On G-equivariant fusion categories

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1 Introduction

A braided monoidal category – also called a "doubly monoidal category" – is the same as a 3-category with only a single object and a single 1-morphism – also called a "doubly stabilized 3-category".

Considerations in quantum field theory have lead people [1, 2] to consider generalizations of braided monoidal categories, where the braiding receives a twist by the action of a (finite) group G. These are called G-equivariant monoidal categories.

Here we discuss which kinds of 3-categories correspond to G-equivariant monoidal categories. We introduce the concept of a G-stabilized 3-category and show that G-stabilized 3-categories are equivalent to G-equivariant monoidal categories.

	ordinary case	**G-equivariant case**
doubly monoidal 1-category	braided monoidal 1-category	G-equivariant monoidal 1-category
doubly stabilized 3-category	3-category with single object and single 1-morphism	3-category with ΣG in lowest degree

Table 1: A braided monoidal category is the same as a 3-category which in degree 0 and 1 "looks like point". We show that a G-equivariant monoidal category is a 3-category which in degree 1 "looks like a group".

2 Strict G-equivariant monoidal structure

We first recall the definition of a G-equivariant category, for the special case that the G-action is strict. Then we reformulate that in terms of Gray 3-categories. (These are briefly reviewed in A).

In the next section we discuss the case where the G-action is non-strict. It is still clear in that case how to pass from G-stabilized 3-categories to G-equivariant categories. But if there is always a procedure going the other way round is less clear in the weak case.

2.1 Strict G-equivariant monoidal categories

Definition 1 (G-equivariant monoidal category)**.** *For G a finite group, a G-equivariant monoidal category is*

- *a monoidal category $(\mathcal{C}, \otimes, 1)$ which is the direct sum (in* Cat*)*
$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$
of full subcategories $\mathcal{C}_g$, one for each element of G, such that the degree map
$$\mathrm{dg} : \mathcal{C} \to G$$
is monoidal, i.e.
$$\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh} \,;$$
- *equipped with a strict monoidal G-action*
$$R : \mathcal{C} \times G \to \mathcal{C}$$
which is such that
$$R_g : \mathcal{C}_h \to \mathcal{C}_{ghg^{-1}} \,;$$
- *and equipped with a coherent G-twisted braiding*
$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ \sigma \downarrow & b \Downarrow \sim & \uparrow \otimes \\ \mathcal{C} \times \mathcal{C} & \xrightarrow[R(\cdot, \mathrm{dg}(\cdot))]{} & \mathcal{C} \times \mathcal{C} \end{array} .$$

Here σ denotes the braiding in Cat.

Remark. Kirillov [2] in addition demands that

- $\mathcal{C}$ is abelian;
- $\mathcal{C}$ is rigid as a monoidal category;
- and that 1 is simple.

Then he calls this structure a *G-equivariant fusion category.* Since these extra conditions do not affect the construction we are after, we will ignore them.

Example (graded vector spaces) The category Vect[G] of G-graded vector spaces is a G-equivariant category with trivial R-action and trivial braiding.

Notice that with our definition of G-equivariant category only homogenoeus G-graded vector spaces are obtainable. If we want to allow for the direct sum of vector spaces in different degree, we have to replace the direct sum

$$\mathcal{C} := \bigoplus_{g \in G} \mathcal{C}_g$$

in Cat with the direct sum in the abelian category of vector spaces, as in Kirillov's definition of G-equivariant fusion categories. A similar remark applies to the next example.

Example (super vector spaces) The category

$$\mathcal{C} = \mathrm{SVect}$$

of (finite dimensional, say) *super vector spaces* with grading-preserving linear maps between them is a $\mathbb{Z}_2$-equivariant monoidal category

$$\mathrm{SVect} = \mathrm{Vect}_{\mathrm{even}} \oplus \mathrm{Vect}_{\mathrm{odd}}\,.$$

It is in fact even a fusion category in Kirillov's sense.

The action R here is trivial. The $\mathbb{Z}_2$-twisted braiding b is that which introduces a sign whenever two odd vector spaces are interchanged in the tensor product.

Example (strict 2-groups) Suppose that the G-equivariant monoidal category $\mathcal{C}$ is discrete, i.e. has only identity morphisms. Then it is just a G-equivariant monoid. If this monoid happens to be a group, H, then it constitutes axactly the same structure as a crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

of groups, where $t = \mathrm{dg}$ and $\alpha = R$. Conversely, every crossed module can be regarded as a G-equivariant monoid this way.

We know that crossed modules are also the same as strict 2-groups, which are, in turn, strict one object 2-groupoids. This is a special case of our general result.

Remark. It seems natural to try to further weaken the concept of a G-equivariant category in various ways. Regarding the previous example we would, for instance, also want to regard weak 2-groups and in particular weak 3-groups as suitably equivariant categories. The Turaev-Kirillov definition excludes this possibility. But our reformulation in terms of stabilized 3-categories indicates the obvious way how to generalize this suitably.

2.2 Stabilized n-categories

2.2.1 Review of k-tuply stabilized n-categories

An n-category with only a single j-morphism for $0 \leq j \leq k-1$ is also called a *k-tuply stabilized n-category.* This is equivalently an $(n-k)$-category which is

- monoidal if $k \geq 1$
- braided monoidal if $k \geq 2$
- symmetric braided monoidal if $k \geq 3$
- "k-tuply" monoidal in general.

Given a k-tuply monoidal n-category C, we write

$$\Sigma C$$

for the corresponding $(k-1)$-tuply monoidal $(n+1)$-category; and generally

$$\Sigma^j C$$

for the corresponding $(k-j)$-tuply monoidal $(n+j)$-category.

Example. The standard example is a monoid G (associative and unital; a monoidal 0-category), which is the same as a once stabilized 1-category, which we write ΣG. If the monoid is abelian, it comes from a 2-category $\Sigma\Sigma G := \Sigma^2 G$ with a single object and a single morphism.

The stabilization hypothesis In fact, an abelian monoid, hence a doubly monoidal 0-category is already also a k-tuply monoidal 0-category for all $k \geq 2$ in that for all $n \geq 2$ an $(n-2)$-tuply stabilized n-category is nothing but an abelian monoid.

A similar phenomenon is observed for k-tuply monoidal 1-categories. For all $k \geq 3$ these are symmetric braided monoidal categories.

The *Baez-Dolan stabilization hypothesis* says that this pattern continues: for all $k \geq n+2$ a k-tuply monoidal n-category is equivalently an $(n+1)$-tuply monoidal n-category.

An braided monoidal category is a 3-category with a single object and a single 1-morphism. We shall recall the mechanism behind this fact now, but generalized to G-equivariant categories.

2.2.2 G-Stabilized Gray 3-categories

We need to slightly generalize the stabilization process of n-categories from the situation where there is just a single j-morphism, to the case where there is a $(j-1)$-group of j-morphisms. In need of some terminology for this situation, we shall make the following definition.

Definition 2 (G-stabilized 3-category)**.** *Let G be a finite group and K be a Gray 3-category*

- *with a single object*
$$\mathrm{Obj}(K) = \{\bullet\}\,;$$
- *such that there is a finite group G with*
$$\mathrm{Hom}_K(\mathrm{Id}_\bullet, -) := \bigoplus_{g\in G} \mathrm{Hom}_K(\mathrm{Id}_\bullet,\ \bullet \xrightarrow{g} \bullet\ )\,,$$
where
$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;=\; \bullet \xrightarrow{g_1 g_2} \bullet\ .$$

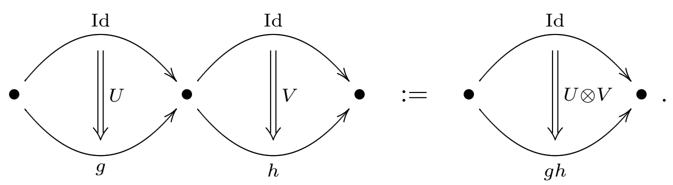
Then we call K a G-stabilized (Gray) 3-category

Our main point then is the following observation.

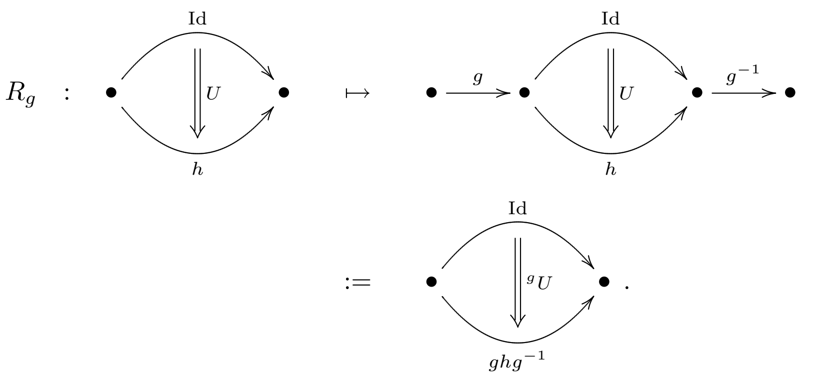
Proposition 1. *For K a G-stabilized Gray 3-category, the (1-)category*
$$\mathcal{C} := \mathrm{Hom}_K(\mathrm{Id}_\bullet, -)$$
is (naturally equipped with the structure of) a G-equivariant abelian monoidal category.

Proof.

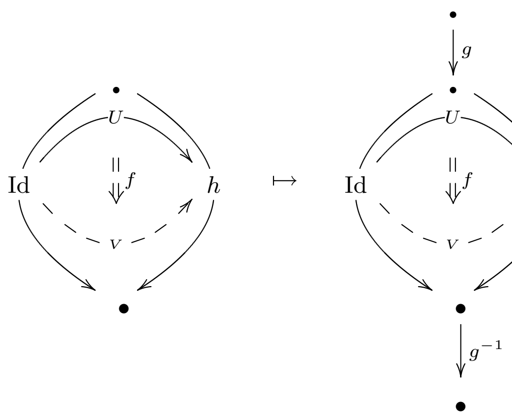
- The tensor product in $\mathcal{C}$ is composition along the single object in K:



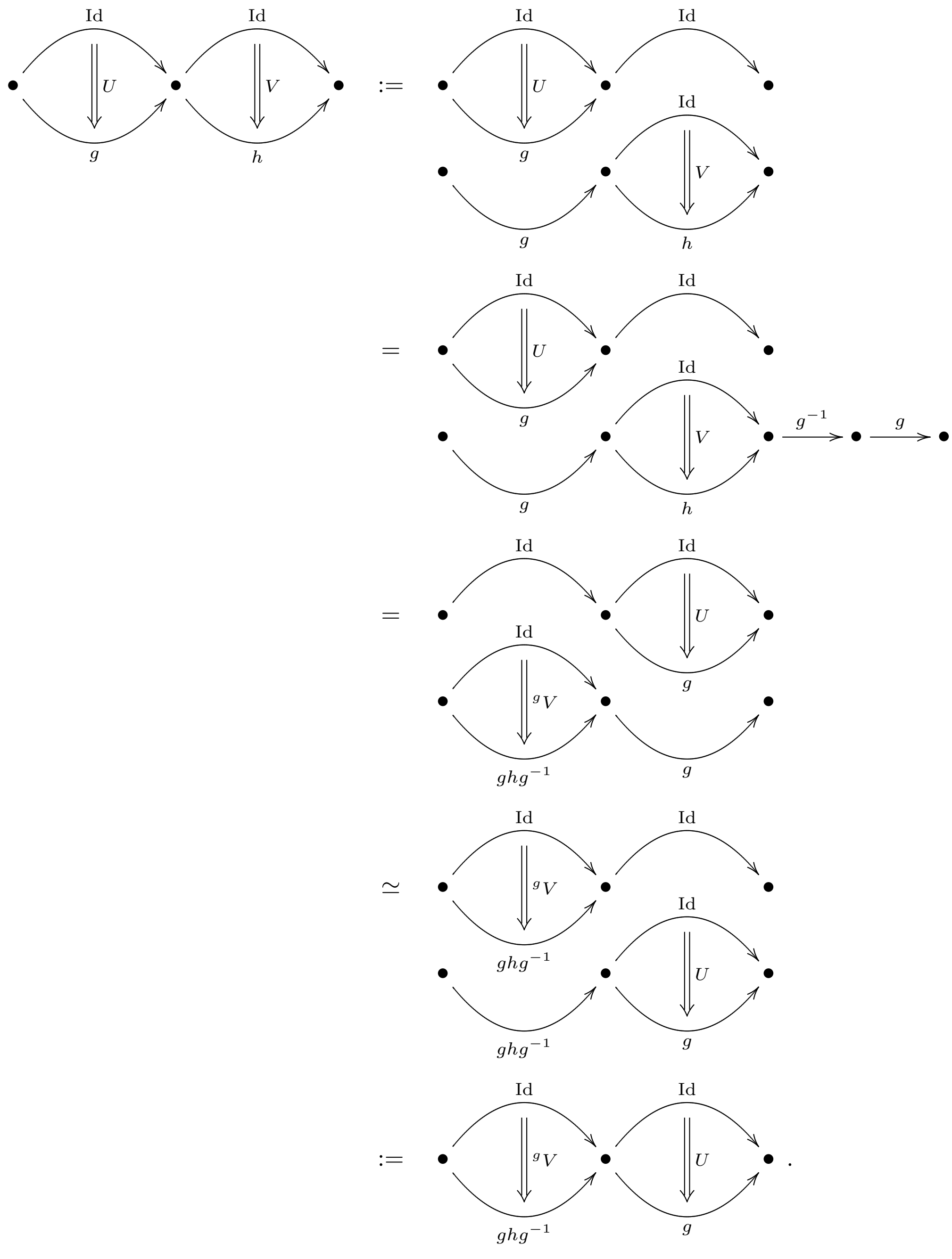
- The G-action R is conjugation with 1-morphisms in K



This extends in the obvious way to a functorial action also on morphisms $f : U \to V$



- The G-twisted braiding isomorphism is the following 3-isomorphism in K:



The first step is just the definition of the horizontal product, described in A. Then the identity morphism is decomposed as gg^{-1} and the definition of the conjugation action R is used. The only non-identity step is then the isomorphism which relates the two ways of horizontally composing 2-morphism, as described in A.

Finally, it is clear that the coherence law in the Gray 3-category ensures the coherence of the resulting G-equivariant category.

□

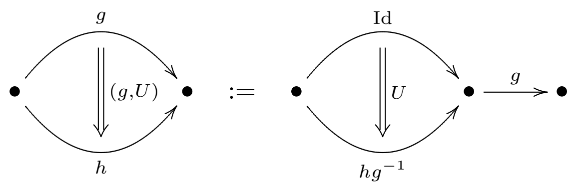
The converse statement is now straightforward.

Proposition 2. *Every G-equivariant monoidal category gives rise to a G-stabilized 3-category.*

Proof. To define the 3-category K given the abelian G-equivariant monoidal category $\mathcal{C}$, let $\mathrm{Hom}_K(\mathrm{Id}_\bullet, -) := \mathcal{C}$ and use the identifications from the proof of proposition 1. All that remains to be constructed are then the Hom-categories $\mathrm{Hom}_K(\bullet \xrightarrow{g} \bullet, \bullet \xrightarrow{h} \bullet)$ for arbitrary $g, h \in G$. But these are already fixed by the fact that postcomposition with 1-morphisms $\bullet \xrightarrow{g} \bullet$ must be an isomorphism of categories

$$\mathcal{C}_h \to \mathcal{C}_{hg} .$$

Therefore the subcategories $\mathrm{Hom}_K(\bullet \xrightarrow{g} \bullet, \bullet \xrightarrow{h} \bullet)$ are canonically isomorphic to $\mathrm{Hom}_K(\mathrm{Id}_\bullet, \bullet \xrightarrow{hg^{-1}} \bullet)$. We write



□

Corollary 1. *There is a bijective correspondence between G-equivariant abelian monoidal 1-categories and G-stabilized abelian Gray 3-categories.*

Proof. The two constructions in proposition 1 and 2 are clearly inverse to each other. □

3 Weak G-equivariant monoidal structure

The original definition of G-equivariant categories in [1, 2] does allow the action

$$R : \mathcal{C} \times G \to \mathcal{C}$$

to respect composition only up to coherent isomorphism. This is of course a natural requirement for the action of a group on a category.

But even more naturally we would allow not just a group action, but a (weak) 2-group action, such that Turaev-Kirillov's definition appears as a special case of that.

In general, n-groups $G_{(n)}$ want to act on $(n-1)$-categories, since the action is an n-functor

$$\rho : \Sigma G_{(n)} \to (n-1)\mathrm{Cat}\,.$$

(Notice that both $\Sigma G_{(n)}$ as well as $(n-1)\mathrm{Cat}$ are n-categories.)

Therefore we now generalize the concept of G-equivariant categories as follows:

Definition 3 ($G_{(2)}$-stabilized 3-category)**.** *Let $G_{(2)}$ be a (possibly weak) 2-group. Let K be a (possibly weak) 3-category. We say that K is $G_{(2)}$-stabilized if*

- $\mathrm{Obj}(K) = \{\bullet\}$
- $\mathrm{Mor}_1(K) = \mathrm{Mor}_1(\Sigma G_{(2)})$
- *there is an inclusion*

$$\Sigma G_{(2)} \hookrightarrow K$$

  which is the identity on 1-morphisms.

Remark. Definition 2 of a G-equivariant category arises again as a special case of a $G_{(2)}$-equivariant category for

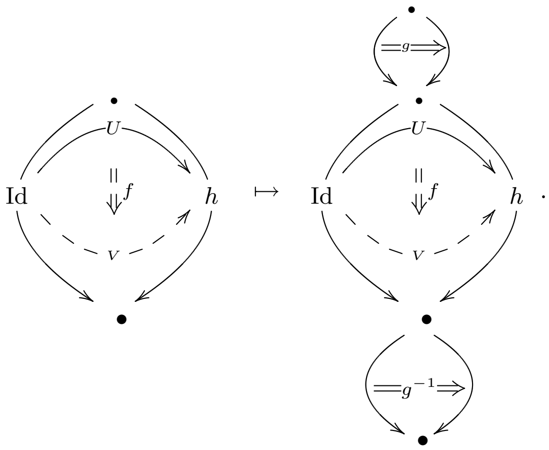
$$G_{(2)} := \mathrm{Disc}(G)\,.$$

Here $\mathrm{Disc}(G)$ denotes the "discrete" 2-group over the (1-)group G, that is the 2-group obtained by regarding G as a category with set of objects being G and only identity morphisms.

(Notice the difference between $\mathrm{Disc}(G)$ and ΣG. The former is always a 2-group, the latter is a 2-group if and only if G is abelian.)

Remark. Apart from the weakening, a qualitatively new aspect of $G_{(2)}$-stabilized as opposed to simply G-stabilized 3-categories is that the former may have two kinds of 2-morphisms: those in the image of the injection $\Sigma G_{(2)} \hookrightarrow K$ and those not in that image.

This leads to a new kind of conjugation action in $G_{(2)}$-stabilized 3-categories, namely conjugation by 2-morphisms $\bullet \overset{}{\underset{}{\Downarrow g}} \bullet$ in $\Sigma G_{(2)}$:



Example. (ein Versuch zu Super-Fusions-Kategorien, als Frage gedacht) Let $\mathcal{C}$ be a abelian braided monoidal category and consider two copies of that, to be called $\mathcal{C}_{\text{even}}$ and $\mathcal{C}_{\text{odd}}$. Form the abelian category

$$\mathcal{C}_{\text{even}} \oplus \mathcal{C}_{\text{odd}}$$

freely generated from these under direct sum and . Take this to be a $\mathbb{Z}_2$-stabilized 3-category, with the nontrivial element σ of $\mathbb{Z}_2$ the degree of $\mathcal{C}_{\text{odd}}$.

Furthermore, fix an object $J \in \mathrm{Obj}(\mathcal{C}_{\text{odd}})$ which is its own weak multiplicative inverse

$$J \otimes J \simeq 1\,.$$

Just for simplicitly I shall assume for the moment this can be strictified, so that I am allowed to write $J \otimes J = 1$.

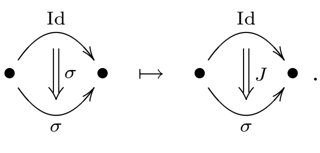
Then we get an injection

$$(\mathbb{Z}_2 \to \mathbb{Z}_2) \hookrightarrow \mathcal{C}$$

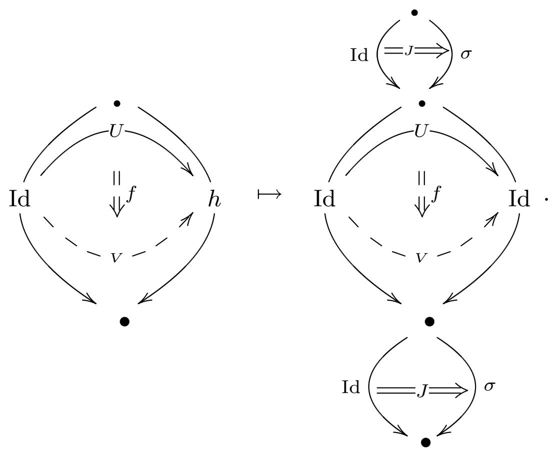
of the strict 2-group coming from the crossed module

$$\mathbb{Z}_2 \xrightarrow{t=\mathrm{Id}} \mathbb{Z}_2 \xrightarrow{\alpha} 1$$

by



This would make K a $(\mathbb{Z}_2 \to \mathbb{Z}_2)$-stabilized 3-category. We might maybe want to take conjugation by J



to act nontrivially, somehow.

Definition 4 ($G_{(2)}$-equivariant monoidal category)**.** *A monoidal category $\mathcal{C}$ is called a $G_{(2)}$-equivariant monoidal category if it arises as*

$$\mathcal{C} = \mathrm{Hom}_K(\mathrm{Id}_\bullet, --)$$

of a $G_{(2)}$-stabilized 3-category K.

Example. For G any finite group, the category $1d\mathrm{Vect}[G]$ of G-graded 1-dimensional vector spaces and isomorphism between these is a weak 2-group, equivalent to $\mathrm{Disc}(G)$, the discrete 2-group over the ordinary group G. The product operation is the ordinary tensor product in Vect. The inversion functor

$$(\cdot)^{-1} : 1d\mathrm{Vect}[G] \to 1d\mathrm{Vect}[G]$$

is

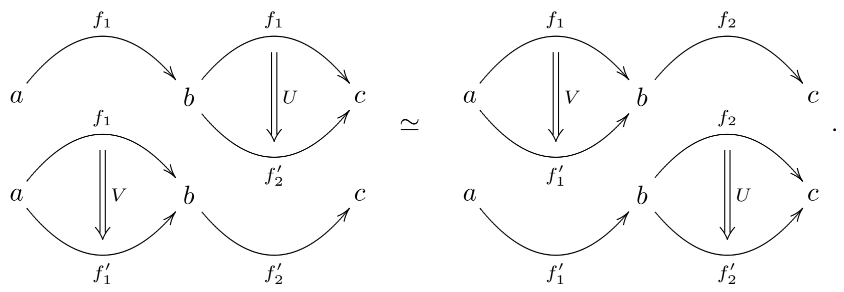
$$(\cdot)^{-1} : (\ V \xrightarrow{f} W\ ) \ \mapsto \ (\ V^* \xrightarrow{f^{*-1}} W^*\ )\,.$$

With V in degree g we have to take V^* to be in degree g^{-1}. Then...

A Gray 3-categories

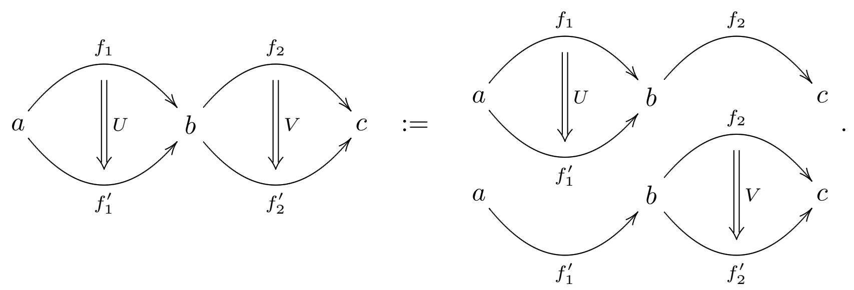
Definition 5 (Gray 3-category)**.** *A Gray 3-category is a 3-category which is strict except possibly for the exchange law for composition of 2-morphisms.*

So the only possibly nontrivial structure morphisms in a Gray 3-category are the 3-isomorphisms



Remark. The relevance of Gray 3-categories is that every weak 3-category is equivalent to some Gray 3-category. (In contrast to weak 2-categories, each of which is equivalent to some strict 2-category.) In this sense Gray 3-categories are "semistrict" – as strict as possible without losing full generality.

Remark. When the exchange law isomorphism is nontrivial, the horizontal composition of 2-morphism has two possible interpretations. We shall agree to read



References

[1] V. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*

[2] A. Kirillov, *On G-equivariant modular categories*