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Abstract

I list a couple of free graded-commutative differential algebras, Koszul dual to various semistrict Lie n-algebras.

Example 1 (Lie algebra)

Let g be a Lie algebra. Choose a basis $\{a^a\}$ of g^* and let the structure constants in the dual basis be $C^a{}_{bc}$. Consider the free graded-commutative algebra $\bigwedge^\bullet g^*$, where g^* is in degree 1. Define a differential on that by

$$da^a + \frac{1}{2} C^a{}_{bc} q^b q^c = 0\,.$$

Nilpotency of d follows from the Jacobi-identity in g

$$\begin{aligned} d^2 a^a &= d(-\frac{1}{2} C^a{}_{bc} a^b a^c) \\ &= \frac{1}{2} C^a{}_{b[c} C^b{}_{de]} a^c a^d a^e \\ &= 0\,. \end{aligned}$$

Example 2 (deRham complex of $\mathbb{R}^n$)

Consider example 1 for $g = \mathrm{Lie}(\mathbb{R}^n)$. Then $\bigwedge^\bullet g^*$ is just the complex of differential forms on $\mathbb{R}^n$ with differential the deRham differential.

Example 3 (trivial flat connection)

An fda-morphism from the Lie algebra g to the deRham complex is a g-valued 1-form on $\mathbb{R}^n$ with vanishing curvature.

Example 4 (differential crossed module)

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Let (g,h) be a differential crossed module. Choose a basis $\{a^a\}$ of g^* and a basis $\{b^i\}$ of h^*. In this basis, let the structure constants of g be $C^a{}_{bc}$. Let the action of g on h have structure constants $\alpha^i{}_{aj}$. Let the morphism from h to g have components $t^a{}_i$.

The fact that α is an action (by derivations)

$$\alpha(a_a)(\alpha(a_b)(b_i)) = \alpha([a_a,a_b])(b_i) + \alpha(a_b)(\alpha(a_a)(b_i))$$

reads in components

$$2\alpha^i{}_{[a|j|}\alpha^j{}_{b]k} = \alpha^i{}_{ck}C^c{}_{ab}\,.$$

One of the conditions on the differential crossed module is

$$t(\alpha(a_a)(b_i)) = [a_a, t(b_i)]\,.$$

In components this reads

$$t^b{}_j\alpha^j{}_{ai} = C^b{}_{ac}t^c{}_i\,.$$

The other condition is

$$\alpha(t(b_i))(b_j) = [b_i,b_j]\,.$$

This says that $\alpha^i{}_{ak}t^a{}_j = \tilde{C}^i{}_{jk}$ are the structure constants of h. In particular, this implies that the expression is antisymmetric in the two lower indices.

Consider the free graded-commutative algebra $\bigwedge^\bullet(g^*\oplus h^*)$ with g^* in degree 1 and h^* in degree 2. Define a differential on that by

$$\begin{aligned} da^a + \frac{1}{2}C^a{}_{bc}a^b a^c + t^a{}_i b^i &= 0 \\ db^i + \alpha^i{}_{aj}a^a b^j &= 0\,. \end{aligned}$$

Nilpotency of d follows from

$$\begin{aligned} d^2a^a &= d(-\frac{1}{2}C^a{}_{bc}a^b a^c - t^a{}_i b^i) \\ &= \frac{1}{2}C^a{}_{bc}C^b{}_{de}a^c a^d a^e - C^a{}_{bc}a^b t^c{}_i b^i + t^a{}_i\alpha^i{}_{bj}a^b b^j \\ &= 0\,, \end{aligned}$$

where the first term vanishes again by the Jacobi identity on g and the second two terms by the first of the two crossed module conditions. Also

$$\begin{aligned} d^2b^i &= d(-\alpha^i{}_{aj}a^a b^j) \\ &= -\alpha^i{}_{aj}(-\frac{1}{2}C^a{}_{bc}a^b a^c - t^a{}_k b^k)b^j + \alpha^i{}_{aj}a^a(-\alpha^j{}_{bk}a^b b^k) \\ &= (\alpha^i{}_{aj}\frac{1}{2}C^a{}_{bc} - \alpha^i{}_{bk}\alpha^k{}_{cj})a^b a^c b^j + \alpha^i{}_{a(j}t^a{}_{k)}b^k b^j \\ &= 0\,. \end{aligned}$$

These two terms vanish by the relations discussed above (α is an action compatible with t).

Example 5 (flat and fake-flat 2-connection)

An fda-morphism from the crossed module $(h \to g)$ to the deRham complex is a g-valued 1-form A and an h-valued 2-form B satisfying $F_A + t(B) = 0$ and $d_A B = 0$.

Example 6 (inner derivation 2-algebra of g)

The Lie 2-algebra of inner derivations of the Lie algebra g is the crossed module $(g \to g)$. Hence, by example 5, an fda-morphism from $(g \to g)$ to the deRham complex is an *arbitrary* g-valued 1-form A on $\mathbb{R}^n$. The 2-form B involved is fixed to be the curvature of the 1-form. The flatness condition $d_A B = 0$ is the Bianchi identity.

—– begin of new part —-

Example 7 (general semistrict Lie 2-algebra)

Baez-Crans have defined semistrict Lie-2-algebras and shown that they are equivalent to 2-term L_∞ algebras. These, in turn, are the same as general differential structures on $\Lambda^\bullet(V_0^* \oplus V_1^*)$. In a basis $\{a^a\}$ of V_0^* and $\{b^i\}$ of V_1^* the most general differential acts as

$$da^a + \frac{1}{2}C^a{}_{bc}a^b a^c + t^a{}_i b^i = 0$$

and

$$db^i + \alpha^i{}_{aj}a^a b^j + \frac{1}{6}r^i{}_{abc}a^a a^b a^c = 0\,.$$

The differential crossed module discussed above is obtained for $r^i{}_{abc} = 0$. In the Lie-2-algebra picture, r encodes a nontrivial **Jacobiator**.

In order to compare this to the Baez-Crans notation, use the following dictionary (where $\{a_a\}$ and $\{b_i\}$ are the bases dual to $\{a^a\}$ and $\{b^i\}$, respectively):

$$\frac{1}{2}C^a{}_{bc}a_a = l_2(a_b, a_c)$$

$$\frac{1}{2}\alpha^i{}_{aj}b_i = l_2(a_b, b_i)$$

$$t^a{}_i a_a = d(b_i)$$

$$\frac{1}{6}r^i{}_{abc} = \frac{1}{2}l_3(a_a, a_b, a_c)\,.$$

Example 8 (morphisms of Lie-2-algebras)

Lie 2-algebras, due to their category-theoretic nature, have a rather obvious notion of 1- and 2-morphisms between them, as explained by Baez-Crans.

A natural notion of 1-morphism between FDAs is a map that is at the same time an algebra homomorphism and a chain map. The respect for the (free) algebra structure says that such a morphism is already fixed once we know its action on the generators.

A natural notion of 2-morphism would be a chain homotopy. But it turns out that, in order to reproduce the 2-morphisms of Baez-Crans Lie 2-algebras this way, we need a slight modification of this, namely a chain homotopy *up to linear order*.

To see this, we will now work out these morphisms explicitly.

First recall the Baez-Crans definition of morphism of 2-term L_∞-algebras.

Definition 1 *A morphism*

$$\varphi : V \to V'$$

of 2-term L_∞-algebras V and V' are maps

$$\phi_0 : V_0 \to V_0'$$

$$\phi_1 : V_1 \to V_1'$$

together with a skew-symmetric map

$$\phi_2 : V_0 \otimes V_0 \to V_1'$$

satisfying

$$\phi_0(d(h)) = d'(\phi_1(h))$$

as well as

$$d(\phi_2(x,y)) = \phi_0(l_2(x,y)) - l_2(\phi_0(x), \phi_0(y))$$

$$\phi_2(x, dh) = \phi_1(l_2(x,h)) - l_2(\phi_0(x), \phi_1(h))$$

and finally

$$\begin{aligned}
&l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x,y,z)) = \\
&\phi_2(x, l_2(y,z)) + \phi_2(y, l_2(z,x)) + \phi_2(z, l_2(x,y)) + \\
&l_2(\phi_0(x), \phi_2(y,z)) + l_2(\phi_0(y), \phi_2(z,x)) + l_2(\phi_0(z), \phi_2(x,y))\,.
\end{aligned}$$

We now show how this is reproduced by a chain map algebra homomorphism between the corresponding dual FDA.

Let $\{a^a\}$ and $\{b^i\}$ as above be a basis for $(\Lambda^\bullet(V_0^* \oplus V_1^*), d_V)$ and let $\{a'^a\}$ and $\{b'^i\}$ be a basis of another Lie 2-algebra. $(\Lambda^\bullet(W_0^* \oplus W_1^*), d_W)$.

Then a morphism

$$q : (\Lambda^\bullet(V_0^* \oplus V_1^*), d_V) \to (\Lambda^\bullet(W_0^* \oplus W_1^*), d_W)$$

reads in that basis

$$q : a^a \mapsto q^a{}_b a'^b$$

and

$$q : b^i \mapsto q^i{}_j b'^j + q^i{}_{ab} a'^a a'^b\,,$$

where the chain map condition demands that the coefficients satisfy

$$-\frac{1}{2}C^a{}_{bc}q^b{}_dq^c{}_ea'^da'^e - t^a{}_iq^i{}_jb'j - t^a{}_iq^i{}_{bc}a'^ba'^c = -q^a{}_d\frac{1}{2}C'^d{}_{bc}a'^ba'^c - q^a_dt'^d{}_ib'^i$$

and

$$\begin{aligned} & -\alpha^i{}_{aj}q^a{}_bq^j{}_ka'^bb'^k - \alpha^i{}_{aj}q^a{}_bq^j{}_{cd}a'^ba'^ca'^d - \frac{1}{6}r^i{}_{abc}q^a{}_dq^b{}_eq^c{}_fa'^da'^ea'^f \\ = \; & d'(q^i{}_jb'^j + q^i{}_{ab}a'^aa'^b) \\ = \; & -q^i{}_j\alpha'^j{}_{ak}a'^ab'^k - \frac{1}{6}q^i{}_jr^j{}_{abc}a'^aa'^ba'^c + q^i{}_{ab}C'^b{}_{cd}a'^aa'^ca'^d + 2q^i{}_{ab}t'^b{}_ja'^ab'^j \end{aligned}$$

hence

$$\frac{1}{2}C^a{}_{de}q^d{}_bq^e{}_c + t^a{}_iq^i{}_{bc} = \frac{1}{2}q^a{}_dC'^d{}_{bc}$$

and

$$t^a{}_iq^i{}_j = q^q{}_dt'^d{}_j$$

and

$$\alpha^i{}_{aj}q^q{}_bq^j{}_k = q^i{}_j\alpha'^j{}_{ak} - 2q^i{}_{ab}t^b{}_j$$

and

$$\alpha^i{}_{dj}q^d{}_{[a}q^j{}_{bc]} + \frac{1}{6}r^i{}_{def}q^d{}_{[a}q^e{}_bq^f{}_{c]} = \frac{1}{6}q^i{}_jr^j{}_{[abc]} - q^i{}_{[a|b|}C'^b{}_{cd]} \,.$$

One can check that this are indeed the equations defining a morphism of Lie-2-algebras.

Definition 2 *A 2-morphism*

$$\tau : \phi \Rightarrow \psi$$

of 1-morphisms of 2-term L_∞-algebras is a map

$$\tau : V_0 \to V'_1$$

such that

$$\psi_0 - \phi_0 = [d,\tau]_0$$
$$\psi_1 - \psi_0 = [d,\tau]_1$$

and

$$\phi_2(x,y) - \psi_2(x,y) = l_2(\phi_0(x),\tau(y)) + l_2(\tau(x),\psi_0(y)) - \tau(l_2(x,y))$$

This is reproduces in terms of FDAs as follows.

Define a 2-morphism $\tau : q \to q'$ between morphisms of FDAs to be a map of degree -1 which on the generators acts as

$$\tau : b^i \mapsto \tau^i{}_aa'^a$$

and

$$\tau : a^a \mapsto 0\,,$$

like a chain homotopy would, but then extend this to a **q-Leibnitz-operator** to all of $\Lambda^\bullet V^*$. This means that, for instance

$$\tau : b^i b^j b^k \mapsto \tau(b^i) q(b^j b^k) + q(b^i)\tau(b^j) q(b^k) + q(b^i b^j)\tau(b^k)\,.$$

This way we get

$$[d,\tau] : a^a \mapsto t^a{}_i \tau^i{}_b a'^b$$

and

$$[d,\tau] : b^i \mapsto -\frac{1}{2}\tau^i{}_a C'^a{}_{bc} a'^b a'^c - \tau^i{}_a t'^a{}_j b'^j + \alpha^i{}_{aj}\tau^j{}_b a'^a a'^b\,.$$

One can check that with these formulas the condition

$$q - q' = [d,\tau]$$

evaluated *on generators* does reproduce the above definition of 2-morphism of 2-term L_∞-algebra.

But then, of course, this condition holds on all of $\Lambda^\bullet V$ only up to terms with at most one occurence of τ.

We might want to address such a chain homotopy up to linear terms a **linearized chain homotopy**.

—– *end of new part* —-

Example 9 (Chern-Simons 3-algebra)

Let g be any Lie algebra. Choose a dual basis $\{a^a\}$, let $C^a{}_{bc}$ be the structure constants in that basis and k_{ab} be the components of the Killing form. Let h be the Lie algebra of $U(1)$ and consider the free graded-commutative algebra $\bigwedge^\bullet(g^* \oplus g^* \oplus h^*)$ with the first summand g^* in degree 1, the second one in degree 2 and with h^* in degree 3. Denote by $\{b^a\}$ the basis of the g^*-summand in degree 2; and by $\{c\}$ a basis of h^*

Invariance of the Killing form

$$k([a_a, a_b], a_c) + k(a_b, [a_a, a_c]) = 0$$

reads in components

$$C_{(ab)c} = 0\,,$$

where $C_{abc} \equiv k_{ad} C^d{}_{bc}$.

On that algebra, define the differential

$$\begin{aligned} &da^a + \frac{1}{2} C^a{}_{bc} a^b a^c + b^a = 0 \\ &db^a + C^a{}_{bc} a^b b^c = 0 \\ &dc + k_{ab} b^a b^b = 0\,. \end{aligned}$$

The first two equations are precisely those of the crossed module $g \to g$ from example 6. So we just need to check that

$$\begin{aligned} d^2 c &= d(-k_{ab} b^a b^b) \\ &= -2t_{ab} C^b{}_{de} a^d b^e b^a \\ &= 2C_{(ae)d} a^d b^e b^a \\ &= 0 \end{aligned}$$

by the invariance of k.

Example 10 (Chern-Simons 3-form)

An fda morphism from the Chern-Simons 3-algebra of example 9 to the deRham complex is a g-valued 1-form A, a g-valued 2-form B and a 3-form C satisfying $F_A + B = 0$ and $dC + t(B \wedge B) = 0$. Hence it is, up to an exact 3-form, really just that 1-form A, which gives rise to its curvature 2-form F_A and its Chern-Simons 3-form $C \propto \mathrm{CS}_A = t(A \wedge dA) + \frac{2}{3} t(A \wedge A \wedge A)$.

Example 11 (String 2-algebra)

For g any Lie algebra and $h = \mathrm{Lie}(\mathbb{R})$, consider $\bigwedge^\bullet (g^* \otimes h^*)$ with g^* in degree 1 and h^* in degree 2. In terms of a dual basis $\{a^a\}$ of g^* and $\{c\}$ of h^* define a differential by

$$\begin{aligned} da^a + \frac{1}{2} C^a{}_{bc} = 0 \\ db + \frac{1}{6} C_{abc} a^a a^b a^c = 0\,, \end{aligned}$$

where $C_{abc} = k_{aa'} C^{a'}{}_{bc}$ as in example 9. This is nilpotent

$$\begin{aligned} d^2 b &= d(-\frac{1}{6} C_{abc} a^a a^b a^c) \\ &= \frac{1}{2} C_{abc} a^a a^b (\frac{1}{2} C^c{}_{de} a^d a^e) \\ &= 0 \end{aligned}$$

due to the Jacobi identity.

The Lie 2-algebra defined by the above fda is known as the skeletal version of the Lie 2-algebra string_g.

Example 12 (inner derivation 3-algebra of string_g)

We want to find something like the 3-algebra of inner derivations of the 2-algebra from example 11.

So we are looking for an FDA defined on $\bigwedge^\bullet (g^* \otimes (g^* \otimes h^*) \otimes h^*)$ with the first g^* in degree 1, the bracket term in degree 2 and the last h^* in degree 3.

Denote by $\{r^a\}$ the chosen dual basis of g^*, interpreted in degree 2 and by c the basis of h^* in degree 3.

The action of the differential on a^a and r^a should be as in example 6

$$\begin{aligned} da^a + \frac{1}{2}C^a{}_{bc}a^b a^c + r^a &= 0 \\ dr^a + C^a{}_{bc}a^b r^c &= 0\,. \end{aligned}$$

This allows to have the following terms in the equation for db

$$db + \frac{1}{6}C_{abc}a^a a^b a^c + uk_{ab}a^a r^b - c = 0\,,$$

where $u \in \mathbb{R}$ is some constant which we use to parameterize our choices. Nilpotency of d requires that

$$\begin{aligned} d^2 b &= d(-\frac{1}{6}C_{abc}a^a a^b a^c - uk_{ab}a^a r^b + c) \\ &= dc + \frac{1}{2}C_{abc}a^a a^b r^c - u\left(-\frac{1}{2}k_{ab}C^a{}_{cd}a^c a^d r^b - k_{ab}r^a r^b + k_{ab}a^a C^b{}_{cd}a^c r^d\right) \\ &= dc + \frac{1}{2}(1-u)C_{abc}a^a a^b r^c + uk_{ab}r^a r^b\,. \end{aligned}$$

Clearly, a special situation is $u = 1$, so let's concentrate on that. Then we have

$$\begin{aligned} da^a + \frac{1}{2}C^a{}_{bc}a^b a^c + r^a &= 0 \\ dr^a + C^a{}_{bc}a^b r^c &= 0 \\ db + \frac{1}{6}C_{abc}a^a a^b a^c + k_{ab}a^a r^b - c &= 0 \\ dc + k_{ab}r^a r^b &= 0\,. \end{aligned}$$

That $d^2 c = 0$ follows as in example 9.

Example 13 (string$_g$**-3-connection)**

An fda morphism of the 3-algebra of example 12 to the deRham complex is a g-valued 1-form A together with a real 3-form B and a real 3-form H satisfying

$$\begin{aligned} H &= dB + \frac{1}{6}C_{abc}A^a \wedge A^b \wedge A^c - k_{ab}A^a F^b \\ &= dB - \left(k_{ab}A^a dA^b + \frac{1}{3}C_{abc}A^a \wedge A^b \wedge A^c\right) \\ &= dB - \mathrm{CS}(A)\,. \end{aligned}$$

Example 14 (inner derivation 3-algebra of $(h \to g)$**)**

Let $(h \to g)$ be a differential crossed module. We want to find the 3-algebra of inner derivations of $(h \to g)$, in analogy to example 6.

I did this computation at the level of groups and then differentiated. The result is the following fda.

Let the vector space be $\bigwedge^\bullet(g^* \otimes (g^* \otimes h^*) \otimes h^*)$, where the first g^* is in degree 1, the $(g^* \otimes h^*)$ is in degree two and h^* is in degree 3.

Denote a chosen basis of g^* in degree 1 by $\{q^a\}$, a basis of g^* in degree 2 by $\{r^a\}$, a basis of h^* in degree 2 by $\left\{s^i\right\}$ and, finally, a basis of h^* in degree 3 by $\left\{t^i\right\}$.

Let $C^a{}_{bc}$, $\alpha^i{}_{aj}$ and $t^i{}_a$ be the tensors characterizing the crossed module $(h \to g)$ as in example 4.

Define a differential by

$$\begin{aligned} &dq^a + \frac{1}{2}C^a{}_{bc}q^bq^c + t^a{}_is^i + r^a = 0 \\ &dr^a + C^a{}_{bc}q^br^c + t^a{}_it^i = 0 \\ &ds^i + \alpha^i{}_{aj}q^as^j - t^i = 0 \\ &dt^i + \alpha^i{}_{aj}q^at^j + \alpha^i{}_{aj}r^as^j = 0 \end{aligned}$$

This looks complicated. But it's really (except for the constants) the only thing that can be written down using only the data provided by the crossed module $(h \to g)$.

Checking that this is indeed nilpotent is - guess what - straightforward but tedious:

$$\begin{aligned} d^2q^a &= d(-\frac{1}{2}C^a{}_{bc}q^bq^c - t^a{}_is^i - r^a) \\ &= C^a{}_{bc}q^b(-\frac{1}{2}C^c{}_{de}q^dq^e - t^c{}_is^i - r^c) + t^a{}_i(\alpha^i{}_{bj}q^bs^j - t^i) + C^a{}_{bc}q^br^c + t^a{}_it^i \\ &= \frac{1}{2}C^a{}_{bc}C^c{}_{de}q^cq^dq^e \\ &\quad +(-C^a{}_{bc}t^c{}_i + t^a{}_j\alpha^j{}_{bi})q^bs^i \\ &= 0 \end{aligned}$$

$$\begin{aligned} d^2r^a &= d(-C^a{}_{bc}q^br^c - t^a{}_it^i) \\ &= C^a{}_{bc}(\frac{1}{2}C^b{}_{de}q^dq^e + t^b{}_is^i + r^b)r^c - C^a{}_{bc}q^b(C^c{}_{de}q^dr^e + t^c{}_it^i) + t^a{}_i(\alpha^i{}_{bj}q^bt^j + \alpha^i{}_{bj}r^bs^j) \\ &= (\frac{1}{2}C^a{}_{bc}C^b{}_{de} - C^a{}_{b[e}C^b{}_{d]c})q^dq^er^c \\ &\quad +(-C^a{}_{bc}t^c{}_j + t^a{}_i\alpha^i{}_{bj})r^bs^j \\ &\quad +C^a{}_{bc}r^br^c \\ &\quad +(-C^a{}_{bc}t^c{}_i + t^a{}_j\alpha^j{}_{bi})q^bt^i \\ &= 0 \end{aligned}$$

$$d^2s^i = d(-\alpha^i{}_{aj}q^as^j + t^i)$$

$$
\begin{aligned}
&= \alpha^i{}_{aj}(\frac{1}{2}C^a{}_{bc}q^bq^c + t^a{}_ks^k + r^a)s^j - \alpha^i{}_{aj}q^a(\alpha^j{}_{bk}q^bs^k - t^j) - \alpha^i{}_{aj}q^at^j - \alpha^i{}_{aj}r^as^j \\
&= (\frac{1}{2}\alpha^i{}_{aj}C^a{}_{bc} - \alpha^i{}_{bk}\alpha^k{}_{cj})q^bq^cs^j \\
&\quad +\alpha^i{}_{aj}t^a{}_ks^ks^j \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
d^2t^i &= d(-\alpha^i{}_{aj}q^at^j - \alpha^i{}_{aj}r^as^j) \\
&= \alpha^i{}_{aj}(\frac{1}{2}C^a{}_{bc}q^bq^c + t^a{}_ks^k + r^a)t^j - \alpha^i{}_{aj}q^a(\alpha^j{}_{bk}q^bt^k + \alpha^j{}_{bk}r^bs^k) \\
&\quad +\alpha^i{}_{aj}(C^a{}_{bc}q^br^c + t^a{}_kt^k)s^j + \alpha^i{}_{aj}r^a(\alpha^j{}_{bk}q^bs^k - t^j) \\
&= (\frac{1}{2}\alpha^i{}_{aj}C^a{}_{bc} - \alpha^i{}_{bk}\alpha^k{}_{cj})q^bq^ct^j \\
&\quad +(\alpha^i{}_{aj}t^a{}_k + \alpha^i{}_{ak}t^a{}_j)s^kt^j \\
&\quad +(-\alpha^i{}_{aj}\alpha^j{}_{bk} + \alpha^i{}_{ck}C^c{}_{ab} + \alpha^i{}_{bj}\alpha^j{}_{ak})q^ar^bs^k \\
&= 0
\end{aligned}
$$

All these expressions vanish after using the identities already listed in example 4.

Example 15 (non-flat 2-connection)

An fda-morphism from the fda of inner derivations of a strict Lie 2-algebra $(h \to g)$ (example 14) to the deRham fda (example 2) is the same as a g-valued 1-form A and an h-valued 2-form B on $\mathbb{R}^n$, which give rise to the g-valued 2-form

$$\beta = F_A + t(B)$$

and the h-valued 3-form

$$H = d_A B$$

satisfying

$$d_A\beta = t(H)$$

and

$$d_A H + \beta \wedge B = 0\,.$$

We may regard this as a flat 3-connection with values in inner derivations of $(h \to g)$. Alternatively, it is an arbitrary 2-connection with 2-curvature β and 3-curvature H, satisfying two Bianchi identities.

Example 16 (morphisms of $\mathrm{Inn}(h \to g)$ connections)

Given two connections (chain maps that respect the algebra structure) Φ : $\mathrm{Inn}(h \to g) \to$ deRham, recall that a 1-morphism between these is a chain homotopy ϵ which is Φ-Leibnitz. For instance, if r^a is of degree 2 as above, then

$$\epsilon\left(r^ar^br^c\right) = \epsilon\left(r^a\right)\Phi\left(r^br^c\right) + \Phi\left(r^a\right)\epsilon\left(r^b\right)\Phi\left(r^c\right) + \Phi\left(r^ar^b\right)\epsilon\left(r^c\right)\,.$$

A 2-morphism between such 1-morphisms is a Φ-Leibnitz homotopy of chain homotopies, and so on.

Let $Q = (d_{\mathrm{Inn}(h\to g)}) \oplus d_{\mathrm{dr}}$ be the differential of the direct sum complex $\mathrm{Inn}(h \to g) \oplus \Omega^\bullet_{\mathrm{dR}}$.

As above, set $\Phi(q^a) = A^a$, $\Phi(r^a) = \beta^a$, $\Phi\left(s^i\right) = B^i$ and $\Phi\left(t^i\right) = H^i$.

For 1-morphisms between Φ and Φ', set $\epsilon(q^a) = \ln g^a$, $\epsilon\left(s^i\right) = a^i$ $\epsilon(r^a) = 0$ $\epsilon\left(t^i\right) = d^i$.

Then $\Phi' - \Phi = [Q, \epsilon]$. And

$$(A' - A)^a = [Q, \epsilon](q^a) = d \ln g^a + C^a{}_{bc} A^b (\ln g^c) - t^a{}_i a^i$$

$$(B' - B)^i = [Q, \epsilon](s^i) = da^i - \alpha^i{}_{aj} (\ln g^a) B^j + \alpha^i{}_{aj} A^a a^j + d^i\,.$$

One sees that this is the first order version of

$$A' = g^{-1} A g + g^{-1} dg - t(a)$$

and

$$B' = g^{-1} B g + d_A a + a \wedge a + d\,,$$

which are the relations familiar from transitions in nonabelian gerbes. Had we not set $\epsilon(r^a) = 0$ but $\epsilon(r^a) = p^a$ then there would have been an additional degree of freedom in the first equation, leading to

$$A' = g^{-1} A g + g^{-1} dg - t(a) - p\,.$$

Let's ignore this extra freedom for the moment and concentrate again on the case $\epsilon(r^a) = 0$.

The transformation of the 2-form and 3-form curvature to first order is found to be

$$(\beta' - \beta)^a = [Q, \epsilon]\,(r^a) = -C^a{}_{bc} (\ln g^b) \beta^c - t^a{}_i d^i$$

and

$$(H' - H)^i = [Q, \epsilon]\left(t^i\right) = dd^i - \alpha^i{}_{aj} (\ln g^a) H^i + \alpha^i{}_{aj} A^a d^j - \alpha^i{}_{aj} \beta^a a^j\,.$$

This is the first order expansion of

$$\beta' = g^{-1} \beta g - t\,(d)$$

and

$$H' = g^{-1} H g + d_A d - \beta(a) \pm [a, d]\,.$$

The last term in the last equation is entirely second order in the transition data, hence cannot be seen in the above first order formalism at all. But including it makes the above indeed reproduce the transition equations for 2- and 3-form curvature known from nonabelian gerbes.