The Globular Extended QFT of the String
propagating on the Classifying Space of a strict
2-Group

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January 26, 2007

Abstract

The general framework of globular extended QFT is applied to the
string propagating on the classifying space of a strict 2-group.

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1 Introduction

This is supposed to be a nontrivial but simple example of a general idea that
goes as follows.

There is a mystery that demands to be understood:

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Mystery 1 The theory of gerbes with connection in terms of local data exhibits a lot of structural resemblance to state sum models of 2-dimensional quantum field theory. Why is that?

Does this point to a deeper pattern that we might want to understand?

1.1 Quantization of parallel n-transport

After a little bit of reflection, I think the pattern is

a) n-Bundles with connection are naturally conceived in terms of parallel transport n-functors.

b) Coupling these n-connections to an n-particle amounts to transgressing these n-functors to a suitable configuration space.

c) Quantizing these charged n-particles amounts to pushing the transgressed n-functors forward to a point.

From this point of view, evolution in the quantum field theory of the charged n-particle is an n-functor that is inherently obtained from the parallel transport n-functor that expresses the background field that the particle propagates in.

Both, the original parallel transport n-functor as well as the resulting quantum propagation n-functor may be locally trivialized. For the former this yields the local description of gerbe holonomy. For the latter this yields the state sum description of QFT.
Figure 1: Quantization, categorification and local trivialization.
Table 1: **The charged $n$-particle and its quantization.** The process begins with a parallel transport $n$-functor $\text{tra}$ for an $n$-bundle with connection, modelling a physical background field. It continues by specifying certain maps into the domain of the parallel transport and transgressing $\text{tra}$ to the configuration space of all these maps. This models the coupling of the background field to a charged $n$-particle (a point particle, a string, a membrane, etc.). Finally, the transgressed $n$-functor may be pushed forward to a point. This yields the quantum theory of the charged $n$-particle coupled to the given background field.
1.2 Example: string on $BG_2$

By the \emph{string on the classifying space of a strict 2-group} we here want to understand the following example of a charged $n$-particle.

Fix a strict 2-group $G_2$ and set

- $n = 2$
- $\text{par} = \Sigma(\mathbb{Z})$
- $\text{tar} = \Sigma(G_2)$, $G_2$ a strict 2-group
- $\text{phas} = \text{Bim}$
- $\text{tra} = 1 \in [\Sigma(G_2), \text{Bim}]$

The fact that $n = 2$ means that we are one categorification step above the ordinary theory of point particles coupled to ordinary vector bundles.

The fact that $\text{par} = \Sigma(\mathbb{Z})$, which is the additive group of integers regarded as a category with a single object, means that we are considering 2-particles that look like strings $\bullet \rightarrow \bullet$ stretching from $\bullet$ to $\bullet$.

The fact that $\text{tar} = \Sigma(G_2)$, which is $G_2$ regarded as a 2-groupoid with a single object, means that these strings propagate on the classifying space $BG_2$ of $G_2$, since the realization $|\Sigma(G_2)|$ of the nerve of $G_2$ is

$$|\Sigma(G_2)| = BG_2.$$  

The fact that $\text{phas} = \text{Bim}$ means that everything takes values in those well-behaved 2-vector spaces that are in the image of the canonical inclusion

$$\text{Bim} \hookrightarrow \text{vect.Mod} := \text{2Vect}.$$  

We assume that we are working over complex vector spaces.

The fact that $\text{tra} = 1$ means that we consider a trivial rank one 2-vector bundle with trivial connection on target space.

Clearly, in particular this last condition can be replaced by something more interesting.

Given this data, we can determine its quantization:

**Proposition 1** The quantum transport

$$q(\text{tra}) : \Sigma(\mathbb{Z}) \rightarrow \text{Bim}$$

of the above system is

$$q(\text{tra}) : (\bullet \rightarrow \bullet) \mapsto \mathbb{C}[\Lambda G_2] \xrightarrow{\text{Id}} \mathbb{C}[\Lambda G_2].$$
Here

\[ \Lambda G_2 = [\Sigma(\mathbb{Z}), \Sigma(G_2)]/\sim \]

is the groupoid obtained by identifying isomorphic 1-morphisms in configuration space, and

\[ \mathbb{C}[^{\Lambda G_2}] \]

is the associated groupoid algebra, over the complex numbers.

This implies that a state \( \psi \) of the system

\[ \psi : 1 \rightarrow q(\text{tra}) \]

is

- a representation \( \psi(\bullet) \) of \( \Lambda G_2 \) over the point \( \bullet \)

- an endomorphism \( \psi(\bullet) \xrightarrow{\psi(\bullet\bullet)} \psi(\bullet) \) of this representation over the string \( \bullet \rightarrow \bullet \).

In string theory terminology, we would call \( \psi(\bullet) \) a D-brane.

Of special interest is the case where \( G \) is a compact, simple and simply connected Lie group, and where

\[ G_2 = \text{String}_k(G) \]

is the string 2-group of \( G \), which comes from the level \( k \in H^3(G, \mathbb{Z}) \).

Then, as discussed elsewhere,

- The state \( \psi(\bullet) \) is an \( \text{AdG} \)-equivariant gerbe module of the gerbe at level \( k \) on \( G \).

As also discussed elsewhere, interaction of strings corresponds to the fusion product on these gerbe modules.
2 The quantum theory of the charged $n$-particle

2.1 The definition

Definition 1 A charged $n$-particle

\[
\begin{array}{c}
\text{par} \\ \\
\gamma \in \text{conf} \\ \\
\text{tar} \\ \\
\text{tra} \\ \\
\text{phas}
\end{array}
\]

is

- an $(n-1)$-category $\text{par}$, called parameter space and thought of as modelling the shape and internal structure of the $n$-particle
- an $n$-category, $\text{tar}$, called target space and thought of as modelling the space that the $n$-particle propagates in
- an $n$-category $\text{phas} = n\text{Vect}$, being the $n$-category of some notion of $n$-vector spaces
- an $n$-functor $\text{tra} : \text{tar} \to \text{phas}$, thought of as encoding the parallel transport in an $n$-bundle with connection on target space
- a choice of sub-$n$-category $\text{conf} \subset [\text{par}, \text{tar}]$, thought of as encoding the configuration space of the $n$-particle.

Given a charged $n$-particle, we obtain the diagram

\[
\begin{array}{c}
\text{conf} \\
\text{par} \\
\text{tar} \\
\text{ev}
\end{array}
\]

where the arrow on the left is the restriction of the canonical evaluation map $\text{ev} : [\text{par}, \text{tar}] \times \text{par} \to \text{tar}$ along the inclusion $\text{conf} \hookrightarrow [\text{par}, \text{tar}]$, and where $p_1$ and $p_2$ are the obvious projection on the first and the second factor, respectively.

There is a corresponding diagram of pullbacks

\[
\begin{array}{c}
[\text{conf}, \text{phas}] \\
[\text{tar}, \text{phas}] \\
[\text{par}, \text{phas}]
\end{array}
\]

\[
\begin{array}{c}
\text{ev}^* \\
p_1^* \\
p_2^*
\end{array}
\]
If the morphisms on the right have adjoints, $\bar{p}_1^*$ and $\bar{p}_2^*$, respectively, we get

\[
\begin{align*}
\text{[conf, phas]} & \\
\text{[tar, phas]} & \xrightarrow{ev^*} \text{[conf $\times$ par, phas]} \\
\text{[par, phas]} & \\
\end{align*}
\begin{align*}
\bar{p}_1^* & \\
\bar{p}_2^* & \\
\end{align*}
\]

The composition of morphisms along the above route is transgression, whereas the composition along the lower route is quantization.

**Definition 2** Given a charged $n$-particle

\[
\begin{align*}
\text{(par $\xrightarrow{\gamma \in \text{conf}}$ tar $\xrightarrow{\text{tra}}$ phas)},
\end{align*}
\]

its (extended, globular) quantum theory is the image

\[
q(\text{tra}) : \text{par} \rightarrow \text{phas}
\]

of $\text{tra}$ under this quantization map.

**Remark.** It is extended because it is an $n$-functor.

It is globular because we think of the globular morphisms in the domain par directly as the extended cobordisms on which the QFT is defined. This means in particular that every $n$-cobordisms in par has the topology of an $n$-disk.

The value of our QFT on topologically nontrivial cobordisms will be taken to be its value on any globular cutting of that cobordisms followed by a suitable trace operation.

**Caveat.** Without further qualification, this definition captures only what would be called the kinematics of the quantum theory.
A charged $n$-particle...

... comes with a configuration space of maps from its parameter space into its target space...

... and a coupling to a transport functor on target space...

... which induces transport functors on configuration space and on parameter space...

... that are known as the transgression and the quantization of the $n$-particle.

Table 2: The story of the charged $n$-particle. A drama in three acts.
2.2 How to compute the space of $n$-sections

For computing the quantization, it is convenient to proceed as follows.

We take the product $\text{conf} \times \text{par}$ to be the adjoint to the internal hom. Then

$$[\text{conf} \times \text{par}, \text{phas}] \simeq [\text{conf}, [\text{par}, \text{phas}]] .$$

The image of $\text{tra} : \text{tar} \rightarrow \text{phas}$ under

$$[\text{tar}, \text{phas}] \xrightarrow{\text{ev}^*} [\text{conf} \times \text{par}, \text{phas}] \xrightarrow{\sim} [\text{conf}, [\text{par}, \text{phas}]]$$

$$\text{tar} \\ \downarrow \rightarrow \text{tar}_*$$

is simply postcomposition with $\text{tra}$:

$$\text{tra}_* : ( \text{par} \bigcirclearrowright \gamma \downarrow \text{tar} ) \rightarrow ( \text{par} \bigcirclearrowright \gamma \text{tar} \rightarrow \text{tas} \bigcirclearrowright \gamma \rightarrow \text{phas} ) .$$

The push-forward

$$[\text{conf} \times \text{par}, \text{phas}] \xrightarrow{\bar{p}_!} [\text{par}, \text{phas}]$$

then corresponds to the push-forward

$$\text{conf} \xrightarrow{p} \{\bullet\}$$

$$\text{conf} \bigcirclearrowright \text{phas} \xrightarrow{\bar{p}^*} \{\bullet\}, [\text{par}, \text{phas}] \xrightarrow{\sim} [\text{par}, \text{phas}]$$

$$\text{tar}_* \rightarrow q(\text{tra})$$

This $q(\text{tra})$ is then defined by

$$\text{Hom}(p^* f_*, \text{tra}_*) \simeq \text{Hom}(f_*, q(\text{tra})) ,$$

for any $f_* : \bullet \rightarrow [\text{par}, \text{phas}]$.

In practice, it is usually sufficient to consider this equivalence for $f_* = 1_*$, the tensor unit in $[\{\bullet\}, [\text{par}, \text{phas}]]$.

This is what we shall do in the following example.
3 Our Example: the string on the classifying space of a 2-group

Here we regard a special case of a globular extended quantum field theory of a charged \( n \)-particle.

3.1 The setup: parameter space, target space, etc.

We set \( n = 2 \) and take parameter space to be

\[
\text{par} = \Sigma(\mathbb{Z})
\]

which we draw like

\[
\Sigma(\mathbb{Z}) = \{ \bullet \to \bullet \}
\]

and think of as modelling a string that stretches from something back to that something. We’ll discover what that something can be as we proceed.

This string is taken to propagate on a target space

\[
\text{tar} = \Sigma(G_2),
\]

which is the 2-category obtained by thinking of a strict 2-group \( G_2 \) as a one-object 2-groupoid.

Noticing that an element

\[
\gamma : \Sigma(\mathbb{Z}) \to \Sigma(G_2)
\]

in configuration space

\[
\text{conf} = [\Sigma(\mathbb{Z}), \Sigma(G_2)]
\]

is, when we send it to the world of topological spaces by taking nerves and geometric realizations

\[
|\gamma| : |\Sigma(\mathbb{Z})| \to |\Sigma(G_2)|
\]

a based loop in the classifying space of \( G_2 \):

\[
|\gamma| : S^1_\bullet \to BG_2\bullet,
\]

we say that \( \text{conf} \) models the configurations of a string that propagates on the classifying space of a strict 2-group.

Notice, however, that this description is somewhat imprecise. In fact, our globular formalism remembers the difference between a circular open string and a closed string. This point will be addressed in detail later on.

All that remains to be specified, now, is the 2-bundle with connection on target space that we wish to couple our 2-particle to. For the moment, we shall be content with understanding the simple case where this 2-bundle is trivial, of rank one and with trivial connection.

This means that we take

\[
\text{tra} = 1 : \Sigma(G_2) \to \text{Bim} \hookrightarrow 2\text{Vect}
\]
to be the tensor unit in the 2-category of all such 2-functors:

\[
\begin{array}{ccc}
1 : & 
\bullet & \rightarrow \\
\downarrow g & & \downarrow h \\
\bullet & \rightarrow & C \\
\downarrow g' & & \downarrow \text{id}_C \\
& & C
\end{array}
\]

3.2 Quantization: sections and push-forward to the point

Quantization of the above system amounts to finding

\[q(\text{tra}_*) : \{\bullet\} \rightarrow [\Sigma(\mathbb{Z}), \text{Bim}]\]

such that

\[
\text{Hom}_{[\text{conf}, \text{par}, \text{phas}]}(1_*, \text{tra}_*) \simeq \text{Hom}_{[\bullet, \text{par}, \text{phas}]}(1, q(\text{tra}_*)).
\]

We call

\[\Gamma(\text{tra}_*) = \text{Hom}(1_*, \text{tra}_*)\]

the space of sections of the \(n\)-bundle on configuration space.

Remember that we want to concentrate on \(\text{tra}_* = 1_*\).

As we have shown elsewhere, we find

**Proposition 2**

\[\Gamma(1_*) = \Lambda\text{Rep}(\Lambda G_2).\]

Here

\[\Lambda G_2 = \text{conf}/\sim\]

is the **loop groupoid** of \(G_2\) obtained by identifying isomorphic 1-morphisms in \(\text{conf} = [\Sigma(\mathbb{Z}), \Sigma(G_2)]\). This is a slight generalization of Willerton's loop groupoid.

Moreover

\[\text{Rep}(\Lambda G_2) = [\Lambda G_2, \text{Vect}]\]

is the category of representations of the loop groupoid and

\[\Lambda\text{Rep}(\Lambda G_2) = [\Sigma(\mathbb{Z}), \text{Rep}(\Lambda G_2)]\]

is the category of loops inside the category of representations of the loop groupoid.

An object in there is an automorphism

\[
\begin{array}{ccc}
\rho & \downarrow L \\
\downarrow \rho
\end{array}
\]
of a representation $\rho$ of $\Lambda G_2$.

The quantization $q(1_\ast)$ that we are after will have to be a functor on the point, with an equivalent space of sections.

We find that, up to equivalence,

$$q(1_\ast) : \{\bullet\} \to \text{par, phas}$$

is given by

$$q(1_\ast)(\bullet) = (\mathbb{C}[\Lambda G_2] \xrightarrow{\text{Id}} \mathbb{C}[\Lambda G_2]),$$

where $\mathbb{C}[\Lambda G_2]$ denotes the groupoid algebra of $\Lambda G_2$.

Notice that under the embedding

$$\text{Bim} \hookrightarrow 2\text{Vect}$$

we have

$$\mathbb{C}[\Lambda G_2] \mapsto \text{Mod}_{\mathbb{C}[\Lambda G_2]} \simeq \text{Rep}(\Lambda G_2).$$

A section of $q(1_\ast)$

$$e : 1_\bullet \to q(1_\ast)$$

is a square

\[
\begin{array}{c}
\mathbb{C} \\
\downarrow_{e_\bullet} \\
\mathbb{C}[\Lambda G_2] \\
\end{array} 
\quad \xrightarrow{\text{Id}} 
\quad 
\begin{array}{c}
\mathbb{C} \\
\downarrow_{e_\bullet} \\
\mathbb{C}[\Lambda G_2] \\
\end{array}
\]

in Bim. Notice that this means that $e_\bullet$ is a module for $\mathbb{C}[\Lambda G]$ and that $e_{\bullet\to\bullet}$ is an automorphism of that representation.

This way we discover that, indeed

$$\text{End}(1_\ast) \simeq [1, q(1_\ast)].$$

$q(1_\ast)$ is the quantization of our 2-bundle on our target space.

In conclusion, we hence find that the globular extended QFT coming from the string propagating “on a 2-group” $G_2$ yields a propagation 1-functor on parameter space

$$q(1_\ast) : \text{par} \to \text{Bim} \hookrightarrow 2\text{Vect}$$

of the form

$$q : (\bullet \xrightarrow{\text{Id}} \bullet) \mapsto (\mathbb{C}[\Lambda G_2] \xrightarrow{\text{Id}} \mathbb{C}[\Lambda G_2]).$$

3.3 Nontrivial topology: traces and states of closed string

A trace is what distinguishes a circular path from a circle.

A trace is what distinguishes a circular open string from a closed string.
Gluing. We need a trace to glue the ends of
\[
\begin{array}{c}
\rho \\
\downarrow L \\
\rho
\end{array}
\]
This gives
\[
\begin{array}{c}
\rho \\
\downarrow L \\
1
\end{array}
\] \[=\]
\[
\begin{array}{c}
1 \\
\downarrow L \\
\rho
\end{array}
\]
where 1 denotes the tensor unit in Rep(ΛG₂), that is the trivial 1-dimensional representation.

The result of the trace is an endomorphism of the trivial representation:
\[
\text{Tr} : \Lambda(\text{Rep}\Lambda G) \to \text{End}_{1_{\text{Rep}\Lambda G}}.
\]
Notice that an endomorphism of the trivial representation of a groupoid is a function on the set of its connected components.

States of open and of closed strings. In conclusion, we find that our globular extended QFT assigns to the open string states that are objects in the category
\[
\Lambda\text{Rep}(\Lambda G).
\]
An object in here
\[
\begin{array}{c}
\rho \\
\downarrow L \\
\rho
\end{array}
\]
can be thought of as the state of an open string that sits on a “ρ-brane”.
(Indeed, as discussed elsewhere, for the case that $G_2 = \text{String}_G$, representations $\rho$ of $\Lambda G_2$ correspond to $G$-equivariant modules of the canonical gerbe on $G$, hence to D-branes on $G$ in the familiar sense.)

To the closed string, on the other hand, it assigns states that are objects in
\[
\text{Tr}(\Lambda\text{Rep}(\Lambda G)) = \text{End}_{1_{\text{Rep}\Lambda G}},
\]
which is the space of functions on connected components of $\Lambda G_2$. 

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3.4 The disk correlator

Given a quantum state of our string on $\Sigma(G_2)$

$$e_1 : 1 \rightarrow q(\text{tra}_*)$$

and given a costate

$$\bar{e}_2 : q(\text{tra}_*) \rightarrow 1$$

we may pair both in a canonical fashion to obtain

$$(\bar{e}_2, e_1) : 1 \xrightarrow{e_1} q(\text{tra}_*) \xrightarrow{\bar{e}_2} 1.$$  

This lives in $\text{End}(1)$, which itself is monoidal. So we may choose a second order section

$$D_1 : \text{Id}_1 \rightarrow (\bar{e}_2, e_1)$$

and a second order cosection

$$\bar{D}_2 : (\bar{e}_2, e_1) \rightarrow \text{Id}_1$$

and form the second order pairing:

$$1 \xrightarrow{e_1} q(\text{tra}_*) \xrightarrow{\bar{e}_2} 1.$$  

As explained elsewhere, this may be addressed as a disk correlator in our QFT.

Concretely, this is a modification of transformations of 2-functors, whose
single component is the 2-cell

\[
\begin{array}{ccc}
  & C & \\
\text{Id} & \downarrow & \text{Id} \\
\varepsilon_1 & \text{Id} & \varepsilon_1 \\
\text{Id} & \downarrow & \text{Id} \\
\varepsilon_2 & \text{Id} & \varepsilon_2 \\
\text{Id} & \downarrow & \text{Id} \\
\bar{e}_2 & \text{Id} & \bar{e}_2 \\
\text{Id} & \downarrow & \text{Id} \\
\bar{e}_1 & \text{Id} & \bar{e}_1 \\
\end{array}
\]

in \text{Bim}.

For simplicity, consider the example where

\[\varepsilon_1(\bullet) = N\]

is any right \(C[\Lambda G_2]\)-module and where

\[\bar{e}_2(\bullet) = N^\vee\]

is the corresponding dual left \(C[\Lambda G_2]\)-module.

Then the duality on these provides canonical choices for \(D_1\) and \(\bar{D}_2\).

If we further abbreviate

\[A := C[\Lambda G_2]\]

and

\[L := \varepsilon \cdot A_A\]

for \(A\) regarded as a right \(A\)-module over itself and

\[R := A \cdot A_C\]

as \(A\) regarded as a left \(A\)-module over itself, then, as discussed elsewhere, the above 2-cell in \text{Bim} may equivalently be rewritten – after applying local trivial-
ization and after passing from globular to dual string diagrams – as