Morphisms of 3-Functors

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August 28, 2006

Abstract

The diagrams defining morphisms of 3-functors.

1 Morphisms of 2-Functors

Definition 1 *Let* $S \xrightarrow{F_1} T$ *and* $S \xrightarrow{F_2} T$ *be two 2-functors. A* **pseudo-natural transformation**

$$S \underset{F_2}{\overset{F_1}{\rightrightarrows}} T \qquad \Downarrow \rho$$

is a map

$$\mathrm{Mor}_1(S) \ni \quad x \xrightarrow{\gamma} y \quad \mapsto \quad \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ \downarrow{\scriptstyle \rho(x)} & \Swarrow{\scriptstyle \rho(\gamma)} & \downarrow{\scriptstyle \rho(y)} \\ F_2(x) & \xrightarrow[F_2(\gamma)]{} & F_2(y) \end{array} \quad \in \mathrm{Mor}_2(T)$$

which is functorial in the sense that

$$\begin{array}{ccccc} F_1(x) & \xrightarrow{F_1(\gamma_1)} & F_1(y) & \xrightarrow{F_1(\gamma_2)} & F_1(z) \\ \downarrow{\scriptstyle \rho(x)} & \Swarrow{\scriptstyle \rho(\gamma_1)} & \downarrow{\scriptstyle \rho(y)} & \Swarrow{\scriptstyle \rho(\gamma_2)} & \downarrow{\scriptstyle \rho(z)} \\ F_2(x) & \xrightarrow[F_2(\gamma_1)]{} & F_2(y) & \xrightarrow[F_2(\gamma_2)]{} & F_2(z) \end{array} \quad = \quad \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma_1 \cdot \gamma_2)} & F_1(z) \\ \downarrow{\scriptstyle \rho(x)} & \Swarrow{\scriptstyle \rho(\gamma_1 \cdot \gamma_2)} & \downarrow{\scriptstyle \rho(z)} \\ F_2(x) & \xrightarrow[F_2(\gamma_1 \cdot \gamma_2)]{} & F_2(z) \end{array}$$

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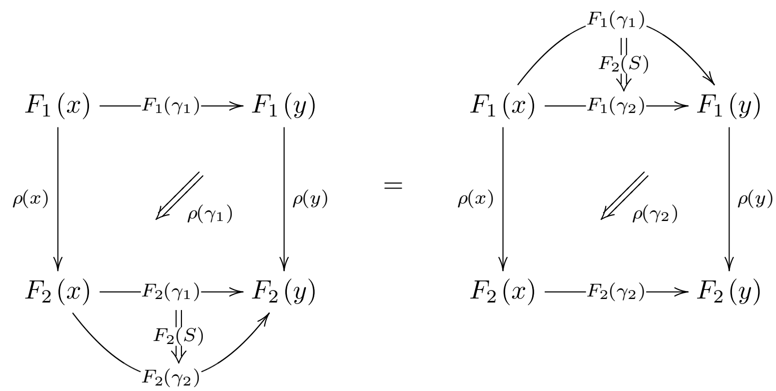
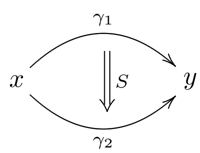
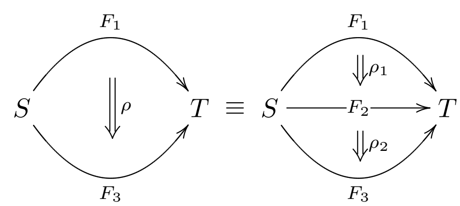
and which makes the pseudonaturality tin can 2-commute



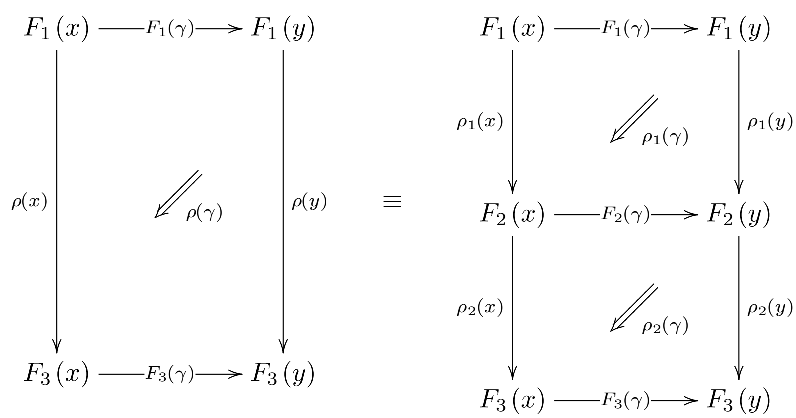
for all


$\in \mathrm{Mor}_2(S)$.

Definition 2 *The vertical* **composition of pseudonatural transformations**

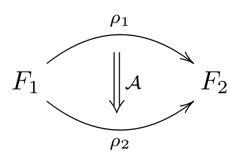


is given by

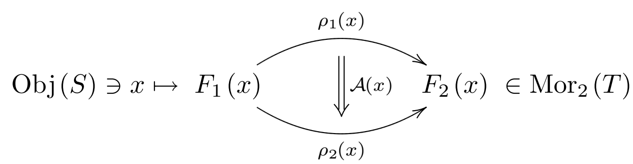


Definition 3 *Let* $F_1 \xrightarrow{\rho_1} F_2 \quad F_1 \xrightarrow{\rho_2} F_2$ *be two pseudonat-*

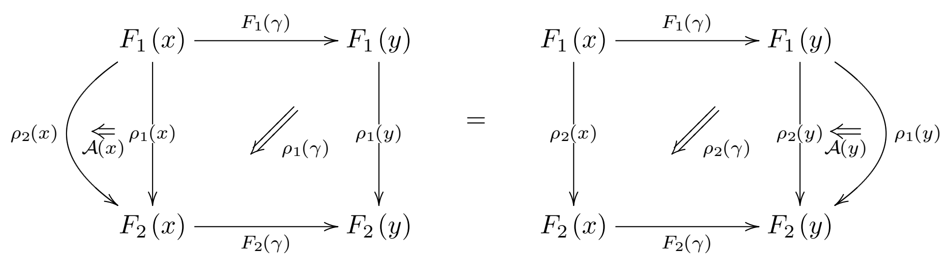
ural transformations. A **modification** *(of pseudonatural transformations)*



is a map



such that

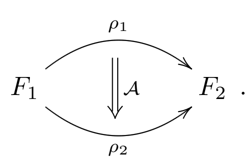


for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition 4 *The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in* $\mathrm{Mor}_2(T)$.

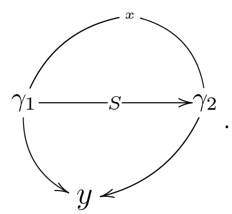
Definition 5 *Let* S *and* T *be two 2-categories. The* **2-functor 2-category** T^S *is the 2-category*

1. *whose objects are functors* $F : S \to T$
2. *whose 1-morphisms are pseudonatural transformations* $F_1 \xrightarrow{\rho} F_2$
3. *whose 2-morphisms are modifications*

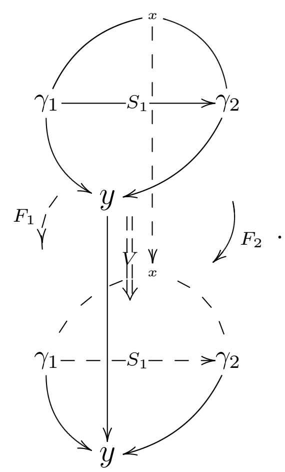


2 Morphisms of 3-Functors

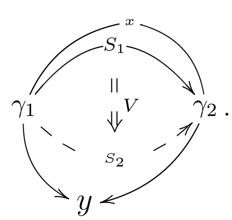
We shall regard 3-categories as special categories internal to 2Cat. From this point of view, a 3-category has a 2-category of objects S, each of which looks like



In a general category internal to 2Cat, we similarly have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like



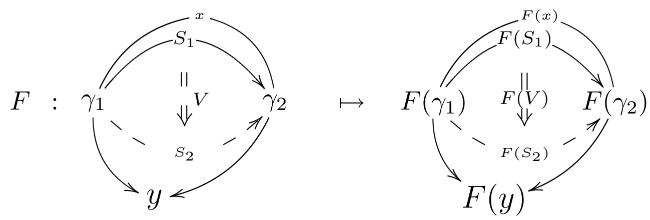
We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like



Instead of saying that V is a morphism of a category internal to 2Cat, we say V is a 3-morphism. Similarly, S_1, S_2 are 2-morphisms, γ_1, γ_2 are 1-morphisms and x and y are objects.

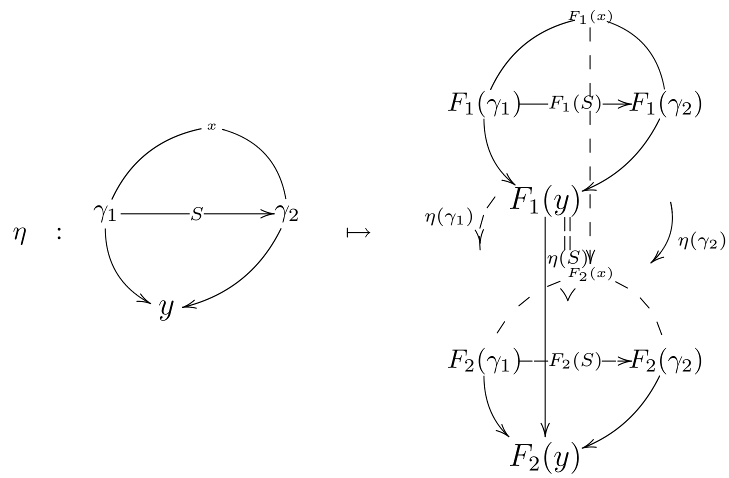
We would have arrived at the same picture had we regarded categories enriched over 2Cat. However, we find that thinking of 3-morphisms as morphisms of a category internal to 2Cat facilitates handling morphisms of 3-functors, to which we now turn.

A 3-functor $F : S \to T$ between 3-categories S and T is a functor internal to 2Cat, hence a map

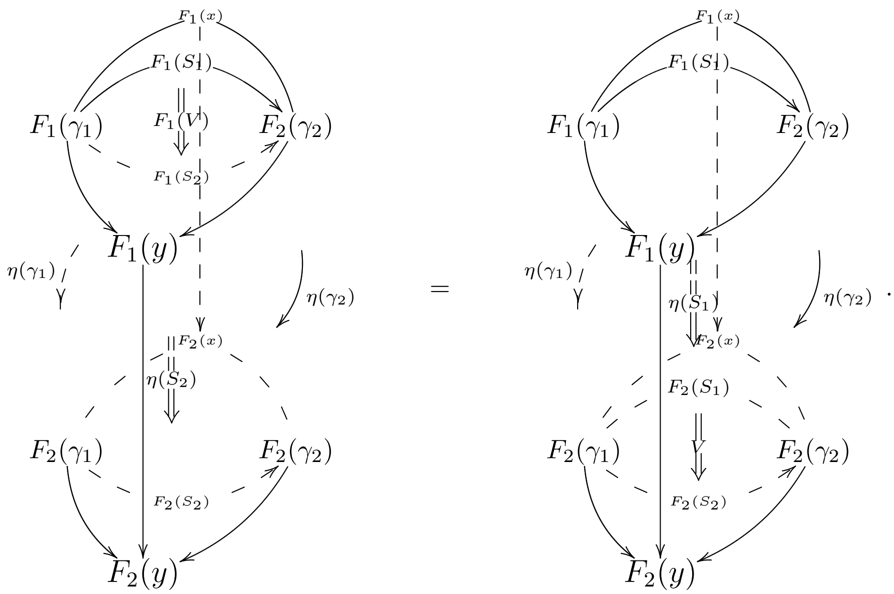


that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism $F_1 \xrightarrow{\eta} F_2$ between two such 3-functors is a natural transformation internal to 2Cat, hence a 2-functor from the object 2-category to the morphism 2-category, hence a 2-functorial assignment

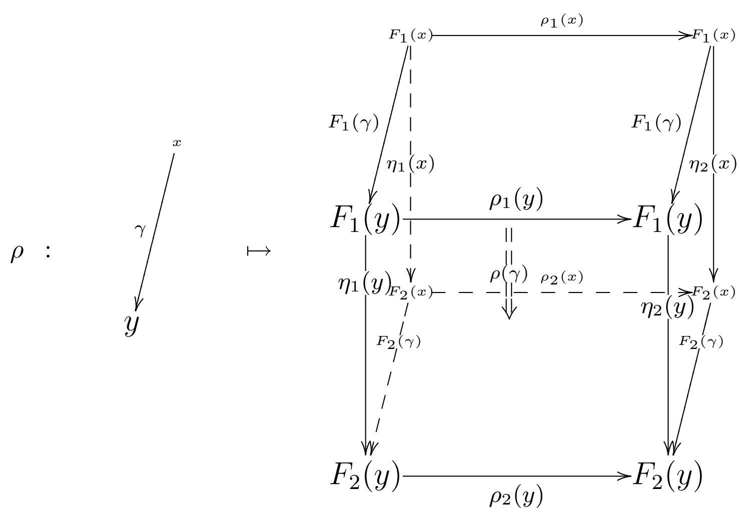


that satisfies the naturality condition

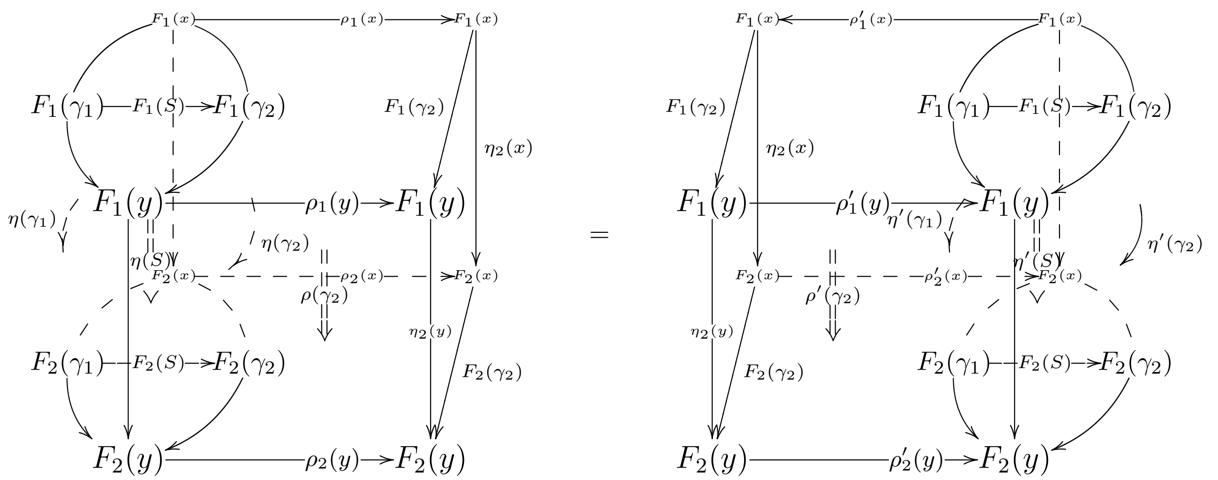


Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors η.

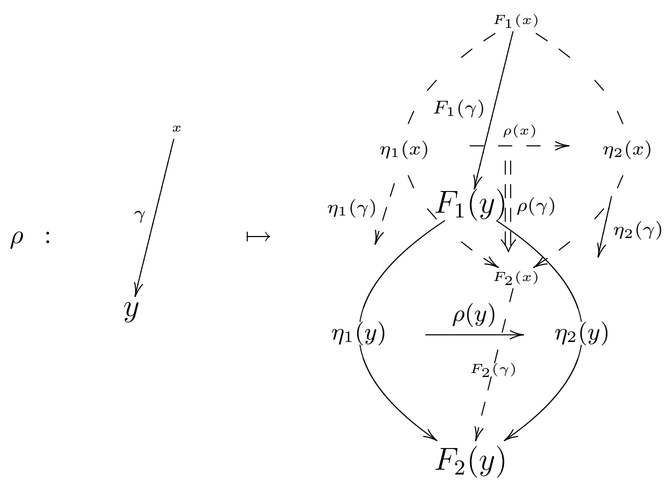
Hence a 2-morphism $\eta \xrightarrow{\rho} \eta'$ of our 3-functors is a 1-functorial assignment



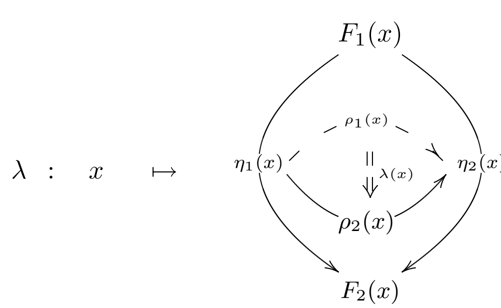
such that



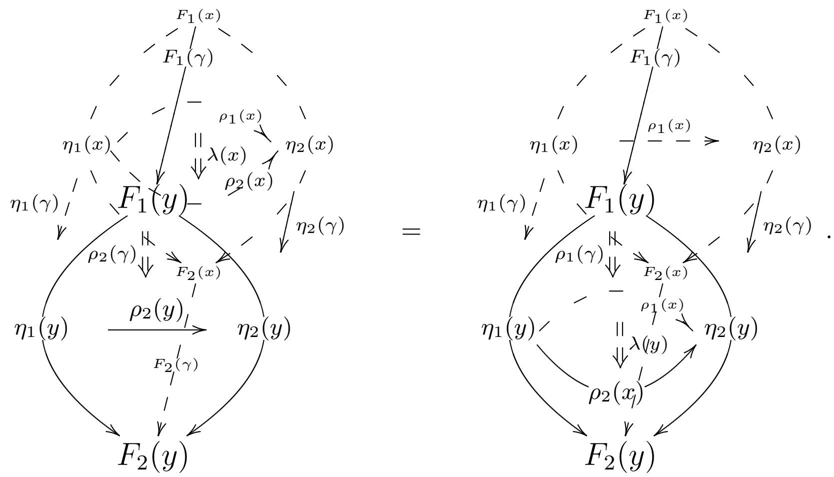
We want to restrict attention to those ρ for which the horizontal 1-morphisms $\rho_1(x)$, $\rho_2(x)$, etc. are identities.



Proceeding this way, a modification $\lambda : \rho_1 \to \rho_2$ of transformations ρ gives us a 3-morphisms of 3-functors. This now is a map



such that



We thus get a 3-category of 3-morphisms of 3-functors.