On nonabelian differential cohomology

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Plan Hallo

Preliminaries

- The idea of differential cohomology
- Overview: concept of nonabelian differential cohomology
- Overview: Chern-Simons application

Nonabelian differential cohomology

1. Thesis: (Nonabelian) Cohomology
2. Anti-Thesis: (Nonabelian) Differential Forms
3. Synthesis: (Nonabelian) Differential Cohomology

A concrete model: L_∞-connections

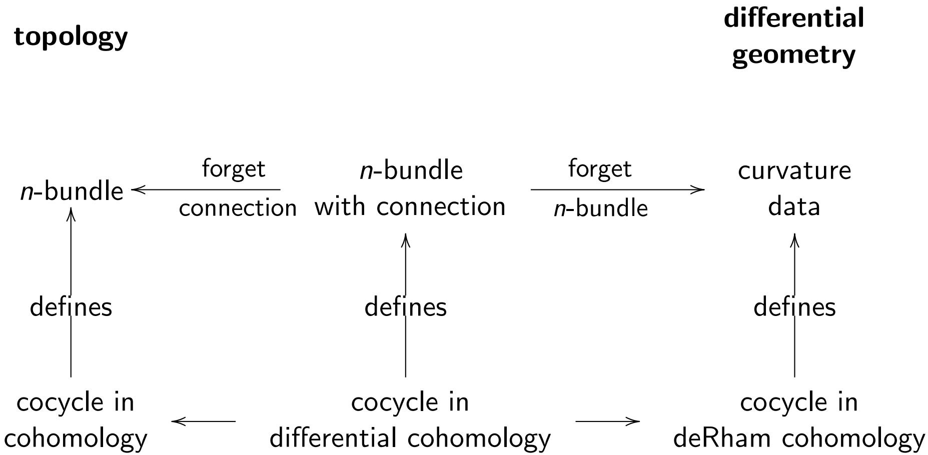
- L_∞-connections
- Application: Obstructing Chern-Simons L_∞-connections

Literature

The idea of differential cohomology

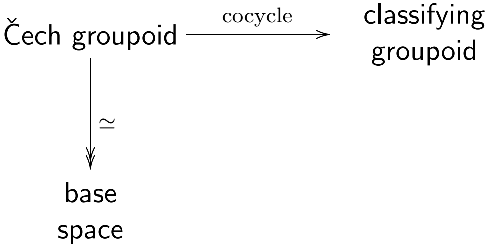
("Generalized") Differential cohomology is about

- refining topology to differential geometry.
- refining cohomology classes by deRham classes approximating them.
- equipping higher generalized bundles with smooth connections.


topology
differential geometry
n-bundle
forget connection
n-bundle with connection
forget n-bundle
curvature data
defines
defines
defines
cycle in cohomology
cycle in differential cohomology
cycle in deRham cohomology

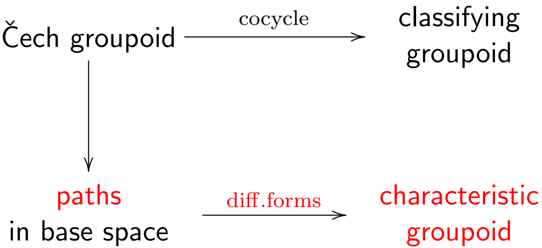
Overview: concept of nonabelian differential cohomology

We recall how *n*-functors



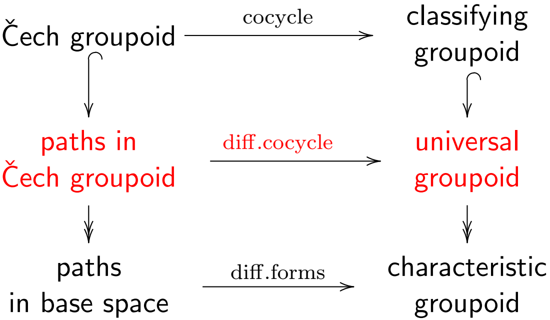
classify higher *G*-bundles and hence yield **(nonabelian) cohomology**.

We <u>describe</u> how *smooth n*-functors



encode **differential forms**.

We merge these two aspects to describe how *n-bundles with connection*
encode **(nonabelian) differential cohomology**.



We propose a concrete way to deal with the fully weak $n = \infty$ situation: L_∞-connections.

This yields diagrams of **smooth spaces** of the form

$$
\begin{array}{ccc}
Y_{\mathrm{vert}} & \xrightarrow{A_{\mathrm{vert}}} & S(\mathrm{CE}(\mathfrak{g})) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{(A, F_A)} & S(\mathrm{W}(\mathfrak{g})) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\{P_i(F_A)\}} & S(\mathrm{inv}(\mathfrak{g}))
\end{array}
$$

L_∞-connections lend themselves to concrete computations.
As an example we can analyze higher Chern-Simons ∞-connections as obstructions to lifts through String-like extension.

- Look at overview of obstructing Chern-Simons connections.
- Skip to main part: Nonabelian differential cohomology.

Overview: Chern-Simons application

- We recall L_∞-algebras, which are a categorified version of ordinary Lie algebras.
- We discuss how Lie algebra cohomology generalizes to L_∞-algebras by looking at their Chevalley-Eilenberg differential algebras.
- We notice that for every L_∞-algebra $\mathfrak{g}$ and every degree n cocycle μ on it, there is an extension

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

  of $\mathfrak{g}$ by $(n-1)$-tuply shifted $\mathfrak{u}(1)$, which includes and generalizes the *String extension*.
- We define for arbitrary L_∞-algebras $\mathfrak{g}$ a notion of higher bundles with L_∞-connection and define characteristic classes for these.

We obtain the following theorem:

Let the degree $(n+1)$ cocycle μ on the L_∞-algebra $\mathfrak{g}$ be in transgression with the invariant polynomial P on $\mathfrak{g}$.

Theorem

The obstruction to lifting a $\mathfrak{g}$-connection (A, F_A) to a $\mathfrak{g}_\mu$-connection $(A', F_{A'})$ is a $b^n\mathfrak{u}(1)$-connection whose single characteristic class is that of

$$P(F_A)\,.$$

Applied to the special case that $\mathfrak{g}$ is an ordinary Lie algebra with bilinear invariant form $\langle\cdot,\cdot\rangle$ and corresponding 3-cocycle $\mu = \langle\cdot,[\cdot,\cdot]\rangle$ we get

Corollary

The lift of an ordinary $\mathfrak{g}$-connection (A, F_A) to a String 2-connection is obstructed by a $b^2\mathfrak{u}(1)$ 3-connection whose local connection 3-form is the Chern-Simons 3-form

$$\mathrm{CS}(A, F_A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle$$

and whose single characteristic class is hence the Pontryagin class of (A, F_A)

$$p_1 = \langle F_A \wedge F_A\rangle \,.$$

Nonabelian differential cohomology

1. Thesis: (Nonabelian) Cohomology
2. Anti-Thesis: (Nonabelian) Differential Forms
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(Nonabelian) Cohomology

Fix

- X — a manifold

and

- $\begin{array}{c} Y \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$ — a surjective submersion

which for the time being we assume to come from

- $\begin{array}{c} Y = \bigsqcup_i U_i \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$ — a good open cover of X by open subsets.

This Y naturally gives rise to

- $Y^\bullet$ — an ∞-groupoid
    - objects are points in Y
    - 1-morphisms are points

      $$\pi_1(y) \longrightarrow \pi_2(y)$$

      in $Y \times_X Y$
    - 2-morphisms are points

      $$\begin{array}{ccc} & \pi_2(y) & \\ \nearrow & \Downarrow & \searrow \\ \pi_1(y) & \longrightarrow & \pi_3(y) \end{array}$$

      in $Y \times_X Y \times_X Y$
    - etc.

We write

- $\mathbf{B}G$ — a one-object ∞-groupoid

and

- G — the corresponding ∞-group.

(nonabelian) cohomology of X

The $\mathbf{B}G$-cohomology of X is

$$\mathrm{Hom}(Y^\bullet, \mathbf{B}G)\,.$$

Examples

- $\mathbf{B}^n U(1)$ — the strict n-groupoid which is trivial everywhere except in degree n, where it has $U(1)$ worth of n-morphisms

A standard fact about Čech cohomology implies that

Fact

$\mathbf{B}^n U(1)$-cohomology is **integral singular cohomology**:

$$\mathrm{Hom}(Y^\bullet, \mathbf{B}^n U(1))_\sim = H^{n+1}(X, \mathbb{Z}) \,.$$

Examples

- $(\mathbf{B}\,U) \times \mathbb{Z}$ — the strict 1-groupoid coming from $\mathbb{Z}$ copies of the stable unitary group U

A standard fact about K-theory implies

Fact

$(\mathbf{B}\,U) \times \mathbb{Z}$-cohomology is **K-theory**:

$$\mathrm{Hom}(Y^\bullet, (\mathbf{B}\,U) \times \mathbb{Z})_\sim = K^0(X)\,.$$

Examples

In general

- $\mathbf{B}G$ — the classifying groupoid of any ∞-group G

Then more or less (depending on taste) by definition:

Fact

$\mathbf{B}G$-cohomology is equivalence classes of principal G-bundles on X:

$$\mathrm{Hom}(Y^\bullet, \mathbf{B}G))_\sim = G\mathrm{Bund}(X)_\sim \,.$$

Interlude

- Since nonabelian cohomology classifies higher **bundles** ...
- ...we can hope to refine cohomology by equipping these bundles with a **connection**.
- Given a connection we expect to obtain its curvature characteristic **differential forms** on base space.

Lie ∞-algebra valued differential forms

Ordinary Lie algebras, the useful perspective

Recall:

- $\mathfrak{g}$ — a finite dimensional Lie algebra

The bracket defines a differential

- $d_{\mathfrak{g}} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$
- $\mathrm{deg}(d) = +1$

The Jacobi identity is equivalent to its nilpotency

- $d^2 = 0$.

The differential graded commutative algebra (DGCA)

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$$

is the Chevalley-Eilenberg algebra of $\mathfrak{g}$.

L_∞ algebras

We turn this around to get

Definition

Every DGCA $(\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$ for $\mathfrak{g}$ an $\mathbb{N}$-graded finite dimensional vector space is the Chevalley-Eilenberg algebra of an L_∞-algebra $\mathfrak{g}$:

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}}) .$$

If $\mathfrak{g}$ is concentrated in the lowest n degrees, we say it is a **Lie n-algebra.**

Examples

- $(n-1)$**-fold shifted** $\mathfrak{u}(1)$: $\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) = (\wedge^\bullet(\overset{n}{c}), d = 0)$
- **String-like extensions**: for $\mathfrak{g}$ any L_∞-algebra and for every closed element $\mu \in \mathrm{CE}^{n+1}(\mathfrak{g})$, $d_{\mathfrak{g}}\mu = 0$, we get a new L_∞-algebra $\mathfrak{g}_\mu$ by "killing" μ – these are shifted central extensions:

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

- **homotopy quotients and crossed modules**: for $\mathfrak{g} \hookrightarrow \mathfrak{h}$ a *normal* sub L_∞-algebra, the homotopy quotient $\mathfrak{h}//\mathfrak{g}$ we denote by $(\mathfrak{g} \hookrightarrow \mathfrak{h})$
- $\mathrm{inn}(\mathfrak{g})$ **and Weil algebra**: in particular, for every $\mathfrak{g}$ we have the Weil algebra

$$W(\mathfrak{g}) := \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) := \mathrm{CE}(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g})\,.$$

Lie ∞-algebra valued forms

Definition

For Y a manifold and $\mathfrak{g}$ an L_∞-algebra, the $\mathfrak{g}$-valued forms on Y are

$$\Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) .$$

The *flat* $\mathfrak{g}$-valued forms are

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y)) .$$

Examples

- For $\mathfrak{g}$ an ordinary Lie algebra, this reproduces the ordinary notion of $\mathfrak{g}$-valued 1-forms.
- $b^{n-1}\mathfrak{u}(1)$-valued forms are just ordinary n-forms:

$$\begin{array}{ccc}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \twoheadleftarrow & \mathrm{W}(b^{n-1}\mathfrak{u}(1)) \\
{\scriptstyle (A,dA=0)}\Big\downarrow & & \Big\downarrow{\scriptstyle (A\in\Omega^n(Y),F_A=dA)} \\
\Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y)
\end{array}$$

Examples

- For $\mathfrak{g}$ an ordinary Lie algebra, μ an $(n+1)$-cocycle, $\mathfrak{g}_\mu$-valued forms are essentially Chern-Simons forms:

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}_\mu) & \twoheadleftarrow & \mathrm{W}(\mathfrak{g}_\mu) \\ {\scriptstyle \substack{A \in \Omega^1_{\mathrm{flat}}(Y,\mathfrak{g}),\, B \in \Omega^n(Y) \\ dB = \mu(A)}} \Big\downarrow & & \Big\downarrow \\ \Omega^\bullet(Y) & \overset{=}{\longrightarrow} & \Omega^\bullet(Y) \end{array}$$

Integration of L_∞-algebras

We can integrate L_∞-algebras to Lie ∞-groupoids.

smooth classifying spaces

- A smooth space is (for us, here) a sheaf on manifolds.
- The smooth classifying space of $\mathfrak{g}$-valued forms is $S(\mathrm{W}(\mathfrak{g})) : U \mapsto \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(U))$

Fact: $\mathfrak{g}$-valued forms are smooth maps into $S(\mathrm{W}(\mathfrak{g}))$

$$\mathrm{Hom}_{\mathrm{smth.Space}}(Y, S(\mathrm{W}(\mathfrak{g}))) \simeq \Omega^\bullet(Y, \mathfrak{g})$$
$$\mathrm{Hom}_{\mathrm{smth.Space}}(Y, S(\mathrm{CE}(\mathfrak{g}))) \simeq \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g})$$

Path n-groupoids

Definition: strict path n-groupoids

Write $\mathcal{P}_{(n)}(Y)$ for the strict globular fundamental n-groupoid of Y: k-morphisms are *thin*-homotopy classes of globular k-paths.
Write $\Pi_n(Y)$ for $\mathcal{P}_n(Y)$ with full homotopy divided out for n-paths.

Paths in the L_∞-algebra classifying space

Definition

The Lie n-group G integrating $\mathfrak{g}$ is $\mathbf{B}G := \Pi_n(S(\mathrm{CE}(\mathfrak{g})))$.

For instance, for $\mathfrak{g}$ an ordinary Lie algebra, and G the simply connected Lie group integrating it, the fact that

$$\mathbf{B}G = \Pi_1(S(\mathrm{CE}(\mathfrak{g})))$$

corresponds to the integration method of Lie algebras in terms of equivalence classes of $\mathfrak{g}$-valued 1-forms on the interval.

Smooth 2-Functors and differential forms

Theorem: ([5], see also [1])

Let G be a strict Lie 2-group and $\mathfrak{g}$ its Lie 2-algebra. Then smooth 2-functors from 2-paths into $\mathbf{B}G$ correspond to $\mathfrak{g}$-valued forms

$$\mathrm{Hom}_{\mathrm{s2Grpd}}(\Pi_2(Y), \mathbf{B}G) = \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{sSpace}}(Y, S(\mathrm{CE}(\mathfrak{g})))$$

$$\mathrm{Hom}_{\mathrm{s2Grpd}}(\mathcal{P}_2(Y), \mathbf{B}G) = \Omega^\bullet_{\mathrm{fake-flat}}(Y, \mathfrak{g}) \subset \mathrm{Hom}_{\mathrm{sSpace}}(Y, S(\mathrm{W}(\mathfrak{g}))) .$$

. . . and using [2]

$$\mathrm{Hom}_{\mathrm{s3Grpd}}(\Pi_3(Y), \mathbf{BE}(G)) = \Omega^\bullet(Y, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{sSpace}}(Y, S(\mathrm{W}(\mathfrak{g})))$$

Remark

If set up correctly, this statement should generalize to any Lie n-algebra and the Lie n-group G integrating it

$$\mathrm{Hom}_{\mathrm{s}\infty\mathrm{Grpd}}(\Pi_\infty(Y), \mathbf{B}G) = \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{sSpace}}(Y, S(\mathrm{CE}(\mathfrak{g})))$$

$$\mathrm{Hom}_{\mathrm{s}\infty\mathrm{Grpd}}(\Pi_\infty(Y), \mathbf{B}\mathrm{INN}_0(G)) = \Omega^\bullet(Y, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{sSpace}}(Y, S(\mathrm{W}(\mathfrak{g})))$$

Punchline

Hence we express ordinary cohomology as well as differential forms as maps from here to there.

- Ordinary cohomology is about n-functors from the Čech n-groupoid $Y^\bullet$.
- DeRham cohomology is about smooth n-functors from the path n-groupoid $\Pi_n(X)$.

To get differential cohomology, we notice that these two aspects merge in a natural way.

(nonabelian) *differential* cohomology

We shall now try to connect nonabelian cohomology with differential forms by completing diagrams of the form

$$
\begin{array}{ccc}
Y^\bullet & \xrightarrow{\text{integral cocycle}} & A \\
\downarrow & & \downarrow \\
?? & \xrightarrow{\text{differential cocycle}} & B \\
\downarrow & & \downarrow \\
\Pi_n(X) & \xrightarrow{\text{differential form}} & C
\end{array}
$$

Example: line bundles

Recall that

- a line bundle is given by a functor

$$Y^\bullet \xrightarrow[{[g]\in H^2(X,\mathbb{Z})}]{g} \mathbf{B}U(1)$$

- a closed 2-form is given by a smooth 2-functor

$$\Pi_2(X) \xrightarrow[{[B]\in H^2_{\mathrm{dR}}(X)}]{B} \mathbf{B}^2U(1)$$

How can these two morphisms be connected?

Definition

Let $\Pi_2^Y(X)$ be something like the codiagonal of

$$\Pi_2(Y^\bullet) = (\cdots \Pi_2(Y^{[3]}) \Rrightarrow \Pi_2(Y^{[2]}) \rightrightarrows \Pi_2(Y)),$$

the simplicial 2-groupoid of 2-paths in the cover.

(Details in [6] and for $n = 1$ in [4]).
This 2-groupoid is generated from

- 2-paths in Y;
- jumps within a fiber;

modulo some relations.

Theorem (corollary of [4] and [5])

Line bundles with connection are equivalent to diagrams

$$
\begin{array}{ccccc}
Y^\bullet & \xrightarrow{\;g\;} & \mathbf{B}U(1) & \qquad & [g] \in H^2(X,\mathbb{Z}) \quad . \\
\downarrow & & \downarrow & & \\
\Pi_2^Y(X) & \xrightarrow{\;(g,A,F_A)\;} & \mathbf{BE}U(1) & \qquad & [g,A] \in \bar{H}^2(X,\mathbb{Z}) \\
\downarrow & & \downarrow & & \\
\Pi_2(X) & \xrightarrow{\;F_A\;} & \mathbf{BB}U(1) & \qquad & [F_A] \in H^2_{\mathrm{dR}}(X)
\end{array}
$$

Here $\mathbf{E}U(1) := \mathrm{INN}\,U(1) = (U(1) \to U(1))$ is the strict 2-group arising as the homotopy quotient of the identity on $U(1)$.

While details haven't been written up, it is clear that the same holds true for all n:

Abelian $(n-1)$-gerbes with connection, nth Deligne cohomology, degree n Cheeger-Simons differential characters are all equivalent to diagrams

$$\begin{array}{ccccc}
Y^\bullet & \xrightarrow{g} & \mathbf{B}^n U(1) & \qquad & [g] \in H^{n+1}(X,\mathbb{Z}) \quad . \\
\downarrow & & \downarrow & & \\
\Pi^Y_{n+1}(X) & \xrightarrow{(g,A,F_A)} & \mathbf{BEB}^{n-1} U(1) & & [g,A] \in \bar{H}^{n+1}(X,\mathbb{Z}) \\
\downarrow & & \downarrow & & \\
\Pi_{n+1}(X) & \xrightarrow{F_A} & \mathbf{BB}^n U(1) & & [F_A] \in H^{n+1}_{\mathrm{dR}}(X)
\end{array}$$

Nonabelian gerbes with connection

We can in principle use any other *n*-group.

Theorem ([6], see also [1])

Let $G = \mathrm{AUT}(H)$ be a strict automorphism Lie 2-group of a Lie group H. Smooth 2-functors

$$\mathcal{P}_2^Y(X) \longrightarrow \mathbf{B}G$$

are the same as the nonabelian differential cocycles on H-gerbes described in [Breen], with vanishing fake curvature.

As before, we get rid of the fake flatness constraint by mapping into $\mathbf{BE}G := \mathbf{B}\mathrm{INN}_0(G)$ instead...

Consequence of [6] and [2]

The full differential cocycles of nonabelian 2-bundles come from diagrams of smooth 3-functors

$$\begin{array}{ccc} Y^\bullet & \xrightarrow{\quad g \quad} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi_3^Y(X) & \xrightarrow{(g, A, F_A)} & \mathbf{BE}G \end{array}$$

- More examples: String connections
- Skip further examples.

String 2-bundles with connection

Another interesting choice is

Definition

For $\mathfrak{h}$ a semisimple Lie algebra and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle, the Lie 3-algebra $\mathfrak{g}_\mu$ integrates, in particular, to a strict Lie 2-group

$$G = \mathrm{String}(H)$$

of the simple, compact, simply connected Lie group H [BCSS].

This leads to String 2-connections.

Consequence of [BCSS], [BBK] [BaezStevenson]

Differential $\mathrm{String}(H)$-cocycles describe String bundles with connection.

To extract the characteristic forms of such a nonabelian differential cocycle we still need to complete the bottom part of this diagram.

$$\begin{array}{ccc} Y^\bullet & \xrightarrow{\;g\;} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi_3^Y(X) & \xrightarrow{(g,A,F_A)} & \mathbf{BE}G \\ \downarrow & & \downarrow{\scriptstyle ?} \\ \Pi_{n+1}(X) & \longrightarrow & ?? \end{array}$$

We expect "$?? = \mathbf{BB}G$", which however only exists if G is sufficiently abelian.

Rational approximation

But there are abelian ∞-groups approximating $\mathbf{B}\mathbf{B}G$: the *rational cohomology of* $B|G|$.

$$
\begin{array}{ccccc}
B|G| & \xrightarrow{\simeq_{\text{rationally}}} & \prod_i K(\mathbb{Z}, n_i) & \xrightarrow{\simeq} & |\prod_i \mathbf{B}^{n_i - 1} U(1)| \\
\downarrow & & \downarrow B & & \downarrow B \\
\text{“}BBG\text{”} & \dashrightarrow & B\prod_i K(\mathbb{Z}, n_i) & \xrightarrow{\simeq} & |\mathbf{B}\prod_i \mathbf{B}^{n_i - 1} U(1)|
\end{array}
$$

Therefore the right answer for general Lie n-group G **should** be

Definition

A nonabelian differential G-cocycle on X is a diagram of smooth ∞-functors

$$
\begin{array}{ccccc}
Y^\bullet & \xrightarrow{g} & \mathbf{B}G & & \text{cocycle} \\
\downarrow & & \downarrow & & \\
\Pi_\infty^Y(X) & \xrightarrow{(g,A,F_A)} & \mathbf{BE}G & & \text{differential cocycle} \\
\downarrow & & \downarrow & & \\
\Pi_\infty(X) & \xrightarrow{\{P_i(F_A)\}} & \mathbf{B}\prod_i \mathbf{B}^{n_i-1}U(1) & & \text{characteristic forms}
\end{array}
$$

where n_i is the degree of the i-th nontrivial rational cohomology group of $B|G|$, the degree of the i-th invariant polynomial of G.

- view two examples for characteristic groupoids
- proceed towards L_∞-connections

Example

Take G an ordinary compact Lie group. Then $H^\bullet(BG,\mathbb{R}) = \mathrm{inv}(\mathfrak{g})$ is generated from the invariant polynomials P_i on $\mathfrak{g}$ and hence

$$
\begin{array}{ccccc}
Y^\bullet & \xrightarrow{\quad g \quad} & \mathbf{B}G & & G\text{-bundle} \\
\downarrow & & \downarrow & & \\
\Pi_2^Y(X) & \xrightarrow{(g, A, F_A)} & \mathbf{BE}G & & G\text{-connection} \\
\downarrow & & \downarrow & & \\
\Pi_\infty(X) & \xrightarrow{P_i(F_A)} & \mathbf{B}\prod_i \mathbf{B}^{n_i - 1}U(1) & & \text{characteristic forms}
\end{array}
$$

Example

Take $G = \mathrm{String}(H)$ the strict String 2-group of a compact, simple, simply connected Lie group H . Then by [BaezStevenson]] $H^\bullet(B|G|, \mathbb{R}) = \mathrm{inv}(\mathfrak{g})/\langle P \rangle$, where P is the suitably normalized Killing form on H.

Hence the characteristic forms of $\mathrm{String}(H)$ 2-bundles are those of the underlying H-bundles, but without the first Pontryagin class.

$$\begin{array}{ccccl} Y^\bullet & \xrightarrow{g} & \mathbf{B}\mathrm{String}(H) & & \text{String 2-bundle} \\ \downarrow & & \downarrow & & \\ \Pi_\infty^Y(X) & \xrightarrow{(g,A,F_A)} & \mathbf{BE}\mathrm{String}(H) & & \text{String connection} \\ \downarrow & & \downarrow & & \\ \Pi_\infty(X) & \xrightarrow{P_i(F_A)} & \mathbf{B}\prod_i \mathbf{B}^{n_i-1}U(1) & & \text{characteristic forms} \end{array}$$

To make progress with understanding how to realize that in detail, it is useful to make the

Observation

If the fibers of Y are n-connected, then $Y^\bullet \simeq \Pi_n^{\mathrm{vert}}(Y)$ and hence we **should** be able to use path groupoids for all *domains*

$$
\begin{array}{ccc}
\Pi_\infty^{\mathrm{vert}}(Y) & \xrightarrow{\quad g \quad} & \mathbf{B}G \\
\downarrow & & \downarrow \\
\Pi_\infty(Y) & \xrightarrow{\quad (g,A,F_A) \quad} & \mathbf{BE}G \\
\downarrow & & \downarrow \\
\Pi_\infty(X) & \xrightarrow{\quad F_A \quad} & \mathbf{B}\prod_i \mathbf{B}^{n_i-1}U(1)
\end{array}
\quad .
$$

What full nonab. differential cohomology **should** be like

But recalling the integration theory of Lie n-algebras, we know that we **should** also be able to use path groupoids for all *codomains*

$$
\begin{array}{ccc}
\Pi_\infty^{\mathrm{vert}}(Y) & \xrightarrow{\quad g \quad} & \Pi_\infty(S(\mathrm{CE}(\mathfrak{g}))) \\
\big\downarrow & & \big\downarrow \\
\Pi_\infty(Y) & \xrightarrow{(g, A, F_A)} & \Pi_\infty(S(\mathrm{W}(\mathfrak{g}))) \\
\big\downarrow & & \big\downarrow \\
\Pi_\infty(X) & \xrightarrow{\quad F_A \quad} & \Pi_\infty(\mathrm{W}(\mathfrak{g})_{\mathrm{basic}})
\end{array}
$$

What full nonab. differential cohomology **should** be like

Finally then, morphisms between path groupoids should be just morphisms of the underlying spaces.
This leads us to study the following objects:

L_∞-connections

Definition: L_∞-connection ([3])

For $\mathfrak{g}$ an L_∞-algebra and X a smooth space, a (generalized Cartan-Ehresmann) $\mathfrak{g}$-connection on X is

- a choice of smooth surjection $Y \twoheadrightarrow X$
- a diagram

$$
\begin{array}{ccc}
Y_{\mathrm{vert}} & \xrightarrow{A_{\mathrm{vert}}} & S(\mathrm{CE}(\mathfrak{g})) & \\
\downarrow & & \uparrow & \\
Y & \xrightarrow{(A, F_A)} & S(\mathrm{W}(\mathfrak{g})) & \text{connection data} \\
\downarrow & & \downarrow & \\
X & \xrightarrow{\{P_i(F_A)\}} & S(\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}) & \text{characteristic forms}
\end{array}
$$

of smooth spaces.

Example: ordinary Cartan-Ehresmann connections

For G an ordinary compact Lie group and $\mathfrak{g}$ its ordinary Lie algebra, let $Y = P \to X$ by a principal G-bundle. A connection on P is given by a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ on $Y = P$ satisfying two conditions which say that the diagram

$$\begin{array}{ccccc}
P_{\mathrm{vert}} & \xrightarrow{A_{\mathrm{vert}}} & S(\mathrm{CE}(\mathfrak{g})) & & \\
\downarrow & & \downarrow & & \\
P & \xrightarrow{(A, F_A)} & S(\mathrm{W}(\mathfrak{g})) & & \text{connection data} \\
\downarrow & & \downarrow & & \\
X & \xrightarrow{\{P_i(F_A)\}} & S(\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}) & & \text{characteristic forms}
\end{array}$$

Notice that $W(\mathfrak{g})_{\mathrm{basic}} = H^\bullet(BG, \mathbb{R})$ is indeed the algebra of invariant polynomials on $\mathfrak{g}$.

Application: Obstructing Chern-Simons connections

Let $\mathfrak{g}$ be an ordinary Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$ and let $\mu = \langle \cdot [\cdot, \cdot] \rangle$ the corresponding cocycle.

Definition

The Chern-Simons 3-bundle (CS 2-gerbe) of a $\mathfrak{g}$-bundle with connection is a $b^3\mathfrak{u}(1)$-connection whose characteristic 4-class is the Pontrjagin 4-class

$$P = \langle F_A \wedge F_A \rangle$$

of the $\mathfrak{g}$-bundle.

Theorem

Chern-Simons 3-bundles are the obstructions to lifting $\mathfrak{g}$-bundles to String 2-bundles, i.e. to $\mathfrak{g}_\mu$-2-bundles.

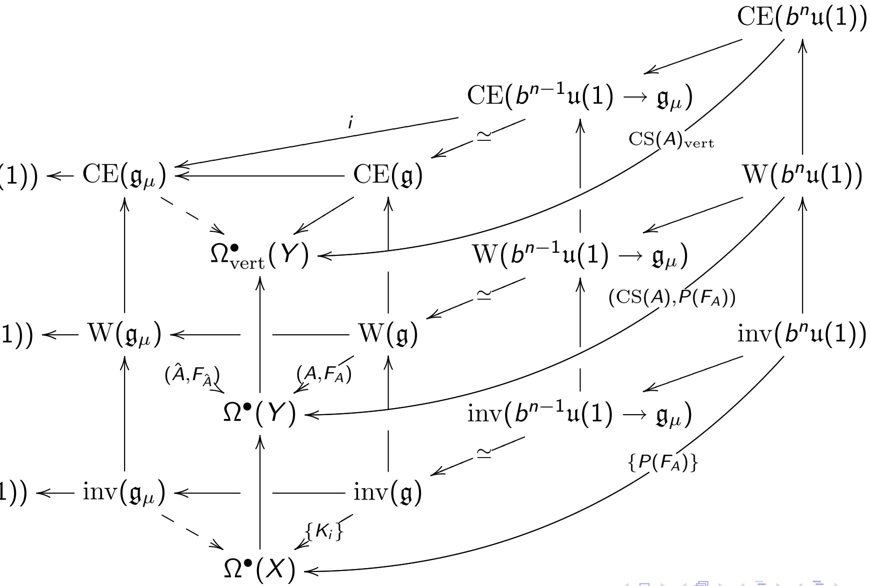
One computes this obstruction in a systematic manner by first lifting into the weak cokernel of

$$(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu)\,,$$

which is always possible, and the projecting out the shifted copy

$$(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \longrightarrow\!\!\!\!\!\rightarrow b^n\mathfrak{u}(1)$$

which contains the failure of the potential lift to just $\mathfrak{g}_\mu$. Applying this procedure to the diagram describing a $\mathfrak{g}$-connection as a whole yields...


CE(b^n u(1))
CE(b^{n-1} u(1) → g_μ)
i
≃
CS(A)_vert
(1)) ← CE(g_μ) ← CE(g)
W(b^n u(1))
Ω•_vert(Y)
W(b^{n-1} u(1) → g_μ)
≃
(CS(A), P(F_A))
1)) ← W(g_μ) ← W(g)
inv(b^n u(1))
(Â, F_Â)
(A, F_A)
Ω•(Y)
inv(b^{n-1} u(1) → g_μ)
≃
{P(F_A)}
1)) ← inv(g_μ) ← inv(g)
{K_i}
Ω•(X)

By chasing the generators of $\mathrm{W}(b^n\mathfrak{u}(1))$ through this diagram one obtains the claimed result.

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- Generalized differential cohomology
- 2-Bundles and String 2-Group
- Connections on nonabelian gerbes
- n-Transport

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