In the discussion following the posts "The Principle of General Tovariance" and "Australian Category Theory", Urs Schreiber, Kea and David Corfield have been mentioning the research work on "categorical non-commutative geometry" that I am carrying on with my collaborators Roberto Conti (now in University of Newcastle - Australia) and Wicharn Lewkeeratiyutkul (in Chulalongkorn University - Bangkok). It is a pleasure to replay with some more detailed information on some of these topics.

Specifically this post is mainly concerned with the "horizontal categorification" (or "oidization/many-objectification" as John Baez prefers to call it) of the notion of (compact Hausdorff topological) space. Let us start with some simple but intriguing questions:

- What might be a good categorical version of the notion of space?
- Might non-commutative geometry provide some guidance towards at least one of the possible answers to the previous question?

In non-commutative geometry (compact Hausdorff) topological spaces are "described" dually as commutative unital C*-algebras making use of the following Gel'fand spectral theorem that, opening the way for the consideration of non-commutative unital C*-algebras as non-commutative compact Hausdorff spaces, is considered the "milestone" of non-commutative geometry (from A. Connes' point of view):

Theorem 1 (Gel'fand)**.** *There exists a duality (i.e. a contravariant equivalence)* $(\Gamma^{(1)}, \Sigma^{(1)})$ *between the category* $\mathscr{T}^{(1)}$*, of continuous maps between compact Hausdorff topological spaces, and the category* $\mathscr{A}^{(1)}$*, of unital homomorphisms of commutative unital C*-algebras. In the specific:*

- $\Gamma^{(1)}$ *is the functor that associates to every compact Hausdorff topological space* X *the unital commutative C*-algebra* $C(X)$ *of complex valued continuous functions on* X *(with pointwise multiplication, conjugation and supremum norm) and that to every continuous map* $f : X \to Y$ *associates the unital* $*$*-homomorphism* $f^\bullet : C(Y) \to C(X)$ *given by the pull-back of continuous functions by* f*;*
- $\Sigma^{(1)}$ *is the functor that associates to every unital commutative C*-algebra* $\mathcal{A}$ *its spectrum* $\mathrm{Sp}(\mathcal{A}) := \{\omega \mid \omega : \mathcal{A} \to \mathbb{C} \text{ is a unital } *\text{-homomorphism}\}$ *(that is a compact Hausdorff space with the weakest topology making continuous all the evaluation maps* $\omega \mapsto \omega(x)$*, for all* $x \in \mathcal{A}$*) and that to every unital* $*$*-homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ *of algebras associates the continuous map* $\phi^\bullet : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$ *given by the pull-back under* ϕ*.*

It is therefore natural to consider "categorifications" of Gel'fand duality as a first step in the process of categorification of A. Connes' non-commutative geometry and the question on the categorified notion of space now becomes:

- What might be a "categorical version of Gel'fand theorem"?

This is an important topic already widely discussed in this blog (see for example the posts What is the Categorified Gelfand-Naimark Theorem? and Categorified Gelfand-Naimark Theorem and Vector Bundles with Connection) in a much more general context.

Our approach here is extremely "minimal" (limited for now only to the many-objectification level) and it makes use of "suitably riassembled" traditional topological techniques (such as Serre-Swan equivalence) in perfect line with the classical Gel'fand result above. In order to provide a "horizontal categorification" of Gel'fand duality, we are looking for:

- "suitable embedding functors" $F : \mathscr{T}^{(1)} \to \mathscr{T}$ and $G : \mathscr{A}^{(1)} \to \mathscr{A}$ of the categories $\mathscr{T}^{(1)}$ (of compact Hausdorff topological spaces) and $\mathscr{A}^{(1)}$ (of unital commutative C*-algebras) into two new categories $\mathscr{T}$ and $\mathscr{A}$;
- an extension of the categorical duality $(\Gamma^{(1)}, \Sigma^{(1)})$ provided by Gel'fand theorem, to a categorical duality (Γ, Σ) between $\mathscr{T}$ and $\mathscr{A}$.

In the light of the following self-explicative table:

Monoids	Small Categories (Monoidoids)
Groups	Groupoids
Associative Unital Rings	Ringoids
Associative Unital Algebras	Algebroids
Unital C*-algebras	C*-categories (C*-algebroids)

we already have a suitable candidate for a horizontal categorification of the category of unital $*$-homomorphisms of unital C*-algebras, namely the category of object-bijective $*$-functors between full C*-categories.

The notion of C*-category has been introduced by J. Roberts (see Ghez P., Lima R., Roberts J. (1985), W*-categories, *Pacific J. Math.* 120 n. 1, 79-109; see also Mitchener P. (2002), C*-categories, *Proceedings of the London Mathematical Society* 84, 375-404) and has been extensively used in algebraic quantum field theory:

Definition 2. *A* ***C*-category*** *is a category $\mathcal{C}$ such that: the sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are complex Banach spaces; the compositions are bilinear maps such that $\|xy\| \leq \|x\| \cdot \|y\|$, $\forall x \in \mathcal{C}_{AB}$, $y \in \mathcal{C}_{BC}$; there is an involutive antilinear contravariant functor $* : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}$, acting identically on the objects, such that $\|x^*x\| = \|x\|^2$, $\forall x \in \mathcal{C}_{BA}$ and such that x^*x is a positive element in the C*-algebra $\mathcal{C}_{AA}$, for every $x \in \mathcal{C}_{BA}$ (i.e. $x^*x = y^*y$ for some $y \in \mathcal{C}_{AA}$).*

In a C*-category $\mathcal{C}$, the "diagonal blocks" $\mathcal{C}_{AA} := \mathrm{Hom}_{\mathcal{C}}(A, A)$ are unital C*-algebras and the "off-diagonal blocks" $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are unital Hilbert C*-bimodules on the C*-algebras $\mathcal{C}_{AA}$ and $\mathcal{C}_{BB}$. We say that $\mathcal{C}$ is full if all the bimodules $\mathcal{C}_{AB}$ are imprimitivity bimodules. In practice, every full C*-category is a "strict-ification" of an equivalence relation in the Picard-Morita groupoid of unital C*-algebras. It is also very useful to see a C*-category as an involutive category fibered over the equivalence relation of its objects: in this way, a (full) C*-category becomes a special case of (saturated) unital Fell bundle over an involutive base category where:

Definition 3. *A unital* ***Fell bundle*** *$(\mathcal{E}, \pi, \mathcal{X})$ is an involutive category $\mathcal{E}$ fibered over the involutive category $\mathcal{X}$ that is also a Banach bundle over $\mathcal{X}$ in such a way that, for every equivalence relation $\mathcal{Y} \subset \mathcal{X}$, its restriction $\pi^{-1}(\mathcal{Y}) \subset \mathcal{E}$ is a C*-category (with the same objects of $\mathcal{Y}$).*

The first problem that we have to face is how to select a suitable full subcategory $\mathscr{A}$ of "commutative" full C*-categories playing the role of horizontal categorification of the category $\mathscr{A}^{(1)}$ of commutative unital C*-algebras. Since we are working in a completely strict categorical environment, our choice is to define a C*-category $\mathcal{C}$ to be commutative if all its diagonal blocks $\mathcal{C}_{AA}$ are commutative C*-algebras.

The second problem is the identification of a good category $\mathscr{T}$ of "spaceoids" playing the role of horizontal categorification of the category $\mathscr{T}^{(1)}$ of continuous maps between compact Hausdorff topological spaces. Making use of Gel'fand duality for the diagonal blocks $\mathcal{C}_{AA}$ and (Hermitian) Serre-Swan equivalence for the off-diagonal blocks $\mathcal{C}_{AB}$ of a commutative full C*-category $\mathcal{C}$, we see that the spectrum of $\mathcal{C}$ identifies an equivalence relation embedded in the Picard groupoid of Hermitian line bundles over the Gel'fand spectra of the diagonal C*-algebras $\mathcal{C}_{AA}$. Finally, riassembling such block-data, we recognize that, globally, the spectrum of a commutative full C*-category can be described as a very special kind of Fell-bundle that we call topological spaceoid:

Definition 4. *A* ***topological spaceoid*** $(\mathcal{E}, \pi, \mathcal{X})$ *is a saturated unital rank-one Fell bundle over the product involutive topological category* $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$ *where* $\Delta_X := \{(p,p) \mid p \in X\}$ *is the minimal equivalence relation of a compact Hausdorff space* X *and* $\mathcal{R}_{\mathcal{O}} := \mathcal{O} \times \mathcal{O}$ *is the maximal equivalence relation of a discrete space* $\mathcal{O}$.

We define **morphism of spaceoids** $(f, F) : (\mathcal{E}_1, \pi_1, \mathcal{X}_1) \to (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ as pairs (f, F) where:

- $f := (f_\Delta, f_\mathcal{R})$ with $f_\Delta : \Delta_1 \to \Delta_2$ a continuous map of topological spaces and $f_\mathcal{R} : \mathcal{R}_1 \to \mathcal{R}_2$ an isomorphism of equivalence relations;
- $F : f^\bullet(\mathcal{E}_2) \to \mathcal{E}_1$ is a fiberwise linear $*$-functor such that $\pi_1 \circ F = \pi_2^f$, where $(f^\bullet(\mathcal{E}_2), \pi_2^f, \mathcal{X}_1)$ denotes the f-pull-back of $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$.

Morphisms of spaceoids can be seen as examples of J. Baez notion of spans (in this case, a span of the Fell bundles of the spaceoids).

Without entering in further technical details, we just say that we can define:

- a **section functor** $\Gamma : \mathscr{T} \to \mathscr{A}$ that to every spaceoids associates a commutative full C*-category of "block-sections",
- a **spectrum functor** $\Sigma : \mathscr{A} \to \mathscr{T}$ that to every commutative full C*-category associates its spectral spaceoid,

in such a way that this duality result holds:

Theorem 5. *The pair of functors* (Γ, Σ) *provides a duality between the category* $\mathscr{T}$ *of object-bijective morphisms between spaceoids and the category* $\mathscr{A}$ *of object-bijective* $*$*-functors between small commutative full C*-categories.*

The usual Gel'fand theorem is easily recovered identifying a compact Hausdorff topological space X with the trivial spaceoid $(\Delta_X \times \{(\bullet, \bullet)\}) \times \mathbb{C}$.

A lot of interesting questions are left open for future investigation:

- Can a full spectral duality theory for non-necessarily commutative C*-categories (or Fell bundles) be developed? And in this case, what is the relation between this horizontal categorified Gel'fand duality and other spectral duality theorems in terms of bundles already developed by R. Cirelli-A. Mania'-L. Pizzocchero and J. Dauns-K. Hofmann?

- What will be the connection between spectral spaceoids and other spectral constructs such as locales and topoi already used in the spectral theorems by C. Mulvey-B. Banachewki and C. Heunen-B. Spitters?

- Are spaceoids only an artifact in order to "force" an extension of Gel'fand categorical duality to the setting of commutative full C*-categories or are they (a part of) an inescapable structure that will emerge in a fundamental way whenever we deal with spectra?

Although we did not have time to explore in a significant way their applications, we are convinced that spaceoids are actually important for a deeper understanding of the notion of spectrum and that this bundle structure naturally associated to the spectrum might be relevant also in algebraic quantum field theory, where it might be used to describe "local gauge structures".

We are now working on a few generalizations of this duality result (for example relaxing the fullness and object-bijectivity conditions) and expecially to an extension of the result to the **strict** higher-categorical setting via a definition of strict n-C*-categories and strict n-Fell bundles: for a more detailed exposition of the several possible extensions of this work (and many other related ideas) it is possible to consult our survey paper "Non-commutative Geometry, Categories and Quantum Physics" and the seminar slides "Categorical Non-commutative Geometry and Quantum Physics". The general picture that is emerging is that the spectrum functor can be seen as an "endofunctor" in the category of commutative unital (higher) Fell-bundles.

As it is clear from the survey paper above, our main source of interest in the study of "categorical" extensions of Gel'fand duality, comes form the need to understand better (and the strict environment provided by C*-categories is just the easiest available playground) the several notions of morphims between non-commutative spaces (A. Connes' spectral triples and their "variants"). In this direction we are now going to settle some of the questions on the definition of non-commutative (spin/Riemannian) submanifolds left open in our previous paper A Category of Spectral Triples and Discrete Groups with Length Function. The formulation of such categorical non-commutative geometry is in turn just only one fundamental step needed in the development of a long term project on "modular algebraic quantum gravity" on which we have been working since 1995 and whose main lines are described in the previous survey paper. Hopefully we will be able to make some progress also on this front.

Thanks for the patience reading a long post ;-)