Curvature

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Abstract

We define a notion of n-curvature suitable for the general concept of n-transport.

Recall that an n**-transport**

$$\mathrm{tra} : \mathcal{P}_n \to T$$

is just an n-functor, where we think of the domain as an n-category describing n-paths of some sort.

If T is an n-category, tra is an ordinary n-functor. If T is an m-category with $m > n$, then tra may accordingly be pseudo, in that it respects composition of n-paths only up to higher coherent equivalence.

We define a notion of n-curvature precisely for those n-transport functors whose codomain is an $(n+1)$-category with at most a single $(n+1)$-morphism between any pair of n-morphisms.

Definition 1 *Let T be an $(n+1)$-category with the property that it has* at most one $(n+1)$*-morphism between any pair of n-morphisms. Then n-transport functors*

$$\mathrm{tra} : \mathcal{P}_n \to T$$

are in bijection with $(n+1)$-transport functors

$$\mathrm{curv}_{\mathrm{tra}} : \mathcal{P}_{n+1} \to T \,.$$

We say $\mathrm{curv}_{\mathrm{tra}}$ *is the n-**curvature** of* tra.

In fact, this bijection should extend to an equivalence of the respective functor categories

$$\mathrm{Hom}(\mathcal{P}_n, T) \simeq \mathrm{Hom}(\mathcal{P}_{n+1}, T) \,.$$

Recall that there is a notion of trivialization and transition of n-transport. This is defined in terms of diagrams in $\mathrm{Hom}(\mathcal{P}_n, T)$. Hence the above equivalence allows us to translate the concept of curvature also to transition data.

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Definition 2 *For any n-category T, we denote by $\tilde{T}$ the $(n+1)$-category obtained from T by declaring there to be a unique $(n+1)$-morphism for every ordered pair of parallel n-morphisms.*

Hence every transport $\mathrm{tra} : \mathcal{P}_n \to T$ has associated to it a curvature $\mathrm{curv}_{\mathrm{tra}} : \mathcal{P}_{n+1} \to \tilde{T}$. The relevance of definition 1 lies in the fact that there are $(n+1)$-categories with unique $(n+1)$-morphisms that do *not* arise this way.

Definition 3 *An n-transport* $\mathrm{tra} : \mathcal{P}_n \to \tilde{T}$ *is called* **flat** *if* $\mathrm{curv}_{\mathrm{tra}}$ *sends all $(n+1)$-morphisms to identities.*

Proposition 1 *For every n-transport* tra *with curvature,* $\mathrm{curv}_{\mathrm{tra}}$ *is a flat $(n+1)$-transport.*

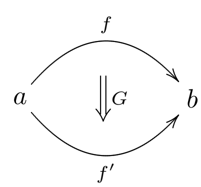
Proof. We have

$$\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}} : \mathcal{P}_{n+2} \to \tilde{\tilde{T}}\,.$$

Notice that $\tilde{\tilde{T}} = \tilde{T}$. All its $(n+2)$-morphisms are identities. □
This is a generalization of the statement known as the **Bianchi identity**.

For working with curvature, it is convenient to introduce the following notation.

Definition 4 *Let $\mathcal{P}_{n+1}$ be a strict $(n+1)$-groupoid. Let*



$$\in \mathrm{Mor}_p(\mathcal{P}_n)$$

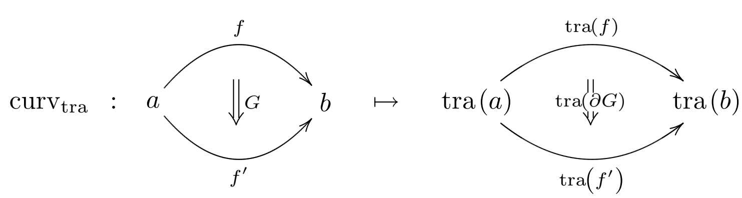
be any p-morphism for $p > 1$, with source $(p-1)$-morphism f and target $(p-1)$-morphism f'. The **boundary** *of G is the $(p-1)$-morphism*

$$\partial G \equiv f^{-1} \circ f'\,.$$

Notice that

$$\partial\partial G = \mathrm{Id}\,.$$

With this notation, we have, for G an $(n+1)$-morphism,



Prop. 1 can now be restated as saying that for all $(n+2)$-morphisms V we have

$$\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}(V) = \mathrm{curv}_{\mathrm{tra}}(\partial V) = \mathrm{tra}(\partial\partial V) = \mathrm{tra}(\mathrm{Id}) = \mathrm{Id}\,.$$

Example 1 (curvature of $\Sigma(G)$-1-transport)

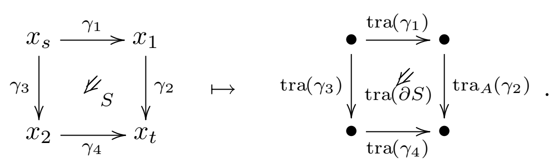
Let G be a Lie group and let X be a manifold. Let $\mathcal{P}_2(X)$ be the 2-groupoid of 2-paths in X. Let $T = \Sigma(G)$.

Smooth 1-transport $\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma(G)$ is in bijection with Lie(G)-valued 1-forms A.

The 2-category $\tilde{T}$ equals $\Sigma(\mathrm{Inn}(G))$, the suspension of the 2-group of inner automorphisms of G, coming from the crossed module $G \stackrel{\mathrm{Id}}{\to} G$.

The curvature of tra is

$$\mathrm{curv}_{\mathrm{tra}} \quad : \quad \mathcal{P}_2(X) \quad \to \quad \Sigma(\mathrm{Inn}(G))$$



Smooth 2-transport $\mathcal{P}_2(X) \to \Sigma(\mathrm{Inn}(G))$ is in bijection with $\mathrm{Lie}(G)$-valued 2-forms $F_A = \mathbf{d}A + A \wedge A$.

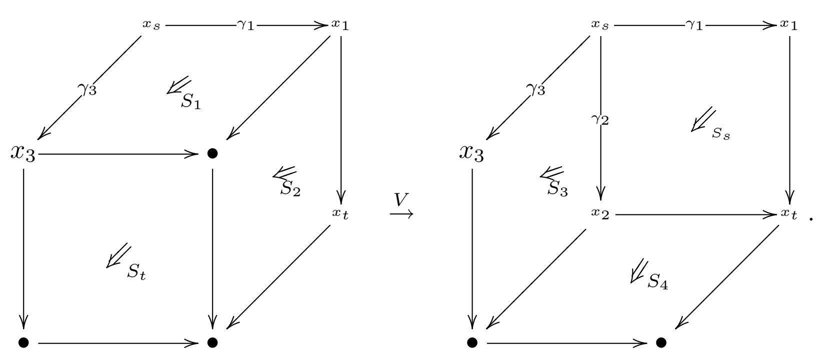
That flatness of $\mathrm{curv}_{\mathrm{tra}}$ implies the ordinary Bianchi identity $d_A F_A = 0$ is a direct consequence of the next example.

Example 2 (curvature of $\Sigma(H \to G)$-2-transport)

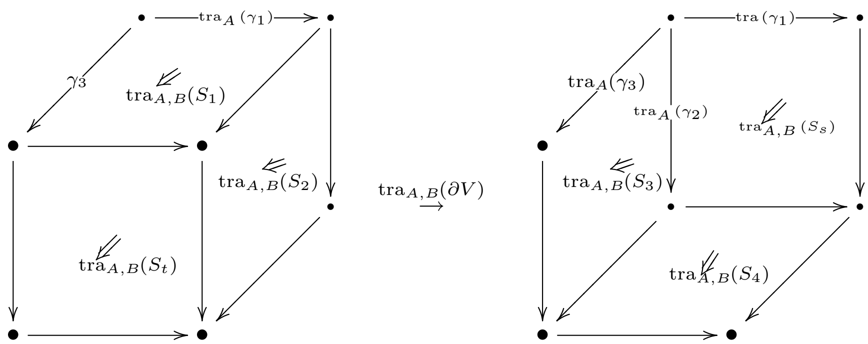
Let $G_2 = (H \stackrel{t}{\to} G)$ be a Lie crossed module, regarded as a strict Lie 2-group. Let $\mathcal{P}_3(X)$ be 3-paths in X and set $T = \Sigma(H \to G)$.

Smooth 2-transport $\mathrm{tra} : \mathcal{P}_2 \to T$ is in bijection with pairs (A, B) in $\Omega^1(X, \mathrm{Lie}(G)) \times \Omega^2(X, \mathrm{Lie}(H))$ such that $F_A + t(B) = 0$.

Let V be cube, a 3-morphism in $\mathcal{P}_3^{\mathrm{cub}}(X)$ spanned by straight paths γ_1, γ_2 and γ_3:



We want to compute $\text{curv}_{\text{tra}}(V)$:



and expand to first order in the length of the three paths. In terms of the universal enveloping algebra of $\text{Lie}(H)$ we use $\text{tra}_{A,B}(S) = 1 + B(S) + \cdots$ as and read off the required re-whiskering from the above diagram. Writing $B(\gamma_1, \gamma_2) = |\gamma_1||\gamma_2|B_{ij}$, etc, the term of order $|\gamma_1||\gamma_2||\gamma_3|$ on the left is

$$\partial_k B_{ij} + A_k(B_{ij}) + \partial_i B_{jk} + A_i(B_{jk}),$$

while on the right it is

$$\partial_j B_{ik} + A_j(B_{ik}).$$

The difference of both is the lowest order term of $\text{tra}(\partial V)$, namely

$$\text{tra}_{A,B}(\partial V) = d_A B(\gamma_1, \gamma_2, \gamma_3) + \mathcal{O}|\gamma_n|^2.$$

We say that

$$H = d_A B$$

is the **curvature 3-form** of (A, B).

The last claim of example 1 now follows by noticing that $\text{curv}_{\text{tra}_A} = \text{tra}_{A,F_A}$.

Similarly, the Bianchi identity of $\text{curv}_{\text{tra}_{A,B}}$ would be a consequence of a general formula for curvature of 3-transport. This is discussed elsewhere.