THE DIFFERENTIABLE SPACE STRUCTURES OF MILNOR CLASSIFYING SPACES, SIMPLICIAL COMPLEXES, AND GEOMETRIC REALIZATIONS

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In this paper we develop a simple and very general concept, called a differentiable space, by means of which one can define smooth functions, differential forms, and the de Rham cohomology on a wide variety of topological spaces without making use of any concept of tangent vectors. Our notion of differentiable space is related to those of J. W. Smith [30] and K. T. Chen [9], but differs in some important respects. We prove some theorems which give sufficient conditions for the de Rham cohomology defined in this way to equal the real singular cohomology of a space. In particular, we show that simplicial complexes, Milnor classifying spaces BG of Lie groups G, and geometric realizations of semi-simplicial manifolds have natural differentiable space structures which yield their correct real cohomology. Unlike J. Dugundji's [14] and C. Watkins' [37] definitions of differential forms on geometric realizations based on their "piecewise smooth" structure, the differentiable space approach permits one to define smooth morphisms from a manifold M to a non-manifold like Milnor's BG. For example, the classifying map $p: M \to BG$ of a G-bundle on M with smooth transition functions is a (smooth) morphism of differentiable spaces, provided that $j$ is constructed using a smooth partition of unity on $M$. Thus not only can one construct explicit characteristic forms on BG (by using a universal connection form on EG), but one can also pull them back at the form level to the de Rham complex of $M$ via $p^*: \Lambda^*(BG) \to \Lambda^*(M)$. In this way one gets a somewhat different perspective on the Chern-Weil homomorphisms which combines topology (classifying spaces) and geometry (connections and curvature).

A differentiable space is defined to be simply a topological space $X$ together with a sheaf $C^\infty(X)$ of germs of continuous real-valued functions on $X$, called smooth functions, satisfying the closure condition

If $f_0, \ldots, f_r$ are smooth functions on $X$, and $g$ is a smooth function on $M$, then $g(f_0, \ldots, f_r)$ is a smooth function on $X$.

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In other words, one simply takes some collection of functions on X and decomposes them to be smooth. For example, on a simplicial complex it is natural to take the barycentric coordinate functions, as well as functions which locally are smooth functions of finitely many barycentric coordinates, to be the smooth functions.

Differential forms on a differentiable space X are defined to be abstract symbols $\eta = \sum_i \eta_i \, dy^i$, $\cdots \times \sum_i \eta_{i} \, dy^{i}$ (locally finite sum), where the $\eta_i$ are smooth functions on X, modulo the equivalence relation defined by calling two such symbols $\eta_1$ and $\eta_2$ equivalent if each smooth map $\phi : E \to X$ (defined on some $\mathbb{R}^n$) pulls back $\eta_1$ and $\eta_2$ to identical forms on $E$.

The contents of this paper are as follows:

$\S 1$ defines differentiable spaces and their morphisms and presents a number of examples, including smooth manifolds, classifying spaces of Lie groupoids, and simplicial spaces.

$\S 2$ defines the de Rham complex of a differentiable space and proves some of its elementary properties, such as the Chain Rule. De Rham cohomology is defined, as usual, as the cohomology of the de Rham complex.

$\S 3$ defines smooth homotopies and shows that de Rham cohomology is invariant under smooth homotopies in the category of differentiable spaces.

$\S 4$ compares our definition of differentiable space with those of J. W. Smith and K. T. Chen, which are related to ours but not equivalent. The approach of Whitney, Sullivan, Thom, et al., who defined differential forms on simplicial complexes to be compatible collections of forms on the simplices, is discussed, as is the similar approach used by C. Winkler and J. Dupont to define differential forms on geometric realizations of semi-simplicial manifolds. We also discuss the de Rham double complex of a semi-simplicial space, as defined by Bott, Shulman, and Staakoff, the Weil algebra, and other related ideas.

In $\S 5$ we prove two theorems which give sufficient conditions for de Rham cohomology to equal real singular cohomology. The first of these (Theorem 5.2) says that if X is paracompact, admissible, and locally smooth and unity subordinate to any open cover, and is locally smoothly contractible, then $H^{\ast}_{\text{dR}}(X) = H^{\ast}_{\text{dR}}(X; \mathbb{R})$.

After discussing the Čech-singular and Čech-simplicial Rham double complexes of a differentiable space we prove (Theorem 5.3) that if a de Rham isomorphism holds on every finite intersection of some open cover of X which admits a smooth partition of unity, then $H^{\ast}_{\text{dR}}(X) = H^{\ast}_{\text{dR}}(X; \mathbb{R})$.

$\S 6$ discusses the problem of when an open cover of a differentiable space admits a smooth partition of unity. This question is important if we want to apply the de Rham theorems mentioned above. The results of this section follow from a recent theorem of H. Toruńczyk, who in the course of studying smooth partitions of unity on Banach spaces gave what amount to necessary and sufficient conditions for a differentiable space whose underlying topological space is metrizable to admit smooth partitions of unity (subordinate to any open cover). An easy consequence of his theorem is that any simplicial complex in the metric topology admits smooth partitions of unity.

In $\S 7$ we prove that $H^{\ast}_{\text{dR}}(X) = H^{\ast}_{\text{dR}}(X; \mathbb{R})$ for any simplicial complex X in either the weak or the strong (metric) topology. We show that our de Rham complexes $A^{\ast}(X; k)$ and $A^{\ast}(X)$ (arising from the weak and strong topologies on X) can be regarded as subcomplexes of the complex $A^{\ast}(X)$ of compatible forms on the simplices of X. (We shall call $A^{\ast}(X)$ "Whitney's complex," though it differs from Whitney's construction [40] in using smooth forms instead of flat cochains.) When X is a finite or locally finite simplicial complex, all three complexes coincide, but otherwise they differ. Thus one can think of $A^{\ast}(X)$ as providing an alternate description of Whitney's complex when X is locally finite.

In $\S 8$ we discuss the differentiable space structures and de Rham cohomologies of classifying spaces, and more generally of geometric realizations of semi-simplicial manifolds or differentiable space. This is most easily done if we choose the Milnor-Buffer-Lor classifying space functor $BG$ and the unwound geometric realization $\mu(X)$, because on these spaces there are defined global barycentric coordinates $y_1, \cdots, y_n$ and other functions $y_i$ or $j_i$ defined on open subsets of $BG$ or $\mu(X)$ which can be chosen to be our smooth functions. We show that if X is a semi-simplicial differentiable space and if $H^{\ast}_{\text{dR}}(X; k) = H^{\ast}_{\text{dR}}(X; k)$ for each $n$, then $H^{\ast}_{\text{dR}}(X; k) = H^{\ast}_{\text{dR}}(X; k)$.

$\S 9$ gives a detailed comparison of six different de Rham complexes defined on geometric realizations, including the double complex of Bohr-Shulman-Staakoff, the Du Pont-Winkler complex $A^{\ast}(X)$ of compatible forms, the unwound versions of these two complexes, and our complexes $A^{\ast}(\mu(X))$ and $A^{\ast}(\mu(X))$ defined using the weak or the strong topology on the unwound geometric realization $\mu(X)$. It is shown that all six complexes are de Rham homotopy equivalent but not isomorphic: The problem of pulling back forms from geometric realizations to manifolds is discussed.

$\S 10$ presents applications of differentiable spaces. One application, mentioned already, is the presentation of characteristic differential forms for G-bundles by explicit universal formulas involving only the transition functions and a smooth partition of unity. Another application is the extension of smooth and $C^\infty$ cohomology theories for spaces with two topologies (studied by the author in [23] and [24]) to non-manifolds like $(BGL - BGL)$, the Haefliger classifying space for foliations in the sheaf and jet topologies [2]. The
C^4 cohomology of (BT → BL) gives rise to characteristic classes of foliations which vary in a C^3 manner when the foliation is varied smoothly. The latter application was outlined in [24], modulo certain details about differentiable spaces which appear in this paper.

The differentiable space approach is very general and can potentially be applied to many spaces other than those discussed in this paper. In future papers the authors plans to present some of these other applications.

Remarks on notation. Smooth always means C^\infty. Manifolds are locally C^\infty diffeomorphic to R^n (n \geq 0) but need not be Hausdorff or second countable. The symbol N denotes the set of nonnegative integers. If X is a semi-simplicial space, then its rth space is denoted either X_r or X[r].

1. Definition and examples of differentiable spaces

Definition. A differentiable space is a topological space X together with, for each open U in X, a collection C^\infty(U) of continuous real-valued functions on U, satisfying the closure conditions:

(i) The rule U → C^\infty(U) defines a sheaf on X (denoted C^\infty(X)).

(ii) For any n, if f_1, …, f_n ∈ C^\infty(U) and g ∈ C^\infty(R^n) (with the usual meaning), then g(f_1, …, f_n) ∈ C^\infty(U).

The elements of C^\infty(X) are called smooth functions on X.

Remark. If one allows g ∈ C^\infty(R^n) (open in R^n), one gets an equivalent definition.

A basic way to define a differentiable space structure is the following. Let X be a topological space, and let (f_i: U_i → M_i) be a collection of continuous functions from open subsets U_i covering X to manifolds M_i. A function f: U → R (U open in X) is said to be locally a smooth function if finitely many of the f_i, for each x ∈ U, there exist a neighborhood W of x in U, a finite set of indices a_1, …, a_n, and a smooth map g: V → R (where V is open in M_{a_1} × M_{a_2} × … × M_{a_n}) such that for each i = 1, …, n, f|_{U_i} is defined on all of W (i.e., U_i ⊆ W).

(ii) We define C^\infty(X) to be the set of all such f. Then (X, (C^\infty(U))) defines a differentiable space structure on X, which we say is generated by (f_i).

Examples.

1. A smooth manifold M with its usual collection of (locally defined) smooth functions is a differentiable space.

2. A topological space X becomes a differentiable space in a (trivial way) if we decree every continuous function on X to be smooth, i.e., C^\infty(U) = (def.) C(U).

3. Let ⌦ be a Lie groupoid — smooth category with inverses [1] (for example, ⌦ is a Lie group G, ⌦ = Haefliger’s ⌦G_N). Then ⌦ is a differentiable space, and so is its Milnor-Buffet-Lor geometric realization BT [5], as we now show. Recall that a point of BT is specified by a collection (t_i, g_i) satisfying

(a) t_i > 0, i ∈ N, and t_i = 0 for all but a finite number of i.

(b) Σ t_i = 1.

(c) g_i ∈ ⌦i, but is defined only on {t_i = 0} (def.) = U_i ⊆ BT. One endows BT with either a weak or a strong topology (see §8 below). In either topology, we can define a differentiable space structure on BT by defining, for each open U in BT,

C^\infty(U) = (f: U → R|f is locally (in the chosen topology) a smooth function of finitely many of the functions g_i: U_i → G and t_i: BT → R).

4. The constructions of Example 3 work even for infinite-dimensional Lie groupoids like ⌦_U = (e-jets of local diffeomorphisms of U, with the C^\infty topology). We denote the functions x’ (coordinates of source), y’ (coordinates of target), and x’(y’ = (x_1, …, x_m, y_1, …, y_n)) = y = partial derivative of y with respect to x) to be smooth functions on ⌦_U; also any function which is locally a smooth function of finitely many of these is called smooth. A function on an open set of ⌦_U is smooth if it is locally a smooth function of finitely many functions of the form t_i or f + g, where f is a smooth function on ⌦_U and g: ⌦_U → ⌦_U is as in Example 3.

5. A simplicial complex X with either the weak or the strong topology (see §7 below) becomes a differentiable space if every function on X which is locally a smooth function of finitely many coordinate functions is smooth.

Definition. A morphism (also called a smooth map) of differentiable spaces is a continuous map which pulls back smooth functions to smooth functions.

That is, X → Y is smooth if

1. X is continuous.

2. For all open U ⊆ X and f ∈ C^\infty(U), f is in C^\infty(Y).

Remark. Condition 1 is superfluous if Y happens to be a logarithm and sets of the form f^{-1}(V) (V open in R^n, f ∈ C^\infty(Y)) is open.

The category of differentiable spaces will be denoted D.

Example 6. If M and N are smooth manifolds, then f: M → N is a morphism of differentiable spaces if and only if it is a smooth map in the usual sense.

Example 7. Let M be a smooth manifold, let ⌦ be a Lie groupoid, let (U_0 ∩ U_1) N be an open cover of M, and let (x_{U_0 ∩ U_1}^U_0 ∩ U_1) be a cocycle
on $M$ with values in $\Gamma$ having smooth transition functions $\gamma_p$. (This is the case, for example, if $\Gamma = T_2$ and $(\gamma_p)$ is the Haefliger cocycle defined by a smooth codimension $q$ foliation on $M$. The cocycle can be classified by the map
\[ f: M \to BT, \]
\[ f(x) = \langle x_0(x), y_p(x) \rangle \]
(see [5]). Since $f$ pulls back $\tilde{t}_i$ to $\tilde{t}_i$ and $\gamma_p$ to $\gamma_p$, it is a morphism of differentiable spaces.

Definition. Let $X$ be a differentiable space, and $Y$ a topological subspace of $X$. One makes $Y$ a differentiable subspace of $X$ by defining

$\mathcal{V}$ open in $Y$ and $f: Y \to \mathbb{R}$, then $f \in C^\infty(Y)$ if and only if for each $y \in Y$ there exist a neighborhood $U$ of $y$ in $X$ and an element $g \in C^\infty(U)$ such that $g \sqcap U \cap \mathcal{V} = f \sqcap U \cap \mathcal{V}$.

We can give a global description of $C^\infty(Y)$ if certain smooth structures of unity exist.

Definition. Let $X$ be a differentiable space, and $(U_1, \mathcal{U}_1)$ an open cover of $X$.

Then a smooth partition of unity subordinate to $(\mathcal{U}_1)$ (as usual) is a collection $(\varphi_j) \in C^\infty(\mathcal{V})$ satisfying:

- $(\varphi_j) > 0$ and $\sum \varphi_j = 1$,
- $\text{supp}(\varphi_j) \subseteq \text{Cl}(\text{def}(f) \cap (0, \infty) \subseteq \mathcal{U}_1$,
- the collection $(\text{supp} \varphi_j)$ is locally finite.

Theorem 1.1. (i) Let $X$ be a differentiable space, and let $Y \subset X$. Suppose that every open neighborhood $W$ of $Y$ admits smooth partitions of unity subordinate to any open cover of $W$. Then $f \in C^\infty(Y)$ if and only if $f$ extends to a smooth function on some open neighborhood of $Y$.

(ii) If $X$ admits smooth partitions of unity, and $Y \subset X$ is closed, then $Y$ admits smooth partitions of unity, and every $f \in C^\infty(Y)$ extends (non-uniquely, in general) to some $f_1 \in C^\infty(X)$.

Proof. (i) For each $y \in Y$ choose a neighborhood $U_j$ of $y$ in $X$ and an $f_j \in C^\infty(U_j)$ such that $f_j(U_j \cap Y) = f_j(U_j \cap Y) = 0$. Let $f = f_j$, and let $(\varphi_j)$ be a smooth partition of unity on $U$ subordinate to $(\mathcal{U}_j)$. Then $\varphi_j f_j$ is a smooth extension of $f_j$ to $U$.

(ii) For the second assertion, proceed as in the proof of (i), but add the open set $X - Y$ to the collection $(\mathcal{U}_j)$ to get a cover of $X$. Choosing a partition of unity, we extend $f$ to $2\sum \varphi_j f_j$ as before. A similar construction shows that $Y$ admits smooth partitions of unity.

Example 8. If $K \subset \mathbb{R}^n$, then $f: K \to \mathbb{R}$ is smooth if and only if $f$ extends to some neighborhood of $K$ in $\mathbb{R}^n$. It is easy to see that if $K$ has an affine or convex structure (e.g., $K = \mathbb{R}^n$, the Euclidean $n$-simplex), then $C^\infty(K)$ is independent of $\mathcal{A}$ and of the affine embedding of $K$ in $\mathbb{R}^n$.

Example 9. IX is any differentiable space and $U$ an open subspace, then $C^\infty(U) = C^\infty(\mathbb{R})$.

2. The de Rham complex of a differentiable space

Rather than defining a notion of tangent vector for differentiable spaces (see [25, 26]), we shall define differential forms as abstract symbols $\xi_1 \wedge \ldots \wedge \xi_{\text{dim}}(\xi_1 \in C^\infty(\mathcal{U}, \mathbb{R} \subset X)$, and compare forms by pulling them back to open subsets of Euclidean space.

Definition. Let $X$ be a differentiable space. Then a plot of $X$ is a smooth morphism $\phi: E \to X$, where $E$ is an open subspace of $\mathbb{R}^m$ for some (finite) $m$. (The terminology is adopted from Chern's [8].)

Note. For convenience, we sometimes drop the requirement that $E$ be open, and allow plots of the form $\phi: I \to X$, $\phi: E \times I \to X$ etc. This will not change any results.

Definition. Let $U$ be a differentiable space, and let $f_j \in C^\infty(U)$, $i = 1, \ldots, p; j = 0, \ldots, q$. Let $\gamma$ denote the symbol $\xi_1 \wedge \ldots \wedge \xi_{p+q}$. Then $f \in C^\infty(U)$, and let $\phi: E \to U$ be a plot. Then $\phi^*\gamma$ denotes the differential form

\[ \phi^*\gamma = \sum_{i=1}^{\text{dim}} (f_i \wedge \ast f_i) \wedge \ldots \wedge \Lambda \, \wedge (f_q \wedge \ast f_q) \in \Omega(X). \]

Let $B^\mathcal{A}(U)$ be the real vector space of symbols of this form ($\phi$ is arbitrary) modulo the equivalence relation:

$\phi_1 \sim \phi_2$ if and only if $\phi_1^*\gamma = \phi_2^*\gamma$ for all plots $\phi: E \to U$.

If $X$ is a differentiable space, then the role $U \to B^\mathcal{A}(U)$ (open in $X$) defines a presheaf of real vector spaces on $X$. Let $\mathcal{A}X$ be the sheaf generated by this presheaf, and let $\mathcal{A}X(U) = \Gamma(A(U), \mathcal{A})$ for sections.

Remark. $B^\mathcal{A}(X)$ contains finite sums of symbols $\xi_1 \wedge \ldots \wedge \xi_n$, but $\mathcal{A}(X)$ also contains finitely finite sums of such symbols.

Lemma 2.1. The canonical homomorphism $B^\mathcal{A}(X) \to \mathcal{A}(X)$ is injective.

Proof. Let $\eta$ be in $\ker$. By sheaf theory we can find an open cover $\mathcal{U}_i$ of each such subset $\eta_i \in B^\mathcal{A}(X)$ of $\eta$ equals 0. Let $\phi: E \to U$ be a plot. It suffices to show that $\phi^*\eta = 0$. Let $\phi_1$ be a plot, then $\phi_1^*\eta_i$ is a plot of $\mathcal{U}_i$, and $\phi_2^*\eta_i = (\phi_2\phi_1)(\phi_1^*\eta_i) = (\phi_1^*\eta_i) = 0$. It follows that $\phi^*\eta = 0$.

Corollary 2.1. If $\phi: E \to X$ is a plot in $X$ and $\phi: E \to X$ then there is a well-defined form $\phi^*\gamma \in \Omega(X)$.
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3. Smooth homotopy invariance

Products. Let X and Y be differential spaces. Then $X \times Y$ is the differential space defined by giving $X \times Y$ the usual Cartesian product topology and defining

for $U$ open in $X \times Y$, $C^\infty(U) = \{ f : U \to \mathbb{R} | f \text{ is locally of the form } g \circ (f_1, \ldots, f_n) \}$ where $g \in C^\infty(\mathbb{R}^n)$ (arbitrary) and each $f_i$ belongs to $C^\infty(Y)$ (for some open $Y \subset X$) or to $C^\infty(W \times \mathbb{R}^n)$.

Example. Let $f$ be the unit interval, regarded as a differentiable subspace of $\mathbb{R}$ with coordinate $t$. Then $f \in C^\infty(X \times I)$ if and only if $f$ is locally of the form $g \circ (t, f_1, \ldots, f_n)$, where $g \in C^\infty(\mathbb{R}^n)$ and $f_i \in C^\infty(U \subset \mathbb{R})$.

Definition. A smooth morphism is a morphism of differential spaces $F : X \to Y$; one says that $f$ and $g$ are smoothly homotopic.

By mimicking the proof of smooth homotopy invariance of de Rham cohomology on manifolds [38], we now prove that $H^n_{\text{dr}}$ is smooth homotopy invariant on the larger category of differential spaces. We start with two technical lemmas which in the manifold case are proved by choosing coordinate charts.

Lemma 3.1. Let $\eta \in A^q(X \times I)$ ($X$ a differential space). Then every $x \in X$ has a neighborhood $U$ in $X$ such that $\eta|U \times I$ can be written as a finite sum of terms of the form $\alpha \wedge df_1 \wedge \cdots \wedge df_m$, and $b(x, t) \in C^\infty(U \times \mathbb{R})$.

Proof. By the definition of $C^\infty(U \times \mathbb{R})$ and the Chain Rule, $\eta$ is locally (on $X \times I$) a finite sum of terms of the desired form. Cover $[x] \times I$ by a finite number of product opens $U_i \times J_i \subset X \times I$ on each of which $\eta$ has such a representation $\eta_i$. Let $U = \cap U_i$ and let $(J_i)$ be a smooth partition of unity subordinate to the cover $(U_i)$ of $I$. Then $\sum_i \eta_i$ is the desired representation of $\eta$ on $U \times I$.

Lemma 3.2. Let $f \in C^\infty(X \times I)$ ($X$ a differential space). Define $F : X \times I \to \mathbb{R}$ by

$$F(x, t) = \int_0^t f(x, s) \, ds.$$

Then $F \in C^\infty(X \times I)$, and hence $\eta \circ F$ is $C^\infty(U \times \mathbb{R})$.

Proof. Choose $x \in X$. By an argument like that used to prove Lemma 3.1, we can find a neighborhood $U$ of $x$ in $X$ such that $f(U \times I) = \mathbb{R}$, where $g \in C^\infty(\mathbb{R}^n)$ and $f \in C^\infty(U \subset \mathbb{R})$. Define $G : U \times I \to \mathbb{R}$ by

$$G(u, t) = \int_0^t g(u, x_0, \ldots, x_n) \, dt.$$
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Then, \( G \in C^\infty(\mathbb{R}^n) \), and \( H(x, y) = G(u(x), \ldots, f(y)) \), so that \( F = G \circ (u_1, \ldots, f_n) \in C^\infty(\mathbb{R}^n \times I) \), q.e.d.

We can now construct a homotopy operator:

\[ L : A^i(\mathbb{R}^n \times I) \to A^{i+1}(\mathbb{R}^n) \]

between \( i \) and \( i+1 \): \( A^i(\mathbb{R}^n \times I) \to A^{i+1}(\mathbb{R}^n) \) (where \( i : \mathbb{R}^n \to \mathbb{R}^n \times I \)).

\[ L(a(x), t) dt \wedge da(x) = \left( \int_0^1 a(x, t) \cdot dt \right) \wedge da(x) = 0 \]

\( \{ f \in C^\infty(U) \) is of order \( i \). To see that this definition of \( L \) is independent of choices, it suffices to pull back via plots of the form \( \phi \times id : E \times I \to \mathbb{R}^n \times I \), and to observe that the operator \( U^* [E(\mathbb{R}^n \times I) \to C^\infty(\mathbb{R}^n)] \) defined by the same formula as \( L \) is known to be well-defined [38]. The same argument shows that
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construction is one way to do this, or one can use special constructions for particular spaces, such as Chen’s iterated integrals on path spaces [3], [9].

The choice of a differentiable space theory depends on the use to be made of it. Chen’s theory is well-suited to studying path spaces \( Y \), since a plot \( \mathbf{K} \) is just a smooth map \( \mathbf{K} \times [0,1] \to Y \). Smith used his theory to study quotients of manifolds. On the other hand, a differentiable space with a distinguished set of functions, for example a simplicial complex with its barycentric coordinates, or \( BG \) with the functions \( s_i \) and \( d_j \) (see §1), is natural to apply Smith’s or our theory. In practice, Smith’s closure condition can be difficult to verify on an explicit collection of functions, and therefore we refer to our own definition for studying simplicial complexes, classifying spaces, and geometric realizations.

The definition which we have given of differentiable space appeared in the work of R. Sikorski [29], but he did not define differential forms in the way we did. Rather, he defined tangent fields on \( X \) as derivations of the ring \( C^\infty(X) \), and did not define a de Rham theory. (A de Rham theory based on the same definition of vector field was worked out by R. Palais [25].) Thus the individual elements of our definition have appeared elsewhere. Nonetheless, we believe that their combination is new, and that the idea of treating simplicial spaces, Milnor’s \( BG \), and geometric realizations of semi-simplicial spaces as differentiable spaces, using a category of differentiable spaces so general that it includes morphisms like \( f : M \to BG \) (\( M \) an ordinary manifold), and the classifying map of a \( G \)-bundle on \( M \), is new.

A different way of putting differential forms on simplicial complexes is to take compatible collections of forms on the simplices. This was done by Whitney [40, p. 226] and Thom [34] (in the special case of a triangulated manifold, the zero-forms obtained are the piecewise smooth (semi-smooth) functions [19, p. 5]), and the idea has been refined by Sullivan [33] and others to compute not only the real cohomology but also the rational homotopy type of a simplicial complex. In a similar vein, J. Dupont [14] and C. Watkiss [57] defined differential forms on stratifying spaces \( BG \) and geometric realizations \([X]\) of semi-simplicial manifolds as differences of differential forms on \( G^\times \Delta^2 \) (resp. \( X \times G \)) compatible under the face maps. The major difference between these theories and ours is that compatible collection theories require working in simple complex or semi-simplicial categories which do not include morphisms like \( f : M \to BG \), while our theory mixes simplicial and non-simplicial constructions easily. For example, in our theory the homomorphism \( f^* : A^*BG \to A^*M \) is defined at the cochain level of differentiable forms, while in the compatible collection approach \( f^* \) is defined only at the cohomology level or as a map of Čech-de Rham double complexes.

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A more detailed comparison of our de Rham complex with those of Whitney, Dupont, and Watkiss will be given in §77 and 9.

So far we have discussed only those de Rham theories based on objects which look like differential forms on the actual spaces in question. For example, in Dupont’s, Watkiss’s, and our theories, a form on \( BG \) involves the barycentric coordinates \( s_i \) as well as the \( G \)-coordinates \( d_j \). If space is equipped with a distinguished set of functions, for example a simplicial complex with its barycentric coordinates, or \( BG \) with the functions \( s_i \) and \( d_j \) (see §1), it is natural to apply Smith’s or our theory. In practice, Smith’s closure condition can be difficult to verify on an explicit collection of functions, and therefore we refer to our own definition for studying simplicial complexes, classifying spaces, and geometric realizations.

While these theories are probably easier to differentiable spaces to compute the cohomology of specific spaces like \( BG \), differentiable spaces are useful for studying how this cohomology maps into the cohomology of manifolds at the cochain level, especially when \( G \)-bundle theory is specified by transition functions. Furthermore, the differentiable space approach, very general, is applicable to other spaces as well.

Indeed, there have been a number of other generalizations of differential forms to non-manifolds. For example, C. D. Marshall [20] defined a de Rham theory for subcartesian spaces, which are Hausdorff spaces locally homeomorphic to (not necessarily open) subspaces of \( \mathbb{R}^n \). In a different direction, there is a de Rham theory for infinite dimensional manifolds. These theories do not apply to Mislov classifying spaces \( BG \) or to non-locally finite simplicial complexes, however, since these spaces are not locally homeomorphic to any set in \( \mathbb{R}^n \) or to any open set in \( \mathbb{R}^n \).

5. Conditions guaranteeing a de Rham isomorphism

Because the concept of differentiable space is so general, it is clear that a de Rham isomorphism \( H^\ast_X(M) \cong \mathbb{H}^\ast_{\text{loc}}(X; \mathbb{R}) \) will hold only if we place some restrictions on \( X \) and \( M \). In this section we present two such sets of restrictions. The first criterion is a local property (smooth, local contractibility) together with requirement of paracompactness and existence of smooth partitions of unity, and is proved using sheaf theory; a virtually identical theorem was proved by Čech for his theory [30]. The second criterion is "non-local": it is de Rham isomorphism holds on all finite intersections of an open cover of \( X \) which admits a smooth partition of unity, then it holds on \( X \).

This is proved by adapting A. Weil’s Čech-de Rham double complex [38]. In the next section we shall prove the existence of smooth partitions of unity on
simplicial complexes. This will allow us to apply the second criterion to
simplicial complexes, classifying spaces, and geometric realizations in §§7
and §8.

To compute de Rham and singular cohomology, we map both of them to
the smooth singular cohomology $H^r_{\text{sm}}$, which is defined as follows.

Definition. A smooth singular $p$-complex of a differentiable space $X$ is a
morphism $\delta^p : X \to \mathbb{R}^p$ of differentiable spaces. The real-valued cochains on
these simplices form a vector space denoted $S^p_{\text{sm}}(X)$, we set $H^r_{\text{sm}}(X) = \text{Hom}(S^r_{\text{sm}}(X), \mathbb{R})$.

Remark. The cochains of $H^r_{\text{sm}}$ are not sheaves.

A map from $\delta^p : X \to \mathbb{R}^p$ is defined, as usual, by integration of
(pulled-back) forms over smooth simplices. This induces a map $H^r_{\text{sm}}(X) \to H^r_{\text{sm}}(X)$.

Lemma 5.1. $H^r_{\text{sm}}$ is invariant under smooth homotopies of differentiable
spaces.

Proof. The proof of homotopy invariance of ordinary singular cohomology
[14, p. 45] can be used without change once one observes that the prism
operator $P : S^p(X) \to S^p_{\text{sm}}(X \times I)$ maps each smooth simplex to a sum of
smooth simplices.

Definition. A differentiable space $X$ is locally smoothly contractible if for
each open $U \subset X$ and each point $x \in U$ there is a neighborhood $V$ of $x$ in
$U$ and a smooth homotopy $F = (f_t) : V \times I \to U$ satisfying

$$f_0 = \text{id} : V \subset U,$$

$$f_1 : V \to \{x\} \subset U \text{ (for some point } y \in U).$$

Theorem 5.2. Let $X$ be a differentiable space which is paracompact and
locally smoothly contractible, and which admits smooth partitions of unity
subordinate to any open cover. Then the natural homomorphisms

$$H^r_{\text{sm}}(X) \to H^r_{\text{sm}}(X) \to H^r_{\text{sm}}(X; R)$$

are isomorphisms.

Proof. Let $S^p$ (resp. $S^p_{\text{sm}}$) be the sheaf on $X$ generated by
$U \to S^p_{\text{sm}}(U; R)$ (resp. $U \to S^p_{\text{sm}}(U)$). Let $\delta^p$ denote the
sheaf $A^p X$ of differential forms (see §2). Now since $X$ is paracompact, the
sheafified and unshaved, singular cohomology theories $H^r_{\text{sm}}$ and $H^r_{\text{sm}}$ are
isomorphic to $H^r_{\text{sm}}$ (see §2). Similarly, $H^r_{\text{sm}}(X) \to H^r_{\text{sm}}(X; R)$.

Now since $X$ admits smooth partitions of unity, the sheaf $C^p_{\text{sm}}$ is fine.
Since $A^p$, $S^p_{\text{sm}}$, and $S^p$ are modules over $C^p_{\text{sm}}$, they, too, are fine.
Now the constant sheaf $R$ injects into $A^p$, $S^p_{\text{sm}}$, and $S^p$. We claim that all
three complexes are resolutions of $R$. Indeed, $X$ is locally smoothly
contractible and all three cohomology theories are smooth homotopy
invariant, it follows that the stalk of the homology (derived) sheaf $A^p, \text{ p of all}
three complexes of sheaves reduces to $H^r(\text{point}) = R$ (in di-

mension $0$) at any point $x \in X$. But the cohomology of global sections of a
resolution of a sheaf $F$ on a paracompact space $X$ by fine sheaves equals the
sheaf cohomology $H^r(X; F)$ [4, pp. 39-50]. It follows that these cohomologies agree, i.e.,

$H^r_{\text{sm}}(X) = H^r_{\text{sm}}(X) = HTS^p_{\text{sm}}(X) = HTS^p = H^r_{\text{sm}}(X; R).$

Q.E.D.

Our next criterion for a singular-de Rham isomorphism is based upon
comparing the cohomology (de Rham, singular, or smooth singular) of a
space $X$ from the cohomology of each finite intersection of some open cover
of $X$. In the de Rham case this entails looking at the Čech-de Rham double
complex of $A_{\nu}$. Weil [38]. In the singular and smooth singular cases we use a
Čech-singular double complex and an extended Mayer-Vietoris theorem.

Definition. Let $X$ be a topological space, and $U$ be an open cover of
$X$, indexed by a totally ordered set $I$. Then $B = (p \in I)$ will denote the
space $\bigcup_{i \in I} U_i$, where $\sigma = (a_0, a_1, \ldots, a_p)$ runs over all strictly ordered
($p + 1$)-tuples of $I$, and $U_{a_0} \cap \cdots \cap U_{a_p}$ maps

$$b_i : U_i \to B_{a_i - 1}, i = 0, \ldots, p,$$

are defined by

$$b_i |_{U_{a_i}} = \text{inclusion} : U_{a_i} \subset U_{a_i}.$$

Here $\sigma = (a_0, a_1, \ldots, a_p, \ldots)$.

If $X$ is a differentiable space, and $B$ an open cover of $X$, then the Čech-de
Rham complex of $X$, denoted $A^p_{\text{Čech}}$ is the double complex $B_{\bullet} \to A^p_{\text{Čech}}$
with the Čech coboundary map

$$\delta = \sum_{j=0}^{p-1} (-1)^j A^j_{\text{Čech}} : A^p_{\text{Čech}} \to A^{p+1}_{\text{Čech}}$$

and the exterior differential $d : A^p_{\text{Čech}} \to A^{p+1}_{\text{Čech}}$. As usual, we make $A^p_{\text{Čech}}$
a cochain complex by defining a total differential $D = d + (-1)^p \delta$ and a total
grading $p + q$.

The Čech-singular complex $S^p_{\text{Čech}}$, and Čech-smooth singular complex
$S^p_{\text{Čech}}$, are defined analogously.

Remark. The relation of this Čech-de Rham complex to Weil's is that if a form $\eta \in A^p_{\text{Čech}}$, can be identified with the alternating Čech
$p$-cochain, with values in $A^p$, defined by $\nu_{a_0 \ldots a_p} = \epsilon \cdot \nu_{a_0 \ldots a_p}$, where the
$\epsilon$ is the signature of the permutation needed to put $a_p \ldots, a_0$ into order.
Theorem 5.3 (Extended Mayer-Vietoris Theorem).  
(i) For any topological space X and any open cover \( \mathcal{U} \), \( H_p(S^1; \mathcal{U}) = H_p^X(X; \mathcal{U}) \); the isomorphism is induced at the cochain level by the restriction map from \( S^1 \) to \( S^1 \in \mathcal{U} \).  
(ii) For any differentiable space \( X \) and any open cover \( \mathcal{U} \), \( H_p(S^1; \mathcal{U}) = H_p^X(X; \mathcal{U}) \) induced by \( S^1 \in \mathcal{U} \).  

Proof. Part (i) is more or less well-known; see [3] or [23, p. 16] for a proof.  
Since the proof is combinatorial in nature, it goes through for the smooth case as well; all we need to check is that the barycentric subdivison of a smooth singular simplex is a sum of smooth simplices, and this is obvious. q.e.d.  
The analog of this theorem for the Čech-de Rham complex requires an additional hypothesis. 

Theorem 5.4. Let \( X \) be a differentiable space, and let \( \mathcal{U} \) be an open cover of \( X \) admitting a subordinate smooth partition of unity. Then  
\[ H_p^X(A, \mathcal{U}) \to H_p^X(X) \]  
induced by the restriction map \( r: A^X \to A^X \in \mathcal{U} \),  

Proof. Let \( (u_i) \) be a smooth partition of unity subordinate to \( \mathcal{U} \). One first shows, as Weil did [38] (in the case where \( X \) is a manifold), that the complex  
\[ 0 \to A^X \xrightarrow{\partial} A^X \to A^X \to \cdots \]  
is exact, by using N. Hamilton's homotopy operator  
\[ N: A^p \to A^p \text{ s.t. } N \partial + \partial N = -1, 0, 1, \cdots \]  
\[ (N(u_0), \ldots, u_p) = \sum_{v \in A^p} u_0 \cdot v \cdot u_0 \cdots u_0 \]  

Here \( \mathcal{U} = X \) which makes sense in our context since \( A^X(X) \) is a module over \( \mathbb{C}^* \)(X) and is closed under locally finite sums. The rest of the proof is exactly the same as Weil's.  

Our second main theorem is now easy to prove.  

Theorem 5.5. Let \( X \) be a differentiable space, and let \( \mathcal{U} \) be an open cover of \( X \) admitting a subordinate smooth partition of unity. Suppose that for every finite intersection \( U_i \in \mathcal{U} \) the maps  
\[ H_p^X(U_i) \to H_p^X(U_i) \to H_p^X(X; \mathcal{U}) \]  
are isomorphisms. Then the maps  
\[ H_p^X(X) \to H_p^X(X) \to H_p^X(X; \mathcal{U}) \]  
are isomorphisms.  

Proof. Consider the maps of double complexes  
\[ A^p \otimes A^q \to A^p \otimes A^q \]  
Filtering by \( p \) we get spectral sequences which map at the \( E_1 \) level by  
\[ H_p(A^q; \mathcal{U}) \to H_p^X(U_i) \to H_p^X(U_i) \to H_p^X(U_i; \mathcal{U}) \]  
By hypothesis these maps are isomorphisms, so  
\[ H_p(A^q; \mathcal{U}) \to H_p^X(U_i) \to H_p^X(U_i; \mathcal{U}) \]  
by the usual spectral sequence arguments. Theorems 5.3 and 5.4 now yield the desired conclusion.  

6. Smooth partitions of unity  
In order to apply Theorems 5.2 and 5.5 we need to know when an open cover of a differentiable space admits a subordinate smooth partition of unity. In this section we find some convenient sufficient conditions for a smooth partition of unity to exist.  
The existence of partitions of unity on Banach spaces has been studied extensively, and some of the results obtained for Banach spaces are general enough to apply to differentiable spaces. In a recent paper [36], H. Toruńczyk gave what amount to necessary and sufficient conditions for a differentiable space whose topology is metrizable to admit smooth partitions of unity. First we need the following definitions.  

Definition. Let \( X \) be a differentiable space. Then \( \mathcal{E}_X \) will denote the collection of open subsets of \( X \) of the form \( f^{-1}(a, b) \) where \( f \in C^*(X), a, b \in R \).  

Definition. A family of subsets is \( C \)-locally finite if \( f \) is the union of a countable number of \( C \)-locally finite subfamilies.  

Definition. Given a set \( A, c(A) \) is the linear space of all \( x = (x_b) \in R^A \) with \( (a \in A, |x_a| > 1/n) \) finite for any integer \( n \geq 1 \); \( c(A) \) is regarded as a Banach space under the norm \( ||c(A)|| = \sup \{|x_a|, a \in A \} \). We make \( c(A) \) a differentiable space by defining \( f \in C^*(A) \) (\( A \) open in \( c(A) \)) if and only if \( f \) is \( C \)-locally a smooth function of finitely many coordinate functions \( x_{a_1}, \ldots, x_{a_k} \).  

Theorem 6.1. (H. Toruńczyk [36]) The following conditions are equivalent for a differentiable space \( X \) whose underlying topological space is metrizable.  
(a) \( X \) admits smooth partitions of unity (subordinate to any open cover).  
(b) \( \mathcal{E}_X \) contains a \( C \)-locally finite base of the topology of \( X \).  
(c) There are a set \( A \) and a homomorphic embedding \( \nu: X \to c(A) \) with \( x \mapsto x \in C^*(X) \) for any \( x \in A \).
Remarks. 1. Condition (b), together with the stipulation that \( C^m(X) \) separates the points of \( X \) (so that \( X \) is \( T_2 \)), implies that \( X \) is metrizable [19, p. 194]. Condition (c) also implies that \( X \) is metrizable since \( C_0(A) \) is a metric space.

2. The map \( u \) in part (a) is a morphism of differentiable spaces.

An important special case of this theorem is the following.

Theorem 6.2. Let \( X \) be a differentiable space. Suppose that \( C^m(X) \) contains a countable collection \( f_0, f_1, f_2, \ldots \) which separates the points of \( X \) and generates the topology of \( X \) (in the sense that the sets \( f_i^{-1}(a, b) \) are a sub-basis of the topology of \( X \)). Then \( X \) admits smooth partitions of unity subordinate to any open cover.

Proof. The collection \( \{f_i^{-1}(a, b) : i \in \mathbb{N}, a, b \text{ rational} \} \) is a countable subbasis for \( X \). The collection \( \{\mathbb{R} \} \) of opens is clopen under finite intersections [36], hence includes the countable subcollection consisting of all finite intersections of \( \{f_i^{-1}(a, b) \} \). Thus \( \{\} \) contains a countable (hence locally finite) basis of \( X \). Now apply Theorem 6.1 and Remark 1.

Applications of Theorems 6.1 and 6.2.

Theorem 6.3. Let \( X \) be a simplicial complex in the strong (metric) topology, regarded as a differentiable space (see §1, Example 3). Then \( X \) admits smooth partitions of unity subordinate to any open cover.

Proof. Let \( A \) be the set of vertices of \( X \), and let \( \{a \in \mathbb{R} \} \) be the barycentric coordinates. Define \( w : X \to C_0(A) \) by \( u(x) = \sum_i \lambda_i(x) \). One verifies easily (see Remark) that \( u \) is a homeomorphic embedding, and it is trivial that \( u^{-1} = \{a \to \mathbb{R} \} \). Now apply Theorem 6.1.

Remark. The strong topology on a simplicial complex \( X \) with vertex set \( A \) is defined in any of the following three equivalent ways:

1. \( (C^\infty(c, e) : c \subseteq A, e \subseteq \mathbb{R}) \) is a sub-basis for the topology of \( X \).
2. \( X \) is a metric space with metric \( d(x, y) = \sum_i |\lambda_i(x) - \lambda_i(y)| \).
3. \( X \) is a metric space with metric \( \lambda(x, y) = \sum_i |\lambda_i(x) - \lambda_i(y)| \).

The equivalence 1 \( \Rightarrow \) 2 is well-known and easily proved; the equivalence 2 \( \Rightarrow \) 3 can be proved easily using the identity \( \Sigma_i |\lambda_i(x)| = 1 \).

Theorem 6.4. The infinite-dimensional Lie groupoid \( f_i \) (see §1, Example 4) admits smooth partitions of unity.

Proof. The coordinates \( u_i, u_i' \) \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \) (bid) are a countable collection of functions on \( f_i \), which separate the points of \( f_i \), and generate its topology. Now apply Theorem 6.2.

Theorem 6.5. Let \( X \) be a differentiable space admitting smooth partitions of unity and having a metrizable topology. Then any topological subspace \( Y \subseteq X \) of \( X \) has its induced structure of differentiable subspace (see §1) admits smooth partitions of unity.

Proof. By Theorem 6.1, \( X \) admits a smooth embedding \( w : X \to C_0(A) \) for some \( A \). Then \( u/Y \) is a smooth embedding of \( Y \) into \( C_0(A) \), so that \( Y \) admits smooth partitions of unity, again by Theorem 6.1.

Remark. For \( Y \) closed this result is obvious (see Theorem 1.1), but Theorem 6.5 applies to any subspace \( Y \).

Although we have already shown that every simplicial complex (in the strong topology) admits smooth partitions of unity, it is of interest to exhibit one explicitly in a special case. In particular, consider the infinite Euclidean simplex \( \Delta^\infty \) in the strong topology with vertices \( 0, 1, 2, \ldots \) and barycentric coordinates \( \lambda_i \). Let \( U \) be any set \( \Delta^\infty \) \( \Delta^\infty \), then \( \{U \} \) is an open cover of \( \Delta^\infty \). Now \( \{U \} \) is a point-finite (but not locally finite) partition of unity, with \( \{U \} \subseteq U \).

Following [10], we can construct a locally finite (but only continuous) partition of unity \( \{u_i \} \) with \( \{u_i > 0 \} \subseteq U \) by setting \( \sum_{i=0}^\infty u_i = \sum_{i=0}^\infty \sum_{j=0}^\infty u_{ij} = 1 \).

If we replace \( x \to \mathbb{R} \) by any smooth function \( f(x) \) which is zero for \( x < 0 \) and positive for \( x > 0 \), then we get a smooth partition of unity \( \{u_i \} \) instead of \( \{\} \), but still only \( \{u_i > 0 \} \subseteq U \). To construct a smooth partition of unity \( \{u_i \} \) subordinate to \( \{U \} \), we must have the stronger condition \( \sum_{i=0}^\infty u_i(x) \subseteq U \). To obtain this, let \( g \) be as before, and set

\[ w_i(x) = g(x - (1/2)^{i+1}) \]

This is defined since \( w_i(x) = 0 \) for all \( i \), then \( u_i(x) \leq (1/2)^{i+1} \), so that

\[ \sum_{i=0}^\infty w_i(x) = \sum_{i=0}^\infty (1/2)^{i+1} = 1 \]

contradiction. We see that

\[ \text{supp}(w_i) = C\{w_i = (1/2)^{i+1} \} \subseteq \{u_i > 0 \} \subseteq U \]

so that \( \{u_i \} \) is subordinate to \( \{U \} \), and \( \supp(w_i) \) is locally finite. Hence \( \{u_i \} \) is a smooth partition of unity on \( \Delta^\infty \) subordinate to \( \{U \} \).

7. de Rham cohomology of simplicial complexes.

In this section we show that the differentiable space de Rham complex of any simplicial complex \( X \) using either the weak or the strong topology on \( X \).
computes the real cohomology of \( X \). We compare these cochain complexes with Whitney’s complex of compatible forms.

We start by recalling some topological facts about simplicial complexes.

Let \( X \) be a simplicial complex, and \( \mathcal{B} \) its set of vertices. As we have remarked, there are two topologies on \( X \): weak and strong; we shall designate these by \( X_{w} \) and \( X_{s} \), respectively. Recall that \( U \subset X_{w} \) is open if and only if \( U \cap \mathcal{B} \) is (relatively) open in \( S \) for every (finite) simplex \( S \) of \( X \). On the other hand, the strong topology on \( X \) is defined as the coarsest topology (finesse open sets) making all the barycentric coordinate functions \( t_{b} (b \in \mathcal{B}) \) continuous; equivalently, it is defined by the metric \( p(x, y) = \max_{b \in \mathcal{B}} |t_{b}(x) - t_{b}(y)| \). \( X \) is path-connected in either topology. It is clear that \( X_{w} \subset X_{s} \) is continuous. Dowker [12] showed that \( X_{w} \) and \( X_{s} \) are homeomorphic complexes. We recall the explicit homotopy inverse \( X_{s} \to X_{w} \) constructed by Milnor [22]. Let \( \mu_{b} \) be the (continuous) partition of unity subordinate to the star open cover \( \{ U_{b} \}_{b \in \mathcal{B}} \) of \( X \) (\( U_{b} = \{ t_{b} > 0 \} \)).

Define \( x_{s} : X_{s} \to X_{w} \) by

\[
u(x) = (\mu_{b}(x))_{b \in \mathcal{B}}
\]

Then \( \nu \) is continuous and maps some open neighborhood of each \( x \in X_{s} \) into some finite subcomplex of \( X_{w} \). It also maps each simplex of \( X_{s} \) onto itself. Let \( h = (h_{t}(0 < t < 1) \) be the linear homotopy with \( h_{0} = \text{id} \) and \( h_{1} = \mu \). Then \( h : X_{s} \times I \to X_{w} \) and \( h : X_{s} \times I \to X_{w} \) are continuous [22], which shows that \( X_{w} \) is indeed a homotopy inverse to \( X_{s} \).

The preceding discussion can be adapted as follows to show that \( X_{w} \to X_{s} \) is a smooth homotopy equivalence of differentiable spaces. In light of Theorem 6.3, we can choose \( \mu_{b} \) to be a smooth partition of unity. Defining \( s \) as before, we see that \( h_{t} = s \), so that \( h \) is a smooth homotopy of \( X_{s} \) into \( X_{w} \). Also, \( t_{b} = (1 - t) \cdot t_{b} + 1 \cdot t_{b} \), which is a smooth function on both \( X_{s} \times I \) and \( X_{w} \times I \). Hence \( h \) is smooth with respect to \( \mathcal{B} \). Thus we have proved

**Theorem 7.1.** Let \( X \) be a simplicial complex. Then \( X_{w} \to X_{s} \) is a smooth homotopy equivalence of differentiable spaces.

With the aid of the results of §5, we can now prove the de Rham theorem for simplicial complexes.

**Theorem 7.2.** Let \( X \) be a simplicial complex, in either the weak or the strong topology. Then the maps

\[
H_{p}^{\text{diff}}(X) \to H_{p}^{\text{scr}}(X) = H_{p}^{\text{diff}}(X)
\]

are isomorphisms.

**Proof.** By Theorem 7.1 and the smooth homotopy invariance of all three cohomology theories, it suffices to consider the case \( X = X_{w} \). Now let \( U_{b} = \{ t_{b} > 0 \} (= \text{star} (p)) \) as before, and for each abstract simplex \( b = (b_{0}, \ldots, b_{k}) \) of \( X \), denote \( \cap_{b \in \mathcal{B}} U_{b} \) by \( U_{b} \). Then \( U_{b} \subset \text{star}(p) \), and there is a linear deformation retraction \( h \) of \( U_{b} \) onto the barycenter \( x_{b} \) of \( S_{b} \) (the geometric simplex corresponding to \( b \)) defined by

\[
h : U_{b} \times I \to U_{b},
\]

\[
h_{t}(x_{b}, t) = \left\{ \begin{array}{ll}
(1 - t) x_{b} + t b & \text{if } b \in \mathcal{B},
(1 - t) x_{b} + t b_{0} & \text{if } b \notin \mathcal{B}.
\end{array} \right.
\]

Furthermore, the explicit formulas show that \( h \) is smooth. It follows by Whitney’s implication that \( H^{n}(U_{b}) = H^{n}(U_{b}) \). If \( B = \text{form} (R^{b}, R^{b}) \), and \( H_{p}^{\text{sc}} \) if we now apply Theorem 3.5, using the open cover \( \{ U_{b} \} \) with subordinate smooth partition of unity \( \{ \mu_{b} \} \), the theorem follows immediately.

Theorem 7.2 shows that \( \mathcal{A}^{\epsilon}(X) \) and \( \mathcal{A}^{\epsilon}(X) \) are commutative cochain models for the real cohomology of any simplicial complex \( X \). We now ask how these cochain complexes are related to each other and to the Whitney model \( \mathcal{A}^{\epsilon}(X) \) of compatible forms on the simplices of \( X \). Recall [39, p. 226] that a cochain \( a \in \mathcal{A}^{\epsilon}(X) \) is defined to be a collection \( (a_{b} : b \in \mathcal{B}(X)) \), where \( a \) runs over the abstract simplices of \( X \). The geometric simplex corresponding to \( b \) and \( (a_{b}) \) satisfies the compatibility condition \( a_{b} \cap \mathcal{B} = a_{b} \) if \( b \in \mathcal{B} \). Actually, Whitney worked with "flat cochains" rather than differentiable forms, but \( H^{p} \mathcal{A}^{\epsilon}(X) \to H^{p} \mathcal{A}^{\epsilon}(X) \) in both cases (see [40] and [7] for proofs).

There are canonical isomorphisms

\[
\mathcal{A}^{\epsilon}(X) \to \mathcal{A}^{\epsilon}(X) = \mathcal{A}^{\epsilon}(X);
\]

\( a \to a^{*} \), where \( a \) is a cochain on \( X \) while \( b \) is defined by

\[
b : a^{*} \to \{ a_{b} \}_{b \in \mathcal{B}(X)}
\]

Although \( a \) and \( b \) induce isomorphisms in cohomology (as can be seen by integrating \( e \)-forms over \( q \)-simplices to map all three theories to the simplicial cohomology of \( X \), they are not in general isomorphisms at the cochain level. The precise relationship between the three cochain complexes is given by the following theorem.

**Theorem 7.3.** (i) For any simplicial complex \( X \), the maps \( a \to a^{*} : \mathcal{A}^{\epsilon}(X) \to \mathcal{A}^{\epsilon}(X) \) and \( b : \mathcal{A}^{\epsilon}(X) \to \mathcal{A}^{\epsilon}(X) \) are injections.

(ii) If \( X \) is locally finite, then \( a \) and \( b \) are isomorphisms.

(iii) If \( X \) is not locally finite, then \( a \) and \( b \) are both strict inclusions.

The proof of this theorem is divided into Lemmas 7.4 through 7.8.
Lemma 7.4. The map \( \alpha : C(A(X)) \rightarrow C(A(Y)) \) is injective for any simplicial complex \( X \).
Proof. Observe first that the weak and strong topologies agree on finite subcomplexes of \( X \), and that every plot \( \phi : E \rightarrow X \) factors locally through a finite subcomplex (since \( E \) is locally compact and every compact subset of \( X \) is contained in a finite subcomplex \([39]\)).

Let \( \omega \in \ker \alpha \) and suppose that \( \omega \neq 0 \). Let \( \phi : E \rightarrow X \) be a plot such that \( \phi_\omega \) is nowhere zero. (We can find such a plot by starting with a plot \( \phi : L \rightarrow X \) with \( \phi \omega = 0 \) and restricting suitably.) Let \( X^{0^0} \) be the \( n \)- skeleton of \( X \) (a closed subset of \( X \)) and let \( C_n \equiv \phi^{0,1,0} \). Then \( \{C_n\} \) is a countable closed cover of \( E \), by the finite category theorem \([19]\), p. 256 (using the local compactness of \( E \)), some \( C_n \) has a nonempty interior \( U \). Then \( \phi(U) \cap X^{0^0} \rightarrow X \) is a plot of \( \omega \), so that \( \phi(U)\phi(U)^\ast \omega = \phi(U)^\ast(U) \). This contradiction shows that \( \ker \alpha = 0 \).

Lemma 7.5. The map \( \beta : C(A(X)) \rightarrow C(A(Y)) \) is injective for any simplicial complex \( X \).
Proof. Let \( \omega \in \ker \beta \) and suppose that \( \omega \neq 0 \). Let \( \phi : E \rightarrow X \) be a plot for which \( \phi_\omega \) is nowhere zero. By restricting \( E \) to some open subset we may assume that \( \phi \) factors through a finite simplicial complex \( Y \subset X \). Assume that \( Y \) has been chosen to be minimal (for the fixed restricted \( \phi \)), and let \( S \) be a top-dimensional simplex of \( Y \). Then \( L \cong \phi_\phi^{-1}(S) \subset X \) is open and nonempty. But \( \phi_\phi \) factors through \( S \), which implies that \( \omega = \phi_\phi^\ast(U) = \phi_\phi(U) \). This contradiction shows that \( \ker \beta = 0 \).

It follows from Lemmas 7.4 and 7.5 that we can regard \( A(X) \) and \( A(Y) \) as subcomplexes of \( A \).

Lemma 7.6. If \( X \) is a finite simplicial complex, then \( A(X) = A(X) \).
Proof. The first equality follows from the fact that the strong and weak topologies agree on finite complexes; we will denote \( A(X) \) and \( A(Y) \) by \( A \). For the second equality, it suffices by (Lemma 7.5) to show that \( X \): \( A(X) \rightarrow A(Y) \) is onto. We use induction on the total number of \( p \) simplices of \( X \). For \( p = 1 \) we have \( X = 0 \) and the result is obvious. Suppose the result has been proven for \( p = 1 \), and that \( X \) has \( p \) simplices. Let \( S \) be a top-dimensional simplex of \( X \), and assume that the vertices of \( S \) are labelled 0, 1, \ldots, \( n \). Now \( Y \approx \Delta^n \), and \( S \) has \( p - 1 \) simplices, so that \( A(Y) = A(Y) \).

Let \( U = \sum_{i=0}^n \xi_i > 0 \) \( U \) is a neighborhood of \( S \). A smooth retraction \( r : U \rightarrow S \) is defined by \( r(x) = \xi(x) = \sum_{i=0}^n \xi_i \xi_i(x) \).

1. \ldots, \( n \). Observe that \( r \) maps \( U \rightarrow S \) to the boundary of \( S \) since \( S \) is top-dimensional.

Let \( \omega \in A(X) \). Then \( \omega \in A(Y) \). By Theorems 1.1 and 6.1.

8. Classifying spaces and geometric realizations

If \( C \) is a topological groupoid (= topological category with inverses), then its Mimin-Buffet-Lor classifying space \( [5] \) is defined to be the space whose points are \( \{\theta \}, \{\theta \} \) \( (\theta, \eta, \xi) \) satisfying

(1) \( \{\} \) is a point of the infinite Euclidean simplex \( \Delta^n \) (i.e., \( n > 0 \), all but finitely many \( i \) are zero, and \( \sum_i^\xi = 1 \)).

(2) \( \xi \in C \) is well-defined only when \( \xi \) is.

Remark. This is not the same as Mimun's and Buffet-Lor's original definition of \( RC \) as \( RC/C \), but is equivalent to it, see \([32]\).

The strong topology \( RC \) on \( RC \) is defined to be the coarsest topology
making the functions $\phi: BC \to R$ and $\epsilon_i : \{ \epsilon_i = 0 \} \to C$ continuous. The weak topology $\beta C$ on $BC$ is defined as follows. Let $NC$ be the nerve [26] of $C$; this is a semi-simplicial space with

$$NC[0] = \{ \text{objects of } C \},$$

$$NC[n] = \{ \text{composable n-tuples } (c_0, \ldots, c_n) \text{ of morphisms in } C \}$$

(i.e., $c_1 \cdot c_2 \cdot \cdots \cdot c_n$ is defined),

and with face maps

$$0: NC[n] \to NC[n-1], \quad i = 0, \ldots, n,$$

defined by omitting or composing morphisms (see [26], [11]). Now if $S(t)$ is the geometric $n$-simplex of $\Delta^n$ with vertices $t_0 < t_1 < \cdots < t_n \in \tau$, then $S(t) \times NC[n]$ maps to $BC$ by

$$((c_0, \ldots, c_n), (c_0, \ldots, c_n)) \mapsto (\langle t_i \rangle, \epsilon_i)$$

where

$$c_i = t_i \quad \text{if } i = \tau,$$

$$c_i = 0 \quad \text{if } i = \nu,$$

$$c_i = c_i+1 \quad \text{if } i = \nu+1 \text{ with } k < m,$$

$$c_i = c_i-1 \quad \text{if } i = \nu \text{ with } k > m,$$

$$c_i = c_{i+1} \quad \text{if } i = \nu \text{ with } k = m,$$

left identity of $c_i$ if $i = \nu+1$, with $k = m+1$,

right identity of $c_i$ if $i = \nu$,

These maps induce an isomorphism of sets

$$\mu_X : \prod_{t \in S(t)} S(t) \times NC[n] / \sim \to BC,$$

where $\sim$ is identification via the face maps:

$$(t, c) \in S(t) \times NC[n] \sim (t, \epsilon c) \in S(t) \times NC[n-1].$$

Here $t \in S(\tau) \in S(\Delta^n) \subset NC[n]$, and $\epsilon c$ means $c$ with $\tau$ omitted. The weak topology $\beta C$ is defined by putting the quotient topology on $BC$ with respect to this map.

More generally, if $X$ is any semi-simplicial space, its unwound geometric realization $\mu(X)$ (called the Milnor geometric realization by tom Dieck [13]) can be defined by

$$\mu(X) = \prod_{t \in S(\tau)} S(t) \times X[\tau] / \sim,$$

where

$$\sim$$

is identification via the face maps:

$$(t, x) \sim (t, \epsilon t x), (t \in S(\tau) \subset S(\tau), x \in X[\tau]).$$

The weak topology $\mu(C)$ on $\mu(X)$ is defined to be the quotient topology under the identification map, which we denote by $\sim$. For example, $\mu(NC) = \beta C$. It is also possible to define a strong topology $\mu(X)$ on $\mu(X)$ in such a way that $\mu(NC) = \beta C$. This works as follows. First observe that the baricentric coordinates $\epsilon : \Delta^n$ pull back to global functions (also denoted $\epsilon$) on $\mu(X)$. Let $U = \epsilon(0, 1) \subset \mu(X)$; the collection $\{ U \}$ is called the canonical open cover of $\mu(X)$. For each $n$ and each abstract n-simplex $\sigma$ of $\Delta^n$, let $U_\sigma = \epsilon_{\sigma(n)}^{-1} U$, and let a map

$$\mu(X) \to [n]$$

be defined by

$$\mu(X) \times X[\sigma] : \tau(x, \epsilon x) \mapsto \mu(x),$$

where $p > n, \tau$ is an abstract $p$-simplex of $\Delta^n$ containing $\sigma$, $t \in S(\tau)$, and $\mu(x) : \tau(x, \epsilon x) \mapsto \mu(x)$. $X[\sigma]$ is the map corresponding to the inclusion $\sigma \subset \tau$ of ordered sets in the semi-simplicial structure of $X$. We define the strong topology $\mu(X)$ on $\mu(X)$ to be the coarsest topology making all the maps $\mu(x)$ continuous. Example. If $X = NC$ and $\sigma = \epsilon(0, 1, \ldots, n)$, then

$$\mu(X) \to [n]$$

Clearly the topologies $\mu(NC)$ and $\beta C$ are the same.

We now discuss the de Rham cohomology of $\mu(X)$, and note that $BC$ is included as a special case.

Definition. Let $X$ be a semi-simplicial object in the category of differentiable spaces. Then $\mu(X)$ (resp. $\mu_n(X)$) is given the smallest differentiable space structure which makes the maps $t : X \to R$ and $\mu(t) : \mu(X) \to [n]$ smooth. Note that these maps are continuous on both $\mu(X)$ and $\mu_n(X)$. Explicitly, the smooth functions on $\mu(X)$ are $t, \mu(t)$ (where $t \in S(\tau)$) open in $\mu(X)$, and all functions which locally (on $\mu(X)$, resp. $\mu_n(X)$) are smooth functions of these.

The de Rham cohomology of $\mu(X)$ can now be studied by techniques similar to those used on simplicial complexes in the preceding section.

Theorem 8.1. Let $X$ be a semi-simplicial object in the category of differentiable spaces. Then $\mu_n(X)$ is a smooth homotopy equivalence.

Proof. Let $(\mu_n)$ be a smooth partition of unity subordinate to the canonical open cover of $\Delta^n = \mu(\emptyset)$ (see §6). Define

$$u : \mu(X) \to \mu(X),$$

where $t \in S(\tau) \subset \Delta^n, \sigma$ is an n-simplex, $u(\sigma(n)) \in S(\tau)$, and $x \in X[\tau]$. The function $u$ is continuous as a map from $\mu_n(X)$ to $\mu_n(X)$, for $t \in \text{int } S(\tau)$.
Filtering by $p$ and computing spectral sequences, we get

$$f^*: H_{d-1}(X_p) \to H_{d-1}(\mathfrak{m}_p) = \bigoplus_{i} H_{d-1}(U_i) = \bigoplus_{i} H_{d-1}(X_p(U_i))$$

at the $E_1$ level. But the last term is just the semi-simplicial unwinding of the co-semi-simplicial module $p : H_{d-1}(X_p)$ (see [33, Appendix A]) and also §9 below). Since unwinding does not change $d$-homology ([23, p. 118], it follows that $f^*$ induces an isomorphism of $E_2$ terms. Since $f^*$ was induced by a cohomology isomorphism, it follows that it induces isomorphism in total (D-) homology.

Corollary 8.4. Let $X$ be a semi-simplicial differentiable space. If the de Rham isomorphism $H_{d-1} = H_{d-1}^\omega$ holds on each $X_p$, then it holds on $p(X)$ and on $p(X)$.

Proof. It is known [23, [3] that $H_{d-1}(\mathfrak{m}_p) = H_{d-1}(S^*(X_p),d)$; in fact, this follows from Theorem 5.3 using an argument similar to that of Theorem 8.5. A similar proof shows that $H_{d-1}(\mathfrak{m}_p) = H_{d-1}(S^*(X_p),d)$. Inspection shows that

$$H_{d-1}(\mathfrak{m}_p) \cong H_{d-1}(\mathfrak{m}_p) \cong H_{d-1}(\mathfrak{m}_p) \cong H_{d-1}(\mathfrak{m}_p) \cong H_{d-1}(\mathfrak{m}_p) \cong H_{d-1}(\mathfrak{m}_p)$$

commutes. Filtration of the three double complexes leads to isomorphic $E_2$ terms, by hypothesis, hence the maps in the diagram are all isomorphisms.

q.e.d.

We observe that in proving Theorem 8.5, we used the Čech-de Rham complex of the canonical open cover of $p(X)$, which always admits a smooth partition of unity no matter what $X$ is. For some applications, however, we will need to know when the sheaf $C^p(X)$ is fine. The following theorem gives a convenient sufficient condition.

Theorem 8.5. Let $X$ be a semi-simplicial differentiable space such that each $X_p$ has a metrisable topology and admits smooth partitions of unity (subordinate to any open cover). Then $\mathcal{P}(X)$ is metrisable and admits smooth partitions of unity, and the same holds for any subspace of $\mathcal{P}(X)$. In particular, the sheaf $C^p(X)$ is fine, as is its restriction to any subspace of $\mathcal{P}(X)$.

Proof. By Theorem 6.1, for each $n$ there exists a set $A_n$ and a smooth embedding $f_n : \mathcal{P}(A_n) \to \mathcal{P}(X)$. Let $(U_i)$ be the canonical open cover of $\mathfrak{m}_p$. For each $n$, a simplex $x_i = \{x_0, \ldots , x_n\}$ in $\mathcal{P}(A_n)$, and let $f_n : \mathcal{P}(A_n) \to \mathcal{P}(X)$ be as before, and let $\lambda_i = \lambda_i^1 \ldots \lambda_i^n : \mathcal{P}(X) \to \mathcal{P}(X)$. For each $k \in N$ define a smooth function $g_k : \mathcal{P}(X) \to [0,1]$ with $g_k(A_n) = (2^{-k})^{2^{-n-1}}$. For each $k$, $n \in N$ and $\lambda$-simplex
Each $f_{\omega}(x)$ is a smooth morphism, and for each $x \in \cal{X}(\mathbb{R})$ only finitely many $f_{\omega}(x)$ are nonzero. Now let $A = N \cup \cup \cup A_\omega$. Define a map

$$F: \cal{X}(\mathbb{R}) \to \mathbb{R}^n$$

by taking the direct product of the maps $\phi_j: \cal{X}(\mathbb{R}) \to \Delta^j \subseteq \mathbb{R}^j$ after $f_{\omega}$ and the maps $\Gamma_{\omega}$. One sees easily that $F$ factors through $\phi_j(\mathbb{R})$ and $F$ is smooth. Furthermore, $F$ is an embedding: this follows from the fact that locally we can always recover the map $\rho_b = \rho_b' \cup \rho_b'' \cup \Gamma_b$ by choosing a suitably. A second application of Theorem 6.5 (together with Remark 1 after is now) shows that $\rho_b(X)$ is meromorphic and admits smooth partitions of unity. Theorem 6.5 shows that subspaces of $\rho_b(X)$ have the same properties.

Examples. $B_0G$ and $B_0G$ admits smooth partitions of unity subordinate to any open cover, but $B_0G$ does not. However, Theorem 8.5 is valid for all three spaces.

9. Comparison of different de Rham theories on geometric realisations

Let $X$ be a semi-simplicial manifold. There are four cochain complexes which can we use to compute $H^n_{\text{dR}}(\cal{X}(\mathbb{R}))$ (which equals $H^n_{\text{dR}}(\cal{X}(\mathbb{R})^n)$ if $X$" satisfies a de Rham isomorphism): 1. The total complex of the double complex $A^\cdot X$ studied by Bott, Shulman, and Stasheff [3] (see above, §4). 2. The complex $A^\bullet(X^\bullet)$ of J. Dupont [14] and C. Watkins [37], in which the $n$-form $\phi$ is defined as a sequence of $m$-forms $\phi^{n\theta} \in A^\bullet(X^\bullet)$ satisfying the compatibilities conditions

$$\partial_i \phi^{n\theta} = (i \partial_i \phi^{n\theta} - (i \partial_i \phi^{n\theta}))$$

on $\Delta^i \times X$ for all $i = 0, \ldots, p$ and all $p = 1, 2, \ldots$, where $\partial_i \Delta^i \times X$ and $\partial_i X \times \Delta^i$ are face maps.

3. The complex $A^\bullet(\Delta^i \times X)$ defined in §8.

4. The complex $A^\bullet(\cal{X}(\mathbb{R}))$ of §8.

In this section we shall compare these four complexes, as well as more complex structures obtained by “unwinding” complexes 1 and 2. We will see that the six complexes are chain homotopy equivalent (c.h.e.), but are not isomorphic.

We first recall the definition of semi-simplicial unwinding [23, Appendix A]. Let $\mathcal{U}$ be the set of strictly increasing $(n+1)$-tuples $\sigma = (\sigma_0, \ldots, \sigma_n)$ of nonnegative integers. One defines face maps $\partial_j: \mathcal{U} \to \mathcal{U}_{-j}$ by $\partial_j(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_n)$, $j = 0, \ldots, n$. If $M$ is a semi-simplicial module, then the unwinding module $M \times \mathcal{U}$ of $M$ is the semi-simplicial module (like in a semi-simplicial module but with only face maps defined) $n \mapsto \mathcal{U} \otimes \mathbb{R}_M$, with the obvious face maps. Finally, if $M$ is a co-semi-simplicial module, then the unwinding module $M \times \mathcal{U}$ is the co-semi-simplicial module, with the obvious face maps.

Theorem 9.1. [23, p. 188] Unwinding does not change the homology of a semi-simplicial or co-semi-simplicial module under the boundary map $d: \mathcal{U} \to \mathcal{U}$. More exactly, if $M$ is a semi-simplicial (resp. co-semi-simplicial) module, then the inclusion $M \to M \times \mathcal{U}$ (resp. projection $M \times \mathcal{U} \to M$) is a chain homotopy equivalence.

(Remark) C. Watkins [37] proved this independently for co-semi-simplicial modules. He uses "simplicial" to mean semi-simplicial or semi-simplicial in our terminology, while he reserves "semi-simplicial" for what we call "facial".

Similarly, if $M$ is a semi-simplicial module, then its unwinding $M \times \mathcal{U}$ is the semi-simplicial module, with its unwinding $M \times \mathcal{U}$ in the face space $n \times \mathcal{U} \times \mathcal{U}_n$. If $M^n$ is the usual homologization of $X$ defined by $M^n = \mathcal{U}^n \times \mathcal{U}_n$, where $(0, x, z) \sim (0, x, z) \times \mathcal{U}^n \times \mathcal{U}_n$. Then $[M \times \mathcal{U}] = [M^n]$. This makes sense even though $M \times \mathcal{U}$ is not a semi-simplicial space, the degeneracy maps not being defined.

Now the Dupont-Watkins construction makes sense on $X \times \mathcal{U}$ (see Watkins [37]), so we can speak of $A^\bullet(X \times \mathcal{U})$, which we can think of as containing collections of forms on the $X \times \mathcal{U}$, where $X$ is a (nondegenerate) geometric $p$-simplex of $\Delta^i$.

Finally, let $\omega_\mathcal{X} \times \mathcal{U}$ be the algebra unwinding of the co-semi-simplicial algebra $A^\bullet(\mathcal{X})$ of Bott, Shulman, and Stasheff.

In [14], Dupont showed that integration of forms over $\Delta^i$ (which lowers the degree of a form by $p$) defines a chain homotopy equivalence (c.h.e.).

The same procedure defines a map

$$f: A^\bullet([X]) \to A^\bullet(\mathcal{X})$$

Dupont’s proof for $f_\mathcal{X}$ also shows that $f_\mathcal{X}$ is a c.h.e. (see also Watkins [37]). On the other hand, the injection

$$f: A^\bullet(\mathcal{X}) \to A^\bullet(\mathcal{X} \times \mathcal{U})$$

is a c.h.e. by Theorem 9.1. It follows that the projection $[X \times \mathcal{U}] \to [X]$ induces a c.h.e.

$$f_\mathcal{X}: A^\bullet([X]) \to A^\bullet([X \times \mathcal{U}])$$
Now Theorem 8.3 and its proof show that $A^\bullet(p_\bullet(X))$ and $A^\bullet(\hat{p}_\bullet(X))$ are chain homotopy equivalent to $A^\bullet(X)$. Thus we see that all six complexes which we are discussing are chain homotopy equivalent.

We now focus our attention on the exact relationship between our complexes $A^\bullet(\hat{p}_\bullet(X))$ and $A^\bullet(p_\bullet(X))$ and the unwound version $A^\bullet(X \times X)$ of Dupont and wears "complexes. The quotient map

$\pi: I \times I \times Z \to X \to p_\bullet(X),$

regarded as a morphism of differentiable spaces, pulls back a form on $p_\bullet(X)$ to a compatible collection of forms on $|I \times I \times Z|$, and hence defines a map

$h: A^\bullet(p_\bullet(X)) \to A^\bullet(|I \times I \times Z|).$

The identity map

$id: p_\bullet(X) \to p_\bullet(X)$

induces a map $A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(p_\bullet(X))$ whose chain homotopy inverse is induced by the smooth map $\nu: \hat{p}_\bullet(X) \to p_\bullet(X)$ which was defined in the proof of Theorem 8.1 using a smooth partition of unity.

The sequence

$A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(p_\bullet(X)) \to A^\bullet(|I \times I \times Z|)$

is superficially analogous to the sequence

$A^\bullet(Y) \to A^\bullet(Y) \to A^\bullet(Y)$

studied in 8.3.1, where $Y$ is a simplicial complex in the strong or weak topology, and $A^\bullet(Y)$ is the Whitney complex of compatible collections of forms on its simplices. We shall imitate our comparison of the terms in the latter sequence to study the former sequence. A major difference between the two cases is that while the strong and weak topologies on a simplicial complex agree when restricted to finite subcomplexes, this is in general not the case geometric realizations of $\hat{p}_\bullet(X)$, where a finite subcomplex of $p_\bullet(X)$ we mean that part of $p_\bullet(X)$ lying over a finite subcomplex of $\Delta^n$. Nonetheless, we have

Theorem 9.2. The maps $A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(p_\bullet(X)) \to A^\bullet(|I \times I \times Z|)$ are strict inclusions. In general, the three inclusions vanish even if restricted to a finite subcomplex of $\Delta^n$.

The proof will be broken up into a sequence of lemmas and discussions.

Lemma 9.3. The map $A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(p_\bullet(X))$ is injective for any semi-simplicial differentiable space $X$.

Proof. Let $\eta \in \hat{p}_\bullet(X)$ and suppose $\eta \neq 0$. As in the proof of Lemma 7.4 we can construct a plot $\phi: E \to \hat{p}_\bullet(X)$ such that $\phi^*\eta = 0$, and by applying the Baer category theorem we can restrict $E$ so that $\phi(E)$ lies over a finite subcomplex of $\Delta^n$. This in itself does not guarantee that $\phi$ factors through $\hat{p}_\bullet(X)$, but if we restrict $E$ further so that $\phi(E)$ lies over the interior of some simplex of $\Delta^n$, then $\phi$ does factor through $\hat{p}_\bullet(X)$, so that $\phi^*\eta = 0$. This contradiction completes the proof.

Lemma 9.4. The map $A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(|I \times I \times Z|)$ is injective for any semi-simplicial differentiable space $X$.

Proof. Imitate the proofs of Lemma 9.3 and Lemma 7.5.

Lemma 9.5. The inclusions $A^\bullet(\hat{p}_\bullet(X)) \to A^\bullet(p_\bullet(X)) \to A^\bullet(|I \times I \times Z|)$ are always strict inclusions.

Proof. If we replace $X$ by the trivial semi-simplicial space $P$ with $P$ a single point for each $n$, then $p(p) = \Delta^n$, and our inclusions reduce to $A^\bullet(\hat{p}_\bullet(X) \to A^\bullet(p_\bullet(X)) \to A^\bullet(|I \times I \times Z|)$, which are strict inclusions by Theorem 7.3 since $\Delta^n$ is not locally finite. The functions exhibited in the proof of Lemma 7.5 to prove this fact pull back to $\hat{p}_\bullet(X)$ and provide examples to prove the present lemma, q.e.d.

Actually, a stronger statement is true. If we restrict the three cochain complexes to a finite subcomplex of $\rho(X)$, they are still not isomorphic, in general (unlike the case of simplicial complexes). This phenomenon can be illustrated by considering the semi-simplicial space $X = NR$, where $R$ is the real line regarded as a topological group under addition. The portion of $\hat{p}(X)$ is $BR$ lying over $\Delta^n = 0$ is then the suspension of $R$, namely

$SR = I \times R/(0, r) \sim (1/s, 1/s)$. For all $r, s \in R$.

where $I = [0, 1]$ is topologized in either the weak topology $SR$, as a quotient of $I \times R$, or the strong topology $SR$, which is the coarsest topology in which the coordinate projections

$t: SR \to I,$

$r: SR \to ^{-1} \to R$

are continuous, where $0^* = \text{image of } (0, r)$ (any $r \in R$), $1^* = \text{image of } (1, r)$, smooth function $f: I \times R \to R$ which is constant on $(0, r) \times R$ and on $(1, r) \times R$, and defines a Deodutta-Warwick-smooth function $f_0$ on $SR$. In order for $f_0$ to lie in $C^\infty (SR)$, however, $f$ must be a (smooth) function $f$ of $x$ alone in some neighborhood of $(0, r) \times R$ and $(1, r) \times R$. While for $f_0$ to lie in $C^\infty (SR)$, $f$ must be a function of $r$ alone on some such neighborhood of the special form $(-\varepsilon, r) \times R$, for example, the function $f$ restricted to $(-\varepsilon, r) \times R$ is Deodutta-Warwick smooth but not smooth, while if $x \in C^\infty (R)$ has support $=[0, \infty)$ then the function $f: SR \to \Delta^n$ is defined
by
\[ f(t, r) = \begin{cases} \frac{r}{1 + t}, & t > 0, \\ 0, & t = 0 \end{cases} \]
is weak-smooth but not strong-smooth.

This example can also be used to illustrate the problem with trying to pull back Watkins-Dupont forms on \( |X| \) or \( |X \times Z| \) to a manifold \( M \) via a smooth map \( f: M \to |X| \) (using some reasonable definition of what it means for \( f \) to be smooth) or via \( M \to |X \times Z| = \mu(X) \) (using our previous definition that \( f \) be a smooth morphism). If such a map happens to factor locally through \( IL, A^* \to X \), (resp. \( IL, A^* \to X \times Z \)), then any Watkins-Dupont form \( \eta \in A^*(|X|) \) or \( A^* (|X \times Z|) \) can be pulled back locally by \( f \). The local pullbacks will agree because they are pulled back from a compatible collection of forms. Therefore a global pull-back \( f^* (\eta) \) on \( M \) is defined in this case. In the general case, however, one can have a smooth morphism \( M \to \mu(X) \) (which does not factor locally in this way), and then one may not be able to pull back Watkins-Dupont forms. For example, consider the map
\[ f: (-0.5, 0.5) \to SR, \]
\[ f(x) = \begin{cases} (x^2, e), & x > 0, \\ (0, x), & x < 0. \end{cases} \]
Now \( f \) is continuous, and it is also smooth because \( f \circ f(x) = x^2 = C^0 (-0.5, 0.5) \), while \( f \) is smooth on \((-0.5, 0.5) - (0) \). (Recall that \( f \) is not defined on \( 0 \in SR \).) The function \( f \) is Watkins-Dupont-smooth on \((-1/2, 1/2) \subset SR \), but its pullback \( (f' \circ f) \) equals \( 4x^2 \) when \( x > 0 \) and \( 3x^2 \) when \( x < 0 \), and is therefore not smooth.

The same map \( \mu(X) \to \mu(X) \) which is a smooth homotopy inverse of \( \mu(X) \to \mu(X) \) (see Theorem 8.1) can be used to construct an explict homotopy inverse
\[ k: A^*(|X \times Z|) \to A^*(\mu(X)) \]
to the map
\[ k: A^*(\mu(X)) \to A^*(|X \times Z|). \]
To do so, we observe, following the proof of Theorem 8.1, that \( f \) factors locally through \( S(e) \times X \), where \( S(e) \) is an \( n \)-simplex of \( A^* \), so that a form in \( A^*(|X \times Z|) \) can be pulled back locally (and hence globally) via \( f \) to a form in \( A^*(\mu(X)) \). This defines the map \( k \). The linear homotopy \( E: \mu(X) \to \mu(X) \) in \( A^*(\mu(X)) \) is defined by \( k \). The linear homotopy \( E: \mu(X) \to \mu(X) \) in \( A^*(\mu(X)) \) is defined by \( k \).
we will only sketch the results and fill in the gaps in that account. The history of this problem begins with Bott and Hasfultzer [2], who defined a singular cohomology theory $H^k_*(X \to Y)$ on spaces $X$ which possess a coarsely topology, based on singular real-valued cochains on $X$ whose values vary continuously when simplified of $(X')$ are moved continuously through $X$. They conjectured that $H^k_*(X \to Y) = H^k_*(X)$; the latter was already known (by the work of Bott, Hasfultzer, Kamber, Tondeur, Goddansen, Vey, et al.) as the algebra of potential characteristic classes for foliations which can be constructed from curvatures and connections by exploiting the Bott vanishing phenomenon. In [23] and [24] we studied the properties of $H_*$ and of other continuous, smooth, and $C^k$ cohomology theories on the category of morphisms of manifolds. In particular, we defined the $C^k$ cohomology theory $T^k_*(X \to Y) = H^k_*\mbox{BEM}(X \to Y)$, where $\Gamma$ is global sections and $\mbox{BEM}(X \to Y)$ is the sheaf on $X$ generated by real-valued cocycles on the $C^k$ singular q-simplices of $X$ whose q-values vary in a $C^k$ manner when simplified of $(X')$ are moved through $X$ in a $C^k$ manner. In order to extend this theory to $(H^k_\Gamma \to B_\Gamma)$ we observed that its definition makes sense in any category of topological spaces on which some notion of $C^k$ functions exist on each $n = 1, 2, \ldots, m$. The category of differentiable spaces is ideally suited for this purpose—once defines a function $f_1 : X \to R$ to be $C^k$ if it is a $C^k$ function of finitely many smooth functions. The theory $T^k_*$, when restricted to morphisms $(X \to X)$ of paracompact manifolds, is invariant under smooth homotopies and satisfies an extended Mayer-Vietoris theorem (analogue to Theorems 5.3 and 5.4 relative to open covers of $X$. Does $T^k_*$ have the same properties on the category of morphisms $(X \to Y)$ of differentiable spaces? To answer this, we observe that the sheaf $\mbox{BEM}(X \to Y)$ is a module over $C^k_\Gamma$. If $X$ is paracompact and admits smooth partitions of unity, then $C^\infty(X)$ is fine, implying that $\mbox{BEM}(X \to Y)$ is fine. The proof of smooth homotopy invariance goes through if $\mbox{BEM}(X \to Y \times I)$ is finite and $X \times I$ paracompact. It follows that $T^k_*$ is smooth homotopy invariant on the category of morphisms $(X \to Y)$ of differentiable spaces on the whole category $T^k_*$, this is proved exactly like Theorem 5.3, using the fact that the homotopy operator $L : \Gamma(X \times I) \to \Gamma(X)$ is natural in $X$. $T^k_*$ satisfy the extended Mayer-Vietoris theorem on $T^k_*$, but only relative to open covers of $X$ which admit smooth partitions of unity. The proof of (Theorem 5.4) can be adapted to $T^k_*$ since image $(\Gamma(X \times I) \to \Gamma(X))$ is a module over $C^\infty_\Gamma(X)$. These two properties are sufficient as in the proof of Theorem 8.3 to prove that $T^k_*(\mu(X) \to \mu(Y) = H_\Gamma(\mbox{BEM}(\Gamma(X \to Y)$ for any morphism $(X \to X)$ of semi-simplification of smooth spaces; here $\mu(X)$ and $\mu(Y)$ may be taken the weak or the strong topology. In case
which equals $\mathcal{H}(\text{WO}_G)$ by an unpublished result of Bott and Haefliger.

Another use of differentiable spaces is to yield universal formulas for differential forms representing characteristic classes of $G$-bundles, where $G$ is a Lie group. We remark that Dupont [15] and Watkin [37] have made similar constructions in their de Rham theories, and that Shulman [27] constructed characteristic forms in the double complex $A^\ast \text{NG}$. Since $H_d^2(\text{BG}; G) = H_d^2(\text{BG}; G)$ (by a Corollary 8.4; $\text{BG}$ can have the weak or the strong topology here), every real characteristic class for $G$-bundles is represented by differential forms in $A^\ast (\text{BG})$. Furthermore, since a $G$-bundle with smooth transition functions on a manifold $M$ is classified by a smooth morphism $p : M \to \text{BG}$ [4, 1], these characteristic forms on $\text{BG}$ can be pulled back to differential forms (not just cohomology classes) on $M$. Since any form $\eta$ on $\text{BG}$ can be expressed in terms of the functions $t_i$ and $g_i$ and their differentials, it follows that $p^\ast \eta$ will be expressed in terms of the transition functions and a smooth partition of unity subordinate to a trivializing open cover of $M$.

To construct explicit characteristic forms on $\text{BG}$, we mimic the Chern-Weil approach using connections and curvatures (see [5]). Let $EG$ be the total space of the universal $G$-bundle over Milnor's $BG$; it is defined by

$$EG = \{ ( collections (t_i, g_i) ) \text{ such that }$$

$$\text{strong topology, the coarsest topology making all } t_i \text{ and } g_i \text{ continuous};$$

$$\text{defined by the homogeneous complex of } G \text{.) The bundle projection } p : EG \to BG \text{ is defined as usual by }$$

$$\text{we make } EG = \mu(\text{PG}) \text{ into a differentiable space in the usual way; this amounts to saying that } p : EG \to BG \text{ is smooth if it is locally a smooth function of }$$

$$\text{Note that the } \theta \text{ is the canonical } g \text{-valued } 1 \text{-form on }$$

$$\text{References}$$

DIFFERENTIABLE SPACE STRUCTURES


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