# Impressions on $\infty$-Lie theory

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**Abstract**

I chat about some of the known aspects of the categorified version of Lie theory – the relation between Lie $\infty$-algebras and Lie $\infty$-groups – indicate how I am thinking about it, talk about open problems to be solved and ideas for how to solve them.

## Contents

# 1 Setting and Plan

For much of the math that pertains to all things more or less *geometric* – in particular for pretty much all the math that is relevant in ("formal high energy") *physics* – Lie theory, which relates

$$\text{Lie groups} \quad \leftrightarrow \quad \text{Lie algebras}\,,$$

is like the bridge between left and right hemisphere.

With the realization that Lie groups and Lie algebras are just the first step of an infinite ladder *all* of which is crucially relevant for the subjects mentioned, the urge is to understand the $\infty$-categorified version of Lie theory, such as to obtain a good understanding of whatever the double arrow in

$$\text{Lie } \infty\text{-groups} \quad \leftrightarrow \quad \text{Lie } \infty\text{-algebras}$$

might mean.

## 1.1 Recent insights

There has recently occurred a crucial step forward with understanding this double arrow.

First, [4] noticed that the age-old construction due to Sullivan [9], going by his name, is secretly precisely the prescription for integrating $L_\infty$-algebras to $\infty$-groupoids.

There are some technicalities in when and how to say that an $\infty$-group is Lie (smooth). The bulk of [4] as well as of [5] are concerned with (different!) ways to realize the Lie structure on the $\infty$-group resulting from this general idea.

Then in [6] and independently in [5], a general systematic method to produce a Lie $\infty$-algebra from a given Lie $\infty$-group has been given.

Little is known at the moment about to which extent these two constructions are mutually inverse.

## 1.2 Main idea

The main idea underlying [9] and hence [4] (as opposed to the technical details of [4] and [5]) is easy and straightforward and deserves to be emphasized.

Recall that a convenient model for an $\infty$-groupoid is a *Kan complex*: a simplicial thing satisfying an extra condition called the "Kan condition". And recall that an $\infty$-group should be nothing but an $\infty$-groupoid with a single object.

Given that, the main idea of integrating a Lie $\infty$-algebra to a Lie $\infty$-group is

**Slogan 1 (Lie $\infty$-algebra $\to$ Lie $\infty$-group)** *The $\infty$-group $\int \mathfrak{g}$ integrating a given Lie $\infty$-algebra $\mathfrak{g}$ is the simplicial thing whose collection of $n$-simplices is* *the collection of flat $\mathfrak{g}$-valued differential forms on the standard $n$-simplex $\Delta^n$:*

$$\left(\int \mathfrak{g}\right)_n := \Omega^\bullet_{\text{flat}}(\Delta^n, \mathfrak{g})\,.$$

This is the age-old construction underlying Sullivan models in rational homotopy theory. Apparently [4] was the first to notice that the resulting simplicial thing actually also satisfies the Kan conditions and hence qualifies as an $\infty$-groupoid.

And the main idea for going the other way is

**Slogan 2 (Lie $\infty$-group $\to$ Lie $\infty$-algebra)** *The Lie $\infty$-algebra* $\mathrm{Lie}(G)$ *of a given $\infty$-group $G$ is the thing whose Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathrm{Lie}(\mathfrak{g}))$ *is the algebra of differential forms on the Kan complex corresponding to $G$.*

This is described in [6] and mentioned in [5]. More precisely, this involves mapping a simplicial space build from the "odd line" into the simplicial space given by $G$, and then considering the space of functions on the resulting graded space. Noticing that functions on maps from the odd line into anything are differential forms on that, the above slogan is obtained.

Currently little is known beyond the mere definition of the constructions corresponding to these slogans and a handful of concrete examples. In particular, the following question seems to be open:

**Question 1** *What is the precise relation between these two constructions?*

## 1.3 A shift in perspective

I will now propose a certain way to think of precisely these two constructions. This is supposed to be a trivial reformulation of the main idea. So if it appears as such to the reader, all the better. Trivial as it is, I am thinking that it is useful. Later I will also make a proposal concerning more the technical details behind the main idea.

### 1.3.1 Lie $\infty$-algebra integration is forming weak fundamental groupoids

**Observation 1 (trivial but helpful)** *There is a generalized smooth space $X_{\mathfrak{g}}$, such that the Lie $\infty$-groupoid $\int \mathfrak{g}$ integrating the Lie $\infty$-algebra $\mathfrak{g}$ according to slogan 1 is its weak fundamental $\infty$-groupoid:*

$$\int \mathfrak{g} = \Pi_\infty(X_{\mathfrak{g}})\,. \qquad (1)$$

Here by a "generalized smooth space" I mean a sheaf on suitable smooth test domains (convex subsets of Euclidean spaces, say). That smooth space is the sheaf which sends each test domain $U$ to the set of flat $\mathfrak{g}$-valued forms on $U$:

$$X_{\mathfrak{g}} : U \mapsto \Omega^\bullet_{\text{flat}}(U, \mathfrak{g})\,.$$

By the "fundamental $\infty$-groupoid" $\Pi_\infty(X)$ of any generalized smooth space $X$ I mean the Kan complex whose space of $n$-simplices is simply the space of singular $n$-simplices in $X$:

$$(\Pi_\infty(X))_n := \text{Hom}(\Delta^n, X)\,.$$

(There is a slight technical issue here with realizing this simplicial smooth space as a Kan complex. I think this is naturally dealt with by using maps with "sitting instants" at the boundary, the way it is done for 1-path groupoids in smooth spaces in [8].)

Then observation 1 is a direct application of the Yoneda lemma, which says that

$$(\Pi_\infty(X_{\mathfrak{g}}))_n := \text{Hom}(\Delta^n, X_{\mathfrak{g}}) \stackrel{\text{Yoneda}}{\simeq} X_{\mathfrak{g}}(\Delta^n) := \Omega^\bullet_{\text{flat}}(\Delta^n, \mathfrak{g}) =: \left(\int \mathfrak{g}\right)_n\,.$$

If we agree that this reformulation of the Kan complexes appearing in slogan 1 as fundamental $\infty$-groupoids of certain smooth spaces is trivial, let's also try to agree that it is helpful.

The first aspect that I find helpful is that this perspective suggests that we also consider other flavors of path $n$-groupoids of a given smooth space.

I think that

**Observation 2 (strict globular fundamental $n$-groupoids of smooth spaces)** *For every generalized smooth space $X$, there is its strict globular fundamental path $n$-groupoid*

$$\Pi_n^{\text{str}}(X)$$

*defined as follows: $(k < n)$-morphisms are maps $[0,1]^k \to X$ modulo thin homotopy, while $n$-morphisms are maps $[0,1]^n \to X$ modulo full homotopy.*

A homotopy $[0,1]^{k+1} \to X$ here is called *thin* if every $(k+1)$-form on $X$ pulled back along it vanishes. Compare [8].

So if our Lie $\infty$-algebra happens to be a Lie $n$-algebra, we need not integrate it to a weak Lie $\infty$-group following slogan 1 and observation 1. We can also integrate it to a *strict globular* $n$-group $G$ by setting

$$\mathbf{B}G := \Pi_n^{\text{str}}(X_{\mathfrak{g}})\,.$$

Here and always, I write $\mathbf{B}G$ for the one-object $n$-groupoid corresponding to the $n$-group $G$.

I think the following examples can be checked:

- For $\mathfrak{g}$ an ordinary Lie algebra, the $G$ in

$$\mathbf{B}G = \Pi_1^{\text{str}}(X_{\mathfrak{g}})$$

is the ordinary simply connected Lie group integrating $G$: by unwrapping the definitions on the right hand, one finds that this amounts to the well-known "integration without integration" of Lie algebras in terms of equivalence classes of parallel transport along intervals.

- For $\mathfrak{g}_\mu$ the String Lie 2-algebra (from [1, 2] with terminology as in [7]) the strict 2-group $G_\mu$ in
$$\mathbf{B}G_\mu = \Pi_2^{\mathrm{str}}(X_{\mathfrak{g}_\mu})$$
is a strict version of the String Lie 2-group essentially being that appearing in [2], but with a different rule for horizontal composition. I am claiming that this strict $G_\mu$ is implicitly used in [3]. (Which is an example for: strict versions of Lie $n$-groups are useful in concrete computations.)

#### 1.3.2 Lie $\infty$-group differentiation is forming the dg-algebra of differential forms

The previous subsection tries to say that it is not so much the Kan complex

$$\int \mathfrak{g}$$

which should first come to mind when integrating Lie $\infty$-algebra $\mathfrak{g}$, but rather the smooth space

$$X_{\mathfrak{g}} .$$

Given any smooth space $X$, we immediately obtain the differential graded commutative algebra of differential forms on it:

$$\Omega^\bullet(X) := \mathrm{Hom}(X, \Omega^\bullet(-)) .$$

**I believe that** *if we don't pass from the smooth space $X_{\mathfrak{g}}$ to its infty-path groupoid $\Pi_\infty(X_{\mathfrak{g}}) = \int \mathfrak{g}$, then the construction of [6] amounts to nothing but sending a smooth space to its algebra of differential forms.*

But I need to check that in more detail.

### 1.4 The main question of $\infty$-Lie theory under this shift of perspective

The above would mean that the construction

$$[4, 5] : \text{Lie } \infty\text{-algebras} \longleftrightarrow \text{Lie } \infty\text{-groups} : [6, 5]$$

is really nothing but the adjunction [10]

$$\begin{array}{ccc} \text{DGCAs} & \longrightarrow\!\!\!\!\dashv & \text{smooth spaces} \\ A & \longmapsto & (X_A := \mathrm{Hom}(A, \Omega^\bullet(-))) \\ \mathrm{Hom}(X, \Omega^\bullet(-)) & \longleftarrow\!\!\!\dashv & X \end{array} \qquad (2)$$

induced by the *ambimorphic* object [10] $\Omega^\bullet(-)$ (a DGCA-valued sheaf), as described in [7].

If so, the main open question induced by 1, is

**Question 2** *To which degree is 2 an equivalence* in cohomology*?*

Notice that there is canonically an inclusion

$$A \hookrightarrow \Omega^\bullet(X_A)$$

and we are asking to which degree this fails to be onto in cohomology.

Notice that Sullivan [9], who is working with something like polynomial forms, does prove that the expected equivalence in his setup does hold. Since we are really just changing perspective on that construction, it should be true that the issue is entirely in understanding how the proof generalizes from polynomial to smooth differential forms.

**Smooth algebras of differential forms** In dicussion with Todd Trimble and Andrew Stacey, it seemed to emerge that the problem that keeps 2 from being an equivalence instead of a mere adjunction might be due to the fact that the DGCA $\Omega^\bullet(X)$ of differential forms on a smooth space does not manifestly remember the smoothness of that space.

If we restrict to 0-forms aka smooth functions, the natural solution seems to be to regard

$$\Omega^0(X) := \mathrm{Hom}(X, \Omega^0(-))$$

not just as an algebra, but actually as a co-pre-sheaf

$$\Omega^0(X) : U \mapsto \mathrm{Hom}(X, \Omega^0(-, U))$$

which we may test on test *co*-domains.

This co-pre-sheaf happens to be monoidal, in that it respects the cartesian product of test codomains. Such monoidal co-pre-sheaves are called $C^\infty$-algebras []. And these enjoy plenty of nice properties which the more naive mere algebra of functions on a smooth space is lacking.

It remains to extend this construction to form $C^\infty$DGCAs. I am thinking that in order to handle this we need to define $C^\infty$-modules and equip $\Omega^n(X)$ with the structure of a $C^\infty$-module over $\Omega^0(X)$. Then we'd might make the definition: a $C^\infty$DGCA is a DGCA whose degree 0 part is a $C^\infty$-algebra and whose degree $n$-parts are each $C^\infty$-modules of this $C^\infty$-algebra.

Then we might have a chance to get an equivalence

$$C^\infty\text{DGCAs} \simeq \text{smooth spaces}\,.$$

If so, lots of nice consequences would result...

## References

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