

Morphisms of 3-Functors

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Abstract

The diagrams defining morphisms of 3-functors.

1 Morphisms of 2-Functors

Definition 1 Let $S \xrightarrow{F_1} T$ and $S \xrightarrow{F_2} T$ be two 2-functors. A **pseudo-natural transformation**

$$\begin{array}{ccc}
 & F_1 & \\
 S & \begin{array}{c} \curvearrowright \\ \Downarrow \rho \\ \curvearrowleft \end{array} & T \\
 & F_2 &
 \end{array}$$

is a map

$$\text{Mor}_1(S) \ni x \xrightarrow{\gamma} y \mapsto \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ \rho(x) \downarrow & \swarrow \rho(\gamma) & \downarrow \rho(y) \\ F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y) \end{array} \in \text{Mor}_2(T)$$

which is functorial in the sense that

$$\begin{array}{ccccc}
 F_1(x) & \xrightarrow{F_1(\gamma_1)} & F_1(y) & \xrightarrow{F_1(\gamma_2)} & F_1(z) \\
 \rho(x) \downarrow & \swarrow \rho(\gamma_1) & \rho(y) \downarrow & \swarrow \rho(\gamma_2) & \rho(z) \downarrow \\
 F_2(x) & \xrightarrow{F_2(\gamma_1)} & F_2(y) & \xrightarrow{F_2(\gamma_2)} & F_2(z)
 \end{array} = \begin{array}{ccc}
 F_1(x) & \xrightarrow{F_1(\gamma_1 \cdot \gamma_2)} & F_1(z) \\
 \rho(x) \downarrow & \swarrow \rho(\gamma_1 \cdot \gamma_2) & \rho(z) \downarrow \\
 F_2(x) & \xrightarrow{F_2(\gamma_1 \cdot \gamma_2)} & F_2(z)
 \end{array}$$

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and which makes the pseudonaturality tin can 2-commute

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F_1(x) & \xrightarrow{F_1(\gamma_1)} & F_1(y) \\
 \rho(x) \downarrow & \swarrow \rho(\gamma_1) & \downarrow \rho(y) \\
 F_2(x) & \xrightarrow{F_2(\gamma_1)} & F_2(y) \\
 & \searrow F_2(S) & \swarrow \\
 & & F_2(\gamma_2)
 \end{array} & = & \begin{array}{ccc}
 & \xrightarrow{F_1(\gamma_1)} & \\
 F_1(x) & \xrightarrow{F_1(\gamma_2)} & F_1(y) \\
 \rho(x) \downarrow & \swarrow \rho(\gamma_2) & \downarrow \rho(y) \\
 F_2(x) & \xrightarrow{F_2(\gamma_2)} & F_2(y)
 \end{array} \\
 \\
 \text{for all } x \begin{array}{ccc}
 & \xrightarrow{\gamma_1} & \\
 & \searrow S & \swarrow \\
 & & \gamma_2
 \end{array} y \in \text{Mor}_2(S).
 \end{array}$$

Definition 2 The vertical composition of pseudonatural transformations

$$\begin{array}{ccc}
 & \xrightarrow{F_1} & \\
 S & \xrightarrow{\quad} & T \\
 & \searrow \rho & \swarrow \\
 & & F_3
 \end{array}
 \equiv
 \begin{array}{ccc}
 & \xrightarrow{F_1} & \\
 S & \xrightarrow{F_2} & T \\
 & \searrow \rho_1 & \swarrow \\
 & & F_2 \\
 & \searrow \rho_2 & \swarrow \\
 & & F_3
 \end{array}$$

is given by

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\
 \rho(x) \downarrow & \swarrow \rho(\gamma) & \downarrow \rho(y) \\
 F_3(x) & \xrightarrow{F_3(\gamma)} & F_3(y)
 \end{array} & \equiv & \begin{array}{ccc}
 F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\
 \rho_1(x) \downarrow & \swarrow \rho_1(\gamma) & \downarrow \rho_1(y) \\
 F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y) \\
 \rho_2(x) \downarrow & \swarrow \rho_2(\gamma) & \downarrow \rho_2(y) \\
 F_3(x) & \xrightarrow{F_3(\gamma)} & F_3(y)
 \end{array}
 \end{array}$$

Definition 3 Let $F_1 \xrightarrow{\rho_1} F_2$ $F_1 \xrightarrow{\rho_2} F_2$ be two pseudonat-

ural transformations. A **modification** (of pseudonatural transformations)

$$\begin{array}{ccc}
 & \rho_1 & \\
 & \curvearrowright & \\
 F_1 & \Downarrow \mathcal{A} & F_2 \\
 & \curvearrowleft & \\
 & \rho_2 &
 \end{array}$$

is a map

$$\text{Obj}(S) \ni x \mapsto F_1(x) \begin{array}{ccc} \xrightarrow{\rho_1(x)} & & \xrightarrow{\rho_1(x)} \\ \Downarrow \mathcal{A}(x) & & \Downarrow \mathcal{A}(x) \\ \xrightarrow{\rho_2(x)} & & \xrightarrow{\rho_2(x)} \end{array} F_2(x) \in \text{Mor}_2(T)$$

such that

$$\begin{array}{ccc}
 \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ \downarrow & \swarrow \rho_1(\gamma) & \downarrow \\ F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y) \end{array} & = & \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ \downarrow & \swarrow \rho_2(\gamma) & \downarrow \\ F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y) \end{array} \\
 \rho_2(x) \begin{array}{c} \curvearrowleft \\ \mathcal{A}(x) \end{array} \rho_1(x) & & \rho_2(y) \begin{array}{c} \curvearrowleft \\ \mathcal{A}(y) \end{array} \rho_1(y)
 \end{array}$$

for all $x \xrightarrow{\gamma} y \in \text{Mor}_1(S)$.

Definition 4 The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in $\text{Mor}_2(T)$.

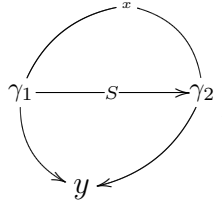
Definition 5 Let S and T be two 2-categories. The **2-functor 2-category** T^S is the 2-category

1. whose objects are functors $F : S \rightarrow T$
2. whose 1-morphisms are pseudonatural transformations $F_1 \xrightarrow{\rho} F_2$
3. whose 2-morphisms are modifications

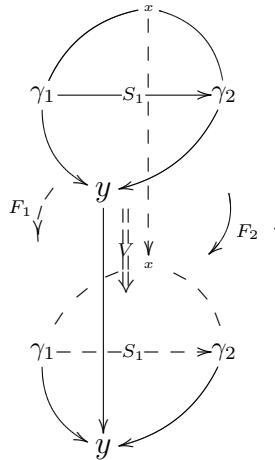
$$\begin{array}{ccc}
 & \rho_1 & \\
 & \curvearrowright & \\
 F_1 & \Downarrow \mathcal{A} & F_2 \\
 & \curvearrowleft & \\
 & \rho_2 &
 \end{array} .$$

2 Morphisms of 3-Functors

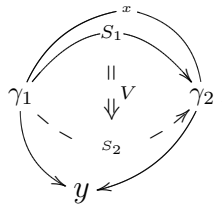
We shall regard 3-categories as special categories internal to 2Cat . From this point of view, a 3-category has a 2-category of objects S , each of which looks like



In a general category internal to 2Cat , we similarly have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like



We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like



Instead of saying that V is a morphism of a category internal to 2Cat , we say V is a 3-morphism. Similarly, S_1, S_2 are 2-morphisms, γ_1, γ_2 are 1-morphisms and x and y are objects.

We would have arrived at the same picture had we regarded categories enriched over 2Cat . However, we find that thinking of 3-morphisms as morphisms

of a category internal to 2Cat facilitates handling morphisms of 3-functors, to which we now turn.

A 3-functor $F : S \rightarrow T$ between 3-categories S and T is a functor internal to 2Cat , hence a map

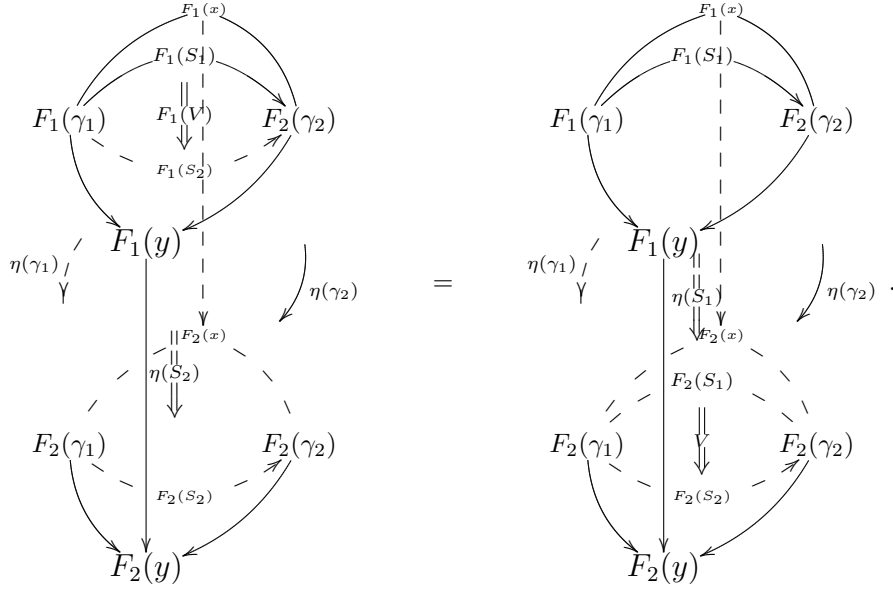
$$F : \begin{array}{ccc} & x & \\ & \curvearrowright S_1 \curvearrowleft & \\ \gamma_1 & \parallel V & \gamma_2 \\ & \downarrow & \\ & S_2 & \\ & \curvearrowleft y \curvearrowright & \end{array} \mapsto \begin{array}{ccc} & F(x) & \\ & \curvearrowright F(S_1) \curvearrowleft & \\ F(\gamma_1) & F \Downarrow V & F(\gamma_2) \\ & \downarrow & \\ & F(S_2) & \\ & \curvearrowleft F(y) \curvearrowright & \end{array}$$

that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism $F_1 \xrightarrow{\eta} F_2$ between two such 3-functors is a natural transformation internal to 2Cat , hence a 2-functor from the object 2-category to the morphism 2-category, hence a 2-functorial assignment

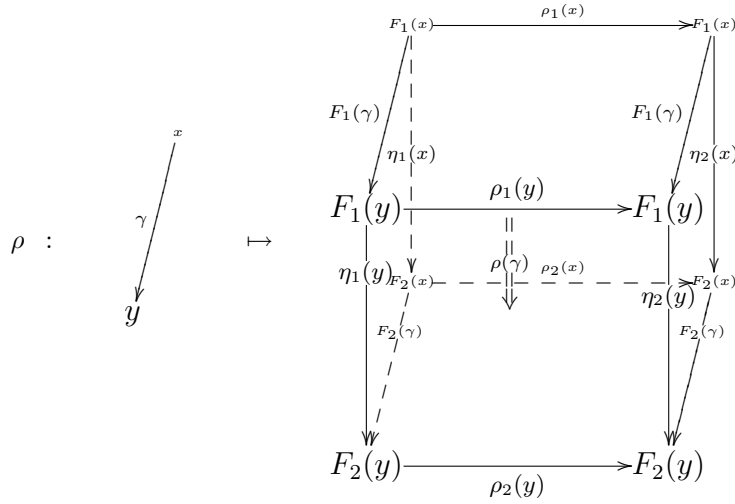
$$\eta : \begin{array}{ccc} & x & \\ & \curvearrowright S \curvearrowleft & \\ \gamma_1 & \xrightarrow{S} & \gamma_2 \\ & \curvearrowleft y \curvearrowright & \end{array} \mapsto \begin{array}{c} \begin{array}{ccc} & F_1(x) & \\ & \curvearrowright F_1(S) \curvearrowleft & \\ F_1(\gamma_1) & \xrightarrow{F_1(S)} & F_1(\gamma_2) \\ & \curvearrowleft F_1(y) \curvearrowright & \end{array} \\ \eta(\gamma_1) \swarrow \downarrow \searrow \eta(\gamma_2) \\ \begin{array}{ccc} & F_2(x) & \\ & \curvearrowright F_2(S) \curvearrowleft & \\ F_2(\gamma_1) & \xrightarrow{F_2(S)} & F_2(\gamma_2) \\ & \curvearrowleft F_2(y) \curvearrowright & \end{array} \end{array}$$

that satisfies the naturality condition

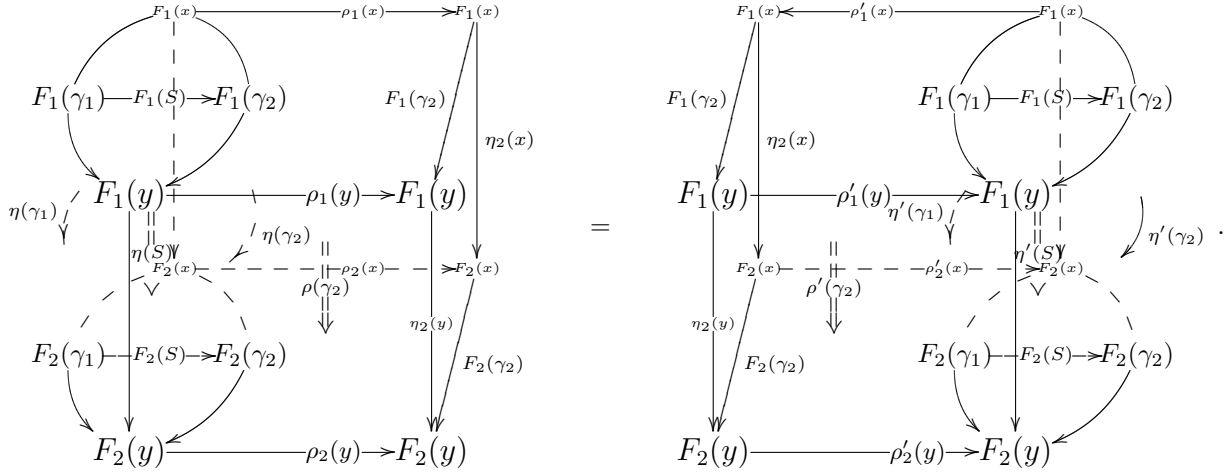


Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors η .

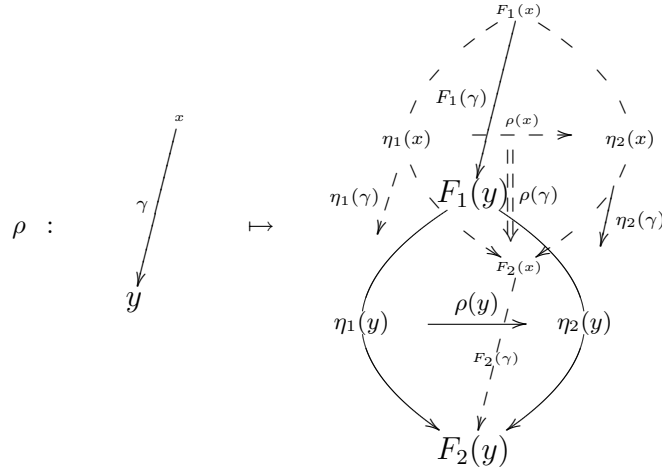
Hence a 2-morphism $\eta \xrightarrow{\rho} \eta'$ of our 3-functors is a 1-functorial assignment



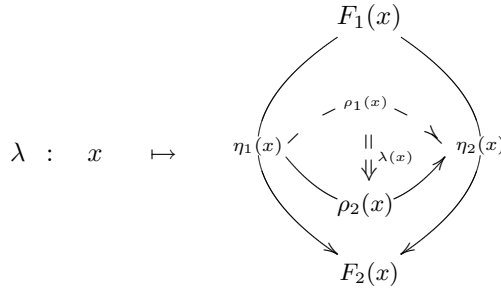
such that



We want to restrict attention to those ρ for which the horizontal 1-morphisms $\rho_1(x)$, $\rho_2(x)$, etc. are identities.



Proceeding this way, a modification $\lambda : \rho_1 \rightarrow \rho_2$ of transformations ρ gives us a 3-morphism of 3-functors. This now is a map



such that

The image shows two commutative diagrams separated by an equals sign. Both diagrams have a central node $F_1(y)$ and a bottom node $F_2(y)$. The left diagram has a top node $F_1(x)$ and a middle node $F_1(\gamma)$. Arrows include $\eta_1(x)$, $\eta_2(x)$, $\eta_1(\gamma)$, $\eta_2(\gamma)$, $\rho_1(x)$, $\rho_2(x)$, $\lambda(x)$, $\rho_2(\gamma)$, $\rho_2(y)$, $F_2(x)$, $F_2(\gamma)$, and $F_2(y)$. The right diagram has a top node $F_1(x)$ and a middle node $F_1(\gamma)$. Arrows include $\eta_1(x)$, $\eta_2(x)$, $\eta_1(\gamma)$, $\eta_2(\gamma)$, $\rho_1(x)$, $\rho_1(\gamma)$, $\rho_2(x)$, $\lambda(y)$, $\rho_2(y)$, $F_2(x)$, and $F_2(y)$.

We thus get a 3-category of 3-morphisms of 3-functors.