gerbe modules from 2-sections

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Abstract

From the point of view of 2-transport, a gerbe with connection is a certain 2-functor. A section of it is a generalized object of that 2-functor.

This note briefly indicates how sections, in this sense, of gerbes transgressed to the configuration space of the open 2-particle (string) associate gerbe modules to the 2-particle's endpoints.

We want to consider an open 2-particle of the form $\mathrm{par} \equiv \{a \to b\}$ propagating on a smooth space X and coupled to an abelian gerbe with connection.

One way to to realize this gerbe with connection is as a smooth 2-functor

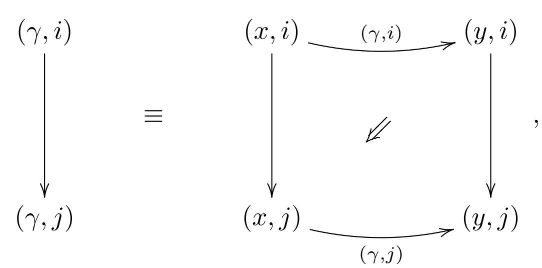
$$\mathrm{tra} : \mathcal{P}_2(U^\bullet) \to \Sigma(1d\mathrm{Vect}) \,,$$

where $\mathcal{P}_2(U^\bullet)$ is the 2-category of 2-paths in the 2-groupoid associated with a chosen surjective submersion $U \to X$.

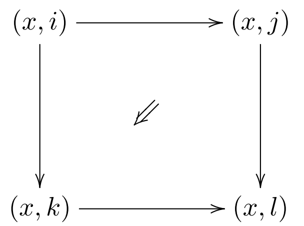
This is hence our target space, $\mathrm{tar} \equiv \mathcal{P}_2(U^\bullet)$.

Correspondingly, configuration space is the sub-2-category $\mathrm{conf} \subset [\mathrm{par}, \mathrm{tar}]$ containing only those morphisms which do not properly translate the 2-particle.

Definition 1 *Objects in* conf *are objects in* $[\mathrm{par}, \mathrm{tar}]$*, i.e. morphisms in* $\mathcal{P}_2(U^\bullet)$*. Morphisms in* conf *are all pseudonatural transformations that are generated from 2-cells in* $\mathcal{P}_2(U^\bullet)$ *of the form*



*E-mail: urs.schreiber at math.uni-hamburg.de *expressing the transition of a path γ from patch i to patch j, as well as those of the form*



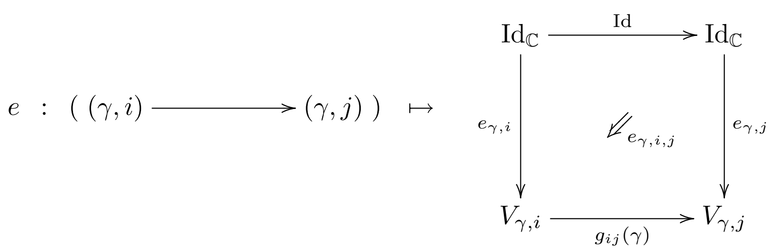
All 2-morphisms in conf *are taken to be identities.*

Let $1 : \mathrm{tra} \to \Sigma(1d\mathrm{Vect})$ be the trivial 2-functor that sends everything to the identity. Let $\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, \Sigma(1d\mathrm{Vect})]$ be the transgression of our gerbe with connection to configuration space.

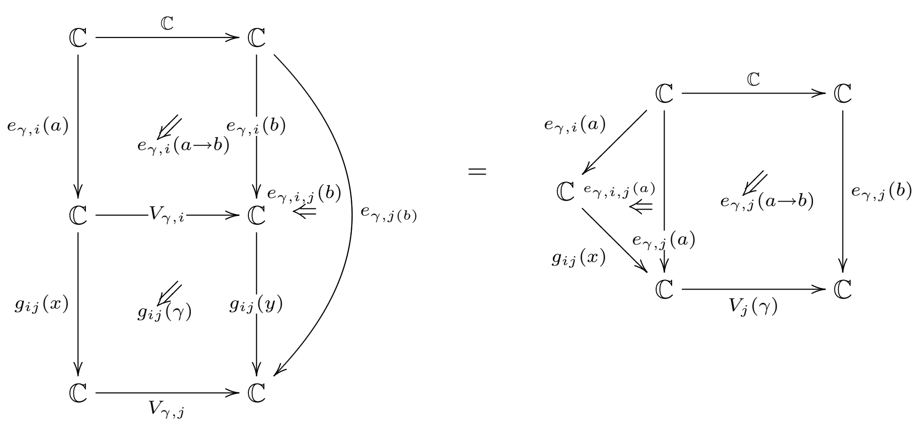
The space of sections is $\mathrm{sect} = [1_*, \mathrm{tra}_*]$.

Observation 1 *A section $e : 1_* \to \mathrm{tra}_*$ is a choice of gerbe module E_a and E_b for the endpoints of the 2-particle, together with a section of the line bundle over path space that fits into a morphism of gerbe modules $E_a \to E_b$.*

Proof. A section, being a pseudonatural transformation of 2-functors, functorially maps $\mathrm{Hom}_1(\mathrm{conf})$ to squares in $[\mathrm{par}, \Sigma(1d\mathrm{Vect})]$.



The existence of the square on the right in turn translates into naturality equations of the form



Here $V_{\gamma,i}$ denotes the fiber over path space, $g_{ij}(x)$ is the fiber of the transition bundle of the gerbe, and $g_{ij}(\gamma)$ is the connection on the transition bundle.

Hence the $e_{\gamma,i,j}(a)$ form precisely a representation of the groupoid $U^{[2]}$ twisted by the bundle gerbe. Functoriality of e translates into functoriality of this groupoid representation. It follows that the $e_{\gamma,i}(a)$ are the fibers of a gerbe module.

Finally, the above equation itself says that $e_{\gamma,i}(a \to b)$ is a morphism of twisted groupoid representations, hence a morphism of gerbe modules. $\square$

Notice that in the case that the gerbe modules involved are trivial (trivial line bundles), $e_{\gamma,i}(a \to b)$ is nothing but an ordinary section of the line bundle V over path space.