Curvature 2-Transport, the Atiyah Sequence and Inner Automorphisms

Urs

June 20, 2007

Abstract

We highlight some aspects of the curvature 2-functor associated to an parallel transport 1-functor coming from a principal bundle with connection.

Contents

1 Introduction

Parallel transport in a smooth principal G-bundle $P \to X$ with connection is a functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

from paths in the base to fiber isomorphisms, which is locally smoothly trivializable.

Elsewhere we had noticed that to every such transport functor is associated a curvature 2-functor

$$\mathrm{curv}(\mathrm{tra}) : \mathcal{P}_2(X) \to T' \,,$$

where T' is some 2-category that is codiscrete at top level.

For many purposes it is useful to regard $\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$ as part of a morphism of short exact sequences, going from the path sequence of X to the integrated Atiyah sequence of P.

$$\begin{tikzcd} \Omega_1(X) \arrow[r, hook] \arrow[d] & \mathcal{P}_1(X) \arrow[r, two heads] \arrow[d, "\mathrm{tra}"] & \Pi_1(X) \arrow[d, "="] \\ \mathrm{Ad}P \arrow[r, hook] & P \times_G P \arrow[r, two heads] & \Pi_1(X) \end{tikzcd} .$$

Here we have another look at the curvature 2-transport from this point of view.

I thank Jim Stasheff for helpful comments on this text.

2 Path sequence and integrated Atiyah sequence

2.1 The path groupoid sequence

Given a manifold X, we have the following versions of groupoids of paths in X.

- The path groupoid $\mathcal{P}_1(X)$. Its morphisms are thin homotopy classes of paths.
- The fundamental groupoid $\Pi_1(X)$. Its morphisms are homotopy classes of paths.
- The loop groupoid $\Omega_1(X)$. Its morphisms are thin homotopy classes of closed loops in X.

These fit into a short exact sequence of groupoids.

$$\Omega_1(X) \hookrightarrow \mathcal{P}_1(X) \twoheadrightarrow \Pi_1(X) \ .$$

This we call the *path sequence* or *path groupoid sequence* of X.

Remark. We call two composable morphisms of groupoids a short exact sequence if the first is mono, the second epi and the preimage of identity morphisms under the second equals the image of the first.

Assumption. We shall assume for the moment for simplicity that X is simply connected. This means that

$$\Pi_1(X) = X \times X = \mathrm{Codisc}(X)$$

is the codiscrete groupoid over X (that is: the pair groupoid – precisely one morphism for every ordered pair of objects).

This assumption makes the following discussion more transparent. For X not simply connected there is a straightforward way to generalize all our constructions.

2.2 The integrated Atiyah sequence

Given a principal G-bundle $P \to X$ over X, we obtain two groupoids associated with that.

- The gauge groupoid $P \times_G P$. Its morphisms are isomorphisms of fibers of P, compatible with the G-action on P.
- The adjoint groupoid $\mathrm{Ad}P$. This is the bundle of groups $P \times_G G$, where G acts on itself by conjugation.

These, too fit into a short exact sequence (remembering that we are assuming X to be simply connected):

$$\mathrm{Ad}P \hookrightarrow P \times_G P \twoheadrightarrow \Pi_1(X)\ .$$

This is the *integrated Atiyah sequence*. Applying the functor

$$\mathrm{Lie} : \mathrm{LieGroupoids} \to \mathrm{LieAlgebroids}$$

to this yields the familiar Atiyah sequence of Lie algebroids over X

$$0 \to \mathrm{ad}P \to TP/G \to TX \to 0\,.$$

2.3 Connections as morphisms from the path sequence to the Atiyah sequence

A connection on P gives rise to a parallel transport functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to P \times_G P\,.$$

Observation 1 (David Roberts) *The parallel transport functor for a principal bundle $P \to X$ with connection provides a morphism from the path sequence of X to the integrated Atiyah sequence of P.*

$$\begin{tikzcd}
\Omega_1(X) \arrow[r, hook] \arrow[d] & \mathcal{P}_1(X) \arrow[r, two heads] \arrow[d, "\mathrm{tra}"] & \Pi_1(X) \arrow[d, "="] \\
\mathrm{Ad}P \arrow[r, hook] & P \times_G P \arrow[r, two heads] & \Pi_1(X)
\end{tikzcd}\ .$$

2.4 Adjoint transport

Definition 1 *For K any skeletal groupoid (all morphisms have same source and target), let*

$$\mathrm{AUT}(K)$$

be the 2-groupoid whose

- *objects are the 1-object sub groupoids of K*

- *morphisms are invertible functors between these*
- *2-morphisms are natural isomorphisms between these functors.*

Definition 2 *For G any groupoid and $K \hookrightarrow G$ the disjoint union of its 1-object subgroupoids, let*

$$\mathrm{INN}_G(K)$$

be the strict 2-groupoid whose

- *objects are the 1-object sub-groupoids of G*
- *morphisms are those of G, regarded as functors between 1-object sub-groupoids given by conjugation*
- *2-morphisms*

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow r"] & b \end{tikzcd}$$

are triangles

$$\begin{tikzcd} a \arrow[rr, "r"] \arrow[dr, "f"'] & & a \arrow[dl, "g"] \\ & b & \end{tikzcd} .$$

Their vertical composition is the obvious one, their horizontal composition is that obtained by regarding r as the component of a natural transformation between conjugation by f and by g.

The following properties of $\mathrm{INN}_G(K)$ are obvious but important.

- We have a canonical inclusion
$$G \hookrightarrow \mathrm{INN}_G(K) .$$
- For $K \hookrightarrow G \twoheadrightarrow B$ a sequence of groupoids with B codiscrete and K skeletal, we have a canonical strict 2-functor
$$\mathrm{INN}_G(K) \to \mathrm{AUT}(K) .$$
  Notice that this is in general neither epi nor mono.

  Its failure to be mono is the crucial aspect of the above definition: two morphisms f and g of G are regarded as different morphisms of $\mathrm{INN}_G(K)$ even if their conjugation action on K is the same.

  This property is responsible for the next one.

- The 2-groupoid $\mathrm{INN}_G(K)$ is codiscrete at top level. (Each Hom-category is codiscrete.)

$$\mathrm{Mor}(\mathrm{INN}_G(K)) = \mathrm{Mor}(G) \times_{s,t} \mathrm{Mor}(G)\,.$$

  In the following section we shall often express this fact by writing, with slight abuse of notation

$$\mathrm{INN}_G(K) = G \times_{s,t} G\,.$$

  This will help make the large diagrams to follow more readable.

  Hence $\mathrm{INN}_G(K)$ is equivalent to the discrete 2-groupoid on the connected components of G. In particular, when G is connected $\mathrm{INN}(G)$ is equivalent to the trivial 2-groupoid.

Definition 3 *Let* SeqGrpd *be the category whose objects are short exact sequences of groupoids*

$$K \hookrightarrow G \twoheadrightarrow B$$

with K skeletal and B codiscrete. Morphisms are the obvious morphisms of such sequences. The above defines a functor

$$\mathrm{INN} : \mathrm{SeqGrpd} \to \mathrm{Grpd}$$

given by

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d] & G \arrow[r, two heads] \arrow[d, "F"] \arrow[ddl, "\mathrm{Ad}"] & B \arrow[d] \\
K' \arrow[r, hook] & G' \arrow[r, two heads] \arrow[ddl, "\mathrm{Ad}"] & B' \\
\mathrm{INN}_G(K) \arrow[d, "\mathrm{INN}(F)"'] & & \\
\mathrm{INN}_{G'}(K') & &
\end{tikzcd}$$

In the case that we are interested in, for

$$\mathrm{tra} : \mathcal{P}_1(X) \to P \times_G P$$

our transport functor, we get a functor

$$\mathrm{Ad}(\mathrm{tra}) : \mathcal{P}_1(X) \to \mathrm{INN}_{P\times_G P}(\mathrm{Ad}P)$$

as

$$\begin{tikzcd}
\Omega_1(X) \arrow[r, hook] \arrow[d] & \mathcal{P}_1(X) \arrow[r, two heads] \arrow[d, "\mathrm{tra}"] \arrow[ddl, "\mathrm{Ad}"] & \Pi_1(X) \arrow[d, "="] \\
\mathrm{Ad}P \arrow[r, hook] & P \times_G P \arrow[r, two heads] \arrow[dddl, "\mathrm{Ad}"] & \Pi_1(X) \\
\mathrm{INN}_{\mathcal{P}_1(X)}(\Omega_1(X)) \arrow[d, "\mathrm{INN}(\mathrm{tra})"'] & & \\
\mathrm{INN}_{P\times_G P}(\mathrm{Ad}P) & &
\end{tikzcd} .$$

Notice that this is an honest 1-functor in that the codomain is a (strict!) 2-category, but the functor just sees its 1-categorical part. Or, in other words, the functor is a strict 2-functor whose domain happens to be a 1-category.

This is crucial for the definition of curvature to follow: curvature of transport is an honest 2-functor with values in $\mathrm{INN}_{P\times_G P}(\mathrm{Ad}P)$.

3 Curvature 2-Functors

The Atiyah sequence admits a splitting

$$\begin{tikzcd}
\mathrm{Ad}P \arrow[r, hook] & P \times_G P \arrow[r, two heads] & \Pi_1(X) \arrow[l, bend right, "\sigma"']
\end{tikzcd}$$

precisely when the bundle P is flat. We may understand the existence of this splitting as the statement that the parallel transport functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to P \times_G P$$

factors through the fundamental groupoid

$$\begin{tikzcd}
\mathcal{P}_1(X) \arrow[r] \arrow[d, "\mathrm{tra}"'] & \Pi_1(X) \arrow[d, "\sigma"] \\
P \times_G P \arrow[r, "="] & P \times_G P
\end{tikzcd} .$$

As we have explained elsewhere, if the connection is not flat, we find that the obstruction for the descent of tra from $\mathcal{P}_1(X)$ to $\Pi_1(X)$ manifests itself in terms of the curvature 2-functor

$$\mathrm{curv} : \Pi_2(X) \to T\,.$$

We now recall the construction of curv and relate it to observation 1.

3.1 Curvature 2-functor as an obstruction

Remark. For convenience, the following diagrams feature labels indicating categories and 2-categories. But they are supposed to be read as diagrams involving the corresponding morphism spaces.

First notice the pullback of the Atiya sequence along itself

$$\begin{tikzcd}
 & & & \mathrm{Ad}P \arrow[d] \\
 & & & P \times_G P \arrow[dd] \\
 & & (P \times_G P)^{[2]} \arrow[ur] \arrow[dl] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rr] & & \Pi_1(X)
\end{tikzcd}$$

By our above discussion, we may equivalently read this as

$$\begin{tikzcd}
 & & & \mathrm{Ad}P \arrow[d] \\
 & & & P \times_G P \arrow[dd] \\
 & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[ur] \arrow[dl] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rr] & & \Pi_1(X)
\end{tikzcd}$$

Now enter the path sequence into this picture

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
$$

and similarly pull it back along itself

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & \Pi_2(X) \arrow[r] \arrow[d] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
$$

Here $\Pi_2(X)$ now denotes one version of the fundamental 2-groupoid of X: objects are the points of X, morphisms are thin homotopy classes of paths in X and 2-morphisms are ordinary homotopy classes of surfaces cobounding these paths.

Notice that for this fundamental 2-groupoid to really be the pullback in the above diagram we need to assume that X is not only 1-connected but also 2-connected. As before, there is a more or less straightforward way to generalize everything to the case where no such assumption on X is made, but for simplicity I will not consider that generalization right now.

While the splitting

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) \arrow[urr, squiggly, "\sigma"] \arrow[ddl, squiggly, "\sigma"] & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
$$

does not exist unless tra is flat, by the universal property of the pullback we always canonically have a morphism

$$\mathrm{curv} : \Pi_2(X) \to \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P)\,.$$

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & \Pi_2(X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[ddrr, bend left, "\mathrm{curv}"] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
.
$$

This is such that if the splitting does exist

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & \Pi_2(X) \arrow[r] \arrow[d] \arrow[ddrr, bend left, "\mathrm{curv}"] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) \arrow[urr, "\sigma"] \arrow[ddl, "\sigma"] & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
$$

then by the universal property curv becomes trivial in that it factors through $\Pi_1(X)$ (i.e. sends every 2-morphism to an identity):

$$
\begin{tikzcd}
 & & \Omega_1(X) \arrow[rr] \arrow[d] & & \mathrm{Ad}P \arrow[d] \\
 & \Pi_2(X) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d] & & P \times_G P \arrow[ddd] \\
\Omega_1(X) \arrow[r] \arrow[dd] & \mathcal{P}_1(X) \arrow[r] \arrow[dd, "\mathrm{tra}"] & \Pi_1(X) \arrow[urr, "\sigma"] \arrow[ddl, "\sigma"] \arrow[dr, bend left, "\mathrm{curv}"] & & \\
 & & & \mathrm{INN}_{P \times_G P}(\mathrm{Ad}P) \arrow[uur] \arrow[dll] & \\
\mathrm{Ad}P \arrow[r] & P \times_G P \arrow[rrr] & & & \Pi_1(X)
\end{tikzcd}
$$

Conversely, if curv is trivial in this sense then the splitting σ exists and tra is flat.