On nonabelian differential cohomology

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Preliminaries
- The idea of differential cohomology
- Overview: concept of nonabelian differential cohomology
- Overview: Chern-Simons application

Nonabelian differential cohomology
1. Thesis: (Nonabelian) Cohomology
3. Synthesis: (Nonabelian) Differential Cohomology

A concrete model: $L_\infty$-connections
- $L_\infty$-connections
- Application: Obstructing Chern-Simons $L_\infty$-connections

Literature
The idea of differential cohomology
(“Generalized”) Differential cohomology is about

- refining topology to differential geometry.
- refining cohomology classes by deRham classes approximating them.
- equipping higher generalized bundles with smooth connections.
On nonabelian differential cohomology

The idea of differential cohomology

Topology

$n$-bundle

connection

defines

cocycle in cohomology

$n$-bundle with connection

forgets

cocycle in differential cohomology

defines

cocycle in deRham cohomology

differential geometry
Overview: concept of nonabelian differential cohomology
We recall how $n$-functors classify higher $G$-bundles and hence yield (nonabelian) cohomology.
We describe how smooth $n$-functors encode differential forms.

\[ \text{Čech groupoid} \xrightarrow{\text{cocycle}} \text{classifying groupoid} \]

\[ \downarrow \]

\[ \text{paths in base space} \xrightarrow{\text{diff. forms}} \text{characteristic groupoid} \]
We merge these two aspects to describe how \textit{n-bundles with connection} encode \textbf{(nonabelian) differential cohomology}. 

\begin{center}
\begin{tikzcd}
\text{Čech groupoid} \arrow[r, \text{cocycle}] \arrow[d, \text{paths in Čech groupoid}] & \text{classifying groupoid} \\
\text{paths in base space} \arrow[d, \text{paths in base space}] & \\
\text{universal groupoid} \arrow[d, \text{characteristic groupoid}] & \\
\text{diff.forms} & \\
\end{tikzcd}
\end{center}
We propose a concrete way to deal with the fully weak $n = \infty$ situation: $L_\infty$-connections.
This yields diagrams of smooth spaces of the form:

\[
\begin{align*}
Y_{\text{vert}} \xrightarrow{A_{\text{vert}}} & S(\text{CE}(\mathfrak{g})) \\
\downarrow \quad \quad & \downarrow \\
Y \xrightarrow{(A,F_A)} & S(\text{W}(\mathfrak{g})) \\
\downarrow \quad \quad & \downarrow \\
X \xrightarrow{\{P_i(F_A)\}} & S(\text{inv}(\mathfrak{g}))
\end{align*}
\]
$L_\infty$-connections lend themselves to concrete computations. As an example we can analyze higher Chern-Simons $\infty$-connections as obstructions to lifts through String-like extension.

- Look at overview of obstructing Chern-Simons connections.
- Skip to main part: Nonabelian differential cohomology.
Overview: Chern-Simons application
- We recall $L_\infty$-algebras, which are a categorified version of ordinary Lie algebras.
- We discuss how Lie algebra cohomology generalizes to $L_\infty$-algebras by looking at their Chevalley-Eilenberg differential algebras.
- We notice that for every $L_\infty$-algebra $\mathfrak{g}$ and every degree $n$ cocycle $\mu$ on it, there is an extension

$$0 \rightarrow b^{n-1}u(1) \rightarrow \mathfrak{g}_\mu \rightarrow \mathfrak{g} \rightarrow 0$$

of $\mathfrak{g}$ by $(n-1)$-tuply shifted $u(1)$, which includes and generalizes the String extension.
- We define for arbitrary $L_\infty$-algebras $\mathfrak{g}$ a notion of higher bundles with $L_\infty$-connection and define characteristic classes for these.

We obtain the following theorem:
Let the degree \((n+1)\) cocycle \(\mu\) on the \(L_\infty\)-algebra \(g\) be in transgression with the invariant polynomial \(P\) on \(g\).

**Theorem**

The obstruction to lifting a \(g\)-connection \((A, F_A)\) to a \(g_\mu\)-connection \((A', F_{A'})\) is a \(b^n u(1)\)-connection whose single characteristic class is that of

\[ P(F_A). \]
Applied to the special case that $\mathfrak{g}$ is an ordinary Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$ and corresponding 3-cocycle $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ we get

**Corollary**

The lift of an ordinary $\mathfrak{g}$-connection $(A, F_A)$ to a String 2-connection is obstructed by a $b^2u(1)$ 3-connection whose local connection 3-form is the Chern-Simons 3-form

$$CS(A, F_A) = \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle$$

and whose single characteristic class is hence the Pontryagin class of $(A, F_A)$

$$p_1 = \langle F_A \wedge F_A \rangle.$$
Nonabelian differential cohomology

1. Thesis: (Nonabelian) Cohomology
3. Synthesis: (Nonabelian) Differential Cohomology
(Nonabelian) Cohomology
Fix

- $X$ — a manifold

and

- $Y \xrightarrow{\pi} X$ — a surjective submersion

which for the time being we assume to come from

$$Y = \bigsqcup_i U_i$$

- $Y \xrightarrow{\pi} X$ — a good open cover of $X$ by open subsets.
This $Y$ naturally gives rise to

- $Y^\bullet$ — an $\infty$-groupoid
  - objects are points in $Y$
  - 1-morphisms are points

\[
\pi_1(y) \rightarrow \pi_2(y)
\]

in $Y \times_X Y$

- 2-morphisms are points

\[
\begin{array}{ccc}
\pi_1(y) & \rightarrow & \pi_2(y) \\
\downarrow & & \downarrow \\
\pi_2(y) & \rightarrow & \pi_3(y)
\end{array}
\]

in $Y \times_X Y \times_X Y$

- etc.
On nonabelian differential cohomology

Nonabelian cohomology

Definition

We write

- $B G$ — a one-object $\infty$-groupoid

and

- $G$ — the corresponding $\infty$-group.

(nonabelian) cohomology of $X$

The $B G$-cohomology of $X$ is

$$\text{Hom}(Y^\bullet, B G).$$
Examples

- $\mathbf{B}^n U(1)$ — the strict $n$-groupoid which is trivial everywhere except in degree $n$, where it has $U(1)$ worth of $n$-morphisms

A standard fact about Čech cohomology implies that

**Fact**

$\mathbf{B}^n U(1)$-cohomology is **integral singular cohomology**:

$$\Hom(Y^\bullet, \mathbf{B}^n U(1))_\sim = H^{n+1}(X, \mathbb{Z}).$$
Examples

- $(BU) \times \mathbb{Z}$ — the strict 1-groupoid coming from $\mathbb{Z}$ copies of the stable unitary group $U$

A standard fact about $K$-theory implies

**Fact**

$(BU) \times \mathbb{Z}$-cohomology is $K$-theory:

$$\text{Hom}(Y^\bullet, (BU) \times \mathbb{Z}) \sim = K^0(X).$$
In general

- $BG$ — the classifying groupoid of any $\infty$-group $G$

Then more or less (depending on taste) by definition:

**Fact**

$BG$-cohomology is equivalence classes of principal $G$-bundles on $X$:

$$\text{Hom}(\mathcal{Y}^\bullet, BG) \sim = GBund(X) \sim.$$
Interlude

- Since nonabelian cohomology classifies higher bundles . . .
- . . . we can hope to refine cohomology by equipping these bundles with a connection.
- Given a connection we expect to obtain its curvature characteristic differential forms on base space.
Lie $\infty$-algebra valued differential forms
Ordinary Lie algebras, the useful perspective

Recall:

- $\mathfrak{g}$ — a finite dimensional Lie algebra

The bracket defines a differential

- $d_{\mathfrak{g}} : \bigwedge \mathfrak{g}^* \to \bigwedge \mathfrak{g}^*$
- $\deg(d) = +1$

The Jacobi identity is equivalent to its nilpotency

- $d^2 = 0$.

The differential graded commutative algebra (DGCA)

$$CE(\mathfrak{g}) := (\bigwedge \mathfrak{g}^*, d_{\mathfrak{g}})$$

is the Chevalley-Eilenberg algebra of $\mathfrak{g}$. 
We turn this around to get

**Definition**

Every DGCA $(\wedge \bullet g^*, d_g)$ for $g$ an $\mathbb{N}$-graded finite dimensional vector space is the Chevalley-Eilenberg algebra of an $L_\infty$-algebra $g$:

$$\text{CE}(g) := (\wedge \bullet g^*, d_g).$$

If $g$ is concentrated in the lowest $n$ degrees, we say it is a **Lie $n$-algebra**.
Examples

- \((n-1)\)-fold shifted \(u(1)\): \(\text{CE}(b^{n-1}u(1)) = (\wedge^n C, d = 0)\)

- **String-like extensions**: for \(g\) any \(L_\infty\)-algebra and for every closed element \(\mu \in \text{CE}^{n+1}(g)\), \(dg\mu = 0\), we get a new \(L_\infty\)-algebra \(g_\mu\) by “killing” \(\mu\) – these are shifted central extensions:

\[
0 \rightarrow b^{n-1}u(1) \rightarrow g_\mu \rightarrow g \rightarrow 0
\]

- **homotopy quotients and crossed modules**: for \(g \hookrightarrow h\) a normal sub \(L_\infty\)-algebra, the homotopy quotient \(h//g\) we denote by \((g \hookrightarrow h)\)

- \(\text{inn}(g)\) and \textbf{Weil algebra}**: in particular, for every \(g\) we have the Weil algebra

\[
\mathcal{W}(g) := \text{CE}(\text{inn}(g)) := \text{CE}(g \xrightarrow{Id} g).
\]
Lie $\infty$-algebra valued forms

**Definition**

For $Y$ a manifold and $g$ an $L_\infty$-algebra, the $g$-valued forms on $Y$ are

$$\Omega^\bullet(Y, g) := \text{Hom}_{DGCA}(W(g), \Omega^\bullet(Y)).$$

The *flat* $g$-valued forms are

$$\Omega^\bullet_{\text{flat}}(Y, g) := \text{Hom}_{DGCA}(CE(g), \Omega^\bullet(Y)).$$
On nonabelian differential cohomology

### Lie ∞-algebra valued differential forms

#### Definition

Examples

- For $g$ an ordinary Lie algebra, this reproduces the ordinary notion of $g$-valued 1-forms.
- $b^{n-1}u(1)$-valued forms are just ordinary $n$-forms:

\[
\begin{align*}
\text{CE}(b^{n-1}u(1)) & \hookrightarrow W(b^{n-1}u(1)) \\
(A, dA = 0) & \hookrightarrow (A \in \Omega^n(Y), F_A = dA) \\
\Omega^\bullet(Y) & = \Omega^\bullet(Y)
\end{align*}
\]
For $g$ an ordinary Lie algebra, $\mu$ an $(n+1)$-cocycle, $g_\mu$-valued forms are essentially Chern-Simons forms:

\[ CE(g_\mu) \hookrightarrow W(g_\mu) \]

\[ A \in \Omega^1_{\text{flat}}(Y, g), B \in \Omega^n(Y) \]

\[ dB = \mu(A) \]

\[ \Omega^\bullet(Y) = \Omega^\bullet(Y) \]
We can integrate $L_\infty$-algebras to Lie $\infty$-groupoids.

**smooth classifying spaces**

- A smooth space is (for us, here) a sheaf on manifolds.
- The smooth classifying space of $g$-valued forms is

\[
S(W(g)) : U \mapsto \text{Hom}_{\text{DGCA}}(W(g), \Omega^\bullet(U))
\]

**Fact: $g$-valued forms are smooth maps into $S(W(g))$**

\[
\text{Hom}_{\text{smth.} \text{Space}}(Y, S(W(g))) \simeq \Omega^\bullet(Y, g)
\]

\[
\text{Hom}_{\text{smth.} \text{Space}}(Y, S(\text{CE}(g))) \simeq \Omega_{\text{flat}}^\bullet(Y, g)
\]
Path $n$-groupoids

**Definition: strict path $n$-groupoids**

Write $\mathcal{P}_n(Y)$ for the strict globular fundamental $n$-groupoid of $Y$: $k$-morphisms are *thin*-homotopy classes of globular $k$-paths.

Write $\Pi_n(Y)$ for $\mathcal{P}_n(Y)$ with full homotopy divided out for $n$-paths.
On nonabelian differential cohomology

- Lie $\infty$-algebra valued differential forms
- Integration

Paths in the $L_\infty$-algebra classifying space

**Definition**

The Lie $n$-group $G$ integrating $\mathfrak{g}$ is $B G := \Pi_n(S(\mathbf{CE}(\mathfrak{g})))$.

For instance, for $\mathfrak{g}$ an ordinary Lie algebra, and $G$ the simply connected Lie group integrating it, the fact that

$$B G = \Pi_1(S(\mathbf{CE}(\mathfrak{g})))$$

corresponds to the integration method of Lie algebras in terms of equivalence classes of $\mathfrak{g}$-valued 1-forms on the interval.
Smooth 2-Functors and differential forms

**Theorem:** ([5], see also [1])

Let $G$ be a strict Lie 2-group and $\mathfrak{g}$ its Lie 2-algebra. Then smooth 2-functors from 2-paths into $BG$ correspond to $\mathfrak{g}$-valued forms

$$\text{Hom}_{s2Grpd}(\Pi_2(Y), BG) = \Omega^\bullet_{\text{flat}}(Y, \mathfrak{g}) = \text{Hom}_{sSpace}(Y, S(CE(\mathfrak{g})))$$

$$\text{Hom}_{s2Grpd}(\mathcal{P}_2(Y), BG) = \Omega^\bullet_{\text{fake-flat}}(Y, \mathfrak{g}) \subset \text{Hom}_{sSpace}(Y, S(W(\mathfrak{g}))).$$

...and using [2]

$$\text{Hom}_{s3Grpd}(\Pi_3(Y), BE(G)) = \Omega^\bullet(Y, \mathfrak{g}) = \text{Hom}_{sSpace}(Y, S(W(\mathfrak{g}))).$$
Remark

If set up correctly, this statement should generalize to any Lie $n$-algebra and the Lie $n$-group $G$ integrating it

\[
\text{Hom}_{\infty\text{-Grpd}}(\Pi_\infty(Y), B G) = \Omega^\bullet_{\text{flat}}(Y, g) = \text{Hom}_{\text{ssSpace}}(Y, S(CE(g)))
\]

\[
\text{Hom}_{\infty\text{-Grpd}}(\Pi_\infty(Y), \text{BInn}_0(G)) = \Omega^\bullet(Y, g) = \text{Hom}_{\text{ssSpace}}(Y, S(W(g)))
\]
Hence we express ordinary cohomology as well as differential forms as maps from here to there.

- Ordinary cohomology is about $n$-functors from the Čech $n$-groupoid $Y^n$.
- DeRham cohomology is about smooth $n$-functors from the path $n$-groupoid $\Pi_n(X)$.

To get differential cohomology, we notice that these two aspects merge in a natural way.
(nonabelian) differential cohomology
We shall now try to connect nonabelian cohomology with differential forms by completing diagrams of the form

\[
\begin{array}{c}
\text{integral cocycle} \\
\downarrow \\
\text{differential cocycle} \\
\downarrow \\
\Pi_n(X) - \text{differential form} \\
\end{array}
\rightarrow
\begin{array}{c}
A \\
\downarrow \\
B \\
\downarrow \\
C \\
\end{array}
\]
Example: line bundles

Recall that

- a line bundle is given by a functor

\[ \mathcal{Y} \xrightarrow{g} B U(1) \]

\[ [g] \in H^2(X, \mathbb{Z}) \]

- a closed 2-form is given by a smooth 2-functor

\[ \Pi_2(X) \xrightarrow{B} B^2 U(1) \]

\[ [B] \in H^2_{dR}(X) \]

How can these two morphisms be connected?
Definition

Let $\Pi^Y_2(X)$ be something like the codiagonal of

$$\Pi_2(Y^\bullet) = (\cdots \Pi_2(Y[3]) \xrightarrow{\sim} \Pi_2(Y[2]) \xrightarrow{\sim} \Pi_2(Y) ),$$

the simplicial 2-groupoid of 2-paths in the cover.

(Details in [6] and for $n = 1$ in [4]).

This 2-groupoid is generated from

- 2-paths in $Y$;
- jumps within a fiber;

modulo some relations.
Theorem (corollary of [4] and [5])

Line bundles with connection are equivalent to diagrams

\[
\begin{array}{ccc}
\prod_2^Y (X) & \xrightarrow{(g,A,F_A)} & BEU(1) \\
\downarrow & & \downarrow \\
\prod_2 (X) & \xrightarrow{F_A} & BBU(1)
\end{array}
\]

\[ [g] \in H^2(X, \mathbb{Z}) \]

\[ [g, A] \in \bar{H}^2(X, \mathbb{Z}) \]

\[ [F_A] \in H^2_{dR}(X) \]

Here \( EU(1) := \text{INN} U(1) = (U(1) \to U(1)) \) is the strict 2-group arising as the homotopy quotient of the identity on \( U(1) \).
While details haven’t been written up, it is clear that the same holds true for all $n$:

Abelian $(n-1)$-gerbes with connection, $n$th Deligne cohomology, degree $n$ Cheeger-Simons differential characters are all equivalent to diagrams

\[ \begin{array}{ccc}
Y \bullet \ar[r]^g & B^n U(1) \\
\Pi^Y_{n+1}(X) \ar[r]^{(g,A,F_A)} & BEB^{n-1} U(1) \\
\Pi_{n+1}(X) \ar[r]^{F_A} & BB^n U(1) \\
\end{array} \]

\[ [g] \in H^{n+1}(X, \mathbb{Z}) \]

\[ [g, A] \in \bar{H}^{n+1}(X, \mathbb{Z}) \]

\[ [F_A] \in H^{n+1}_{dR}(X) \]
On nonabelian differential cohomology

Differential cohomology

Example: non-abelian differential 2-coycles

Nonabelian gerbes with connection

We can in principle use any other $n$-group.

Theorem ([6], see also [1])

Let $G = \text{AUT}(H)$ be a strict automorphism Lie 2-group of a Lie group $H$. Smooth 2-functors

$$\mathcal{P}^Y_2(X) \longrightarrow \mathbf{B}G$$

are the same as the nonabelian differential cocycles on $H$-gerbes described in [Breen], with vanishing fake curvature.

As before, we get rid of the fake flatness constraint by mapping into $\mathbf{B}E_G := \mathbf{B}\text{INN}_0(G)$ instead...
Consequence of [6] and [2]

The full differential cocycles of nonabelian 2-bundles come from diagrams of smooth 3-functors

\[ \begin{align*}
Y^\bullet & \xrightarrow{g} B G \\
\Pi_3^Y(X) & \xrightarrow{(g,A,F_A)} B E G
\end{align*} \]

- More examples: String connections
- Skip further examples.
String 2-bundles with connection

Another interesting choice is

**Definition**

For $\mathfrak{h}$ a semisimple Lie algebra and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle, the Lie 3-algebra $\mathfrak{g}_\mu$ integrates, in particular, to a strict Lie 2-group

$$G = \text{String}(H)$$

of the simple, compact, simply connected Lie group $H$ [BCSS].

This leads to String 2-connections.

**Consequence of [BCSS], [BBK] [BaezStevenson]**

Differential $\text{String}(H)$-cocycles describe String bundles with connection.
To extract the characteristic forms of such a nonabelian differential cocycle we still need to complete the bottom part of this diagram.

\[
\begin{array}{ccc}
\mathcal{Y}^\bullet & \xrightarrow{g} & \mathcal{B}G \\
\downarrow & & \downarrow \\
\Pi_3^\mathcal{Y}(X) & \xrightarrow{(g,A,F_A)} & \mathcal{B}\mathcal{E}G \\
\downarrow & & \downarrow \\
\Pi_{n+1}(X) & \rightarrow & ??
\end{array}
\]

We expect “?? = \(\mathcal{B}\mathcal{B}G\)”, which however only exists if \(G\) is sufficiently abelian.
But there are abelian $\infty$-groups approximating $BBG$: the *rational cohomology of $B|G|$.*
Therefore the right answer for general Lie $n$-group $G$ should be

**Definition**

A nonabelian differential $G$-cocycle on $X$ is a diagram of smooth $\infty$-functors

\[ Y \Rightarrow g \Rightarrow B G \]

\[ \Pi_\infty(Y) \Rightarrow (g, A, F_A) \Rightarrow B \prod_i B^{n_i-1} U(1) \]

where $n_i$ is the degree of the $i$-th nontrivial rational cohomology group of $B|G|$, the degree of the $i$-th invariant polynomial of $G$. 
view two examples for characteristic groupoids
proceed towards $L_\infty$-connections
Example

Take $G$ an ordinary compact Lie group. Then $H^\bullet(BG, \mathbb{R}) = \text{inv}(g)$ is generated from the invariant polynomials $P_i$ on $g$ and hence

$$
\begin{array}{ccc}
Y^\bullet & \xrightarrow{g} & BG \\
\downarrow & & \downarrow \\
\Pi_2^Y(X) & \xrightarrow{(g,A,F_A)} & BEG \\
\downarrow & & \downarrow \\
\Pi_\infty(X) & \xrightarrow{P_i(F_A)} & B \prod_i B^{n_i-1} U(1) \\
\end{array}
$$

$G$-bundle  
$G$-connection  
characteristic forms
Example

Take $G = \text{String}(H)$ the strict String 2-group of a compact, simple, simply connected Lie group $H$. Then by [BaezStevenson]

\[ H^\bullet(B|G|, \mathbb{R}) = \text{inv}(g)/\langle P \rangle, \]

where $P$ is the suitably normalized Killing form on $H$.

Hence the characteristic forms of $\text{String}(H)$ 2-bundles are those of the underlying $H$-bundles, but without the first Pontryagin class.

\[ \begin{array}{ccc}
\mathcal{Y}^\bullet & \xrightarrow{g} & B\text{String}(H) \\
\downarrow & & \downarrow \\
\prod^\mathcal{Y}(X) & \xrightarrow{(g,A,F_A)} & BE\text{String}(H) \\
\downarrow & & \downarrow \\
\prod_\infty(X) & \xrightarrow{P_i(F_A)} & B\prod_i B^{n_i-1}U(1)
\end{array} \]

String 2-bundle

String connection

characteristic forms
To make progress with understanding how to realize that in detail, it is useful to make the

**Observation**

If the fibers of $Y$ are $n$-connected, then $Y^\bullet \simeq \Pi^\text{vert}_n(Y)$ and hence we **should** be able to use path groupoids for all *domains*.

\[
\begin{align*}
\Pi^\text{vert}_\infty(Y) & \quad \xrightarrow{g} \quad BG \\
\Pi_\infty(Y) & \quad \xrightarrow{(g,A,F_A)} \quad B\oplus G \\
\Pi_\infty(X) & \quad \xrightarrow{F_A} \quad B\prod_i B^{n_i-1}U(1)
\end{align*}
\]
What full nonab. differential cohomology should be like

But recalling the integration theory of Lie $n$-algebras, we know that we should also be able to use path groupoids for all codomains.
Finally then, morphisms between path groupoids should be just morphisms of the underlying spaces. This leads us to study the following objects:
Definition: $L_\infty$-connection ([3])

For $g$ an $L_\infty$-algebra and $X$ a smooth space, a (generalized Cartan-Ehresmann) $g$-connection on $X$ is

- a choice of smooth surjection $Y \rightarrow X$
- a diagram

\[
\begin{array}{ccc}
Y_{\text{vert}} & \xrightarrow{A_{\text{vert}}} & S(CE(g)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{(A,F_A)} & S(W(g)) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\{P_i(F_A)\}} & S(W(g)_{\text{basic}})
\end{array}
\]

connection data

characteristic forms of smooth spaces.
Example: ordinary Cartan-Ehresmann connections

For $G$ an ordinary compact Lie group and $\mathfrak{g}$ its ordinary Lie algebra, let $Y = P \to X$ by a principal $G$-bundle. A connection on $P$ is given by a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ on $Y = P$ satisfying two conditions which say that the diagram

\[
\begin{array}{ccc}
P_{\text{vert}} & \xrightarrow{A_{\text{vert}}} & S(\text{CE}(\mathfrak{g})) \\
\downarrow & & \downarrow \\
P & \xrightarrow{(A, F_A)} & S(\text{W}(\mathfrak{g})) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\{P_i(F_A)\}} & S(\text{W}(\mathfrak{g})_{\text{basic}})
\end{array}
\]

connection data

Notice that $\text{W}(\mathfrak{g})_{\text{basic}} = H^\bullet(BG, \mathbb{R})$ is indeed the algebra of invariant polynomials on $\mathfrak{g}$. 
Application: Obstructing Chern-Simons connections
Let $g$ be an ordinary Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$ and let $\mu = \langle \cdot [\cdot, \cdot] \rangle$ the corresponding cocycle.

**Definition**

The Chern-Simons 3-bundle (CS 2-gerbe) of a $g$-bundle with connection is a $b^3 u(1)$-connection whose characteristic 4-class is the Pontrjagin 4-class

$$P = \langle F_A \wedge F_A \rangle$$

of the $g$-bundle.

**Theorem**

Chern-Simons 3-bundles are the obstructions to lifting $g$-bundles to String 2-bundles, i.e. to $g_\mu$-2-bundles.
One computes this obstruction in a systematic manner by first lifting into the weak cokernel of

$$(b^{n-1}u(1) \to g_\mu),$$

which is always possible, and the projecting out the shifted copy

$$(b^{n-1}u(1) \to g_\mu) \to b^n u(1)$$

which contains the failure of the potential lift to just $g_\mu$. Applying this procedure to the diagram describing a $g$-connection as a whole yields...
On nonabelian differential cohomology

$L_\infty$-connections

The computation

$CE(b^n u(1)) \leftarrow CE(b^{n-1} u(1) \rightarrow g_\mu)$

$CS(A)_{vert}$

$W(b^n u(1))$
By chasing the generators of $W(b^n u(1))$ through this diagram one obtains the claimed result.
Literature

- Generalized differential cohomology
- 2-Bundles and String 2-Group
- Connections on nonabelian gerbes
- $n$-Transport
Differential cohomology

1. Daniel S. Freed; *Dirac Charge Quantization and Generalized Differential Cohomology*, [arXiv:hep-th/0011220]

2-Bundles and String 2-Group

- Baas, Böckstedt, Kro
Connections on nonabelian gerbes

$n$-Transport


