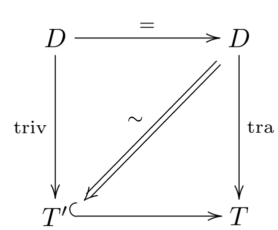
On Correlators of the 2-Particle

August 7, 2007

Contents

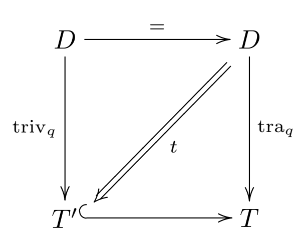
0.1 Trivialization of quantum transport n-functors

A trivialization of classical transport n-functors (parallel transport in n-bundles with connection), is an *equivalence*



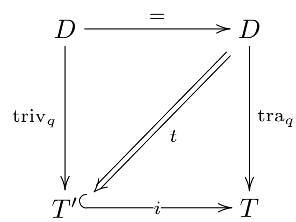
of the given n-functor tra with one which factors through an inclusion $i : T' \hookrightarrow T$.

For quantum transport n-functors, the requirement of equivalence is too strong. Rather, one has to settle for transformations

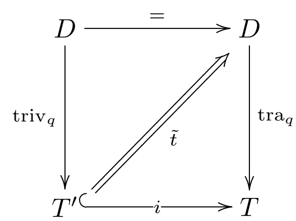


which have the property that they allow to *solve the components of* tra_q *for those of* triv_q *and* t.

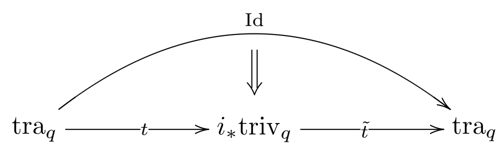
Definition 1 *We say that a morphism*



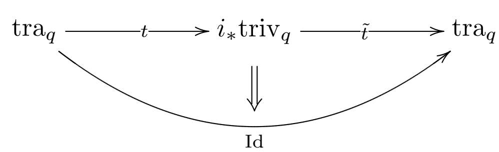
of 2*-functors is an* i*-trivialization, if there is a morphism*



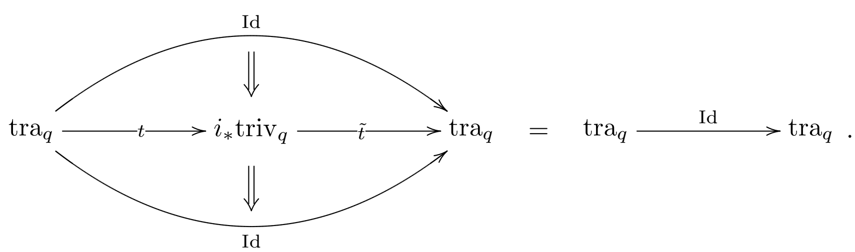
and 2-morphisms



and

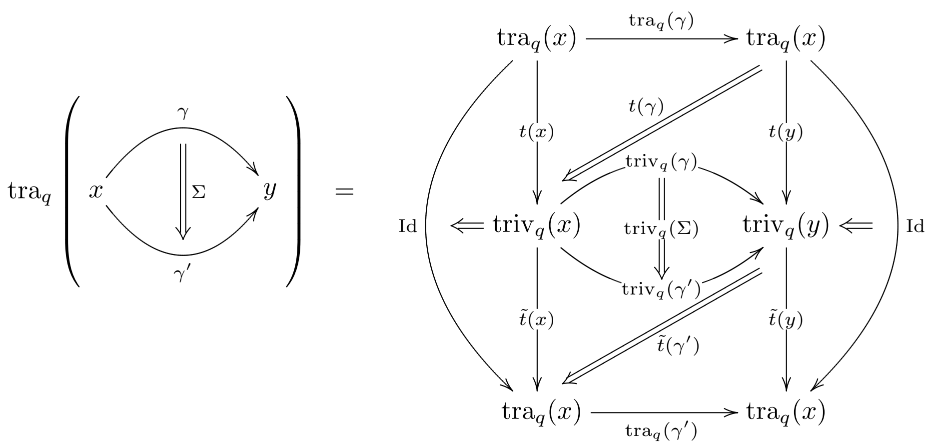


which are one-sided inverse of each other, in that



We say the choice of triv_q, t, $\tilde{t}$ and of the corresponding 2-morphisms is a *choice of* i*-trivialization* of tra_q. There is an obvious generalization of this to *local* trivializations, which however we will not need at the moment.

Proposition 1 *If* $\mathrm{tra}_q \xrightarrow{t} \mathrm{triv}_q$ *is a choice of i-trivialization of* tra_q*, then the component map of* tra_q *is expressible in terms of those of* triv_q *and the trivialization morphisms as follows:*



for all 2-morphisms Σ *in the domain* C*.*

0.2 Defect lines from trivialized bimodules

A *defect line* in 2-dimensional quantum field theory is a line labeled by an object in a monoidal category $\mathcal{C}$ which is equipped with the structure of an internal bimodule.

For our purposes it is crucial to carefully distinguish the defect line itself from the bimodule object $\dot{K}$ it is labeled by, and to distinguish that object, in turn, from the corresponding morphism K in the category $\mathrm{Bim}(\mathcal{C})$.

We now discuss how the defect line is to be thought of as arising from the $(\Sigma\mathcal{C} \hookrightarrow \mathrm{Bim}(\mathcal{C}))$ -trivialization of the functor which assigns the bimodule K to edges.

Proposition 2 *Let* $D = \mathrm{par}_2$ *be the 1-categorical interval*

$$\mathrm{par}_2 := \{ \bullet_1 \longrightarrow \bullet_2 \}$$

and let $\mathcal{C}$ *be any monoidal category. Then a sufficient condition for a 2-functor*

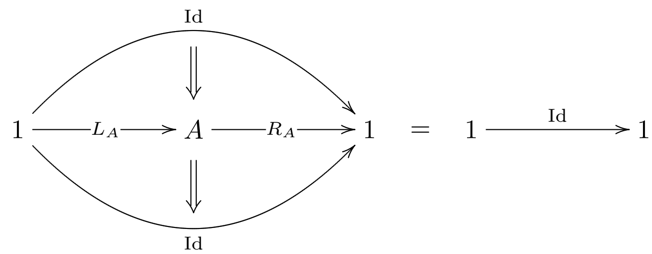
$$\mathrm{tra}_q : D \to \mathrm{Bim}(\mathcal{C})$$

to be $(\Sigma\mathcal{C} \overset{i}{\hookrightarrow} \mathrm{Bim}(\mathcal{C}))$*-trivializable is that it sends* $\bullet_1$ *and* $\bullet_2$ *to special Frobenius algebra objects in* $\mathcal{C}$*.*

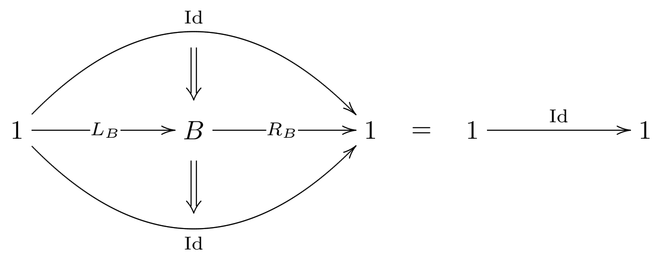
Proof. Write

$$\mathrm{tra}_q(\bullet_1 \longrightarrow \bullet_2) = A \xrightarrow{N} B$$

for the value of our 2-functor on the edge. *Choose* two special ambidextrous adjunctions



and



corresponding to the special Frobenius algebras A and B, respectively. We write

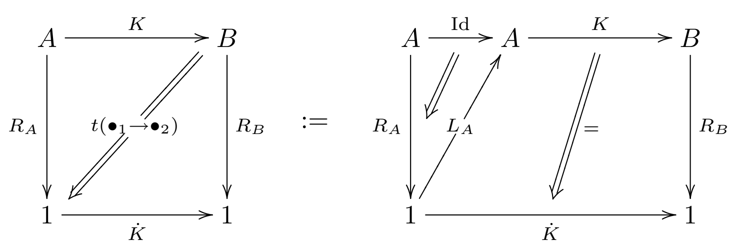
$$\dot{K}_{A,B} := \dot{K} := 1 \xrightarrow{L_A} A \xrightarrow{K} B \xrightarrow{R_B} 1 \ ,$$

for the corresponding object part of K relative to these chosen ambijunctions.

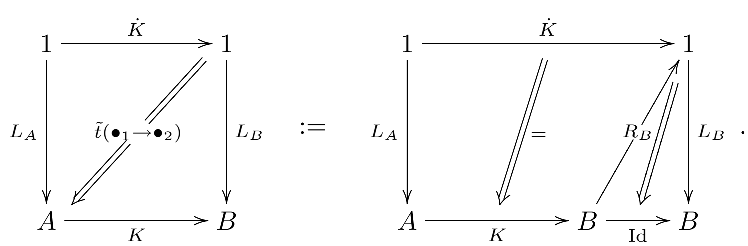
Let the i-trivial 2-functor be given by

$$\mathrm{triv}_q : (\bullet_1 \longrightarrow \bullet_2) \mapsto 1 \xrightarrow{\dot{K}} 1 \ .$$

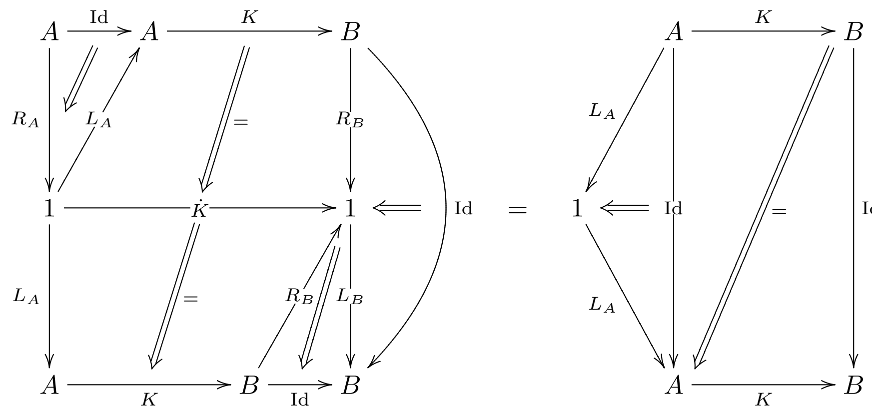
Take the component map of $t : \mathrm{tra}_q \to \mathrm{triv}_q$ to be



and that of $\tilde{t} : \mathrm{triv}_q \to \mathrm{tra}_q$ to be



Then we have



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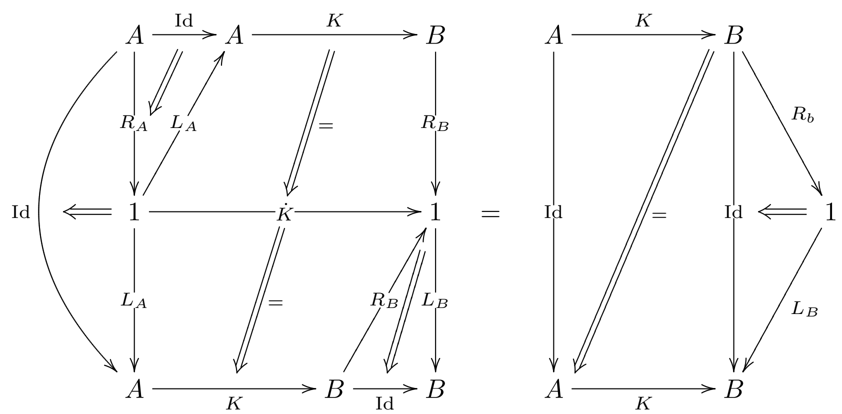
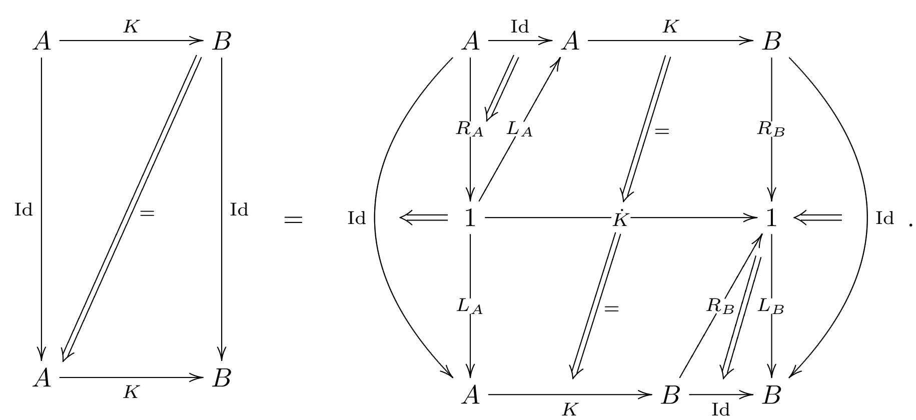


□

Corollary 1 *The identity on K may be re-expressed as*



0.3 Disk diagram from 2-functors

0.3.1 Quantum n-Particle concept formation

We establish some terminology, useful for describing the situation which we want to look at.

The open 2-particle is the 2-category $\mathrm{par}_2 = \{ \bullet_1 \longrightarrow \bullet_2 \}$. A 2-functor

$$\mathrm{tra}_q : \mathrm{par}_2 \to \mathrm{Bim}(\mathcal{C}) \subset 2\mathrm{Vect}_{\mathcal{C}}$$

is a *2-space of states* of the 2-particle, where the algebras

$$A := \mathrm{tra}_q(\bullet_1)$$

and

$$B := \mathrm{tra}_q(\bullet_2)$$

are to be thought of, under the embedding $\mathrm{Bim}(\mathcal{C}) \hookrightarrow 2\mathrm{Vect}_{\mathcal{C}}$ as the 2-vector space of states over the endpoints of the string.

A morphism

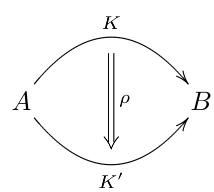
$$U : \mathrm{tra}_q \to \mathrm{tra}_q$$

or, more generally

$$U : \mathrm{tra}_q \to \mathrm{tra}'_q$$

is a *propagator* of the 2-particle over a strip ("time evolution operator").

We shall concentrate on a propagator whose component map is a bigon

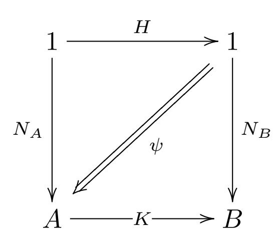
$$U(\bullet_1 \longrightarrow \bullet_2) \;:=$$


in $\mathrm{Bim}(\mathcal{C})$ with A and B special Frobenius algebras.

For $I : \mathrm{par}_2 \to \mathrm{Bim}(\mathcal{C})$ an $(\Sigma\mathcal{C} \overset{i}{\hookrightarrow} \mathrm{Bim}(\mathcal{C}))$-trivial transport, a (Schrö)dinger *state* of a the 2-particle is a morphism

$$|\psi\rangle : I \to \mathrm{tra}_q \,.$$

Its component map is hence a 2-cell in $\mathrm{Bim}(\mathcal{C})$ of the form

$$|\psi\rangle(\bullet_1 \longrightarrow \bullet_2) \;=$$


.

The modules N_A and N_B are called the *D-branes* to which the endpoints of the 2-particle in this state are attached.

We assume $\mathcal{C}$ to be rigid and hence to have duals on objects. Then the state of the 2-particle has an adjoint state

$$\langle\psi| : \mathrm{tra}_q \to I$$

given by the component map

$$\langle\psi|(\bullet_1 \longrightarrow \bullet_2) \;=\; \begin{array}{ccc} A & \xrightarrow{K} & B \\ {\scriptstyle N_A^\vee}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \psi^\dagger} & \downarrow{\scriptstyle N_B^\vee} \\ 1 & \xrightarrow[H^\vee]{} & 1 \end{array} .$$

A *Heisenberg state* is a functor

$$\phi : \mathrm{End}(\mathrm{tra}_q) \to \mathrm{End}(I)$$

which sends operators on the space of states to correlators.

The Hom-functor sends Schrödinger states to the corresponding Heisenberg states.

$$\mathrm{Hom}\left(\begin{array}{c} 1 \\ {\scriptstyle |\psi_1\rangle}\downarrow \\ \mathrm{tra}_q \end{array} \;,\; \begin{array}{c} 1 \\ {\scriptstyle \langle\psi_2|}\uparrow \\ \mathrm{tra}_q \end{array} \right) \;:\; \mathrm{End}(\mathrm{tra}_q) \to \mathrm{End}(I)\,.$$

(Here we are, for simplicity, ignoring some details, like the freedom to have different tra_q and I and the issue of whether and how to identify $H \simeq H^\vee$).

We take

$$\mathrm{End}_0(I) \overset{j}{\hookrightarrow} \mathrm{End}(I)$$

to be the sub-category whose objects are bigons (instead of rectangles) and demand that the Hom-pairing has a j-trivialization

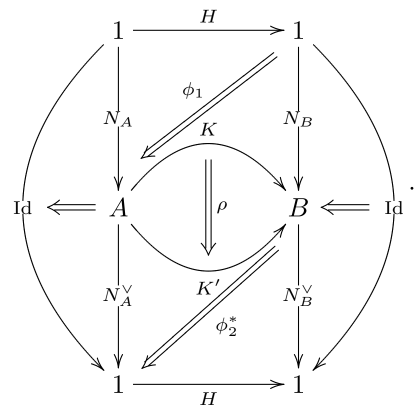
$$\begin{array}{ccc} \mathrm{End}(\mathrm{tra}_q) & \xrightarrow{=} & \mathrm{End}(\mathrm{tra}_q) \\ {\scriptstyle \mathrm{corr}(\psi_1,\psi_2)}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle b} & \downarrow{\scriptstyle \mathrm{Hom}(\psi_1,\psi_2)} \\ \mathrm{End}_0(I) & \overset{j}{\hookrightarrow} & \mathrm{End}(I) \end{array}$$

allowing to solve $\mathrm{corr}(\psi_1, \psi_2)$ for $\mathrm{Hom}(\psi_1, \psi_2)$. The trivializing morphism here is the *boundary condition* on the 2-particle.

0.3.2 The disk correlator from a pairing of 2-states

This way the *correlator* of two 2-states of the 2-particle over the strip in the above setup is a 2-cell in $\mathrm{Bim}(\mathcal{C})$ of the form

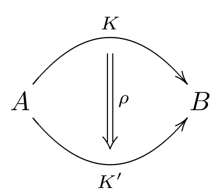
$$\mathrm{corr}(\psi_1, \psi_2) := \langle \psi_2 | U | \psi_1 \rangle_b := \quad (1)$$



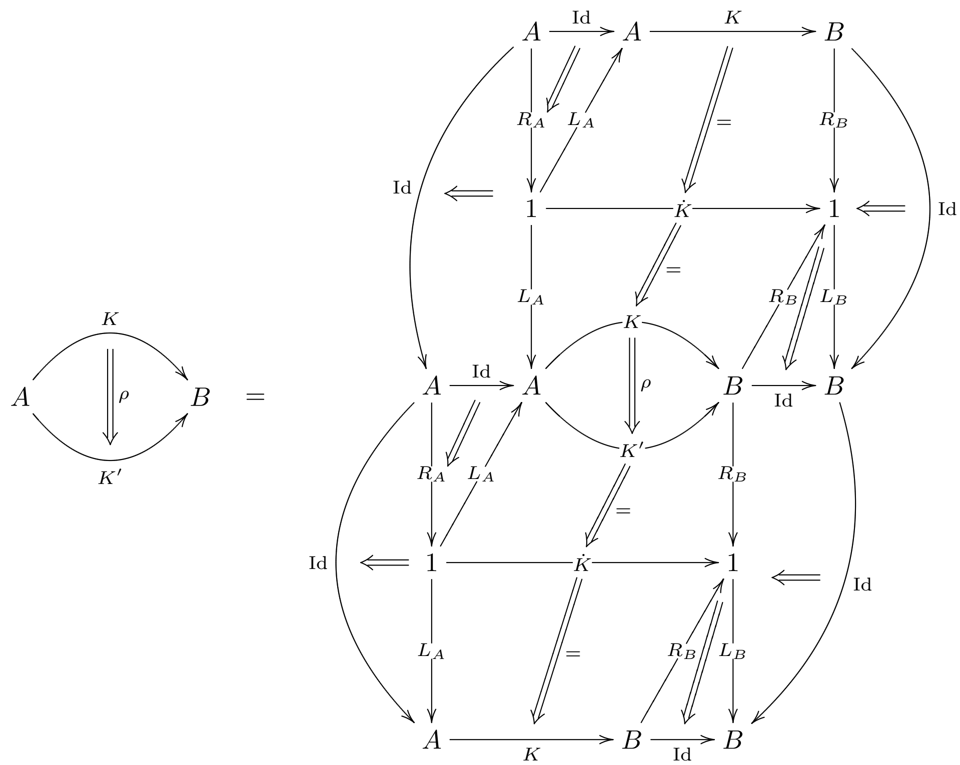
0.3.3 The local trivialization of the disk correlator

We now use local ($\Sigma\mathcal{C} \hookrightarrow \mathrm{Bim}(\mathcal{C})$)-trivialization of the correlator (1) to rewrite it identically such that its interior becomes a pasting diagram entirely in $\Sigma\mathcal{C}$. The string diagram Poincaré-dual to this globular pasting diagram is the FRS disk diagram for a disk correlator with a bulk field insertion, two boundary field insertions and a defect line.

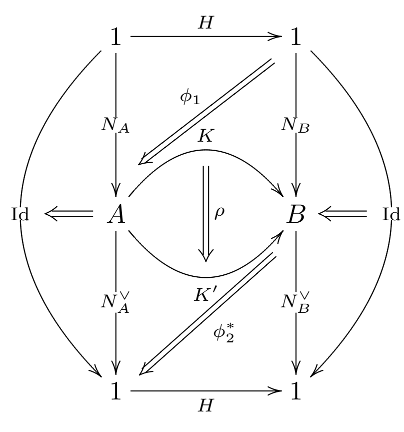
Given any morphism of bimodules



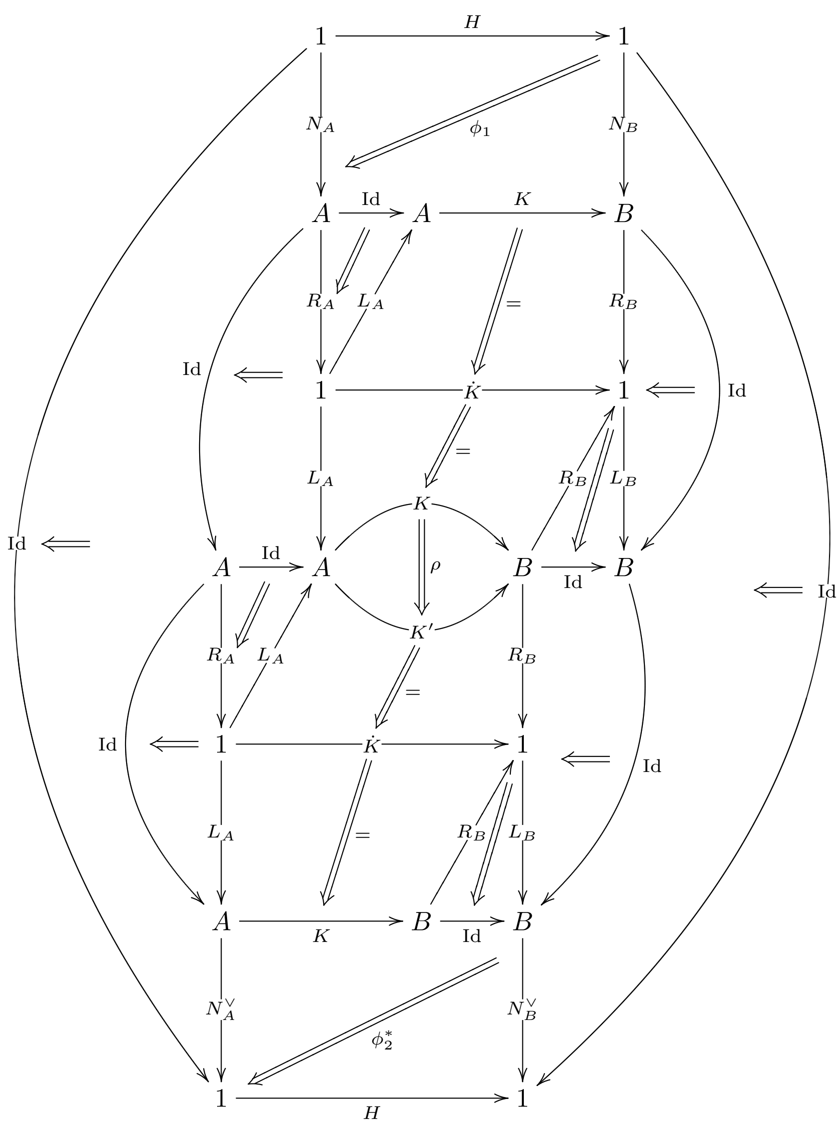
we may, using corollary 1, identically rewrite it as



After performing the same operation inside the correlator (1), one can merge the incoming and outgoing states each with one half of the inserted identity 2-cells:



$$=$$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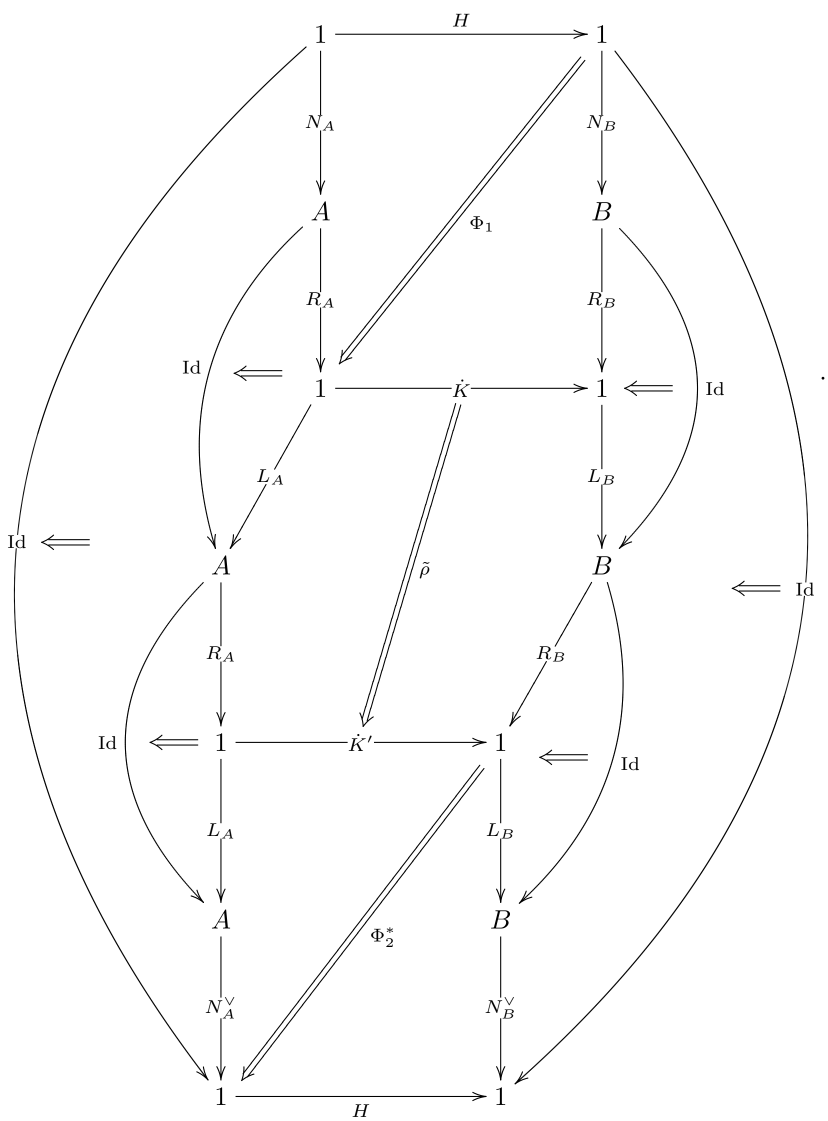

This way a pasting diagram entirely internal to $\Sigma\mathcal{C}$ is obtained. Accordingly, its Poincaré-dual string diagram is a tangle diagram in $\mathcal{C}$.