Lie algebra cohomology and $\mathrm{inn}(\mathfrak{g})$

May 29, 2007

Abstract

How Lie algebra cocycles, invariant polynomials and transgression elements manifest themselves in terms of the dual of the Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ associated with every Lie algebra $\mathfrak{g}$. And how that helps to see that every transgression element gives rise to an exact sequence of Lie $(2n+1)$-algebras

$$0 \to \mathfrak{g}_\mu \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0$$

Contents

1 Characteristic classes in terms of $\mathrm{inn}(\mathfrak{g})^*$ cohomology

Lie algebra cohomology, invariant polynomials and Chern-Simons elements can all be conveniently conceived in terms of the quasi-free differential graded algebra corresponding to the Lie 2-algebra

$$\mathrm{inn}(\mathfrak{g})$$

of inner derivations of the Lie algebra $\mathfrak{g}$.

The relation to the more common formulation of these phenomena in terms of the cohomology of the universal G-bundle comes from the fact that this universal bundle is the realization of the nerve of $\mathrm{INN}(G)$.

1.1 Formulation in terms of the cohomlogy of EG

Let G be a compact, simply connected simple Lie group.

The classical formulation of

- Lie algebra cocycles
- invariant polynomials
- transgression induced by Chern-Simons elements

is the following.

Consider the fibration corresponding to the universal principal G-bundle:

$$G \longrightarrow EG \xrightarrow{p} BG \,.$$

- A Lie algebra $(2n+1)$-cocycle μ (with values in a trivial module) is an element
$$\mu \in H^{2n+1}(\mathfrak{g}, \mathbb{R})\,.$$
By compactness of G, this is the same as an element in de Rham cohomology of G:
$$\mu \in H^{2n+1}(G, \mathbb{R})\,.$$
- An invariant polynomial k of degree $n+1$ represents an element in
$$k \in H^{2n+2}(BG, \mathbb{R})\,.$$
- A transgression form mediating between μ and k is a cochain $cs \in \Omega^{2n+1}(EG)$ such that
$$cs|_G = \mu$$
and
$$d\,cs = p^* k\,.$$

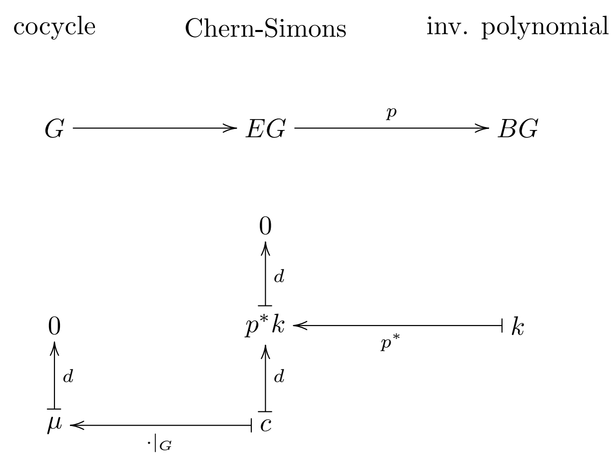

Figure 1: **Lie algebra cocycles, invariant polynomials and transgression forms** in terms of cohomology of the universal G-bundle.

1.2 Formulation in terms of cohomology of $\mathrm{inn}(\mathfrak{g})^*$

The universal G-bundle may be obtained from the sequence of groupoids

$$\mathrm{Disc}(G) \to \mathrm{INN}(G) \to \Sigma G$$

by taking geometric realizations of nerves:

$$\begin{array}{ccccc} \mathrm{Disc}(G) & \longrightarrow & \mathrm{INN}(G) & \longrightarrow & \Sigma G \\ \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} \\ G & \longrightarrow & EG & \longrightarrow & BG \end{array} .$$

$\mathrm{Disc}(G)$ and $\mathrm{INN}(G)$ are strict 2-groups, coming from the crossed modules

$$\mathrm{Disc}(G) = (1 \to G)$$

and

$$\mathrm{INN}(G) = (\mathrm{Id} : G \to G) .$$

On the other hand, ΣG is a 2-group only if G is abelian.

1.2.1 Cocycles, invariant polynomials and Chern-Simons elements

Differentially, this corresponds to the sequence

$$\begin{array}{ccccc} \mathrm{Disc}(G) & \longrightarrow & \mathrm{INN}(G) & \xrightarrow{p} & \Sigma G \\ \downarrow{\scriptstyle \mathrm{Lie}} & & \downarrow{\scriptstyle \mathrm{Lie}} & & \downarrow \\ \bigwedge^\bullet s\mathfrak{g}^* & \longleftarrow & \bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*) & \xleftarrow{p^*} & \bigwedge^\bullet (ss\mathfrak{g}^*) \end{array} .$$

In terms of this, we have

- A Lie algebra $(2n+1)$-cocycle μ (with values in a trivial module) is an element
$$\mu \in \bigwedge^{(2n+1)}(s\mathfrak{g}^*)$$
$$d_{\mathfrak{g}}\mu = 0\,.$$

- An invariant polynomial k of degree $n+1$ is an element
$$k \in \bigwedge^{n+1}(ss\mathfrak{g}^*)$$
$$d_{\mathrm{inn}(\mathfrak{g})}k = 0\,.$$

- A transgression form cs inducing a transgession between a $(2n+1)$-cocycle μ and a degree $(n+1)$-invariant polynomial is a degree $(2n+1)$-element
$$cs \in \bigwedge(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$
such that
$$cs|_{\bigwedge^\bullet(s\mathfrak{g}^*)} = \mu$$
and
$$d_{\mathrm{inn}(\mathfrak{g})}cs = p^*k\,.$$

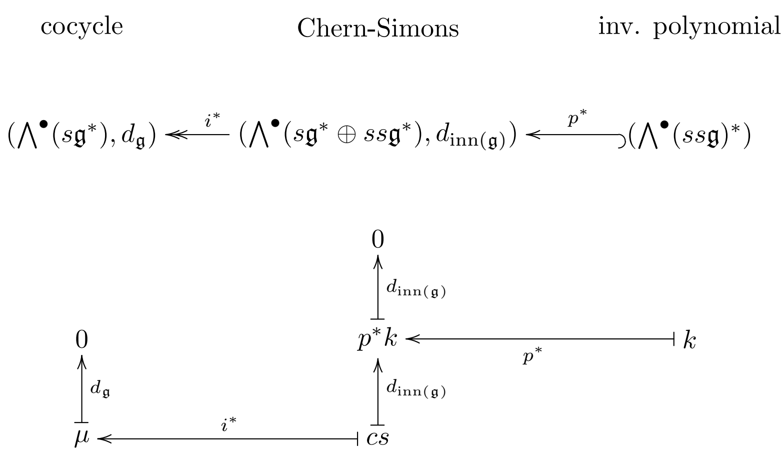

Figure 2: **Lie algebra cocycles, invariant polynomials and transgression elements** in terms of cohomology of $\mathrm{inn}(\mathfrak{g})$.

1.2.2 Transgression and the trivializability of $\mathrm{inn}(\mathfrak{g})$

It is important that

EG is contractible

$\Leftrightarrow$ $\mathrm{INN}(G)$ is trivializable

$\Leftrightarrow$ the cohomology of $\mathrm{inn}(\mathfrak{g})^* = (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})})$ is trivial

$\Leftrightarrow$ there is a homotopy $\tau : 0 \to \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}$, i.e. $[d_{\mathrm{inn}(\mathfrak{g})}, \tau] = \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}$.

This implies that if

$$cs$$

is to be a transgression element mediating between μ and k, then we have

$$cs = \tau(p^*k) + d_{\mathrm{inn}(\mathfrak{g})}q \,.$$

So for every invariant polynomial k

$$d_{\mathrm{inn}(\mathfrak{g})}k = 0$$

a "potential" c does exist. The nontrivial condition is then that cs restricted to $\mathfrak{g}$ is a cocyle.

cocycle Chern-Simons inv. polynomial

$$(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) \overset{i^*}{\twoheadleftarrow} (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \overset{p^*}{\hookleftarrow} (\bigwedge^\bullet(ss\mathfrak{g})^*)$$

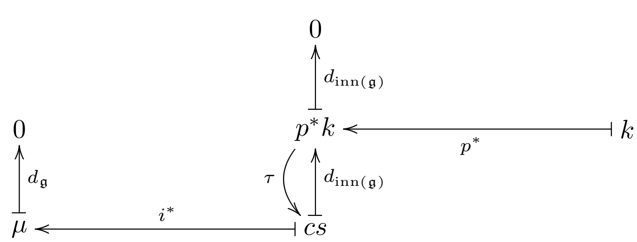

Figure 3: **The homotopy operator** τ exists due to the trivializability of $\mathrm{inn}(\mathfrak{g})$.

1.3 Formulation in terms of components

From the $\mathrm{inn}(\mathfrak{g})$-description it is easy to read off the properties of cocycles and invariant polynomials in terms of components:

Fix a Lie algebra $\mathfrak{g}$ and a basis $\{X_a\}$ with dual basis $\{t^a\}$, regarded as a basis of $s\mathfrak{g}^*$ and $\{r^a\}$, regarded as a basis of $ss\mathfrak{g}^*$.

- A Lie $(2n+1)$-cocylce is a completely antisymmetric tensor

$$\mu = \mu(t) = \mu_{a_1 \cdots a_{2n+1}} t^{a_1} \wedge \cdots t^{a_{2n+1}}$$

  such that

$$\sum_{i=1^{2n+1}} (-1)^i \mu_{[a_1 \cdots a_i \cdots a_{2n+1}} C^{a_i}{}_{bc]} = 0 \,.$$

- A degree $n+1$ symmetric invariant polynomial is a completely symmetric tensor

$$k = k(r) = k_{a_1 \cdots a_{n+1}} r^{a_1} \wedge \cdots \wedge r^{a_{n+1}}$$

  such that

$$\sum_{i=1^{2n+1}} k_{a_1 \cdots a_i \cdots a_{n+1}} C^{a_i}{}_{bc} = 0 \,.$$

By explicitly computing the homotopy operator τ (compare Chern and Simons [**?**]), using the theory of derivation homotopies, we find that the restriction

$$\tau(k(r))|_{\bigwedge^\bullet(s\mathfrak{g}^*)}$$

has components proportional to

$$k_{a_1 a_2 \cdots a_{n+1}} t^{a_1} \wedge (d_{\mathfrak{g}} t^{a_1}) \wedge \cdots (d_{\mathfrak{g}} t^{a_{n+1}}) \,.$$

2 Lie $(2n+1)$-algebras from characteristic classes

Lie cocycles, invariant polynomials and Chern-Simons elements induce Lie $(2n+1)$-algebra extensions of Lie algebras.

2.1 Lie n-algebras of Baez-Crans type

For each $(n+1)$-cocycle μ of a Lie algebra $\mathfrak{g}$ we obtain a Lie n-algebra

$$\mathfrak{g}_\mu$$

of Baez-Crans type.

In words. This has the same objects as $\mathfrak{g}$ and the only nontrivial morphisms live in a 1-dimensional space of n-morphisms. The coherence of the Jacobiator at that level is precisely the cocycle μ.

In L_∞-language. On the graded commutative coalgebra

$$S^s(s\mathfrak{g} \oplus s^n\mathbb{R})$$

we have the nilpotent degree -1 codifferential

$$D = d_2 + d_{n+1}$$

with

$$d_2(sX, sY) = s[X, Y]$$

and

$$d_n(sX_1, \cdots, sX_{n+1}) = s^n\mu(X_1, \cdots, X_{n+1})$$

for all $X, Y, X_i \in \mathfrak{g}$.

In differential coalgebra language. On the dual graded commutative algebra

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus s^n\mathbb{R}^*)$$

‘ we have the nilpotent degree +1 differential

$$d_{\mathfrak{g}_\mu}$$

which is such that

$$d_{\mathfrak{g}_\mu}|_{\bigwedge^\bullet(s\mathfrak{g}^*)} = d_{\mathfrak{g}}$$

and

$$db = -\mu\,,$$

where b is the canonical basis of $s^n\mathbb{R}$.

2.2 Lie $(2n+1)$-algebras of Chern type

For each degree $(n+1)$ invariant polynomial k of a Lie algebra $\mathfrak{g}$ we obtain a Lie $2n+1$-algebra

$$\mathrm{ch}_k(\mathfrak{g})$$

of Chern type.

In words. This has the same objects and 1-morphisms as $\mathrm{inn}(\mathfrak{g})$ The only further nontrivial morphisms live in a 1-dimensional space of $2n+1$-morphisms.

In L_∞-language. On the graded commutative coalgebra

$$S^s(s\mathfrak{g} \oplus ss\mathfrak{g} \oplus s^{2n+1}\mathbb{R})$$

we have the nilpotent degree -1 codifferential

$$D = d_1 + d_2 + d_{n+1}$$

with

$$d_1(ssX) = sX$$
$$d_2(sX, sY) = s[X, Y]$$
$$d_2(sX, ssY) = ss[X, Y]$$

and

$$d_{n+1}(ssX_1, \cdots, ssX_{n+1}) = s^{2n+1}k(X_1, \cdots, X_{n+1})$$

for all $X, Y, X_i \in \mathfrak{g}$.

In differential coalgebra language. On the dual graded commutative algebra

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n+1}\mathbb{R}^*)$$

' we have the nilpotent degree +1 differential

$$d_{\mathrm{ch}_k(\mathfrak{g})}$$

which is such that

$$d_{\mathrm{ch}_k(\mathfrak{g})}|_{\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

and

$$dc = k(r) = k_{a_1 \cdots a_{n+1}} r^{a_1} \wedge \cdots \wedge r^{a_{n+1}} ,$$

where c is the canonical basis of $s^{2n+1}\mathbb{R}$ and where $\{r^a\}$ is a basis of $ss\mathfrak{g}^*$.

2.3 Lie $(2n + 1)$-algebras of Chern-Simons type

For each Chern-Simons element cs of degree $(2n + 1)$, relating an invariant polynomial k of degree $n + 1$ with a cocycle μ_k of degree $2n + 1$ we obtain a Lie $2n + 1$-algebra

$$\mathrm{cs}_k(\mathfrak{g})$$

of Chern-Simons type.

In words. This has the same objects and 1-morphisms as $\mathrm{inn}(\mathfrak{g})$ The only further nontrivial morphisms live in a 1-dimensional space of $2n$-morphisms and in a 1-dimensional space of $2n + 1$-morphisms.

In differential coalgebra language. On the dual graded commutative algebra

$$\bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus \oplus s^{2n}\mathbb{R}^* \oplus s^{2n+1}\mathbb{R}^*)$$

we have the nilpotent degree +1 differential

$$d_{\mathrm{cs}_k(\mathfrak{g})}$$

which is such that

$$d_{\mathrm{cs}_k(\mathfrak{g})}|_{\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

and

$$db = -cs + c$$

$$dc = k(r)\,.$$

Here $\{b\}$ is the canonical basis of $s^{2n}\mathbb{R}^*$, $\{c\}$ is the canonical basis of $s^{2n+1}\mathbb{R}^*$ and $\{r^a\}$ is a basis of $ss\mathfrak{g}^*$.

Theorem 1

- *For each Chern-Simons element cs relating a degree $(n+1)$ invariant polynomial k on a Lie algebra $\mathfrak{g}$ with a $(2n+1)$-cocycle μ_k we have an exact sequence of Lie $(2n+1)$-algebras*

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{cs}_k(\mathfrak{g}) \to 0\,.$$

- *The Lie $(2n+1)$-algebra $\mathrm{cs}_k(\mathfrak{g})$ is trivializable*

$$\mathrm{cs}_k(\mathfrak{g}) \simeq \mathrm{inn}(\mathfrak{g}_{\mu_k})\,.$$

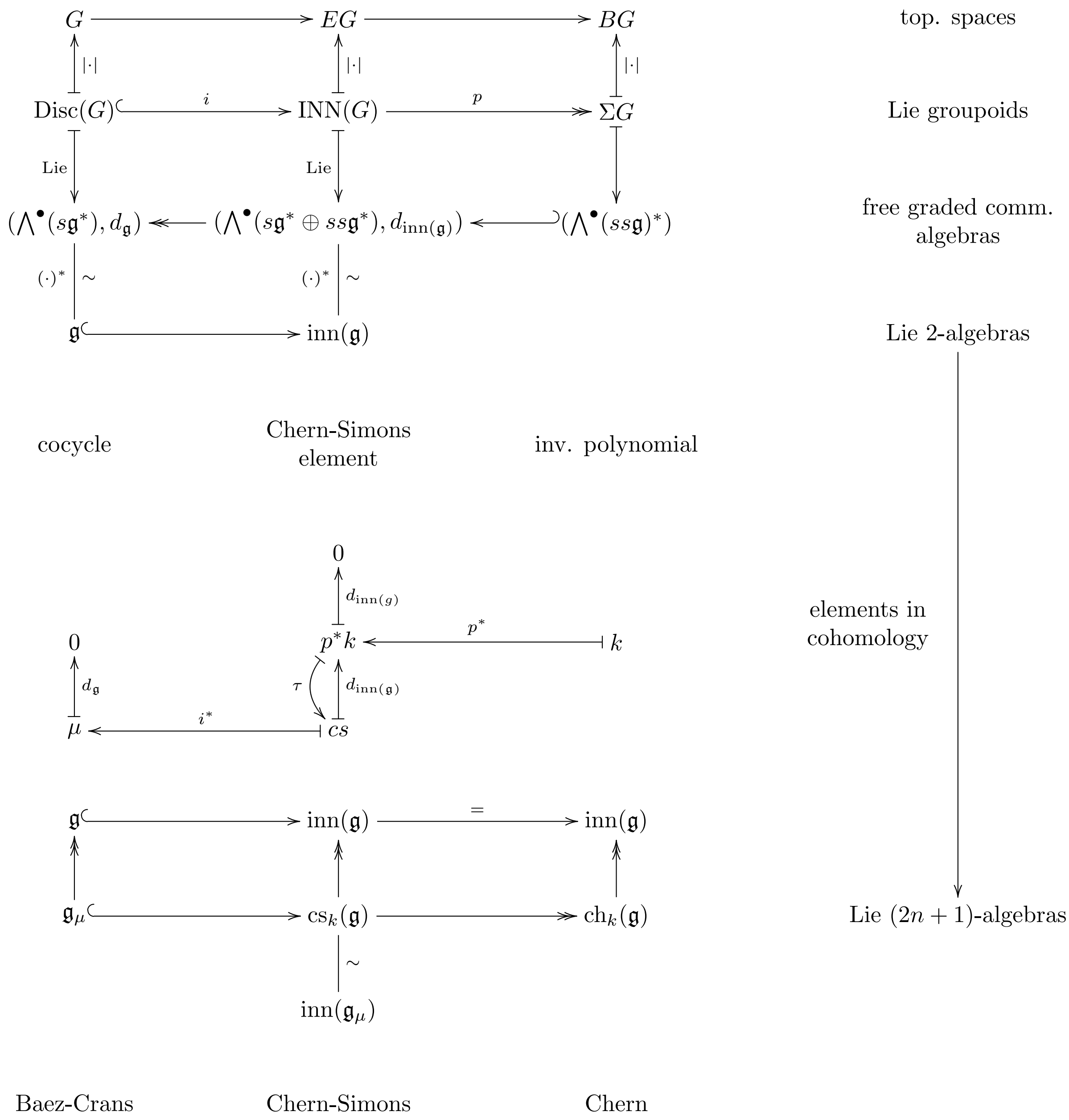

Figure 4: **Chern Lie $(2n+1)$-algebras**: for each Lie algebra $(n+1)$ cocycle μ which is related by transgression to an invariant polynomial k we obtain an exact sequence of Lie $(2n+1)$-algebras.