The Charged Quantum n-Particle: Kinematics and Dynamics

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Abstract

Arrow theory of n-dimensional quantum objects charged under an n-transport on their target space.

Contents

1 The concept

There is a mystery that demands to be understood:

Mystery 1 *The theory of gerbes with connection in terms of local data exhibits a lot of structural resemblance to state sum models of 2-dimensional quantum field theory.*

Why is that?

Does this point to a deeper pattern that we might want to understand?

After a little bit of reflection, I think the pattern is

a) n-Bundles with connection are naturally conceived in terms of parallel transport n-functors.

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b) Coupling these n-connections to an n-particle amounts to transgressing these n-functors to a suitable configuration space.

c) Quantizing these charged n-particles amounts to pushing the transgressed n-functors forward to a point.

From this point of view, evolution in the quantum field theory of the charged n-particle is an n-functor that is inherently obtained from the parallel transport n-functor that expresses the background field that the particle propagates in.

Both, the original parallel transport n-functor as well as the resulting quantum propagation n-functor may be locally trivialized. For the former this yields the local description of gerbe holonomy. For the latter this yields the state sum description of QFT.

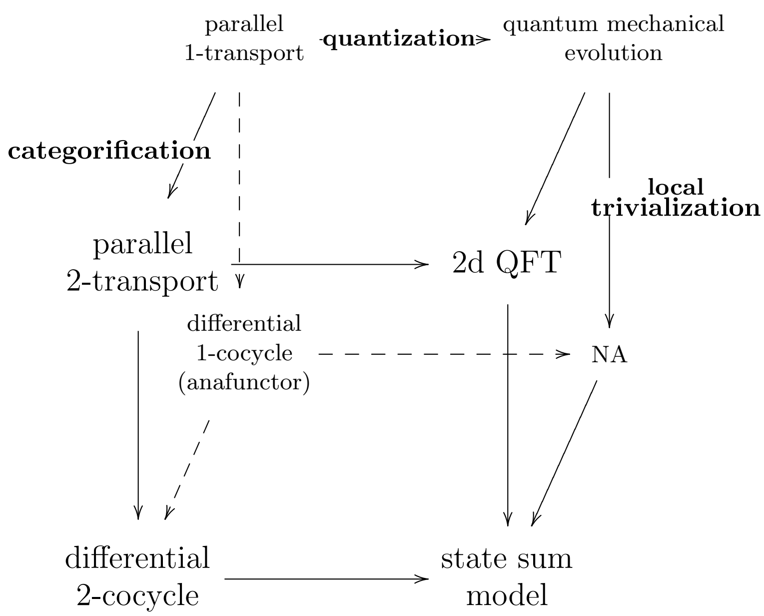


Figure 1: **Quantization, categorification and local trivialization**.

	classical data		**quantum theory**
	background field	**n-particle**	
name of **n-functor**	parallel transport	action	quantum propagation
image of **n-functor**	monodromy	classical phases	quantum amplitudes
	with values in $\mathrm{phas} = n\mathrm{Vect}$		
domain	on target space tar	on configuration space $\mathrm{conf} \subset [\mathrm{par}, \mathrm{tar}]$	on parameter space par
in **symbols**	$\mathrm{tra} : \mathrm{tar} \to \mathrm{phas}$	$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, \mathrm{phas}]$	$q(\mathrm{tra}) : \mathrm{par} \to \mathrm{phas}$
operation **in physics terms**	coupling $\curvearrowright$	quantization $\curvearrowright$	
correspondence	$\mathrm{tar} \xleftarrow{\mathrm{ev}} \mathrm{conf} \times \mathrm{par} \xrightarrow{p} \mathrm{par}$		
operation **in symbols**	$[\mathrm{tar}, \mathrm{phas}] \xrightarrow{\mathrm{ev}^*} [\mathrm{conf} \times \mathrm{par}, \mathrm{phas}] \xrightarrow{\bar{p}^*} [\mathrm{par}, \mathrm{phas}]$ coupling, quantization $\mathrm{tar} \longmapsto \mathrm{ev}^*\mathrm{tar} \longmapsto \bar{p}^*\mathrm{ev}^*\mathrm{tar}$		
elements	flat sections $e : 1 \to \mathrm{tra}$ in $\Gamma(\mathrm{tra}) = \mathrm{Hom}(1, \mathrm{tra})$	states $\psi : 1_\bullet \to q(\mathrm{tra})$ in $\mathrm{Hom}(1_*, \mathrm{tra}_*) \simeq \mathrm{Hom}(1_\bullet, q(\mathrm{tra}))$	
pairing of **elements**	holonomy		correlator

Table 1: **The charged n-particle and its quantization**. The process begins with a parallel transport n-functor tra for an n-bundle with connection, modelling a physical background field. It continues by specifying certain maps into the domain of the parallel transport and transgressing tra to the configuration space of all these maps. This models the coupling of the background field to a charged n-particle (a point particle, a string, a membrane, etc.). Finally, the transgressed n-functor may be pushed forward to a point. This yields the quantum theory of the charged n-particle coupled to the given background field.

2 Definitions

kinematics	dynamics
vector bundle $V \to X$	connection ∇
space of states	evolution operator
H	$U(t) : H \to H$
objects	morphisms
space of sections	path integral

Table 2: **Quantization** involves a kinematical and a dynamical aspect.

2.1 Kinematics

Definition 1 *A* **charged n-particle**

$$\left(\mathrm{par} \xrightarrow{\gamma\in\mathrm{conf}} \mathrm{tar} \xrightarrow{\mathrm{tra}} \mathrm{phas} \right)$$

is

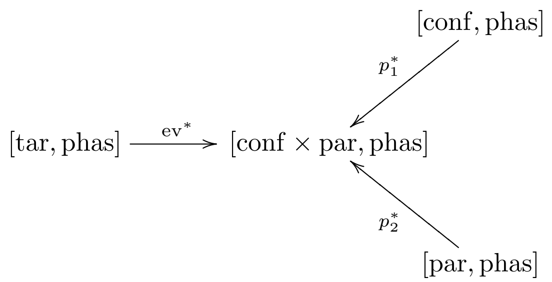
- *an* $(n-1)$*-category* par*, called* **parameter space** *and thought of as modelling the shape and internal structure of the n-particle*
- *an n-category,* tar*, called* **target space** *and thought of as modelling the space that the n-particle propagates in*
- *an n-category* $\mathrm{phas} = n\mathrm{Vect}$*, being the n-category of some notion of n-vector spaces*
- *an n-functor* $\mathrm{tra} : \mathrm{tar} \to \mathrm{phas}$*, thought of as encoding the parallel* **transport** *in an n-bundle with connection on target space*
- *a choice of sub-n-category* $\mathrm{conf} \subset [\mathrm{par}, \mathrm{tar}]$*, thought of as encoding the* **configuration space** *of the n-particle.*

Given a charged n-particle, we obtain the diagram

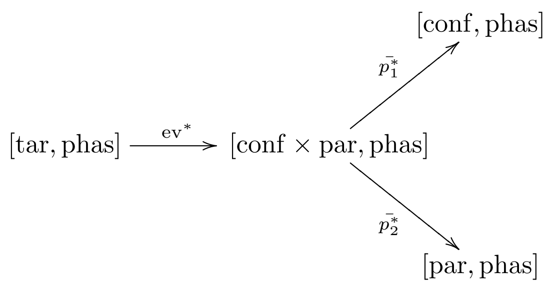
$$\begin{array}{ccccc} & & & & \mathrm{conf} \\ & & & \nearrow^{p_1} & \\ \mathrm{tar} & \xleftarrow{\mathrm{ev}} & \mathrm{conf}\times\mathrm{par} & & \\ & & & \searrow_{p_2} & \\ & & & & \mathrm{par} \end{array} ,$$

where the arrow on the left is the restriction of the canonical evaluation map $\mathrm{ev} : [\mathrm{par}, \mathrm{tar}] \times \mathrm{par} \to \mathrm{tar}$ along the inclusion $\mathrm{conf} \hookrightarrow [\mathrm{par}, \mathrm{tar}]$, and where p_1 and p_2 are the obvious projection on the first and the second factor, respectively.

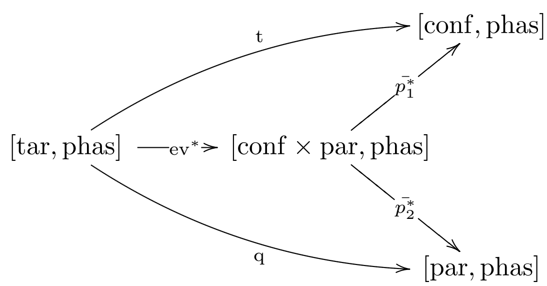
There is a corresponding diagram of pullbacks



If the morphisms on the right have adjoints, $\bar{p_1^*}$ and $\bar{p_2^*}$, respectively, we get



The composition of morphisms along the above route is **transgression**, whereas the composition along the lower route is **quantization**.



Definition 2 *Given a charged n-particle*

$$\left(\mathrm{par} \xrightarrow{\gamma \in \mathrm{conf}} \mathrm{tar} \xrightarrow{\mathrm{tra}} \mathrm{phas} \right),$$

the kinematic part of its (extended, globular) **quantum theory** *is the image*

$$q(\mathrm{tra}) : \mathrm{par} \to \mathrm{phas}$$

of tra *under this quantization map.*

Remark. It is *extended* because it is an n-functor.

It is *globular* because we think of the globular morphisms in the domain par directly as the extended cobordisms on which the QFT is defined. This means in particular that every n-cobordisms in par has the topology of an n-disk.

The value of our QFT on topologically nontrivial cobordisms will be taken to be its value on any globular cutting of that cobordisms followed by a suitable trace operation.

We then have the following terminology:

Definition 3 (sections and states) *Let* $1 : \mathrm{par} \to \mathrm{phas}$ *denote the tensor unit in the respective functor category. By abuse of notation, we also write* 1 *for its pullback to* $\mathrm{conf} \times \mathrm{par}$ *and* 1_* *for the corresponding functor from* conf *to* $[\mathrm{par}, \mathrm{phas}]$.

By definition, we have an isomorphism

$$\mathrm{Hom}_{[\mathrm{conf},[\mathrm{par},\mathrm{phas}]]}(1_*, \mathrm{tra}_*) \simeq \mathrm{Hom}_{[\mathrm{par},\mathrm{phas}]}(1, q(\mathrm{tra}))\,.$$

An object on the left

$$e : 1_* \to \mathrm{tra}_*$$

is a **section** *of the n-bundle that the n-particle couples to.*

An object on the right

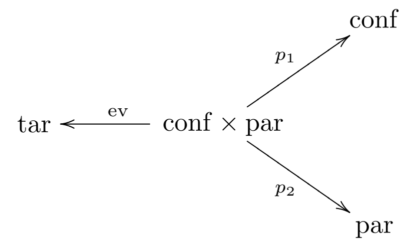
$$\psi : 1 \to q(\mathrm{tra})$$

is a **state** *of the quantum n-particle.*

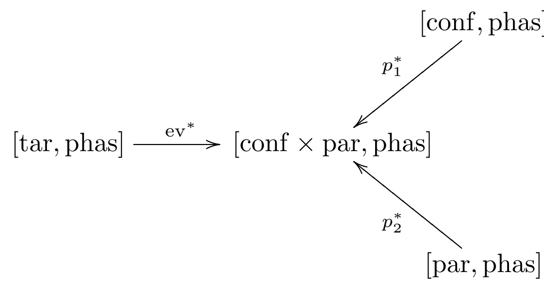
A charged n-particle...

$$\left(\mathrm{par} \xrightarrow{\gamma\in\mathrm{conf}} \mathrm{tar} \xrightarrow{\mathrm{tra}} \mathrm{phas}\right)$$

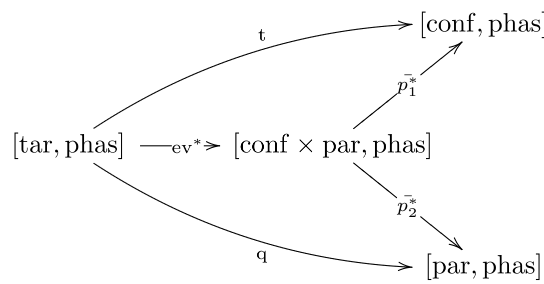
... comes with
a configuration space of maps
from its parameter space
into its target space...



... and a coupling to
a transport functor
on target space...



...which induces transport functors
on configuration space
and on parameter space...



...that are known as the
transgression
and the quantization
of the n-particle.

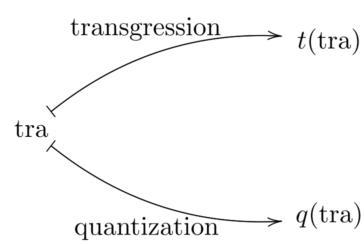


Table 3: **The story of the charged n-particle.** A drama in three acts.

2.2 Dynamics

Definition 4 *A* **worldvolume** *or* **diagram** *of a charged n-particle*

$$\left(\mathrm{par} \xrightarrow{\gamma \in \mathrm{conf}} \mathrm{tar} \xrightarrow{\mathrm{tra}} \mathrm{phas} \right)$$

is

- *an n-category* worldvol
- *a collection of n-functors*

  $$\mathrm{in}_i : \mathrm{par} \to \mathrm{worldvol}$$

  for $i = 1, 2, \cdots n_{\mathrm{in}}$
- *a collection of n-functors*

  $$\mathrm{out}_i : \mathrm{par} \to \mathrm{worldvol}$$

  for $i = 1, 2, \cdots n_{\mathrm{out}}$.

Definition 5 *Given a worldvolume of an n-particle, as above, a choice of subcategory*

$$\mathrm{hist} \subset [\mathrm{worldvol}, \mathrm{tar}]\,,$$

which is compatible with the choice of configuration space in that

$$\mathrm{in}_i^* \mathrm{hist} \simeq \mathrm{conf}$$

and

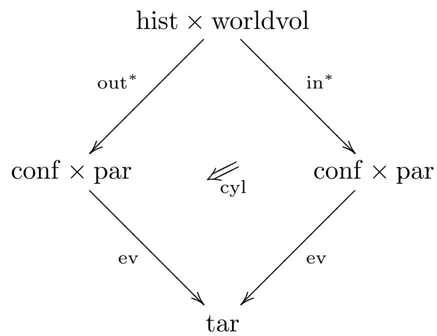
$$\mathrm{out}_i^* \mathrm{hist} \simeq \mathrm{conf}$$

for all ingoing and outgoing copies of the n-particle, is called a **space of histories***, or* **space of trajectories***, or* **space of paths** *of the n-particle, over the given worldvolume.*

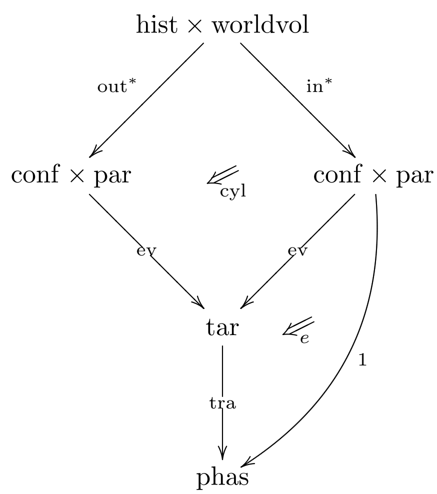
Of particular interest are worldvolumes that are **cylinders** over parameter space. We say a diagram $(\mathrm{worldvol}, \mathrm{in}, \mathrm{out})$ is a cylinder, if there is a unique transformation

$$\mathrm{par} \underset{\mathrm{out}}{\overset{\mathrm{in}}{\Downarrow}} \mathrm{worldvol}\ .$$

Notice that this induces a transformation



We can regard this as a **correspondence for states** that involves a pullback along in*, then a composition with cyl



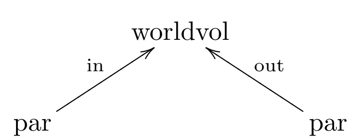
and finally a push-forward along out*.

The pullback here is canonically defined. All the subtlety is within the definition of the push-forward along out*.

For $n = 1$, the space of sections is just a 0-category (a set) and no notion of adjoint functors is available to define the push-formward.

However, we can naturally push-forward in the world of sets when we have the structure of a *measure* available.

Definition 6 (propagation by path integral) *Given an n-particle* par *and a cylinderical worldvolume*



and given that the category hist *of paths is internal to measure spaces the* **path integral propagator** *along* worldvol *is the map of sections*

$$\mathrm{Hom}(1, \mathrm{tra}) \to \mathrm{Hom}(1, \mathrm{tra})$$

defined first pulling back along in^*, *then transporting with* cyl *and the pushing forward, using the measure* $d\mu$ *on* hist, *along* out^*

$$e \mapsto \int_{(\mathrm{out}^*)^{-1}} \mathrm{cyl}^* \mathrm{tra}(\mathrm{in}^* e) \; d\mu$$

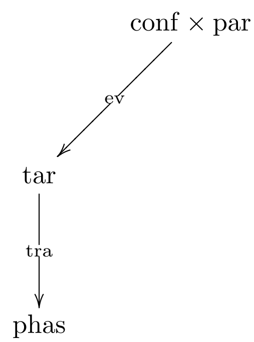
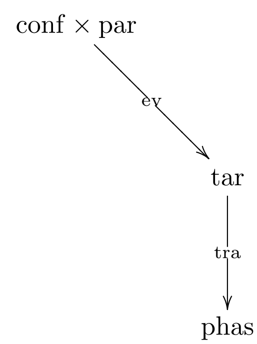
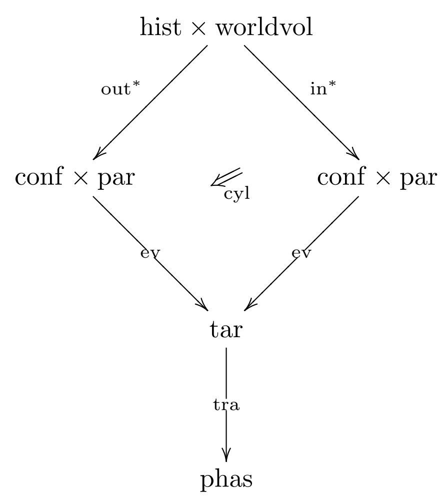
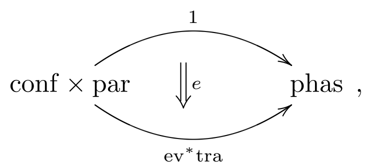
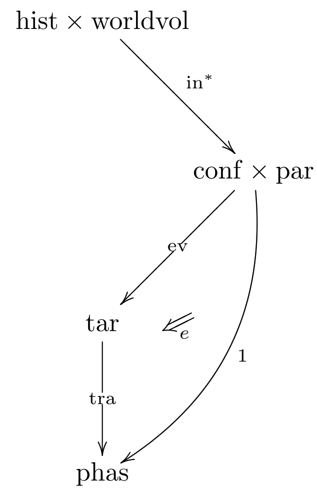
From one copy of an n-particle	
and another copy	
that are incoming and outgoing on a worldvolume cyl...	

Table 4: **A cobordism** between two copies of an n-particle...

...we can form the pull back of a state e
of the incoming n-particle,

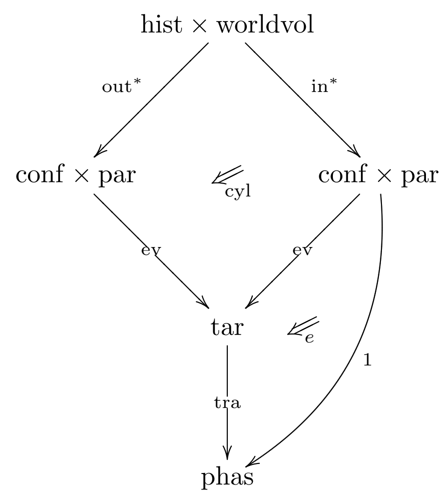


to the worldvolume,
$e \mapsto \mathrm{in}^* e$,



then transport it over the worldvolume

$\mathrm{in}^* e \mapsto \mathrm{cyl}^* \mathrm{tra}(\mathrm{in}^* e)$,



and finally push it forward
along out* to the outgoing n-particle

$$\mathrm{cyl}^* \mathrm{tra}(\mathrm{in}^* e) \mapsto e' := \int_{(\mathrm{out}^*)^{-1}} \mathrm{cyl}^* \mathrm{tra}(\mathrm{in}^* e)\, d\mu$$

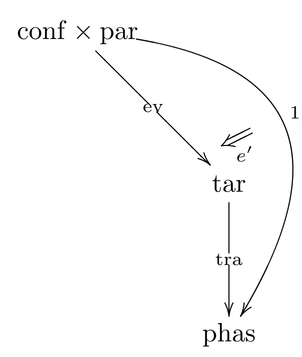


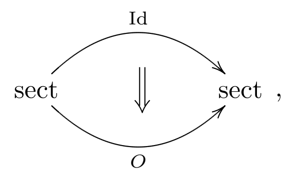
Table 5: ... allows to **propagate** incoming to outgoing states by means of a **path integral**.

2.3 Observables

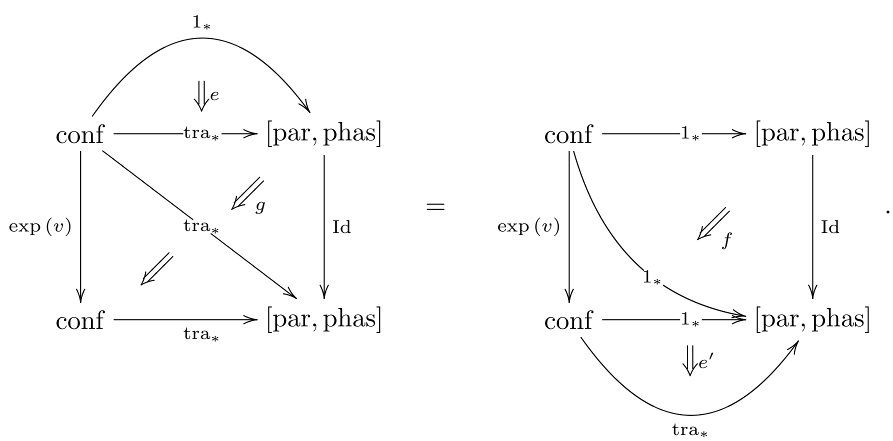
Definition 7 *The* **algebra of observables** *of an n-particle is that submonoid of the monoid of endomorphisms of the space of sections*

$$\mathrm{sect} \xrightarrow{O} \mathrm{sect}$$

which contains the elements connected to the identity in the sense that



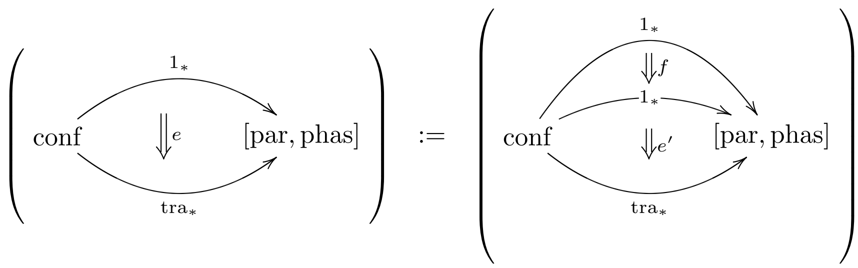
where we regard sect *as an n-category with two objects, i.e. those that are given in components by*



- The endomorphisms of the trivial functor

$$\mathrm{posobs} = \mathrm{End}(1_*)$$

act by precomposition

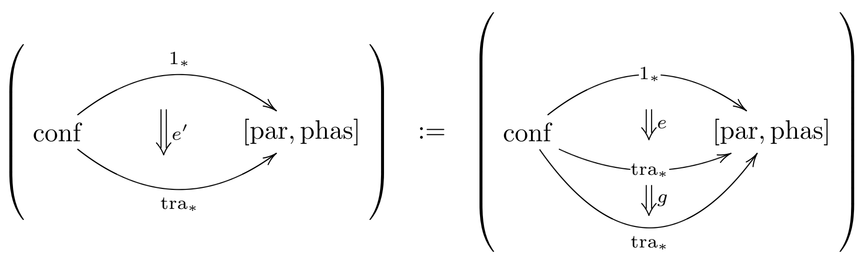


This is the monoid of **position operators**.

- The automorphisms of the transport functor

$$G = \mathrm{Aut}(\mathrm{tra}_*)$$

act by postcomposition.



This is the **group of local gauge transformations**.

- Invertible flows act as **translation operators**:

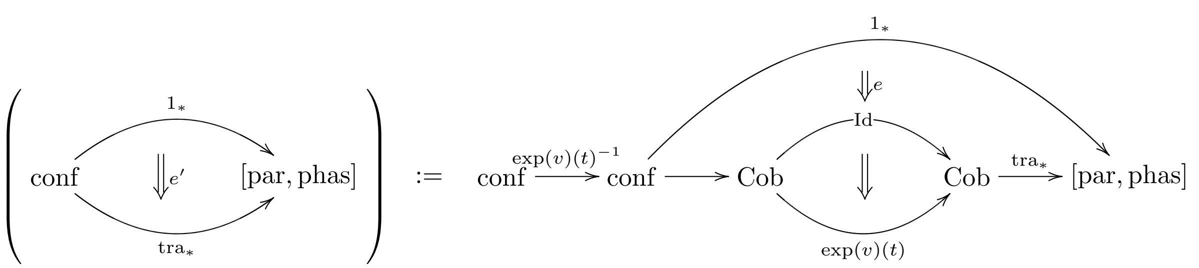


Table 6: **Monoids acting on the space of sections**, $\mathrm{sect} = \mathrm{Hom}_{[\mathrm{conf},[\mathrm{par},\mathrm{phas}]]}(1_*, \mathrm{tra}_*)$.

3 Supplementary Concepts

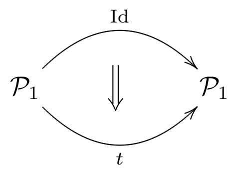
3.1 Vector Fields and Flows

We formulate the arrow theory of a **flow along a vector field**.

Let $\mathcal{P}_1$ be a category. Let

$$\mathrm{F}(\mathcal{P}_1) \subset \Sigma(\mathrm{Aut}(\mathcal{P}))$$

be the category whose single object is $\mathcal{P}_1$, and whose morphisms are natural transformations

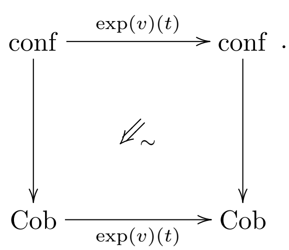


with composition being horizontal composition of natural transformations.

Definition 8 *For R some group, an R-**flow** on $\mathcal{P}_1$ is a functor*

$$\exp(v) : \Sigma(R) \to F(\mathcal{P}_1)\,.$$

An R-flow on Cob is compatible with the configuration space symmetries if



In that case, the R-flow $\exp(v)$ defines, for any $t \in R$, a **translation operator**

$$\exp(v)(t) : \mathrm{sect} \to \mathrm{sect}$$

on the space of states, which sends any section e to