mathbb{R}^{p|q}$ and extend it patchwise to integration over arbitrary supermanifolds. Then we notice that integrals over Lie derivatives vanish and deduce from that Stokes' theorem for supermanifolds.

Plan.

- integration over $\mathbb{R}^{p|q}s$;
- extend patchwise to integration over supermanifolds;
- notice that the intgerals over Lie derivatives vanish;
- from that and some natural defintions follows Stokes' theorem.

2.1 Integration over $\mathbb{R}^{p|q}s$

Just as supermanifolds are structures modeled on $\mathbb{R}^{p|q}s$, integration over them is modeled by integration over $\mathbb{R}^{p|q}$.

Definition 1 *Let*

$$\int |\mathrm{vol}_{\mathbb{R}^{p|q}}| : \; C^\infty_{\mathrm{cpt}}(\mathbb{R}^{p|q}) \longrightarrow C^\infty(\mathbb{R}^{0|0}) = \mathbb{R}$$

be the $\mathbb{R}$*-linear map given by*

$$\int |\mathrm{vol}_{\mathbb{R}^{p|q}}| : \; f_I(x^1, \cdots, x^p)\theta^I \longmapsto \int f_{12\cdots q}(x^1, \cdots, x^p) dx^1\, dx^2 \, \cdots \, dx^n \; .$$

where on the right we have the ordinary multi-variable integral over $\mathbb{R}^n$ (Riemann, or Lebesgue, for instance) of the coefficient $f_{1\cdots q}$ of the top odd-function power in f.

Remark: Berezinian integral. The fact that we project onto top odd-function powers is usually called the *Berezinian integration* over odd coordinates. On $\mathbb{R}^{n|1}$ the Berezinian integral is often written as

$$\int (g_0(x) + g_1(x)\theta)\; d\theta = g_1(x)$$

and motivated by observing that it is

- "normalized": $\int \theta d\theta = 1$
- it is "translation invariant" in that $\int (g+\epsilon) d\theta = \int g$ for ϵ "another odd parameter" (can be made precise but is not really relevant here).

We want to generalize this to things that *locally* look like $\mathbb{R}^{p|q}$, i.e. to supermanifolds. But if something is not *uniquely* identified, locally, with $\mathbb{R}^{p|q}$, but only up to isomorphism, we need to make sense of the integration measure $|\mathrm{vol}_{\mathbb{R}^{p|q}}|$.

2.2 Super determinant lines

Let A be a superalgebra and L a free A module of rank $p|q$. This means that there exists an isomorphism of A-modules

$$e: \; A^{p|q} \xrightarrow{\simeq} L \; .$$

This is a choice of basis of L over A. But there is in general not just one, but many. Any other such

$$e': \; A^{p|q} \xrightarrow{\simeq} L$$

is related to the previous one by an automorphism of $A^{p|q}$:

$$e^{-1} \circ e': \; A^{p|q} \xrightarrow{e'} L \xrightarrow{e^{-1}} A^{p|q} \; .$$

For any such automorphism we have the super determinant (Berezinian) defined by

$$\mathrm{Det}(e^{-1} \circ e') := \mathrm{Ber}(e^{-1} \circ e') \in \mathrm{GL}_A(A^{1|0}) = A^\times$$

with

$$\mathrm{Det} \begin{pmatrix} K & L \\ M & N \end{pmatrix} = \det(K - LN^{-1}M)\det(N)^{-1} \, .$$

If we are just interested in the supervolume spanned by the elements of the basis, we should consider the $A^\times$-torsor of equivalence classes

$$\mathrm{Det}'(L) := \left\{ [\; A^{p|q} \underset{\simeq}{\xrightarrow{e}} L \;] = \frac{1}{\mathrm{Det}(F)} [\; A^{p|q} \underset{\simeq}{\xrightarrow{F}} A^{p|q} \underset{\simeq}{\xrightarrow{e}} L \;] \right\} \, .$$

To get a full A-module we set

$$\mathrm{Det}(L) := A \otimes_{A^\times} \mathrm{Det}'(L) \, .$$

This $\mathrm{Det}(L)$ is, manifestly, a free A-module of total rank 1, which we take to be of rank $1|0$ if q is even and of rank $0|1$ if q is odd.

This Det is a functor from the category of free A-modules with isomorphisms between them to the category of A-lines (free A-modules of rank $1|0$ of $0|1$):

$$\mathrm{Det} : A\mathrm{Mod}_{\mathrm{free,isos}} \to A\mathrm{Lines} :$$

$$\mathrm{Det} : (\; L \xrightarrow{f} L' \;) \mapsto (\; \mathrm{Det}(L) \xrightarrow{\mathrm{Det}(f)} \mathrm{Det}(L') \;) \, ,$$

where

$$\mathrm{Det}(f) : [\; A^{p|q} \underset{\simeq}{\xrightarrow{e}} L \;] \mapsto [\; A^{p|q} \underset{\simeq}{\xrightarrow{e}} L \underset{\simeq}{\xrightarrow{f}} L' \;] \, .$$

If $A = C^\infty(X)$ and L is a locally free module over A – (the sections of) a vector bundle –, we can perform this construction locally to get (the sections of) a line bundle $\mathrm{Det}(L)$, i.e.

$$\Gamma(U, \mathrm{Det}(L)) \simeq \mathrm{Det}_{C^\infty(U)}((C^\infty(U))^{p|q}) \, .$$

$$\mathrm{Det} : \mathrm{VectBund}_{\mathrm{isos}} \to \mathrm{LineBund} \, .$$

2.3 Volume forms and densities

In particular, let $L = \Omega^1(X)$, then we write

$$\mathrm{Vol}(X) := \mathrm{Det}(\Omega^1(X))$$

for the line of *oriented* volume elements over X. If X is orientable, then the sections of $\mathrm{Vol}(X)$ are the *volume forms* on X.

We want to divide out the orientation: let $\mathrm{or}(X)$ be the orientation bundle of $|X|$ regarded as a $\{+1,-1\} \subset C^\infty(X)$-module by letting -1 act by orientation reversal.

Then define:

$$|\mathrm{Vol}(X)| := \mathrm{Vol}(X) \otimes_{\{1,-1\}} \mathrm{or}(X)\,.$$

The sections of $|\mathrm{Vol}(X)|$ are the *densities* over X.

On $X = \mathbb{R}^{p|q}$ we have a canonical section of $|\mathrm{Vol}(X)|$, namely

$$\mathrm{vol}_{\mathbb{R}^{p|q}} := dt^1 \wedge dt^2 \wedge \cdots dt^p \wedge d\theta^1 \wedge \cdots d\theta^q \otimes (+)\,,$$

where $(+)$ denotes the orientaton represented by $dt^1 \wedge \cdots \wedge dt^p$.

Example. Notice how tensoring with the orientation bundle enforces the absolute value $|\cdot|$ idea: let $dt^1 \wedge dt^2 \otimes +$ be standard density on $\mathbb{R}^2$. Under the orientation reversing diffeomorphism

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$
$$f(t^1) = t^2\,, f(t^2) = t^1\,.$$

Then

$$\begin{aligned}
& f^*(dt^1 \wedge dt^2 \otimes_{C^\infty(\mathbb{R}^2)^\times} (+)) \\
= \quad & dt^2 \wedge dt^1 \otimes_{C^\infty(\mathbb{R}^2)^\times} (-) \\
= \quad & -dt^1 \wedge dt^2 \otimes_{C^\infty(\mathbb{R}^2)^\times} (-) \\
= \quad & dt^1 \wedge dt^2 \otimes_{C^\infty(\mathbb{R}^2)^\times} (+)
\end{aligned}$$

Proposition 1 *There is a unique $\mathbb{R}$-linear map*

$$\int_X : \Gamma_{\mathrm{cpt}}(|\mathrm{Vol}(X)|) \to \mathbb{R}$$

with the property that for

$$\phi : X|_U \to \mathbb{R}^{p|q}$$

any chart of X, and for all

$$f|\mathrm{vol}_{\mathbb{R}^{p|q}}| := (f\mathrm{vol}_{\mathbb{R}^{p|q}}, +) \in \Gamma_{\mathrm{cpt}}(|\mathrm{Vol}(\mathbb{R}^{\mathrm{p|q}})|)$$

we have

$$\int_X \phi^*(f|\mathrm{vol}_{\mathbb{R}^{p|q}}|) = \int_{\mathbb{R}^{p|q}} f|\mathrm{vol}_{\mathbb{R}^{p|q}}|\,.$$

Using a partition of unity

$$\{h_i \in C^\infty_{\mathrm{Cpt}}(X)\},$$

$$\sum_i h_i = 1\,,$$

every compactly supported section of $|\mathrm{Vol}(X)|$ may be patched together by pulled back sections like this:

$$\int_X \omega = \int_X \sum_i h_i \omega = \sum_i \int_X h_i \omega = \sum_i \int_X \phi^*(f_i |\mathrm{vol}_{\mathbb{R}^{p|q}}|) = \sum_i \int_{\mathbb{R}^{p|q}} f_i |\mathrm{vol}_{\mathbb{R}^{p|q}}|\,.$$

If $|M|$ is oriented it means we have chosen a global section $\sigma \in \Gamma(\mathrm{or}(X))$. Fixing that in the above formulas makes the integral then a map on volume forms

$$\int_X : \mathrm{Vol}(X) \to \mathbb{R}\,.$$

2.4 Lie derivatives

We define

$$L : \Gamma(TX) \otimes \Omega^\bullet(X) \to \Omega^\bullet(X)$$

by Cartan's formula

$$L_v := [d, \iota_v]\,.$$

Given an *even* vector field $v \in \Gamma(TX)$ its flow $\exp(v) : X \to X$ is an orientation preserving diffeomorphism and the Lie derivative along v is

$$L_v \omega = \frac{d}{dt}(\exp(tv)^* \omega - \omega)\,.$$

The integral is invariant under orientation preserving diffeomorphisms

$$f : \; X \xrightarrow{\simeq} X$$

in that

$$\int_X f^* \omega = \int_X \omega$$

for all $\omega \in \Gamma_{\mathrm{cpt}}|\mathrm{Vol}(X)|$. Hence it follows that

$$\int_X L_v \omega = 0$$

for all even $v \in \Gamma(TX)$ and $\omega \in \mathrm{Vol}(X)$ on X.

This is extended to odd vector field simply by noticing that multiplying any odd vector field with an odd scalar yields an even vector field.

2.5 Integral forms

The main point of the discussion here is that there is a difference between top forms and volume forms (the former don't even exist in general) and that for integration purposes it is the volume forms which matter.

Just as the even deRham complex is generated from any top form by contractions, we can generate a complex from any volume form by contractions. The resulting complex is called that of *integral forms.*

Definition 2 *Let X be a supermanifold of dimension $p|q$ such that $|X|$ is oriented. The complex $I^\bullet(X)$ is, as a graded A-module*

$$I^n(X) := \begin{cases} \wedge^{p-n}\Gamma(TX)\otimes \mathrm{Vol}(X) & \text{for } n \le p \\ 0 & \text{for } n > p \end{cases} .$$

We write wedging with $v \in \Gamma(TX)$ as

$$\iota_v : I^n(X) \to I^{n-1} .$$

To define the differential on $I^\bullet(X)$, first define an operation

$$L : \Gamma(TX)\otimes I^\bullet(X) \to I^\bullet(X)$$

by

- *letting L_v act on $I^p(X)$ as the Lie derivative on ω (already defined):*

$$L_v|_{I^p(X)} : \omega \mapsto L_v\omega ;$$

- *setting*

$$[L_v, \iota_w] := \iota_{[v,w]} .$$

The differential d on the complex is then defined by

- *setting*

$$d|_{I^p} = 0 ;$$

- *setting*

$$[d, \iota_v] = L_v .$$

Using the canonical identification

$$I^p(X) \simeq \mathrm{Vol}(X)$$

we get a map

$$\int_X : I^p(X) \to \mathbb{R} .$$

2.6 Stokes' theorem

For X a supermanifold of dimension $p|q$, $|X|$ oriented and for any

$$\lambda \in I^{p-1}_{\mathrm{cpt}}(X)$$

we have

$$\int_X d\eta = 0\,.$$

Because by definition there is $v \in \Gamma_{\mathrm{cpt}}(TX)\}$ such that $\lambda = \iota_v \omega$ and hence

$$\int_X d\eta = \int_X d\iota_v \omega = \int_X [d, \iota_v]\omega = \int_X L_v \omega = 0\,.$$