quantum n-transport

September 15, 2006

Abstract

Given an n-particle charged under an n-vector n-transport, I would like to describe its quantum mechanics in terms of a universal construction on n-transport functors.

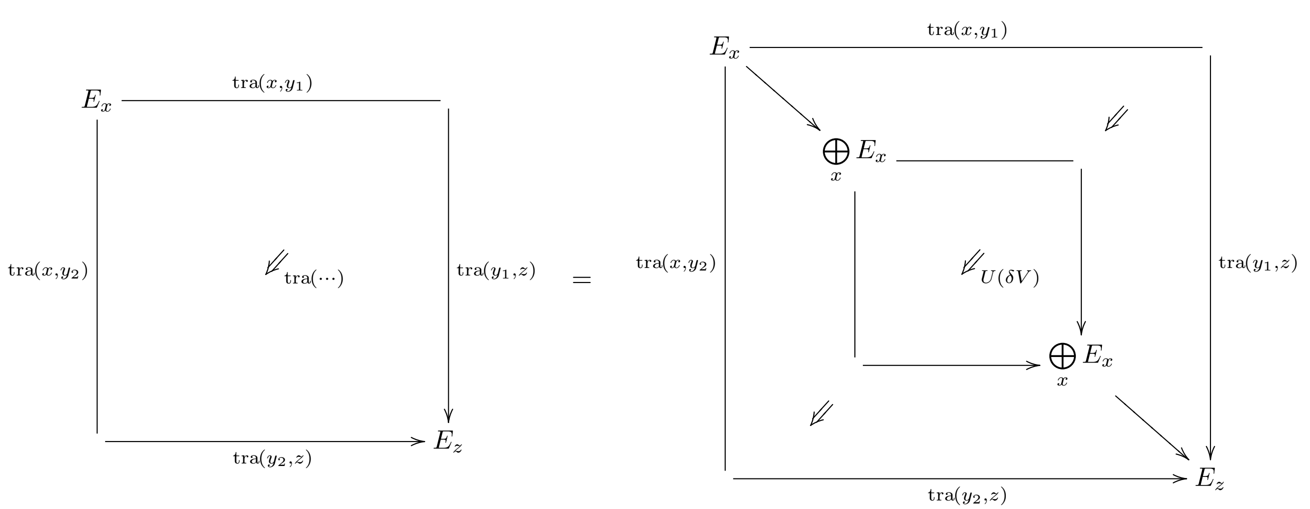
From Parallel Transport to Propagators. Let $\mathrm{tra} : \mathcal{P}_n^{\mathrm{cub}}(X) \to n\mathrm{Hilb}$ be an n-vector transport. Assume, as a model for the smooth case, that $\mathcal{P}_n^{\mathrm{cub}}(X)$ is generated from a collection of "infinitesimal" n-cubes, with X a finite set.

Let, furthermore, C_n be the n-category with just a single nontrivial n-morphism (i.e two objects, two nontrivial 1-morphisms, etc.). Think of this as the archetype of an n-cube.

We might have a chance of defining a "path integral" for n-particles charged under tra, if there is an n-functor

$$U(\delta V) : C_n \to n\mathrm{Hilb}$$

such that tra factors through U over every elementary n-cube of $\mathcal{P}_n^{\mathrm{cub}}(X)$, and such that U is, in some sense, the the smallest n-functor with this property.



For $n = 1$ it is easy to see what U should be like. I want to understand the property that U is the *smallest* functor with the desired property in terms of some universal construction. It should be some kind of coproduct in a suitable category.

Below I show how for $n = 1$ we can understand U as something very much, but still apparently not quite like a coproduct.

Realization. As a warmup, here are some considerations for $n = 1$.

Let G be a graph and C be a category.

Definition 1 *A morphism*

$$\gamma : G \to C$$

shall be a map sending edges of G to morphisms of C, such that sequences of edges are sent to sequences of morphisms. No map from vertices to objects is required.

For instance

$$\gamma \;:\; \begin{array}{c} \bullet \\ \downarrow{\scriptstyle a} \\ \bullet \\ \downarrow{\scriptstyle b} \\ \bullet \end{array} \quad\mapsto\quad \begin{array}{c} V_1 \\ {\scriptstyle f_1}\downarrow \\ V_2 \\ {\scriptstyle f_2}\downarrow \\ V_3 \end{array} .$$

Define morphisms between such morphisms by

Definition 2 *A transformation of two such morphisms*

$$G \underset{\gamma'}{\overset{\gamma}{\rightrightarrows}} C, \qquad \rho : \gamma \Rightarrow \gamma'$$

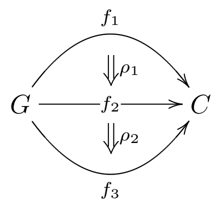
is a pair of assignments

$$\rho, \tilde{\rho} : \mathrm{Obj}\,(C) \to \mathrm{Mor}\,(C)$$

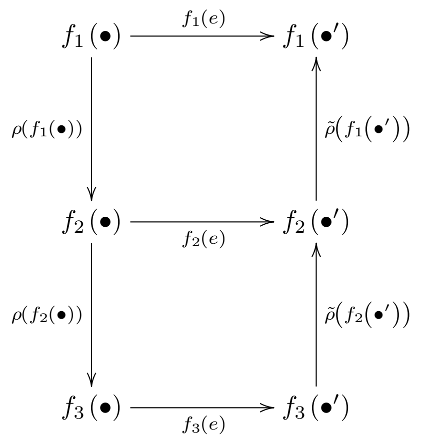
such that for all $(\bullet \xrightarrow{e} \bullet') \in \mathrm{Mor}\,(G)$ *we have*

$$\begin{array}{ccc} f\,(\bullet) & \xrightarrow{f(e)} & f\,(\bullet') \\ {\scriptstyle \rho(f(\bullet))}\downarrow & & \uparrow{\scriptstyle \tilde{\rho}(f(\bullet'))} \\ f'\,(\bullet) & \xrightarrow[f'(e)]{} & f'\,(\bullet') \end{array} .$$

Composition of such transformations is just composition of the vertical morphisms:



corresponding to



Definition 3 *Denote by $[G, C]$ the category whose objects are morphisms $G \to C$ and whose morphisms are as in def. 2.*

The motivating **example** for this definition is the following:

Let X be some finite set, and let $\mathcal{P}_1(X)$ be the groupoid freely generated from the free directed graph on X. Hence morphisms in $\mathcal{P}_1(X)$ are sequences of elements of x, like $x_1 \to x_2 \to x_3$, composition is concatenation and the only relation is $(x \to y \to x) = \mathrm{Id}_x$.

Let P be the *graph*

$$\cdots \bullet \xrightarrow{n} \bullet \xrightarrow{n+1} \bullet \xrightarrow{n+2} \bullet \cdots \,.$$

A **trajectory** in X is a morphism

$$\gamma : P \to \mathcal{P}_1(X)\,.$$

Fix a functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathbf{Vect}\,.$$

By composition with a trajectory, we get a graph map

$$\gamma^*\mathrm{tra} : \; P \xrightarrow{\gamma} \mathcal{P}_1(X) \xrightarrow{\mathrm{tra}} \mathbf{Vect}\,.$$

Assume we want to understand the coproduct

$$\bigoplus_{\gamma} \gamma^* \mathrm{tra}$$

in $[P, \mathcal{P}_1(X)]$.

This is the object of $[P, \mathcal{P}_1(X)]$ with the property that for all γ_i there exists a morphism

$$\gamma_i^* \mathrm{tra} \xrightarrow{f_i} \bigoplus_{\gamma} \gamma^* \mathrm{tra}$$

and such that for every other object Q with morphisms

$$\gamma_i^* \mathrm{tra} \xrightarrow{\tilde{f}_i} Q$$

there is a unique

$$\bigoplus_{\gamma} \gamma^* \mathrm{tra} \dashrightarrow Q$$

such that for all γ_i

$$\begin{array}{ccc} \gamma^* \mathrm{tra} & & \\ \downarrow f_i & \searrow^{\tilde{f}_i} & \\ \bigoplus_{\gamma} \gamma^* \mathrm{tra} & \dashrightarrow & Q \end{array} .$$

What I would like to obtain is the cocone over the $\gamma_i^* \mathrm{tra}$ given by

$$\begin{array}{cccc} U : & P & \to & \mathbf{Vect} \\ & (\bullet \xrightarrow{n} \bullet) & \mapsto & \left(\bigoplus_{x \in X} E_x\right) \xrightarrow{U(\delta t)} \left(\bigoplus_{x \in X} E_x\right) \end{array} ,$$

where $U(\delta t)$ is the $|X| \times |X|$-matrix whose (x, y)-entry is

$$U(\delta t)_{x,y} = \mathrm{tra}(x \to y) .$$

Here the morphisms f_i are given by the injections

$$\begin{array}{c} E_x \\ \downarrow f_i(E_x) \\ \bigoplus_{w \in X} E_w \end{array}$$

and projections

$$\begin{array}{c} E_x \\ \uparrow \tilde{f}_i(E_x) \\ \bigoplus_{w \in X} E_w \end{array}$$

coming from the coproduct of vector spaces.

Clearly, all the squares

$$\begin{array}{ccc} E_x & \xrightarrow{\mathrm{tra}(x\to y)} & E_y \\ \big\downarrow & & \big\uparrow \\ \left(\bigoplus_{w\in X} E_w\right) & \xrightarrow[U(\delta t)]{} & \left(\bigoplus_{w\in X} E_w\right) \end{array}$$

commute.

While I think this U has a morphism into any other Q which is a cocone over all the $\gamma_i^*\mathrm{tra}$, this morphism will not in general be a morphism of cocones. So maybe I should just drop this requirement. Or I should find a variation of the above setup.