Lie ∞-Connections and applications to String- and Chern-Simons n-transport

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What we want to do.

- We want to understand n-dimensional QFTs that arise as Σ-**models**: those that come from transgression of n-dimensional parallel transport;
  - for instance the **charged** $(n-1)$-**brane**, but also (higher) **gauge theory**.
  - **finite description**: n-bundles ($(n-1)$-gerbes) with connection in terms of *n-groupoid valued parallel transport n-functors* [Baez-S., S.-Waldorf I, II , II]
  - **differential description**: n-bundles with connection in terms of *L_∞-algebra-valued connections* [Sati-S.-Stasheff].
- Then use this to describe
  - (generalized) Chern-Simons forms;
  - (generalized) Chern-Simons $(n+1)$-bundles (n-gerbes)
  - possibly related phenomena like Green-Schwarz mechanism.

fundamental object	**background field**
$(n-1)$-brane	$(n-1)$-gerbe
n-particle	n-bundle

Table: **The two schools of counting** higher dimensional structures. Here n is in $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Plan

1. The result to be discussed
2. Parallel n-transport
   1. Σ-models
   2. Background fields
   3. Parallel n-transport
3. L_∞-connections
   1. L_∞-algebras
   2. L_∞-valued differential forms
   3. L_∞-connections
4. Applications
   1. Obstructing $(n+1)$-bundles
5. Literature

The result to be discussed

- We recall L_∞-algebras, which are a categorified version of ordinary Lie algebras.
- We discuss how Lie algebra cohomology generalizes to L_∞-algebras by looking at their Chevalley-Eilenberg differential algebras.
- We notice that for every L_∞-algebra $\mathfrak{g}$ and every degree n cocycle μ on it, there is an extension

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

  of $\mathfrak{g}$ by $(n-1)$-tuply shifted $\mathfrak{u}(1)$, which includes and generalizes the *String extension*.
- We define for arbitrary L_∞-algebras $\mathfrak{g}$ a notion of higher bundles with L_∞-connection and define characteristic classes for these.

We obtain the following theorem:

Let the degree $(n+1)$ cocycle μ on the L_∞-algebra $\mathfrak{g}$ be in transgression with the invariant polynomial P on $\mathfrak{g}$.

Theorem

The obstruction to lifting a $\mathfrak{g}$-connection (A, F_A) to a $\mathfrak{g}_\mu$-connection $(A', F_{A'})$ is a $b^n\mathfrak{u}(1)$-connection whose single characteristic class is that of

$$P(F_A)\,.$$

Applied to the special case that $\mathfrak{g}$ is an ordinary Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$ and corresponding 3-cocycle $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ we get

Corollary

The lift of an ordinary $\mathfrak{g}$-connection (A, F_A) to a String 2-connection is obstructed by a $b^2\mathfrak{u}(1)$ 3-connection whose local connection 3-form is the Chern-Simons 3-form

$$\mathrm{CS}(A, F_A) = \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle$$

and whose single characteristic class is hence the Pontryagin class of (A, F_A)

$$p_1 = \langle F_A \wedge F_A \rangle \, .$$

Parallel n-transport

One way of understanding what we are after here is to ask:

An n-dimensional Σ-*model* is a quantum field theory which comes from assigning phases to maps of some n-dimensional parameter space into some target space, but –

- What is a Σ-model, *really*?

Such phase assignments come from **background fiels** like gauge connections, Kalb-Ramond fields, supergravity 3-forms fields –

- What is a background field, *really*?

Our answer to these questions is:

- It is a parallel n-transport.

Finally, an L_∞-connection is the differential description of parallel n-transport: skip parallel transport and jump to L_∞-connections

To set the scene:

What is a Σ-Model?

The charged $(n-1)$-brane

The input

target space: $\mathbf{X}$

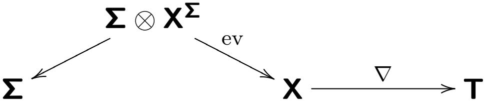
background field: $\nabla : \mathbf{X} \to \mathbf{T}$

parameter space $\mathbf{\Sigma}$

The output

config. space: $\mathbf{X}^{\mathbf{\Sigma}}$

transgression: $\nabla^{\mathbf{\Sigma}} : \mathbf{X}^{\mathbf{\Sigma}} \to \mathbf{T}^{\mathbf{\Sigma}}$



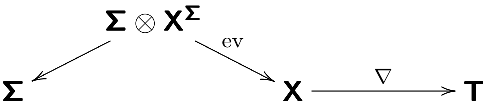
The charged $(n-1)$-brane

The input

target space:	$\mathbf{X}$	(an n-groupoid)
background field:	$\nabla : \mathbf{X} \to \mathbf{T}$	(an n-functor)
parameter space	$\mathbf{\Sigma}$	(an n-groupoid)

The output

config. space:	$\mathbf{X}^{\mathbf{\Sigma}}$	(internal hom object)
transgression:	$\nabla^{\mathbf{\Sigma}} : \mathbf{X}^{\mathbf{\Sigma}} \to \mathbf{T}^{\mathbf{\Sigma}}$	(internal hom morphism)



Example: the charged particle

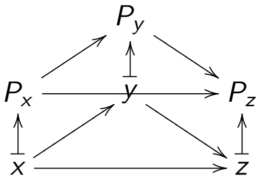
The input

target space:	$\mathbf{X} = \mathcal{P}_1(X)$	(paths in spacetime X)
background field:	$\nabla : \mathcal{P}_1(X) \to \mathrm{Vect}$	(gauge connection)
parameter space	$\mathbf{\Sigma} = \Pi_1(S^1)$	(paths in the circle)

The output

config. space:	$\mathbf{X}^{\mathbf{\Sigma}} = \mathcal{P}_1(LX)$	(thin paths in loop space)
transgression:	$\nabla^{\mathbf{\Sigma}} : \mathcal{P}_1(LX) \to \Lambda\mathrm{Vect}$	(holonomy)

$$\begin{array}{c} \mathrm{Vect} \\ \uparrow \nabla \\ \mathcal{P}_1(X) \end{array}$$



In an analogous manner one can describe

- the (Kalb-Ramond) charged string;
- the (C_3-field) charged membrane;

and also

- gauge theory.

(skip further examples)

Example: the charged string

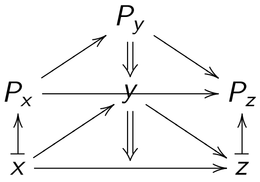
The input

target space:	$\mathbf{X} = \mathcal{P}_2(X)$	(2-paths in spacetime X)
background field:	$\nabla : \mathcal{P}_2(X) \to 2\mathrm{Vect}$	(Kalb-Ramond field)
parameter space	$\mathbf{\Sigma} = \Pi_1(S^1)$	(path in the circle)

The output

config. space:	$\mathbf{X}^{\mathbf{\Sigma}} = \mathcal{P}_1(LX)$	(paths in loop space)
transgression:	$\nabla^{\mathbf{\Sigma}} : \mathcal{P}_1(LX) \to \Lambda 2\mathrm{Vect}$	(connection on loop space)

$$\begin{array}{c} 2\mathrm{Vect} \\ \uparrow \nabla \\ \mathcal{P}_2(X) \end{array}$$



Example: the charged membrane

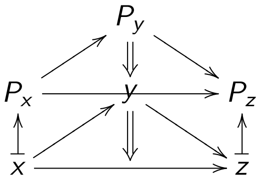
The input

target space:	$\mathbf{X} = \mathcal{P}_3(X)$	(3-paths in spacetime X)
background field:	$\nabla : \mathcal{P}_3(X) \to 3\mathrm{Vect}$	(SUGRA C_3-field)
parameter space	$\mathbf{\Sigma} = \Pi_2(\Sigma)$	(2-paths in Σ)

The output

config. space:	$\mathbf{X}^{\mathbf{\Sigma}} = \mathcal{P}_2(X^{\Sigma})$	(paths in Σ-space)
transgression:	$\nabla^{\mathbf{\Sigma}} : \mathcal{P}_1(X^{\Sigma}) \to \Lambda 3\mathrm{Vect}$	(connection on Σ-space)

$$\begin{array}{c} 3\mathrm{Vect} \\ \uparrow \nabla \\ \mathcal{P}_3(X) \end{array}$$



Example: gauge theory

The input

target space:	$\mathbf{X} = \mathbf{B}G$	(G regarded as groupoid)
background field:	$\nabla : \mathbf{B}G \to \mathbf{T}$	(field on $\mathbf{B}G$, eg. CS 3-bundle)
parameter space	$\mathbf{\Sigma} = \mathcal{P}_1(\Sigma)$	(paths in Σ)

The output

config. space:	$\mathbf{X}^{\mathbf{\Sigma}} = \mathrm{Bund}_\nabla(G)$	(G-bundles with connection)
transgression:	$\nabla^{\mathbf{\Sigma}} : \mathrm{Bund}_\nabla(G) \to \mathbf{T}^{\mathbf{\Sigma}}$	(gauge field action)

$$
\begin{array}{ccccc}
 & & \mathcal{P}_1(\Sigma) \otimes \mathbf{B}G^{\mathcal{P}_1(\Sigma)} & & \\
 & \swarrow & & \searrow^{\mathrm{ev}} & \\
\mathcal{P}_1(\Sigma) & & & & \mathbf{B}G \xrightarrow{\nabla} \mathbf{T}
\end{array}
$$

More precisely, in this context:

What is a background field?

The notion of background field

The background field is a mechanism to consistently assign

$$\nabla : \text{worldvolumes} \to \text{phases}$$

Familiar from

- Cheeger-Simons differential characters;
- $\simeq$ Deligne cohomology;
- $\simeq$ bundle gerbes with connection.

But here we want a little more:

- **localization** to all d-dimensional submanifolds;
- generalization to **arbitrary gauge n-groups**.

We want

- **parallel transport n-functors**.

What we have to do is:

- Task: *Characterize those n-functors that qualify as parallel transport.*

The solution we find is:

- An n-functor is a parallel transport if it is
  - smoothly;
  - locally trivializable;
  - with respect to a structure Lie n-group $G_{(n)}$.

Local trivialization of n-transport

We shall say that an $(n+1)$-functor

$$\begin{array}{c} \Pi_{n+1}(X) \\ \downarrow \\ T \end{array}$$

is a *parallel transport* with respect to a structure n-group $G_{(n)}$ if it admits a *smooth local trivialization* in the following sense.

n-Transport

Definition

An *n*-**transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is. . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\Pi_{n+1}(X)$$

on the fundamental $(n+1)$-groupoid of $X\ldots$

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\Pi_{n+1}(X) \xrightarrow{\nabla}$$

an $(n+1)$-functor...

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\Pi_{n+1}(X) \xrightarrow{\nabla} T$$

with values in some $(n+1)$-category of **fibers**...

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\begin{array}{ccc} \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\ & & \Big\downarrow \nabla \\ & & T \end{array}$$

such that locally, when pulled back to $Y \xrightarrow{\pi} X$ with n-connected fibers. . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccc}
\Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
& \swarrow^{\simeq} & \downarrow \nabla \\
& & T
\end{array}
$$

it is
equivalent to. . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccc}
\Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
\Big\downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \swarrow_{\simeq} & \Big\downarrow{\scriptstyle \nabla} \\
 & & T
\end{array}
$$

to a locally defined n-functor . . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccc}
\Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
\downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \overset{\simeq}{\Leftarrow} & \downarrow{\scriptstyle \nabla} \\
\mathbf{B}\mathrm{INN}_0(G_{(n)}) & & T
\end{array}
$$

with values in inner automorphisms of the structure n-group. . . . . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\begin{array}{ccc} \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\ \downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \swarrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \nabla} \\ \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T \end{array}$$

which are embedded into the fibers. . .

n-Transport

Definition

An n-**transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccccc}
\Pi^{\mathrm{vert}}_{n+1}(Y) & \hookrightarrow & \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
 & & \Big\downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \swarrow\!\!\swarrow{\scriptstyle \simeq} & \Big\downarrow{\scriptstyle \nabla} \\
 & & \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T
\end{array}
$$

such that restricted to vertical paths...

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\begin{array}{ccccc}
\Pi^{\mathrm{vert}}_{n+1}(Y) & \hookrightarrow & \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
\downarrow g & & \downarrow \nabla_{\mathrm{loc}} & \swarrow \simeq & \downarrow \nabla \\
\mathbf{B}G_{(n)} & & \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T
\end{array}$$

it factors through the structure group itself. . .

n-Transport

Definition

An *n*-**transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccccc}
\Pi_{n+1}^{\mathrm{vert}}(Y) & \hookrightarrow & \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \swarrow\!\!\swarrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \nabla} \\
\mathbf{B}G_{(n)} & \hookrightarrow & \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T
\end{array}
$$

which sits canonically inside the inner automorphisms.

n-Transport

Definition

An n-**transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$\begin{array}{ccccc} \Pi^{\mathrm{vert}}_{n+1}(Y) & \hookrightarrow & \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\ \downarrow g & & \downarrow \nabla_{\mathrm{loc}} & \swarrow \simeq & \downarrow \nabla \\ \mathbf{B}G_{(n)} & \hookrightarrow & \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T \end{array}$$

The vertical part is the cocycle/ descent data. . .

n-Transport

Definition

An ***n*-transport**

- on the smooth space X
- with structure Lie n-group $G_{(n)}$

is

$$
\begin{array}{ccccc}
\Pi^{\mathrm{vert}}_{n+1}(Y) & \hookrightarrow & \Pi_{n+1}(Y) & \xrightarrow{\pi} & \Pi_{n+1}(X) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \nabla_{\mathrm{loc}}} & \swarrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \nabla} \\
\mathbf{B}G_{(n)} & \hookrightarrow & \mathbf{B}\mathrm{INN}_0(G_{(n)}) & \xrightarrow{i} & T
\end{array}
$$

While the horizontal part is the connection/ curvature data.

One checks that this definition encompasses the examples one would expect:

- ordinary bundles with connection
- (nonabelian) gerbes with connection

(skip further details)

Parallel transport and bundles with connection

Parallel 1-transport and its equivalence to bundles with connection is discussed in [S.-Waldorf I].

Here we need the following version of the statement:

Parallel transport and bundles with connection

Theorem

For G simply connected, the category of G-bundles with connection on X is equivalent to diagrams

$$\begin{array}{ccccc} \Pi_2^{\mathrm{vert}}(Y) & \xrightarrow{g} & \mathbf{B}G & & \text{cocycle/descent data} \\ \downarrow & & \downarrow & & \\ \Pi_2(Y) & \xrightarrow{(A,F_A)} & \mathbf{B}\mathrm{INN}(G) & & \text{local connection data} \\ \downarrow & \nearrow_{\nabla} & & & \\ \Pi_2(X) & & & & \end{array}$$

of smooth 2-functors, for $Y \to X$ having connected and simply connected fibers.

Parallel transport and 2-bundles with connection

Parallel 2-transport and its equivalence to bundles with connection is discussed in [Baez-S.,S.-Waldorf II, S.-Waldorf III].

Here we need the following version of the statement:

Parallel transport and 2-bundles with connection

Theorem (unpublished)

For $G_{(2)} = \mathrm{AUT}(G)$, the category of G-gerbes with connection on X is equivalent to diagrams

$$\begin{array}{ccccc} \Pi_3^{\mathrm{vert}}(Y) & \xrightarrow{g} & \mathbf{B}G_{(2)} & & \text{cocycle/descent data} \\ \downarrow & & \downarrow & & \\ \Pi_3(Y) & \xrightarrow{(A,F_A)} & \mathbf{B}\mathrm{INN}_0(G_{(2)}) & & \text{local connection data} \\ \downarrow & \nearrow_{\nabla} & & & \\ \Pi_3(X) & & & & \end{array}$$

of smooth 3-functors, for $Y \to X$ having 2-connected fibers.

The sequences of n-groups

$$G_{(n)} \hookrightarrow \mathrm{INN}(G_{(n)}) \twoheadrightarrow \mathbf{B}G_{(n)}$$

appearing here is noteworthy.
Its L_∞-version will play a crucial role.

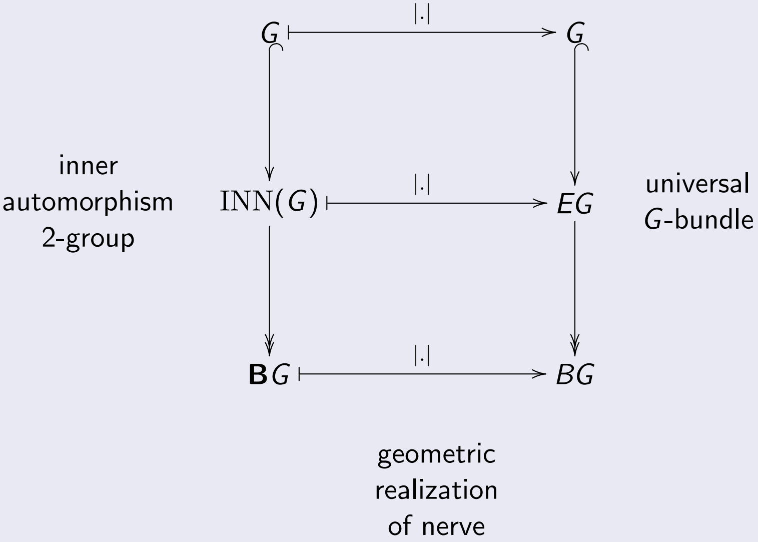
$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})$$

(skip further details)

Universal n-bundles

in their n-groupoid incarnation

Theorem (Segal, interpreted following Roberts-S.)



Theorem (Roberts-S., Baez-Stevenson, Roberts-Stevenson)

$$\begin{array}{ccccc} & G_{(2)} & \xrightarrow{|\cdot|} & G_{(2)} & \\ & \downarrow & & \downarrow & \\ \text{inner automorphism 3-group} & \mathrm{INN}_0(G_{(2)}) & \xrightarrow{|\cdot|} & EG_{(2)} & \text{universal } G_{(2)}\text{-2-bundle} \\ & \downarrow & & \downarrow & \\ & \mathbf{B}G_{(2)} & \xrightarrow{|\cdot|} & BG_{(2)} & \end{array}$$

geometric realization of nerve

Strategy from here on.

We will now pass from Lie n-groups and their morphisms to Lie n-algebras ($\simeq L_\infty$-algebras) and their morphisms.

This will make many things more powerfully tractable, at the cost of potentially losing "integral" information.

L_∞-connections

From n-transport to Lie ∞-connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$ **differential algebras** (qDGCAs)
$\mathrm{BINN}(G)$		$\mathrm{inn}(\mathfrak{g})$	$\mathrm{W}(\mathfrak{g})$
$\uparrow \nabla_{\mathrm{loc}}$		$\uparrow$	$\downarrow (A, F_A)$
$\Pi_{n+1}(Y)$		TY	$\Omega^\bullet(Y)$

Parallel n-transport is a morphism of Lie $(n+1)$-groupoids.

From n-transport to Lie ∞-connections

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Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	**differential** $\simeq$ **algebras** (qDGCAs)
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$\uparrow \nabla_{\mathrm{loc}}$		$\uparrow$	$\downarrow (A, F_A)$
$\Pi_{n+1}(Y)$		TY	$\Omega^\bullet(Y)$

This morphism may be differentiated. . .

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$\Pi_{n+1}(Y)$		TY	$\Omega^\bullet(Y)$

This morphism may be differentiated...

From n-transport to Lie ∞-connections

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$\mathbf{B}\mathrm{INN}(G)$		$\mathrm{inn}(\mathfrak{g})$	$\mathrm{W}(\mathfrak{g})$
$\uparrow \nabla_{\mathrm{loc}}$		$\uparrow$	$\downarrow (A, F_A)$
$\Pi_{n+1}(Y)$		TY	$\Omega^\bullet(Y)$

...to produce a morphism of Lie $(n+1)$-algebras.

From n-transport to Lie ∞-connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	Lie n-algebras ($\simeq$ n-term L_∞-algebras)	$\simeq$	differential algebras (qDGCAs)
$\mathbf{B}\mathrm{INN}(G)$		$\mathrm{inn}(\mathfrak{g})$		$\mathrm{W}(\mathfrak{g})$
$\uparrow \nabla_{\mathrm{loc}}$		$\uparrow$		$\downarrow (A, F_A)$
$\Pi_{n+1}(Y)$		TY		$\Omega^\bullet(Y)$

…to produce a morphism of Lie $(n+1)$-algebras.

From n-transport to Lie ∞-connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$ **differential algebras** (qDGCAs)
$\mathbf{B}\mathrm{INN}(G)$		$\mathrm{inn}(\mathfrak{g})$	$\mathrm{W}(\mathfrak{g})$
$\uparrow \nabla_{\mathrm{loc}}$		$\uparrow$	$\downarrow (A, F_A)$
$\Pi_{n+1}(Y)$		TY	$\Omega^\bullet(Y)$

These are best handled in terms of their dual maps,

From n-transport to Lie ∞-connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	**differential** $\simeq$ **algebras** (qDGCAs)

$$
\begin{array}{ccc}
\mathbf{B}\mathrm{INN}(G) & \mathrm{inn}(\mathfrak{g}) & \mathrm{W}(\mathfrak{g}) \\
\uparrow \nabla_{\mathrm{loc}} & \uparrow & \downarrow (A, F_A) \\
\Pi_{n+1}(Y) & TY & \Omega^\bullet(Y)
\end{array}
$$

which are morphisms of quasi-free differential-graded algebras.

L_∞-algebras

Ordinary Lie algebras as codifferential coalgebras

L_∞-algebras are easiest understood by way of the following

Observation

A bracket

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

induces a degree -1 codifferential

$$D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g}$$

on the free graded-cocommutative coalgebra $\vee^\bullet \mathfrak{g}$ (with $\mathfrak{g}$ regarded as being in degree 1) and the Jacobi identity is equivalent to

$$D^2 = 0.$$

L_∞-algebras are quasi-free codifferential coalgebras

Definition

An L_∞-algebra is a $\mathbb{N}_+$-graded vector space $\mathfrak{g}$ together with a degree -1 codifferential

$$D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g}$$

such that

$$D^2 = 0\,.$$

The original definition in terms of k-ary brackets can be seen to be equivalent to this concise definition [LadaStasheff, LadaMarkl] using the fact that codifferentials on free coalgebra are fixed by their action on "cogenerators".

L_∞-algebras are Lie ∞-algebras

Fact

L_∞-algebras generated in degrees $1, 2, \cdots n$	$\simeq$	(semistrict) **Lie n-algebras** $\simeq$ n-vector space with skew and coherently Jacobi bracket functor

Towards an ∞-Lie theorem

L_∞-algebras are to Lie ∞-groupoids as ordinary Lie algebras are to ordinary Lie groups.

- L_∞-algebras may be integrated to Lie ∞-groupoids [Getzler,Henriques]
- Lie ∞-groupoids may be differentiated to yield Lie ∞-algebras [Ševera].

Chevalley-Eilenberg-algebras of L_∞-algebras

Definition (L_∞-Chevalley-Eilenberg algebra)

For $\mathfrak{g}$ a finite dimensional L_∞-algebra, its Chevalley-Eilenberg algebra

$$\mathrm{CE}(\mathfrak{g})$$

is the free graded-commutative algebra $\wedge^\bullet \mathfrak{g}^*$ equipped with the degree $+1$ differential

$$d_{\mathrm{CE}(\mathfrak{g})} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

given by

$$d_{\mathrm{CE}(\mathfrak{g})}\omega = \omega(D(\cdot))\,.$$

Remark on terminology

- We say *quasi-free differential graded commutative algebras* (**qDGCA**s) for DGCAs which are free as GCAs but not necessarily as DGCAs.
- In the physics literature these qDGCAs are, somewhat imprecisely, often addressed as "free differential algebras" (**FDA**s).

Weil-algebras of L_∞-algebras

The Weil algebra

$$\mathrm{W}(\mathfrak{g}) = \left(\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{\mathrm{W}(\mathfrak{g})} = \begin{pmatrix} d_{\mathrm{CE}(\mathfrak{g})} & 0 \\ \sigma & \sigma \circ d_{\mathrm{CE}(\mathfrak{g})} \circ \sigma^{-1} \end{pmatrix} \right)$$

of a (finite dimensional) L_∞-algebra is

- the mapping cone of the identity on $\mathrm{CE}(\mathfrak{g})$;
- the CE-algebra of $\mathrm{inn}(\mathfrak{g})$;
- the Lie $(n+1)$-algebra of $\mathrm{INN}(G_{(n)})$.

L_∞-algebra valued forms

L_∞-algebra valued forms

Definition

For $\mathfrak{g}$ any L_∞-algebra, a $\mathfrak{g}$-valued form on Y is a DGCA morphism

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \; .$$

A flat $\mathfrak{g}$-valued form is such a morphism which factors through the canonical projection

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g})$$

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}) & \twoheadleftarrow & \mathrm{W}(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle (F,F_A=0)} & & \Big\downarrow{\scriptstyle (A,F_A)} \\ \Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y) \end{array}$$

Example

shiftet $\mathfrak{u}(1)$: $b^{n-1}\mathfrak{u}(1)$

$b^{n-1}\mathfrak{u}(1)$-valued forms are just ordinary n-forms.

$$
\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) & \twoheadleftarrow & \mathrm{W}(\mathfrak{g}) \\
\Big\downarrow{\scriptstyle A, F_A = 0} & & \Big\downarrow{\scriptstyle (A \in \Omega^n(Y), F_A = dA)} \\
\Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y)
\end{array}
$$

Example

strict Lie 2-algebras $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) & \twoheadleftarrow & \mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \\ \Big\downarrow {\scriptstyle (A,B)} \; {\scriptstyle F_A + t(B) = 0} & & \Big\downarrow {\scriptstyle (A,B,\beta,H)} \; {\scriptstyle \beta = F_A + t(B),\; H = d_A B} \\ \Omega^\bullet(Y) & = & \Omega^\bullet(Y) \end{array}$$

$A \in \Omega^1(Y, \mathfrak{g})$, $B \in \Omega^2(Y, \mathfrak{h})$

Example

String Lie n-algebras

$$
\begin{array}{ccccccc}
 & & \text{String-like} & & \text{Chern-Simons} & & \text{Chern} \\
1 & & 2n & & 2n+1 & & 2n+1 \\
\mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathfrak{g}_\mu) & \twoheadleftarrow & \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) & \hookleftarrow & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \\
\downarrow {\scriptstyle (A)}\ {\scriptstyle F_A=0} & & \downarrow {\scriptstyle (A,B)}\ {\scriptstyle F_A=0,\ dB+\mathrm{CS}_k(A)=0} & & \downarrow {\scriptstyle (A,B,C)}\ {\scriptstyle C=dB+\mathrm{CS}_P(A)} & & \downarrow {\scriptstyle (A,C)}\ {\scriptstyle dC=k((F_A)^{n+1})} \\
\Omega^\bullet(Y) & = & \Omega^\bullet(Y) & = & \Omega^\bullet(Y) & = & \Omega^\bullet(Y)
\end{array}
.
$$

$A \in \Omega^1(Y,\mathfrak{g})$, $B \in \Omega^{2n}(Y)$, $C \in \Omega^{2n+1}(Y)$

Examples

The ordinary String Lie 2-algebra

$$
\begin{array}{ccccccc}
\mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathrm{string}(\mathfrak{g})) & \twoheadleftarrow & \mathrm{W}(\mathrm{string}_k(\mathfrak{g})) & & \\
\| & & \|\simeq & & \|\simeq & & \\
\mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathfrak{g}_\mu) & \twoheadleftarrow & \mathrm{CE}(\mathrm{cs}_k(\mathfrak{g})) & \hookleftarrow & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \\
\downarrow {\scriptstyle (A)}\; {\scriptstyle F_A=0} & & \vdots\; {\scriptstyle (A,B)}\; {\scriptstyle F_A=0,\; dB+\mathrm{CS}_P(A)=0} & & \downarrow {\scriptstyle (A,B,C)}\; {\scriptstyle C=dB+\mathrm{CS}_P(A)} & & \downarrow {\scriptstyle (A,C)}\; {\scriptstyle dC=\langle F_A\wedge F_A\rangle} \\
\Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y)
\end{array}
$$

L_∞-connections

Differentiating

$$\infty\mathrm{Grpd} \xrightarrow{\mathrm{Lie}} L_\infty \xrightarrow{(\cdot)^*} \mathrm{qDGCAs}$$

we obtain from the definition of n-transport:

Definition

For $\mathfrak{g}$ an L_∞-algebra and X a smooth space, a $\mathfrak{g}$-connection descent object with respect to $Y \twoheadrightarrow X$ is a diagram

$$(\cdot)^* \circ \mathrm{Lie} \left(\begin{array}{ccc} \Pi^{\mathrm{vert}}_{n+1}(Y) & \xrightarrow{g} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi_{n+1}(Y) & \xrightarrow{\nabla_{\mathrm{loc}}} & \mathbf{B}\mathrm{INN}(G) \\ \downarrow & & \downarrow \\ \Pi_{n+1}(X) & \longrightarrow & T \end{array} \right)$$

Differentiating

$$\infty\mathrm{Grpd} \xrightarrow{\mathrm{Lie}} L_\infty \xrightarrow{(\cdot)^*} \mathrm{qDGCAs}$$

we obtain from the definition of n-transport:

Definition

For $\mathfrak{g}$ an L_∞-algebra and X a smooth space, a $\mathfrak{g}$-connection descent object with respect to $Y \twoheadrightarrow X$ is a diagram

$$(\cdot)^* \left(\begin{array}{ccc} T_{\mathrm{vert}} Y & \xrightarrow{dg} & \mathfrak{g} \\ \downarrow & & \downarrow \\ TY & \xrightarrow{d\nabla_{\mathrm{loc}}} & \mathrm{inn}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ TX & \longrightarrow & k \end{array} \right)$$

Differentiating

$$\infty\mathrm{Grpd} \xrightarrow{\mathrm{Lie}} L_\infty \xrightarrow{(\cdot)^*} \mathrm{qDGCAs}$$

we obtain from the definition of n-transport:

Definition

For $\mathfrak{g}$ an L_∞-algebra and X a smooth space, a $\mathfrak{g}$-connection descent object with respect to $Y \twoheadrightarrow X$ is a diagram

$$\begin{array}{ccc}
\Omega^\bullet_{\mathrm{vert}}(Y) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\
\uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
\Omega^\bullet(y) & \xleftarrow{(A, F_A)} & \mathrm{W}(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\Omega^\bullet(X) & \xleftarrow{\{P_i\}} & \mathrm{inv}(\mathfrak{g})
\end{array}$$

$$\begin{array}{ccc}
\Omega^\bullet_{\mathrm{vert}}(Y) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\
\Big\uparrow{\scriptstyle i^*} & & \Big\uparrow \\
\Omega^\bullet(Y) & \xleftarrow{(A,F_A)} & W(\mathfrak{g}) \\
\Big\uparrow{\scriptstyle \pi^*} & & \Big\uparrow \\
\Omega^\bullet(X) & \xleftarrow{\{K_i\}} & \mathrm{inv}(\mathfrak{g}) \\
 & & \\
H^\bullet_{\mathrm{dR}}(X) & \xleftarrow[\{[K_i]\}]{} & H^\bullet(\mathrm{inv}(\mathfrak{g}))
\end{array}$$

descent data

first Cartan-Ehresmann condition

connection data

second Cartan-Ehresmann condition

characteristic forms

Chern-Weil homomorphism

Example

Ordinary Cartan-Ehresmann connections

Let $P \to X$ be an ordinary principal G-bundle and $A \in \Omega^1(P, \mathfrak{g})$ a Cartan-Ehresmann connection 1-form on the total space.
Choosing $Y := P$, this is a $\mathfrak{g}$-connection descent object

Example

Interesting examples of $(n+1)$ $\mathfrak{g}$-connection descent objects arise as obstruction to lifting an ordinary 1-connection to a String-like n-connection.

These obstructing $(n+1)$-bundles with connection are (generalized) Chern-Simons $(n+1)$-bundles.

Application: Obstructing $(n+1)$-bundles

Let $\mathfrak{g}$ be an ordinary Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$ and let $\mu = \langle \cdot [\cdot, \cdot] \rangle$ the corresponding cocycle.

Definition

The Chern-Simons 3-bundle (CS 2-gerbe) of a $\mathfrak{g}$-bundle with connection is a $b^3\mathfrak{u}(1)$-connection whose characteristic 4-class is the Pontrjagin 4-class

$$P = \langle F_A \wedge F_A \rangle$$

of the $\mathfrak{g}$-bundle.

Theorem

Chern-Simons 3-bundles are the obstructions to lifting $\mathfrak{g}$-bundles to String 2-bundles, i.e. to $\mathfrak{g}_\mu$-2-bundles.

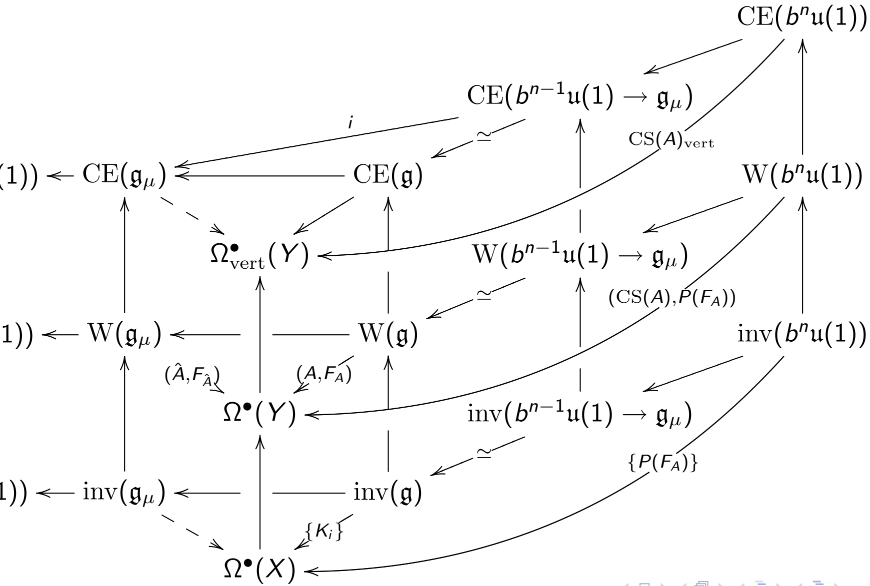
One computes this obstruction in a systematic manner by first lifting into the weak cokernel of

$$(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu)\,,$$

which is always possible, and the projecting out the shifted copy

$$(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \longrightarrow\!\!\!\!\!\rightarrow b^n\mathfrak{u}(1)$$

which contains the failure of the potential lift to just $\mathfrak{g}_\mu$. Applying this procedure to the diagram describing a $\mathfrak{g}$-connection as a whole yields...


$\mathrm{CE}(b^n\mathfrak{u}(1))$
$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu)$
i
$\simeq$
$\mathrm{CS}(A)_{\mathrm{vert}}$
$(1)) \leftarrow \mathrm{CE}(\mathfrak{g}_\mu)$
$\mathrm{CE}(\mathfrak{g})$
$\mathrm{W}(b^n\mathfrak{u}(1))$
$\Omega^\bullet_{\mathrm{vert}}(Y)$
$\mathrm{W}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu)$
$(\mathrm{CS}(A), P(F_A))$
$1)) \leftarrow \mathrm{W}(\mathfrak{g}_\mu)$
$\mathrm{W}(\mathfrak{g})$
$\mathrm{inv}(b^n\mathfrak{u}(1))$
$(\hat{A}, F_{\hat{A}})$
(A, F_A)
$\Omega^\bullet(Y)$
$\mathrm{inv}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu)$
$\{P(F_A)\}$
$1)) \leftarrow \mathrm{inv}(\mathfrak{g}_\mu)$
$\mathrm{inv}(\mathfrak{g})$
$\{K_i\}$
$\Omega^\bullet(X)$

By chasing the generators of $\mathrm{W}(b^n\mathfrak{u}(1))$ through this diagram one obtains the claimed result.

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