Chern-Simons and string_G Lie-3-algebras

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Abstract

We show that the Chern-Simons Lie-3-algebra $\text{cs}(g)$, for any Lie algebra g, sits inside the Lie-3-algebra of inner derivations of the strict Lie 2-algebra $(\hat{\Omega}_k g \to Pg)$:

$$\text{cs}(g) \subset \text{inn}(\text{string}_g)\,.$$

1 Introduction

For any Lie algebra g and fixed Killing form $k\langle\cdot,\cdot\rangle$ (for $\langle\cdot,\cdot\rangle$ some fixed normalization of the Killing form and $k \in \mathbb{R}$ the **level**) there is a semistrict Lie-3-algebra $\text{cs}(g)$ such that 3-connections

$$d\text{tra} : \text{Lie}(\mathcal{P}_1(X)) \to \text{cs}(g)$$

(i.e. algebroid morphisms from the pair algebroid of X to the 3-algebroid $\text{cs}(g)$) are given by a g-valued 1-form A, its curvature 2-form and its Chern-Simons 3-form

$$\text{CS}(A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle$$

on X.

Another Lie 3-algebra canonically associated to $(g, k\langle\cdot,\cdot\rangle)$ is obtained as follows: The semistrict Baez-Crans Lie 2-algebra g_k is equivalent to the strict Lie 2-algebra

$$\text{string}_g := (\hat{\Omega}_k g \to Pg)\,.$$

For any strict Lie 2-algebra $(r \xrightarrow{\delta} s)$, the Lie-3-algebra $\text{inn}(r \xrightarrow{\delta} s)$ of its inner derivations is characterized by the fact that 3-connections

$$d\text{tra} : \text{Lie}(\mathcal{P}_1(X)) \to \text{inn}(r \to s)$$

are given by an s-valued 1-form A, an r-valued 2-form B such that with

$$\beta := F_A + \delta(B)$$

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and

$$H = d_A B$$

we have

$$d_A \beta = \delta(H)$$

and

$$d_A H + \beta \wedge B = 0\,.$$

Had we in addition required that $\beta = 0$, then this would characterize $(r \to s)$ itself. For $r = \hat{\Omega}_k g$ and $s = Pg$ we write

$$\mathrm{string}_g := (\hat{\Omega}_k g \to Pg)\,.$$

It is known that

$$\mathrm{string}_g \simeq g_k\,.$$

Here we are after a generalization of this equivalence when passing from string_g to $\mathrm{inn}(\mathrm{string}_g)$.

We fall short of actually proving an equivalence. Instead we construct a morphism

$$\mathrm{cs}(g) \to \mathrm{inn}(\mathrm{string}_g)$$

and a morphism

$$\mathrm{inn}(\mathrm{string}_g) \to \mathrm{cs}(g)$$

such that

$$\mathrm{cs}(g) \to \mathrm{inn}(\mathrm{string}_g) \to \mathrm{cs}(g)$$

is the identity on $\mathrm{cs}(g)$.

We will work throughout in terms of the Koszul dual description of semistrict Lie-n-algebras. Ever Lie-n-algebra is encoded in a free differential graded algebra, and morphisms of Lie-n-algebras are given by maps between FDAs that are at the same time chain maps and algebra homomorphisms.

2 The details

Definition 1 (the Lie 3-algebra $\mathrm{inn}(\hat{\Omega}_k g \to Pg)$)

The FDA of the Lie-3-algebra of $\mathrm{inn}(\hat{\Omega}_k g \to Pg)$ is spanned in degree 1 by the basis $\{t^a(\sigma)\}$ spanning $(Pg)^*$, in degree 2 by $\{q^a(\sigma)\}$ also spanning a $(Pg)^*$ as well as by $\{(s, s^a)\}$, spanning $(\hat{\Omega}_k g)^*$. In degree 3 we have a basis $\{(h, h^a(\sigma))\}$ spanning again a $(\hat{\Omega}_k g)^*$.

The graded differential d is defined on this basis by the following equations.

$$dt^a(\sigma) + \frac{1}{2} C^a{}_{bc} t^b(\sigma) t^c(\sigma) + s^a(\sigma) + q^a(\sigma) = 0$$

$$ds^a(\sigma) + C^a{}_{bc} t^b(\sigma) s^c(\sigma) - h^a(\sigma) = 0$$

$$ds + 2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\, d\sigma - h = 0$$

$$dq^a(\sigma) + C^a{}_{bc}t^b(\sigma)q^c(\sigma) + h^a(\sigma) = 0$$

$$dh^a(\sigma) + C^a{}_{bc}t^b(\sigma)h^c(\sigma) + C^a{}_{bc}q^b(\sigma)s^c(\sigma) = 0\,.$$

$$dh + 2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}h^b(\sigma)\, d\sigma + 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\, d\sigma = 0\,.$$

Here $C^a{}_{bc}$ are the structure constants of g in the chosen basis and k_{ab} are the components of $\langle\cdot,\cdot\rangle$ in that basis.

Remark. I take loops to be parameterized by $[0, 2\pi]$. So in particular $h^a(2\pi) = 0$ and $s^a(2\pi) = 0$ (but $t^a(2\pi)$ and $q^a(2\pi)$ are nonvanishing).

Definition 2 (the Lie 3-algebra $cs(g)$)

The FDA of cs(g) is spanned in degree 1 by a basis $\{t^a\}$ of g^* and in degree 2 by a basis $\{r^a\}$, also spanning g^*, as well as by an element $\{b\}$ spanning $\mathbb{R}^*$. Another element $\{c\}$ is in degree 3 and spans another $\mathbb{R}^*$.

The differential d in this basis is defined by the following equations.

$$dt^a + \frac{1}{2}C^a{}_{bc}t^bt^c + r^a = 0$$

$$dr^a + C^a{}_{bc}t^br^c = 0$$

$$db - k\left(\frac{1}{6}C_{abc}t^at^bt^c + k_{ab}t^ar^b\right) - c = 0$$

$$dc - k(k_{ab}r^ar^b) = 0\,.$$

Definition 3 (the morphism $\mathrm{cs}(g) \to \mathrm{inn}(\hat{\Omega}_k g \to Pg)$**))**

For any smooth function

$$f : [0, 2\pi] \to \mathbb{R}$$

such that $f(2\pi) = 1$ define an FDA algebra homomorphism $\mathrm{cs}(g) \to \mathrm{inn}(\hat{\Omega}_k g \to Pg))$ on our basis elements as follows.

$$t^a(\sigma) \mapsto f(\sigma)t^a$$

$$q^a(\sigma) \mapsto f(\sigma)r^a$$

$$s^a(\sigma) \mapsto (f - f^2)(\sigma)\frac{1}{2}C^a{}_{bc}t^bt^c$$

$$s \mapsto b$$

$$h^a(\sigma) \mapsto (f - f^2)(\sigma)C^a{}_{bc}t^br^c$$

$$h \mapsto k(k_{ab}t^ar^b) + c$$

Remark. Notice that the condition $f(2\pi) = 1$ ensures that for instance $(f - f^2)(\sigma)C^a{}_{bc}t^b r^c$ indeed vanishes for $\sigma = 2\pi$.

Proposition 1 *The above morphism is indeed a chain map.*

Proof. We check on all basis elements that the morphism commutes with the differential.

$$\begin{aligned} dt^a(\sigma) &= -\frac{1}{2}C^a{}_{bc}t^b(\sigma)t^c(\sigma) - s^a(\sigma) - q^a(\sigma) \\ &\mapsto -f^2(\sigma)\frac{1}{2}C^a{}_{bc}t^b t^c - (f - f^2)(\sigma)\frac{1}{2}C^a{}_{bc}t^b t^c - f(\sigma)r^a \end{aligned}$$

$$\begin{aligned} dt^a(\sigma) &\mapsto f(\sigma)dt^a \\ &= -f(\sigma)\frac{1}{2}C^a{}_{bc}t^b t^c - f(\sigma)r^a \end{aligned}$$

$$\begin{aligned} ds^a(\sigma) &= -\frac{1}{2}C^a{}_{bc}t^b(\sigma)s^c(\sigma) + h^a(\sigma) \\ &\mapsto f(f - f^2)(\sigma)\frac{1}{2}C^a{}_{bc}t^b C^c{}_{de}t^d t^e + (f - f^2)(\sigma)C^a{}_{bc}t^b r^c \end{aligned}$$

$$\begin{aligned} ds^a(\sigma) &\mapsto (f - f^2)(\sigma)d\frac{1}{2}C^a{}_{bc}t^b t^c \\ &= (f - f^2)(\sigma)C^a{}_{bc}t^b r^c \end{aligned}$$

$$\begin{aligned} ds &= -2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\, d\sigma + h \\ &\mapsto -k\int_0^{2\pi} f(\sigma)\frac{d}{d\sigma}(f - f^2)\, d\sigma\; k_{ab}t^a C^b{}_{de}t^d t^e + k(k_{ab}t^a r^b) + c \\ &= k\frac{1}{6}C_{abc}t^a t^b t^c + k(k_{ab}t^a r^b) + c \end{aligned}$$

$$\begin{aligned} ds &\mapsto db \\ &= k\left(\frac{1}{6}C_{abc}t^a t^b t^c + k_{ab}t^a r^b\right) + c \end{aligned}$$

$$\begin{aligned} dq^a(\sigma) &= -C^a{}_{bc}t^b(\sigma)q^c(\sigma) - h^a(\sigma) \\ &\mapsto -f^2(\sigma)C^a{}_{bc}t^b r^c - (f - f^2)(\sigma)C^a{}_{bc}t^b r^c \end{aligned}$$

$$\begin{aligned} dq^a(\sigma) &\mapsto f(\sigma)dr^a \\ &= -f(\sigma)C^a{}_{bc}t^b r^c \end{aligned}$$

$$
\begin{aligned}
dh^a(\sigma) &= -C^a{}_{bc}t^b(\sigma)h^c(\sigma) - C^a{}_{bc}q^b(\sigma)s^c(\sigma) \\
&\mapsto -f(f-f^2)(\sigma)C^a{}_{bc}t^bC^c{}_{de}t^br^c - f(f-f^2)(\sigma)\frac{1}{2}C^a{}_{bc}r^bC^c{}_{de}t^dt^e(\sigma) \\
&= -f(f-f^2)(\sigma)dC^a{}_{bc}t^br^c = 0
\end{aligned}
$$

$$
\begin{aligned}
dh^a(\sigma) &\mapsto (f-f^2)(\sigma)dC^a{}_{bc}t^br^c \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
dh &= -2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}h^b(\sigma)\,d\sigma - 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\,d\sigma \\
&\mapsto -2k\int_0^{2\pi} f(\sigma)\frac{d}{d\sigma}(f-f^2)(\sigma)\,d\sigma\; k_{ab}t^aC^b{}_{de}t^dr^e - k\int_0^{2\pi} f(\sigma)\frac{d}{d\sigma}(f-f^2)(\sigma)\,d\sigma\; k_{ab}r^aC^b{}_{de}t^bt^c \\
&= k\frac{1}{3}C_{abc}t^at^br^c + k\frac{1}{6}C_{abc}r^at^bt^c \\
&= k\frac{1}{2}C_{abc}t^at^br^c
\end{aligned}
$$

$$
\begin{aligned}
dh &\mapsto d\,k(k_{ab}t^ar^b) + dc \\
&= -\frac{k}{2}k_{ab}C^a{}_{de}t^dt^er^b - k(k_{ab}r^ar^b) + k(k_{ab}t^aC^b{}_{de}t^dr^e) + k(k_{ab}r^ar^b) \\
&= \frac{k}{2}k_{ab}C^a{}_{de}t^dt^er^b
\end{aligned}
$$

□

Definition 4 (the morphism $\mathrm{inn}(\hat{\Omega}_k g \to Pg)) \to \mathrm{cs}(g)$**)**

Define an FDA algebra homomorphism $\mathrm{inn}(\hat{\Omega}_k g \to Pg)) \to \mathrm{cs}(g)$ on our basis by the following assignments.

$$t^a \mapsto t^a(2\pi)$$

$$r^a \mapsto q^a(2\pi)$$

$$b \mapsto s + k\frac{1}{2}\int k_{ab}t^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma - k\frac{1}{2}\int k_{ab}t^b(\sigma)\frac{d}{d\sigma}t^a(\sigma)\,d\sigma$$

$$c \mapsto h - 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma$$

Proposition 2 *The above morphism is indeed a chain map.*

Proof. We check on all basis elements that the morphism commutes with the differential.

$$\begin{aligned} dt^a &= -\frac{1}{2}C^a{}_{bc}t^bt^c - r^a \\ &\mapsto -\frac{1}{2}C^a{}_{bc}t^b(2\pi)t^c(2\pi) - q^a(2\pi) \end{aligned}$$

$$\begin{aligned} dt^a &\mapsto dt^a(2\pi) \\ &= -\frac{1}{2}C^a{}_{bc}t^b(2\pi)t^c(2\pi) - q^a(2\pi) \end{aligned}$$

$$\begin{aligned} dr^a &= -C^a{}_{bc}t^br^c \\ &\mapsto -C^a{}_{bc}t^b(2\pi)q^c(2\pi) \end{aligned}$$

$$\begin{aligned} dr^a &\mapsto dq^a(2\pi) \\ &= -C^a{}_{bc}t^b(2\pi)q^c(2\pi) \end{aligned}$$

$$\begin{aligned} db &= k\left(\frac{1}{6}C_{abc}t^at^bt^c + k_{ab}t^ar^b\right) + c \\ &\mapsto k\left(\frac{1}{6}C_{abc}t^a(2\pi)t^b(2\pi)t^c(2\pi) + k_{ab}t^a(2\pi)q^b(2\pi)\right) + h - 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma \end{aligned}$$

$$\begin{aligned} db &\mapsto ds + d\frac{1}{2}k\int k_{ab}t^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma - d\frac{1}{2}k\int k_{ab}(\frac{d}{d\sigma}t^a(\sigma))t^b(\sigma)\,d\sigma \\ &= -2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\,d\sigma + h \\ &\quad +k\frac{1}{6}C_{abc}t^a(2\pi)t^b(2\pi)t^c(2\pi) + 2k\int_0^{2\pi} k_{ab}t^a(\sigma)\frac{d}{d\sigma}s^b(\sigma)\,d\sigma - 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma \\ &\quad +k\,k_{ab}t^a(2\pi)q^b(2\pi) \\ &= h + k\frac{1}{6}C_{abc}t^a(2\pi)t^b(2\pi)t^c(2\pi) - 2k\int_0^{2\pi} k_{ab}q^a(\sigma)\frac{d}{d\sigma}t^b(\sigma)\,d\sigma \\ &\quad +k\,k_{ab}t^a(2\pi)q^b(2\pi) \end{aligned}$$

$$
\begin{aligned}
dc &= k\left(k_{ab} r^a r^b\right) \\
&\mapsto k\left(k_{ab} q^a(2\pi) q^b(2\pi)\right)
\end{aligned}
$$

$$
\begin{aligned}
dc \mapsto\ & dh - 2k\, d \int_0^{2\pi} k_{ab} q^a(\sigma) \frac{d}{d\sigma} t^b(\sigma)\, d\sigma \\
=\ & -2k \int_0^{2\pi} k_{ab} t^a(\sigma) \frac{d}{d\sigma} h^b(\sigma)\, d\sigma - 2k \int_0^{2\pi} k_{ab} q^a(\sigma) \frac{d}{d\sigma} s^b(\sigma)\, d\sigma \\
& +2k \int_0^{2\pi} k_{ab} C^a{}_{de} t^d(\sigma) q^e(\sigma) \frac{d}{d\sigma} t^b(\sigma)\, d\sigma + 2k \int_0^{2\pi} k_{ab} h^a(\sigma) \frac{d}{d\sigma} t^b(\sigma)\, d\sigma \\
& k \int_0^{2\pi} k_{ab} q^a(\sigma) \frac{d}{d\sigma} C^b{}_{de} t^d(\sigma) t^e(\sigma)\, d\sigma \\
& +2k \int_0^{2\pi} k_{ab} q^a(\sigma) \frac{d}{d\sigma} q^b(\sigma)\, d\sigma + 2k \int_0^{2\pi} k_{ab} q^a(\sigma) \frac{d}{d\sigma} s^b(\sigma)\, d\sigma \\
=\ & k\left(k_{ab} q^a(2\pi) q^b(2\pi)\right)
\end{aligned}
$$

□

Proposition 3 *The composition*

$$\mathrm{cs}(g) \to \mathrm{inn}(\mathrm{string}_g) \to \mathrm{cs}(g)$$

is strictly the identity morphism on cs(g).

Proof. This is immediate. □