String and Chern-Simons Lie 3-Algebras

Urs Schreiber

with
Jim Stasheff

based in parts on work with
John Baez
Alissa Crans
David Roberts
Danny Stevenson
Konrad Waldorf

August 17, 2007

Plan

The Talk

Further material

Thanks

to

- John Baez
- Bruce Bartlett
- Christoph Schweigert
- Jim Stasheff

for helpful comments on earlier versions of this talk.

Motivation

How can we understand quantum Chern-Simons theory as a 3-functorial Quantum Field Theory
– such that it allows us to derive the TFT construction of 2-dimensional CFT [Fuchs,Runkel,Schweigert] from first principles
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Strategy

Proceed in two steps:

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2. Quantize.

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- Atiyah and Segal have famously axiomatized *d*-dimensional QFTs
- as functors

$$Z : n\mathrm{Cob}_S \to \mathrm{Vect}$$

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$$Z \;:\; \left(\partial_{\mathrm{in}}\Sigma \xrightarrow{(\Sigma,g)} \partial_{\mathrm{out}}\Sigma \right) \mapsto \left(H_{\mathrm{in}} \xrightarrow{U(\Sigma,g)} H_{\mathrm{out}} \right).$$

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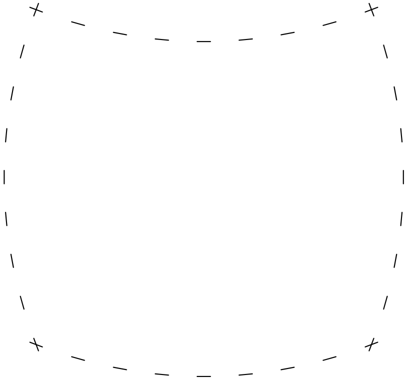
$\langle \phi | U(\Sigma) | \psi \rangle$



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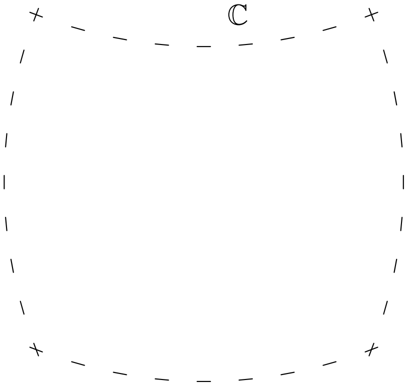

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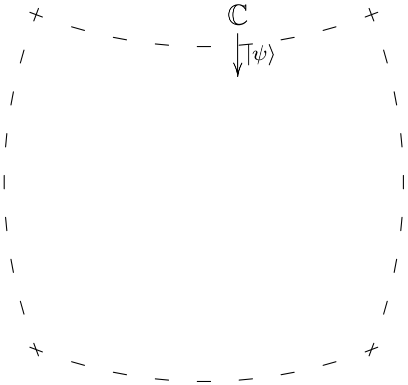

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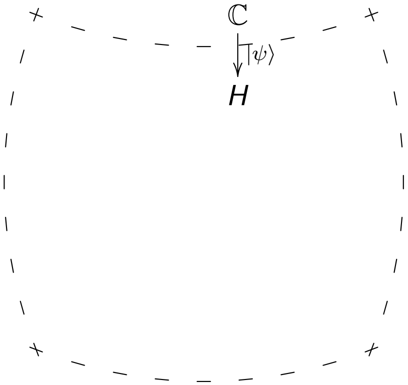

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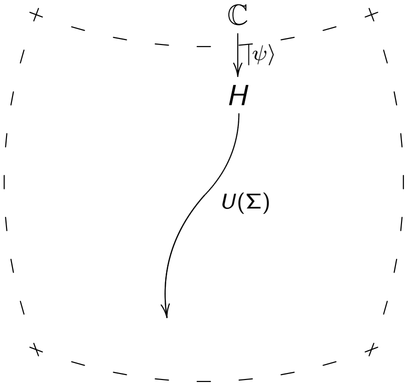

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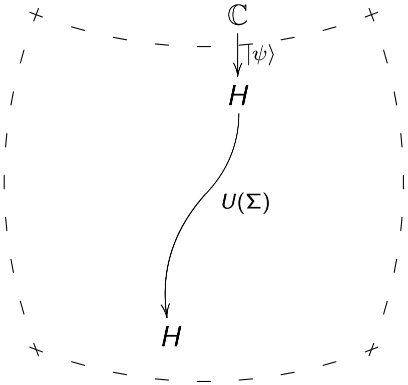

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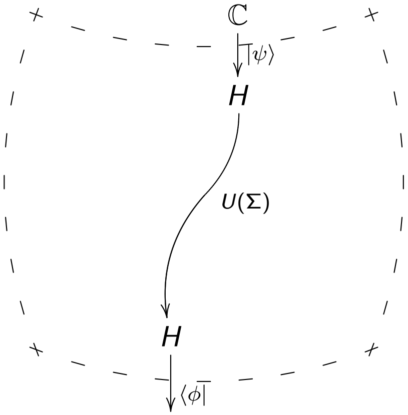

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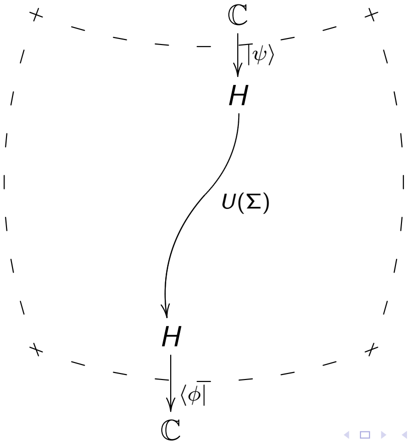

A Quantum Field Theory is an n-Functor

But later it was noticed that this is too imprecise if we want to be able to talk about

crucial requirements on QFT description

- *locality*
- *boundary conditions.*

Instead:

refined picture

An n-dimensional QFT should be an n-functor.
[Freed, Hopkins, Stolz, Teichner]

(remark on n-categories)

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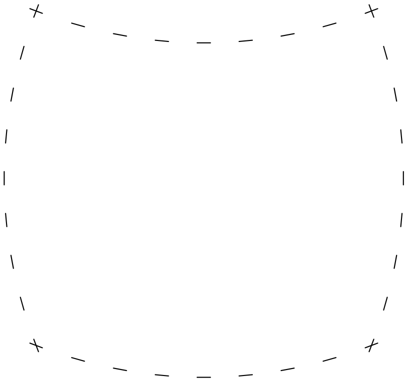

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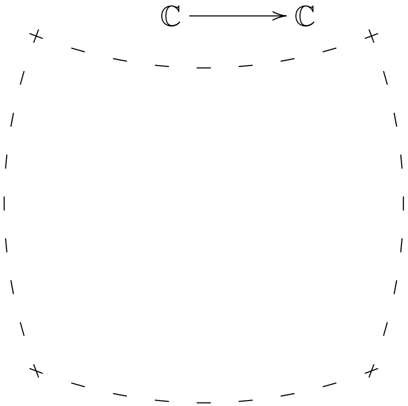

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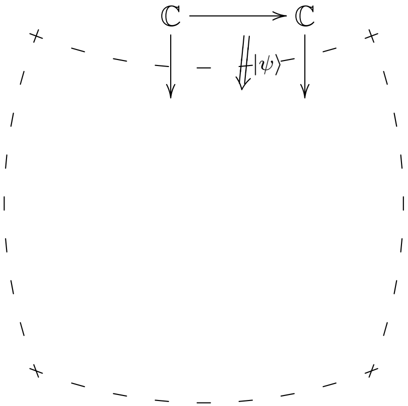

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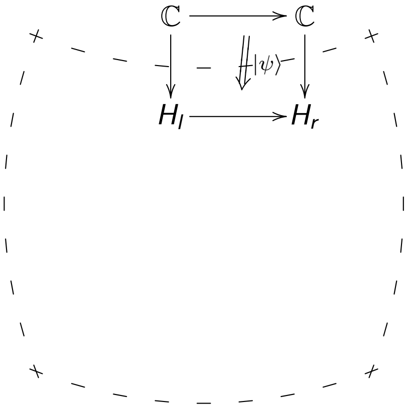

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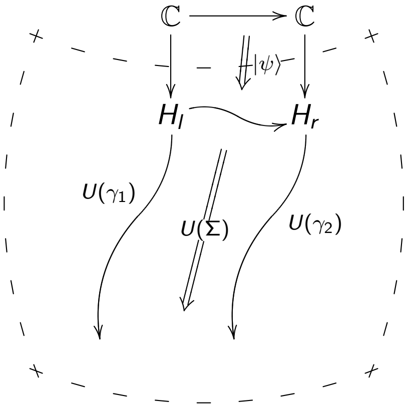

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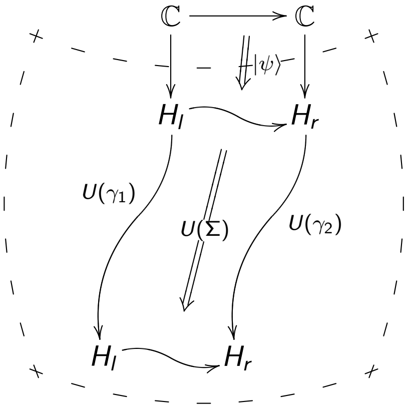

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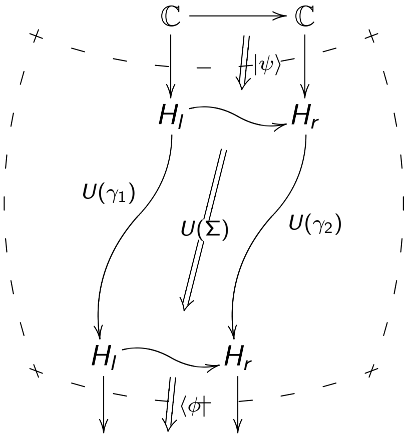

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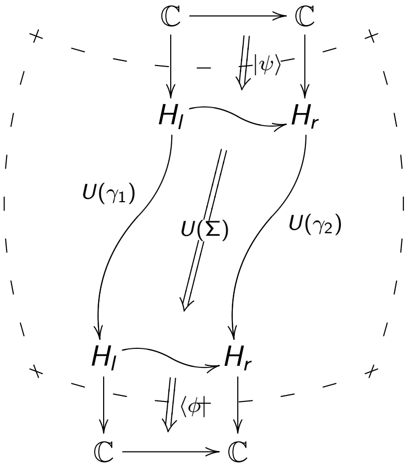

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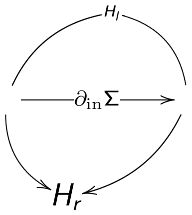

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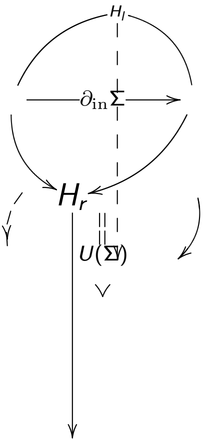

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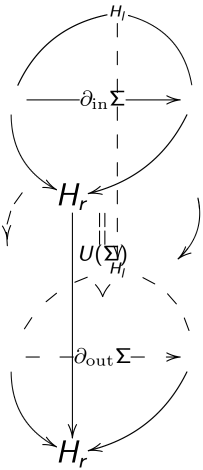

n-Particles and $(n-1)$-Branes

It follows that the action of the *n-particle*...

n-Particle

- $n = 1$: the point particle
- $n = 2$: the string
- $n = 3$: the membrane
- n-particle $\simeq$ $(n-1)$-brane

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n-Bundles and $(n-1)$-Gerbes

It follows that the action of the *n-particle* charged under an *n-bundle with connection*. . .

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Parallel n-Transport

It follows that the action of the *n-particle* charged under an *n-bundle with connection*

is itself an n-functor

- $$\mathrm{tra}_1 : \left(x \xrightarrow{\gamma} y \right) \mapsto \left(V_x \xrightarrow{P\exp\left(\int_\gamma A\right)} V_y \right)$$

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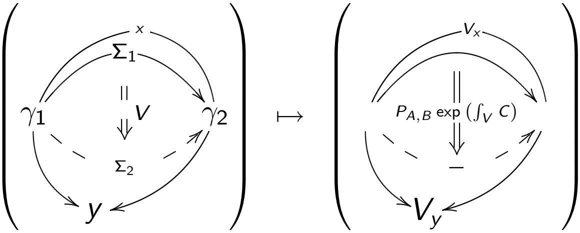

Parallel n-Transport

A parallel n-transport is (locally) an n-functor from the path n-groupoid to the structure n-group.

$$\mathrm{tra}_n : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

$(n+1)$-Curvature

Its $(n+1)$-curvature is (locally) an $(n+1)$-functor from the fundamental $(n+1)$-groupoid to the inner automorphism $(n+1)$-group of $G_{(n)}$.

$$d\mathrm{tra}_n := \mathrm{curv}_{(n+1)} : \Pi_{n+1}(X) \to \Sigma(\mathrm{INN}\, G_{(n)})$$

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Parallel n-Transport

A parallel n-transport is (locally) an n-functor from the path n-groupoid to the structure n-group .

$$\mathrm{tra}_n : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

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We will re-encounter these crucial facts in their Lie n-algebra incarnation shortly.
[U.S., David Roberts]
(on tangent categories) (on inner automorphisms)

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electrically charged 1-particle: $G_{(1)} = U(1)$

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Tough question. Let's pass to the differential picture.

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Finding the Chern-Simons Lie 3-algebra

Problem

Identify that class of 3-transport – given by its structure 3-group – which evaluates to the Chern-Simons functional on 3-dimensional morphisms.

Strategy

- Differentiate. Pass from Lie n-groups to Lie n-algebras.
- Find that Lie 3-algebra $cs_k(g)$ with the property that connections taking values in it, $\mathrm{Vect}(-, cs_k(g))$, correspond to triples (A, B, C) of forms such that $C = CS_k(A) + dB$.

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From paralllel n-transport to Lie n-algebra valued connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	Lie n-algebras ($\simeq$ n-term L_∞-algebras)	$\simeq$	differential algebras (qDGCAs)

$$\Sigma(\mathrm{INN}(G_{(n)})) \xleftarrow{} \; \Pi_{n+1}(X) \quad (F)$$

$$\mathrm{inn}(\mathfrak{g}_{(n)}) \xleftarrow{} \; \mathrm{Vect}(X) \quad (f)$$

$$(\bigwedge^\bullet(s\mathfrak{g}_n^* \oplus ss\mathfrak{g}_{(n)}^*), d) \xrightarrow{f^*} (\Omega^\bullet(X), d)$$

Parallel n-transport is a morphism of Lie $(n+1)$-groupoids.

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These are best handled in terms of their dual maps,

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which are morphisms of quasi-free differential-graded algebras.

The basic concepts

Lie 2-algebras

- Baez and Crans consider 2-vector spaces as categories internal to vector spaces.
- This are nothing but 2-term chain complexes – but interpreted suitably.
- They then define a Lie 2-algebra to be a Lie algebra internal to these 2-vector spaces.
- With strict skew symmetry but weak Jacobi identity.
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Lie n-algebras, L_∞-algebras and qDGCAs

- Baez and Crans showed that their Lie 2-algebras are equivalent to 2-term L_∞-algebras.
- In general, semistrict Lie n-algebras should be equivalent to n-term L_∞-algebras.
- n-Term L_∞-algebras are, in turn, equivalent to quasi-free differential graded algebras (qDGCAs) concentrated in the first n-degrees.
- The L_∞- and qDGCA-formulation are good for computations. The categorical Lie n-algebra formulation is helpful conceptually.

Example: ordinary Lie algebras

For $\mathfrak{g}$ an ordinary Lie algebra, and with $\{t_a\}$ a chosen basis with structure constants $\{C^a{}_{bc}\}$, the corresponding qDGCA is the graded-commutative exterior algebra

$$\wedge^\bullet(s\mathfrak{g}^*)$$

with $s\mathfrak{g}^*$ denoting $\mathfrak{g}^*$ in degree 1, on which

$$dt^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c$$

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The $\mathrm{inn}(\cdot)$-construction

Definition. (Inner derivation Lie $(n+1)$-algebra)

$\mathrm{inn}(\mathfrak{g}_{(n)})$ is the mapping cone of the identity on $\mathfrak{g}_{(n)}$
$\mathrm{inn}(\mathfrak{g}_{(n)}) \simeq (\bigwedge(s\mathfrak{g}_{(n)}) \oplus ss\mathfrak{g}_{(n)}), d')$

Proposition

- There is a canonical injection $\mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g})$.
- $\mathrm{inn}(\mathfrak{g}_{(n)})$ is *contractible*
- $(\bigwedge(s\mathfrak{g}_{(1)}) \oplus ss\mathfrak{g}_{(1)}), d')$ is the *Weil algebra* of $\mathfrak{g}_{(1)}$

Remark.

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The qDGCA of $\mathrm{inn}(\mathfrak{g})$: the Weil algebra

$\mathrm{inn}(\mathfrak{g}) \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d)$ is spanned by generators $\{t^a\}$ in degree 1 and $\{r^a\}$ in degree 2, with differential

$$dt^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c - r^a$$
$$dr^a = -C^a{}_{bc} t^b \wedge r^c \,.$$

We will now

- express the Lie algebra cohomology of $\mathfrak{g}$ in terms of the cohomology of the qDGCA underlying $\mathrm{inn}(\mathfrak{g})$.
- use the insight gained thereby to describe three families of Lie n-algebras: one for each cocycle, one for each invariant polynomial and one for each transgression element.
- then show that for the canonical 3-cocycle on a semisimple Lie algebra, connections with values in the Lie 3-algebra obtained this way describe the Chern-Simons parallel transport which we are after.

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Lie algebra cohomology in terms of $\mathrm{inn}(\mathfrak{g})$

- A Lie algebra n-cocycle μ is

$$d|_{\bigwedge^\bullet(s\mathfrak{g}^*)}\mu = 0\,.$$

- An invariant degree n-polynomial k is

$$d|_{\bigwedge^\bullet(ss\mathfrak{g}^*)}k = 0\,.$$

- A transgression element cs is

$$\begin{aligned}\mathrm{cs}|_{\bigwedge^\bullet s\mathfrak{g}^*} &= \mu \\ d\mathrm{cs} &= k\,.\end{aligned}$$

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The homotopy operator

- Recall that we said that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is trivializable.
- This means there is a homotopy

$$\mathrm{inn}(\mathfrak{g}_{(n)}) \underset{\mathrm{Id}=[d,\tau]}{\overset{0}{\rightrightarrows}} \mathrm{inn}(\mathfrak{g}_{(n)}) \qquad \Downarrow \tau$$

- Since we have τ, we have an effective algorithm to always solve $k = d\mathrm{cs}$ as

$$\mathrm{cs} := \tau(k) + dq\,.$$

- The only nontrivial condition is hence $\mathrm{cs}|_{\bigwedge^\bullet s\mathfrak{g}^*} = \mu$.

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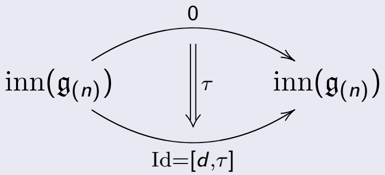

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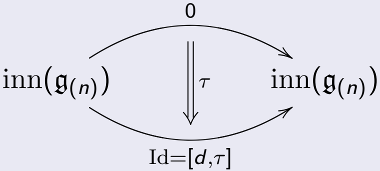

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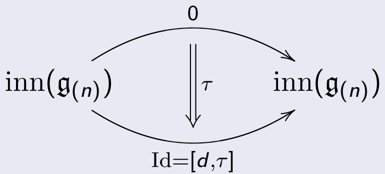

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A map of the cocycle situation

cocycle	Chern-Simons	inv. polynomial

$$\left(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}\right) \xleftarrow{\;i^*\;} \left(\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}\right) \xleftarrow{\;p^*\;} \left(\bigwedge^\bullet(ss\mathfrak{g})^*\right)$$

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow{\scriptstyle d_{\mathrm{inn}(\mathfrak{g})}} & & \\
0 & & p^*k & \xleftarrow{\;p^*\;} & k \\
\uparrow{\scriptstyle d_{\mathfrak{g}}} & & {\scriptstyle \tau}\downarrow \;\; \uparrow{\scriptstyle d_{\mathrm{inn}(\mathfrak{g})}} & & \\
\mu & \xleftarrow{\;i^*\;} & cs & &
\end{array}$$

Baez-Crans Lie n-algebras from cocycles

Definition and proposition [Baez,Crans]

For every Lie algebra $(n+1)$-cocycle μ of the Lie algebra $\mathfrak{g}$ there is a skeletal Lie n-algebra

$$\mathfrak{g}_\mu \,.$$

Construction.

Set $\mathfrak{g}_\mu \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus s^n\mathbb{R}^*), d)$ such that the differential is given by

$$\begin{aligned} dt^a &= -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \\ db &= -\mu \end{aligned}$$

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Chern Lie n-algebras from invariant polynomials

Definition and proposition

For every degree $(n+1)$ Lie algebra invariant polynomial k of the Lie algebra $\mathfrak{g}$ there is a Lie $(2n+1)$-algebra

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Set $\mathrm{ch}_k(\mathfrak{g}) \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{(2n+1)}\mathbb{R}^*), d)$ such that we have

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Theorem

Whenever they exist, these Lie $(2n+1)$-algebras form a (weakly) short exact sequence:

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0\,.$$

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Definition

For our purposes, an *n*-connection with values in the Lie *n*-algebra $\mathfrak{g}_{(n)}$ on a space X is a morphism

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Ordinary connection 1-forms

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n=1

$$\begin{array}{c} \mathfrak{g} \\ {\scriptstyle (A)}\uparrow {\scriptstyle F_A = 0} \\ \mathrm{Vect}(X) \end{array}$$

for $A \in \Omega^1(X, \mathfrak{g})$.

Morphisms into $\mathfrak{g}_{(1)}$ come from *flat* connection 1-forms.

Ordinary connection 1-forms

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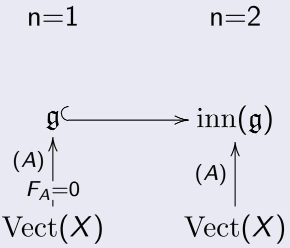

for $A \in \Omega^1(X, \mathfrak{g})$.

Morphisms into $\mathrm{inn}(\mathfrak{g}_{(1)})$ come from *arbitrary* connection 1-forms.

General Chern-Simons-like connections

Theorem

For every degree $(2n+1)$ Lie algebra transgressive element, $(2n+1)$-connections with values in $\mathrm{cs}_k(\mathfrak{g})$ are in bijection with $\mathfrak{g}$-Chern-Simons forms.

This means...

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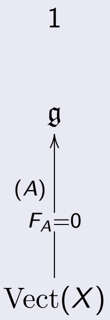
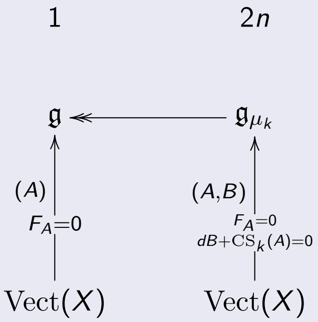
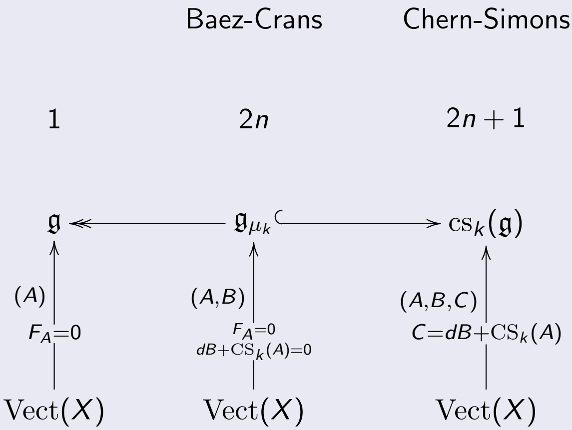


General Chern-Simons-like connections

Baez-Crans



General Chern-Simons-like connections



General Chern-Simons-like connections

	Baez-Crans	Chern-Simons	Chern
1	$2n$	$2n+1$	$2n+1$
$\mathfrak{g} \twoheadleftarrow$	$\mathfrak{g}_{\mu_k} \hookrightarrow$	$\mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow$	$\mathrm{ch}_k(\mathfrak{g})$
$\uparrow (A)$, $F_A=0$	$\uparrow (A,B)$, $F_A=0$, $dB+\mathrm{CS}_k(A)=0$	$\uparrow (A,B,C)$, $C=dB+\mathrm{CS}_k(A)$	$\uparrow (A,C)$, $dC=k((F_A)^{n+1})$
$\mathrm{Vect}(X)$	$\mathrm{Vect}(X)$	$\mathrm{Vect}(X)$	$\mathrm{Vect}(X)$

The standard Chern-Simons 3-connection

Finally: the case we wanted to understand

Let now $\mathfrak{g}$ be semisimple and let

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle$$

be the canonical 3-cocycle.

Theorem (Baez, Crans, S, Stevenson)

The corresponding Baez-Crans Lie 2-algebra $\mathfrak{g}_\mu$ is equivalent to that of the corresponding String 2-group

$$\mathfrak{g}_\mu \simeq \mathrm{Lie}(\mathrm{String}_k(G)) \,.$$

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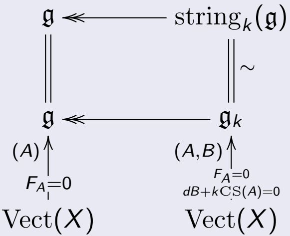
$$
\begin{array}{c}
\mathfrak{g} \\
\Big\| \\
\mathfrak{g} \\
{\scriptstyle (A)}\Big\uparrow {\scriptstyle F_A = 0} \\
\mathrm{Vect}(X)
\end{array}
$$

The standard Chern-Simons 3-connection



The standard Chern-Simons 3-connection

$$\begin{array}{ccccc}
\mathfrak{g} & \twoheadleftarrow & \mathrm{string}_k(\mathfrak{g}) & \hookrightarrow & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) \\
\| & & \|\wr & & \|\wr \\
\mathfrak{g} & \twoheadleftarrow & \mathfrak{g}_k & \hookrightarrow & \mathrm{cs}_k(\mathfrak{g}) \\
{\scriptstyle (A)}\uparrow & & {\scriptstyle (A,B)}\uparrow & & {\scriptstyle (A,B,C)}\uparrow \\
{\scriptstyle F_A=0} & & {\scriptstyle F_A=0,\ dB+k\mathrm{CS}(A)=0} & & {\scriptstyle C=dB+k\mathrm{CS}(A)} \\
\mathrm{Vect}(X) & & \mathrm{Vect}(X) & & \mathrm{Vect}(X)
\end{array}$$

The standard Chern-Simons 3-connection

$$
\begin{array}{ccccccc}
\mathfrak{g} & \twoheadleftarrow & \mathrm{string}_k(\mathfrak{g}) & \hookrightarrow & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) & & \\
\| & & \|\sim & & \|\sim & & \\
\mathfrak{g} & \twoheadleftarrow & \mathfrak{g}_k & \hookrightarrow & \mathrm{cs}_k(\mathfrak{g}) & \twoheadrightarrow & \mathrm{ch}_k(\mathfrak{g}) \\
(A)\uparrow & & (A,B)\uparrow & & (A,B,C)\uparrow & & (A,C)\uparrow \\
F_A=0 & & \begin{matrix} F_A=0 \\ dB+k\mathrm{CS}(A)=0 \end{matrix} & & C=dB+k\mathrm{CS}(A) & & dC=\langle F_A \wedge F_A \rangle \\
\mathrm{Vect}(X) & & \mathrm{Vect}(X) & & \mathrm{Vect}(X) & & \mathrm{Vect}(X)
\end{array}
$$

Conclusion

1. Every degree $(2n+1)$ transgressive element in Lie algebra cohomology gives rise to an exact sequence

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0$$

of Lie $(2n+1)$-algebras.

2. Connections with values in these Lie $(2n+1)$-algebras are precisely the corresponding Chern-Simons functionals, "localized" as a $(2n+1)$-transport.

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- Which Lie 3-group does our Chern-Simons Lie 3-algebra $\mathrm{cs}_k(\mathfrak{g})$ integrate to?
- How does the Chern-Simons 3-group help to understand the relation between Chern-Simons theory and Wess-Zumino-Witten theory?
- How does the Chern-Simons 3-transport help to understand String bundles with connection?
- How are the Chern Lie $(2n+1)$-algebras related to Roytenberg's Lie n-algebras with weak skew-symmetry?

The following slides contain brief comments on this.

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Integrating the Chern-Simons Lie 3-algebra

We have found the Lie 3-algebra

$$\mathrm{cs}_{\langle\cdot,\cdot\rangle}(\mathfrak{g})$$

which characterizes Chern-Simons 3-transport. Due to the isomorphism

$$\mathrm{cs}_{\langle\cdot,[\cdot,\cdot]\rangle} \simeq \mathrm{inn}(\mathfrak{g}_{\langle\cdot,[\cdot,\cdot]\rangle})$$

and the equivalence

$$\mathfrak{g}_{\langle\cdot,[\cdot,\cdot]\rangle} \simeq \mathrm{string}_{\langle\cdot,\cdot\rangle}(\mathfrak{g})$$

the corresponding Lie 3-group should be

$$\mathrm{INN}_0(\mathrm{String}_{\langle\cdot,\cdot\rangle})(G)\,.$$

What does $\mathrm{INN}_0(\mathrm{String}_k(G))$ know about Wess-Zumino-Witten theory?

By the *n-functorial holographic principle* we expect that the WZW 2-transport arises as *transformations* of the Chern-Simons 3-transport.

Indeed, one can see that the 2-category of cylinders in the 3-group $\mathrm{INN}_0(\mathrm{String}_k(G))$ has the right properties for this.

(To see this, recall that, by [Baez,Crans,S,Schreiber], the 2-group $\mathrm{String}_k(G)$ is, as a groupoid, precisely the canonical gerbe on G which enters the WZW theory.)

How does the Chern-Simons Lie 3-algebra help to understand String-bundles with connections?

Ask Google the question: "*How to get a spinning string from here to there?*"

Which role do weakly skew-symmetric Lie n-algebras play?

Baez and Crans considered Lie n-algebras which have a strictly skew-symmetric bracket, which however satisfies the Jacobi identity only weakly.

One can see that the Lie n-algebras $\mathfrak{g}_\mu$ for each $(n+1)$-cocycle μ which we discussed are ordinary Lie algebras except that their $(n-2)$nd coherence for the Jacobiator is nontrivial, and in fact precisely given by this cocycle.

Now, Roytenberg considered the case that skew-symmetry is weakened, too. It seems that one finds in his description that the higher coherences then are given by *symmetric* forms on the Lie algebra. In particular, for $n = 2$ the Killing form may be regarded as the "skew-symmetrizator".

This seems to suggest the following picture:

Which role do weakly skew-symmetric Lie n-algebras play?

This seems to suggest the following picture:

weak Jacobi weak skew symmetry

$$\mathfrak{g}_{\mu_k} \longrightarrow \mathrm{cs}_\mu(\mathfrak{g}) \longrightarrow \mathrm{ch}_\mu(\mathfrak{g})$$

and might thus give an even more fundamentel interpretation of the exact sequence which we found.
But this needs to be better understood.

More details

On the following slides are collected a couple of definitions and explanations which were omitted from the main part.

n-Categories

Our 3-Categories are Gray-Categories

Whenever details are relevant, we shall restrict to $n = 3$ and use 3-categories which are *Gray-categories*. In such 3-categories everything is strict, except possibly the exchange law for 2-morphisms.

Our 3-Groupoids are Gray-Groupoids

Given a notion of n-category, an n-groupoid is an n-category in which all k-morphisms are equivalences.

(back)

n-Groups

Definition

An n-group is an n-groupoid with a single object.

Notation

We write $G_{(n)}$ when we think of an n-group as a monoidal $(n-1)$-groupoid. Then we write $\Sigma G_{(n)}$ for the corresponding one-object n-groupoid.

Example

For $G_{(2)}$ any strict 2-group, $\mathrm{INN}_0(G_{(2)})$ is a 3-group coming from a Gray-groupoid.

(back)

Tangent n-Categories

Definition

For C any n-category, let $\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \}$ be the "fat point", and let

$$TC \subset \mathrm{Hom}(\mathbf{pt}, C)$$

be that maximal sub-n-category which collapses to a 0-category when pulled back along the inclusion $\{\bullet\} \hookrightarrow \mathbf{pt}$.
We call TC the *tangent n-category* to C.

The general concept of tangent categories is important for the bigger picture underlying our dicussion, but for the present purpose many of the details are not crucial.

Properties of the tangent *n*-category

Proposition

For C any n-category, we find that the n-bundle

$$TC \to \mathrm{Obj}(C)$$

is a "deformation retract" of $\mathrm{Obj}(C)$ in that

$$TC \simeq \mathrm{Obj}(C)\,.$$

Properties of the tangent n-category

Proposition

For C any n-category, we find that the n-bundle

$$TC \to \mathrm{Obj}(C)$$

sits inside an exact sequence

$$\mathrm{Mor}(C) \to TC \to \mathrm{C}\,.$$

When $C = \Sigma G_{(n)}$ comes from an n-groupoid, this “is” the universal $G_{(n)}$-bundle.

Inner automorphism $(n+1)$-Groups

Definition

Given any n-groupoid C, we say that

$$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{AUT}(C))$$

is the inner automorphism $(n+1)$-group of C.

proposition

$\mathrm{INN}(G_{(n)})$ sits inside the exact sequence

$$1 \to Z(G_{(n)}) \to \mathrm{INN}(G_{(n)}) \to \mathrm{AUT}(G_{(n)}) \to \mathrm{OUT}(G_{(n)}) \to 1$$

of $(n+1)$-groups. Here $Z(G_{(n)})$ is the (suspension of) the categorical center of $\Sigma G_{(n)}$.

Inner automorphism $(n+1)$-Groups

Proposition

There is a canonical inclusion

$$TC \hookrightarrow T_{\mathrm{Id}_C}(\mathrm{AUT}(C)\,.$$

Definition

The image under this inclusion we call

$$\mathrm{INN}_0(C) \subset \mathrm{INN}(C)\,.$$

(back)