

transport of sections

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Abstract

Given a 2-vector 2-transport, we identify those notions that are necessary to define the holonomy over a disk with specified boundary condition and with up to two given boundary insertions.

Definition 1 *The 2-point is the category*

$$p_2 \equiv \{ \bullet \longrightarrow \circ \}$$

consisting of two objects and a single nontrivial morphism.

Definition 2 *The category of 2-point cobordisms is the 2-functor 2-category*

$$\text{Cob}_2 \equiv [p_2, \mathcal{P}_2(X)].$$

*The discrete category over the collection of objects is the **configuration space***

$$\text{Conf}_2 \equiv \text{Disc}(\text{Obj}(\text{Conf}_2))$$

of the 2-point.

Let \mathcal{C} be a monoidal category, and let

$$T \equiv \text{Bim}(\mathcal{C})$$

be the 2-category of bimodules internal to \mathcal{C} .

Fix a 2-transport

$$\text{tra} : \mathcal{P}_2(X) \rightarrow T.$$

Denote by

$$\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}}$$

the unique 2-transport on $\mathcal{P}_2(X)$ that sends everything to the identity on the tensor unit $\mathbb{1}$.

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Definition 3 A section of the 2-transport tra is a morphism

$$\begin{array}{ccc}
 & (\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_* & \\
 \text{Conf}_2 & \begin{array}{c} \curvearrowright \\ \Downarrow e \\ \curvearrowleft \end{array} & [p_2, T] . \\
 & \text{tra}_* &
 \end{array}$$

Accordingly, the space of sections is the category

$$\Gamma \equiv [(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*, \text{tra}_*].$$

A cosection of the 2-transport tra is a morphism

$$\begin{array}{ccc}
 & \text{tra}_* & \\
 \text{Conf}_2 & \begin{array}{c} \curvearrowright \\ \Downarrow f \\ \curvearrowleft \end{array} & [p_2, T] . \\
 & (\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_* &
 \end{array}$$

Accordingly, the space of cosections is the category

$$\Gamma \equiv [\text{tra}_*, (\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*].$$

Notice that, since Conf_2 is a discrete category, a section e is a collection of morphisms

$$\begin{array}{ccc}
 & \gamma^*(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}}) & \\
 p_2 & \begin{array}{c} \curvearrowright \\ \Downarrow e(\gamma) \\ \curvearrowleft \end{array} & T , \\
 & \gamma^* \text{tra} &
 \end{array}$$

one for each $\gamma : p_2 \rightarrow \mathcal{P}_2(X)$.

Definition 4 The algebra of observables is the monoid

$$A \equiv \text{End}(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}}) .$$

Notice that this is the monoid of bundles with connection on X whose fibers are objects of C . It is commutative in as far as C is braided.

Definition 5 The space of sections Γ is equipped with a left A -action, and the space of cosections with a right A -action in the obvious way, by pre- and

postcomposition, respectively:

$$\begin{array}{ccc}
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T] \\
 \Downarrow ae & & \Downarrow \\
 \text{Conf}_2 & \xrightarrow{\text{tra}_*} & [p_2, T]
 \end{array}
 \equiv
 \begin{array}{ccc}
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T] \\
 \Downarrow a_* & & \Downarrow e \\
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T] \\
 \Downarrow e & & \Downarrow \\
 \text{Conf}_2 & \xrightarrow{\text{tra}_*} & [p_2, T]
 \end{array}$$

and

$$\begin{array}{ccc}
 \text{Conf}_2 & \xrightarrow{\text{tra}_*} & [p_2, T] \\
 \Downarrow \tilde{e}a & & \Downarrow \\
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T]
 \end{array}
 \equiv
 \begin{array}{ccc}
 \text{Conf}_2 & \xrightarrow{\text{tra}_*} & [p_2, T] \\
 \Downarrow \tilde{e} & & \Downarrow \\
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T] \\
 \Downarrow a_* & & \Downarrow \\
 \text{Conf}_2 & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})_*} & [p_2, T]
 \end{array}
 .$$

Definition 6 A two-point disk transport associated to a cobordism

$$\begin{array}{ccc}
 p_2 & \xrightarrow{\gamma_1} & \mathcal{P}_2(X) \\
 \Downarrow D & & \Downarrow \\
 p_2 & \xrightarrow{\gamma_2} & \mathcal{P}_2(X)
 \end{array}
 ,$$

as well as to a section e_1 and a cosection \tilde{e}_2 is the morphism

$$\begin{array}{ccc}
 p_2 & \xrightarrow{\gamma_1} & \mathcal{P}_2(X) \\
 \Downarrow D & & \Downarrow \\
 p_2 & \xrightarrow{\gamma_2} & \mathcal{P}_2(X) \\
 \xrightarrow{\text{tra}} & & T \\
 \Downarrow e_1 & & \Downarrow \tilde{e}_2 \\
 \mathcal{P}_2(X) & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})} & T \\
 \Downarrow \tilde{e}_2 & & \Downarrow \\
 \mathcal{P}_2(X) & \xrightarrow{(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{1}})} & T
 \end{array}
 .$$

Example 1

This two-point disk transport is given by a 2-morphism in T of the form

$$\begin{array}{ccc}
 \mathbb{I} & \xrightarrow{\text{Id}} & \mathbb{I} \\
 e_1(x_1) \downarrow & \swarrow e_1(\gamma_1) & \downarrow e_1(y_1) \\
 A_{x_1} & \xrightarrow{\text{tra}(\gamma_1)} & A_{y_1} \\
 \text{tra}(\gamma_-) \downarrow & \swarrow \text{tra}(D) & \downarrow \text{tra}(\gamma_+) \\
 A_{x_2} & \xrightarrow{\text{tra}(\gamma_2)} & A_{y_2} \\
 \bar{e}_2(x_2) \downarrow & \swarrow \bar{e}_2(\gamma_2) & \downarrow \bar{e}_2(y_2) \\
 \mathbb{I} & \xrightarrow{\text{Id}} & \mathbb{I}
 \end{array}$$

This describes a section coming in, propagating along D , and being projected on the section coming out.

Definition 7 *Let*

$$\begin{array}{ccc}
 & \gamma_1^*(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{I}}) & \\
 & \curvearrowright & \\
 p_2 & \begin{array}{c} \mathbb{I} \\ \xrightarrow{c} \\ \mathbb{I} \\ \vee \end{array} & T \\
 & \curvearrowleft & \\
 & \gamma_2^*(\mathcal{P}_2(X) \rightarrow \text{Id}_{\mathbb{I}}) &
 \end{array}$$

be the peusonatural transformation given by the 2-morphism

$$\begin{array}{ccc}
 \mathbb{I} & \xrightarrow{\text{Id}} & \mathbb{I} \\
 \text{Id} \downarrow & \swarrow c & \downarrow \text{Id} \\
 \mathbb{I} & \xrightarrow{\text{Id}} & \mathbb{I}
 \end{array}$$

A **boundary condition** b for a two-point disk transport is a morphism

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \gamma_1^*(\mathcal{P}_2(X) \rightarrow \mathbb{1}) & & \\
 \curvearrowright & & \\
 p_2 & \xrightarrow{b} & p_2 \\
 \curvearrowleft & & \\
 \gamma_2^*(\mathcal{P}_2(X) \rightarrow \mathbb{1}) & & \\
 \text{Id}_{\mathbb{1}} \xrightarrow{c} \text{Id}_{\mathbb{1}} & & \\
 \text{V} & & \text{V} \\
 \text{Id}_{\mathbb{1}} & & \text{Id}_{\mathbb{1}}
 \end{array}
 & &
 \begin{array}{ccc}
 & \gamma_1 & \\
 & \curvearrowright & \\
 & p_2 & \xrightarrow{\text{tra}} \mathcal{P}_2(X) & \xrightarrow{\text{tra}} & T \\
 & \curvearrowleft & & & \\
 & \gamma_2 & & & \\
 \text{Id}_{\mathbb{1}} & & \text{Id}_{\mathbb{1}} & & \text{Id}_{\mathbb{1}} \\
 \text{V} & & \text{V} & & \text{V} \\
 \text{Id}_{\mathbb{1}} & & \text{Id}_{\mathbb{1}} & & \text{Id}_{\mathbb{1}} \\
 & e_1 & & & \tilde{e}_2 \\
 & \text{V} & & & \text{V} \\
 & \text{Id}_{\mathbb{1}} & & & \text{Id}_{\mathbb{1}}
 \end{array}
 \end{array}$$

Here $\text{Id}_{\mathbb{1}} \xrightarrow{c} \text{Id}_{\mathbb{1}}$ is the **two-point disk holonomy** of the two-point disk transport for the given boundary condition b .

Example 2

Assume that everything takes values not in arbitrary bimodules, but just in right induced bimodules

$$\text{RIBim}(\mathcal{C}) \subset \text{Bim}(\mathcal{C}) ,$$

and that the sections involved are such that

$$(\mathbb{1} \xrightarrow{e_1(x)} A_x) = (\mathbb{1} \xrightarrow{A_x} A_x) ,$$

as well as

$$(A_x \xrightarrow{\tilde{e}_2(x)} \mathbb{1}) = (A_x \xrightarrow{A_x} \mathbb{1}) ,$$

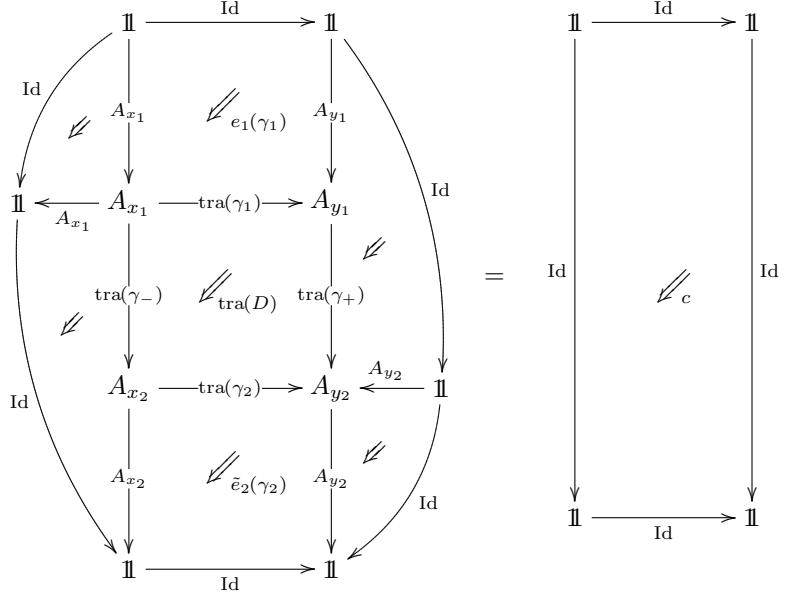
for all $x \in X$.

Then a boundary condition b for this is given by the modification of pseudo-natural transformations that looks like

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathbb{1} & \xrightarrow{\text{Id}} & \mathbb{1} \\
 \downarrow A_{x_1} & \searrow e_1(\gamma_1) & \downarrow A_{y_1} \\
 A_{x_1} & \xrightarrow{\text{tra}(\gamma_1)} & A_{y_1} \\
 \downarrow \text{tra}(\gamma_-) & \searrow \text{tra}(D) & \downarrow \text{tra}(\gamma_+) \\
 A_{x_2} & \xrightarrow{\text{tra}(\gamma_2)} & A_{y_2} \\
 \downarrow A_{x_2} & \searrow \tilde{e}_2(\gamma_2) & \downarrow A_{y_2} \\
 \mathbb{1} & \xrightarrow{\text{Id}} & \mathbb{1}
 \end{array}
 & = &
 \begin{array}{ccc}
 \mathbb{1} & \xrightarrow{\text{Id}} & \mathbb{1} \\
 \downarrow A_{x_1} & \searrow & \downarrow A_{x_1} \\
 A_{x_1} & \xrightarrow{\text{tra}(\gamma_-)} & A_{x_2} \\
 \downarrow A_{x_2} & \searrow & \downarrow A_{x_2} \\
 \mathbb{1} & \xrightarrow{\text{Id}} & \mathbb{1}
 \end{array}
 \end{array}$$

The unlabeled 2-morphisms here are the canonical ones.

This modification has a one-sided inverse, which allows to deduce that the two-point disk holonomy is



If we now let tra be the 2-connection on a $U(1)$ -gerbe and remove the two insertion points by setting $\gamma_1 = \text{Id}$ and $\gamma_2 = \text{Id}$, then this reproduces the diagram for the disk holonomy of a $U(1)$ -gerbe.