Morphisms of 3-Functors

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Abstract
The diagrams defining morphisms of 3-functors.

1 Morphisms of 2-Functors

Definition 1 Let $S \xrightarrow{F_1} T$ and $S \xrightarrow{F_2} T$ be two 2-functors. A pseudo-natural transformation

\[
\begin{array}{ccc}
S & \xrightarrow{\rho} & T \\
\downarrow & & \downarrow \\
F_2 & & F_1
\end{array}
\]

is a map

\[
\begin{array}{ccc}
F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\
\rho(x) & & \rho(y) \\
F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y)
\end{array}
\in \text{Mor}_2(T)
\]

which is functorial in the sense that

\[
\begin{array}{ccc}
F_1(x) & \xrightarrow{F_1(\gamma_1 \cdot \gamma_2)} & F_1(z) \\
\rho(x) & & \rho(z) \\
F_2(x) & \xrightarrow{F_2(\gamma_1 \cdot \gamma_2)} & F_2(z)
\end{array}
\]

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and which makes the pseudonaturality tin can 2-commute

\[
\begin{align*}
F_1(x) & \xrightarrow{F_1(\gamma_1)} F_1(y) \\
\rho(x) & \xrightarrow{\rho(\gamma_1)} \rho(y) \\
F_2(x) & \xrightarrow{F_2(\gamma_1)} F_2(y) \\
F_3(x) & \xrightarrow{F_3(\gamma_1)} F_3(y)
\end{align*}
\]

\[
\begin{align*}
F_1(x) & \xrightarrow{F_1(\gamma_2)} F_1(y) \\
\rho(x) & \xrightarrow{\rho(\gamma_2)} \rho(y) \\
F_2(x) & \xrightarrow{F_2(\gamma_2)} F_2(y) \\
F_3(x) & \xrightarrow{F_3(\gamma_2)} F_3(y)
\end{align*}
\]

for all \( x \xrightarrow{\gamma_1} S \xrightarrow{\gamma_2} y \in \text{Mor}_2(S) \).

**Definition 2** The vertical composition of pseudonatural transformations

\[
\begin{align*}
S & \xrightleftharpoons[\rho]{\sim} T \\
F_1 & \xrightarrow{F_2} F_2
\end{align*}
\]

is given by

\[
\begin{align*}
F_1(x) & \xrightarrow{F_1(\gamma)} F_1(y) \\
\rho(x) & \xrightarrow{\rho(\gamma)} \rho(y) \\
F_2(x) & \xrightarrow{F_2(\gamma)} F_2(y) \\
F_3(x) & \xrightarrow{F_3(\gamma)} F_3(y)
\end{align*}
\]

\[
\begin{align*}
F_1(x) & \xrightarrow{F_1(\gamma)} F_1(y) \\
\rho(x) & \xrightarrow{\rho(\gamma)} \rho(y) \\
F_2(x) & \xrightarrow{F_2(\gamma)} F_2(y) \\
F_3(x) & \xrightarrow{F_3(\gamma)} F_3(y)
\end{align*}
\]

**Definition 3** Let \( F_1 \xrightarrow{\rho_1} F_2 \) and \( F_1 \xrightarrow{\rho_2} F_2 \) be two pseudonat-
ural transformations. A **modification** (of pseudonatural transformations)

\[
\begin{array}{c}
F_1 \downarrow \rho_1 \\
\downarrow A \\
F_2 \end{array}
\]

is a map

\[
\begin{array}{c}
\text{Obj}(S) \ni x \mapsto F_1(x) \downarrow \rho_1(x) \rightarrow F_2(x) \in \text{Mor}_2(T)
\end{array}
\]

such that

\[
\begin{array}{ccc}
F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\
\downarrow \rho_2(x) & \xleftarrow{A(x)} \rho_1(x) \rho_1(y) & \downarrow \rho_2(y) \\
F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y)
\end{array}
\]

\[
\begin{array}{ccc}
F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\
\downarrow \rho_2(x) & \xleftarrow{A(y)} \rho_2(y) \rho_1(y) & \downarrow \rho_2(y) \\
F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y)
\end{array}
\]

for all \( x \xrightarrow{\gamma} y \in \text{Mor}_1(S) \).

**Definition 4** The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in \( \text{Mor}_2(T) \).

**Definition 5** Let \( S \) and \( T \) be two 2-categories. The **2-functor 2-category** \( T^S \) is the 2-category

1. whose objects are functors \( F : S \rightarrow T \)

2. whose 1-morphisms are pseudonatural transformations \( F_1 \xrightarrow{\rho} F_2 \)

3. whose 2-morphisms are modifications

\[
\begin{array}{c}
F_1 \downarrow \rho_1 \\
\downarrow A \\
F_2 \end{array}
\]
2 Morphisms of 3-Functors

We shall regard 3-categories as special categories internal to 2Cat. From this point of view, a 3-category has a 2-category of objects $S$, each of which looks like

$$
\gamma_1 \xrightarrow{S} \gamma_2 \xrightarrow{y}.
$$

In a general category internal to 2Cat, we similarly have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like

$$
\gamma_1 \xrightarrow{S_1} \gamma_2 \xrightarrow{F_1 \downarrow \downarrow \downarrow \downarrow} \xrightarrow{V} \xrightarrow{\downarrow} \xrightarrow{F_2} \xrightarrow{\downarrow} \gamma_2.
$$

We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like

$$
\gamma_1 \xrightarrow{V} \xrightarrow{s_2} \gamma_2 \xrightarrow{y}.
$$

Instead of saying that $V$ is a morphism of a category internal to 2Cat, we say $V$ is a 3-morphism. Similarly, $S_1, S_2$ are 2-morphisms, $\gamma_1, \gamma_2$ are 1-morphisms and $x$ and $y$ are objects.

We would have arrived at the same picture had we regarded categories enriched over 2Cat. However, we find that thinking of 3-morphisms as morphisms
of a category internal to 2Cat facilitates handling morphisms of 3-functors, to which we now turn.

A 3-functor \( F : S \to T \) between 3-categories \( S \) and \( T \) is a functor internal to 2Cat, hence a map

\[
F : \gamma_1 \overset{\eta}{\underset{\psi}{\Rightarrow}} \gamma_2 \quad \mapsto \quad F(\gamma_1) \overset{\eta'(\gamma_2)}{\underset{\psi'}{\Rightarrow}} F(\gamma_2)
\]

that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism \( F_1 \overset{\eta}{\Rightarrow} F_2 \) between two such 3-functors is a natural transformation internal to 2Cat, hence a 2-functor from the object 2-category to the morphism 2-category, hence a 2-functorial assignment

\[
\eta : \gamma_1 \overset{\gamma}{\Rightarrow} \gamma_2 \quad \mapsto \quad F_1(\gamma_1) \overset{\eta'(\gamma_2)}{\underset{\psi'}{\Rightarrow}} F_2(\gamma_2)
\]
that satisfies the naturality condition

\[
\begin{array}{c}
F_1(\gamma_1) \downarrow \quad F_1(y) \downarrow \\
\eta(\gamma_1) \quad \quad \eta(\gamma_2) \\
F_2(\gamma_1) \downarrow \quad F_2(y) \downarrow \\
F_2(\gamma_2) \quad \quad F_2(\gamma_2)
\end{array}
\]

Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors \(\eta\).

Hence a 2-morphism \(\eta \xrightarrow{\rho} \eta'\) of our 3-functors is a 1-functorial assignment

\[
\begin{array}{c}
\rho : \gamma \\
\downarrow \quad \downarrow \\
y \\
\end{array}
\]
such that

\[
\begin{array}{c}
\xymatrix{
F_1(\gamma_1) \ar[r]^{F_1(\delta)} & F_1(\gamma_2) \\
\eta(\gamma_1) \ar[u]_{\eta(\gamma_1)} & \rho_1(y) \ar[r] & F_1(y) \\
F_2(\gamma_1) \ar[r]_{F_2(S)} & F_2(\gamma_2) \\
F_2(y) & \rho_2(y) \ar[r] & F_2(y)
}
\end{array}
\]

\[
= \begin{array}{c}
\xymatrix{
F_1(\gamma_2) \\
\eta_2(x) \\
F_2(\gamma_2) \\
F_2(y) & \rho_2(y) \ar[r] & F_2(y)
}
\end{array}
\]

We want to restrict attention to those \(\rho\) for which the horizontal 1-morphisms \(\rho_1(x), \rho_2(x),\) etc. are identities.

Proceeding this way, a modification \(\lambda : \rho_1 \rightarrow \rho_2\) of transformations \(\rho\) gives us a 3-morphisms of 3-functors. This now is a map

\[
\begin{array}{c}
\xymatrix{
F_1(x) \\
\eta_1(x) \ar[u]_{\eta_1(x)} & \rho_1(x) \ar[r] & F_1(x) \\
\eta_2(x) \ar[u]_{\eta_2(x)} & \rho_2(x) \ar[r] & F_2(x)
}
\end{array}
\]
such that

\[
\begin{array}{c}
\eta_1(y) \rightarrow F_1(y) \\
\downarrow \quad \downarrow \\
\eta_2(y) \leftarrow F_2(y)
\end{array}
\]

\[
\begin{array}{c}
\rho_1(x) \quad \rho_2(y) \quad \eta_1(\gamma) \\
\downarrow \quad \downarrow \quad \downarrow \\
\psi_1(x) \quad \psi_2(y) \quad \eta_2(\gamma)
\end{array}
\]

\[
\begin{array}{c}
\lambda(x) \quad \lambda(y) \\
\downarrow \quad \downarrow \\
\rho_1(x) \quad \rho_2(y)
\end{array}
\]

We thus get a 3-category of 3-morphisms of 3-functors.