On Lie ∞-modules and the BV complex.

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Abstract

Some tentative remarks on generalizations of Chevalley-Eilenberg algebras from Lie algebras and their modules to Lie ∞-algebras and their modules, with an eye towards understanding the Batalin-Vilkovisky complex.

Contents

1 Introduction

The Chevalley-Eilenberg algebra of a Lie ∞-algebra $\mathfrak{g}$ is a cochain complex in *non-negative* degree.

An ω-vector space B is a cochain complex in *non-positive* degree.

The Batalin-Vilkovisky complex is a complex in *arbitrary*, positive and negative, degree, which contains as a subcomplex in non-negative degree the CE-algebra of an L_∞-algebra, and as a subcomplex in non-positive degree an ω-vector space.

Under duality in cochain complexes, negative and positive degrees are interchanged. Hence

$$\mathfrak{g}^* \oplus B$$

is in arbitrary degree.

Compare table 3.

In the case that $\mathfrak{g}$ is just an ordinary Lie algebra (in degree 1) and B just an ordinary module for it, $\mathfrak{g}^* \oplus B$ would support the structure of a Lie-Rinehart pair.

We propose the obvious generalization of this to a structure on arbitrarily graded $\mathfrak{g}^* \oplus B$, which we think of as a Lie-Rinehart ∞-pair. Then we show that the BV-complex is indeed a special case of this.

This we do for a simplified special case, compare table 5.

Main point The main point is this:

As is well known, a dual L_∞ structure on a positively graded cochain complex $\mathfrak{g}^*$ of vector space is an extension of the differential of $\mathfrak{g}^*$ to one on

$$\Lambda\mathfrak{g}^*$$

which makes $\Lambda\mathfrak{g}^*$ into a monoid internal to chain complexes under the obvious tensor product.

Then let A be a commutative algebra and B be a *non-positively* graded cochain complex of A-modules, $B \in \mathbf{Ch}^\bullet(A)$ which is A in degree 0. Assume that $\mathfrak{g}^*$ is also a complex of A-modules, $\mathfrak{g} \in \mathbf{Ch}^\bullet(A)$. Then we first form the graded-symmetric tensor algebra

$$\wedge_A^\infty(\mathfrak{g}^* \oplus B)$$

over A, then forget the A-module structure and just remember the underlying vector space structure over our ground field K, thus obtaining

$$F(\wedge_A^\infty(\mathfrak{g}^* \oplus B)) \in \mathbf{Ch}^\bullet(K)$$

which remembers the fact that it comes from A-modules by having a canonical monoid structure induced from that. This makes it a “free-over-A” graded commutative differerential algebra.

The (dual) Lie-Rinehart ∞-pair structure is essentially an extension of the differential on $F(\wedge^\infty(\mathfrak{g}^*\oplus B))$ such that this monoid structure remains respected.

Warning. Careful with these unfinished notes.

Discussion. I talked about this stuff a lot with Zoran Škoda and with Danny Stevenson. Of course all imperfections in the following are mine. With Zoran I am involved in a project on Lie-Rinehart 2-pairs, and the following thoughts pertain to that, but don't include the main point of that project, which is to find the right definition of a Lie-Rinehart n-pair such that its Chevalley-Eilenberg algebra is the one - probably - I consider here. Danny was most helpful with providing help at various points.

Much of my thinking about these matters is influenced in one way or another by interaction with Jim Stasheff. I am also thankful for some comments by Johannes Huebschmann. Possibly what I am trying to say here has already been said by him or somebody else, and it's just me being slow.

Plan.

1. basics on chain complexes
2. the Koszul complex
3. the Tate construction
4. examples in the BV context

2 Cochain complexes

The objects that we shall be concerned with here are **differential graded algebras** (dg-algebras). The right way to think of a dg-algebra is as a monoid in a category of cochain complexes.

dg-algebra	**monoid in $\mathbf{Ch}^\bullet(A)$**
wedge product	tensor product
graded-commutativity	nontrivial symmetric braiding

Table 1: **Realizing dg-algebras as monoids in chain complexes**.

We recall very basic facts about the category of chain complexes and the homological algebra one can do with it.

2.1 A-modules

A module is for an algebra precisely what a representation is for a group.

Definition 1 *For A any algebra over k, a* **module** *N for A is a k-vector space N together with an action of A on N by linear operators, namely an algebra homomorphism*

$$\rho : A \to \mathrm{End}_{\mathrm{Vect}_k}(N)\,.$$

Examples.

- A module for the ground field k regarded as an algebra over itself is nothing but an ordinary k-vector space.
- A module for the polynomial algebra $k[X]$ over a single variable is a vector space with one singled out endomorphism $\rho(X) \in \mathrm{End}(N)$ of it.
- The space of sections of a k-vector bundle $E \to X$ over some space X is a module over the algebra of k-valued functions on X.

The last example is the crucial one in the context of the BV formalism. Therefore we recall the statement underlying it in full detail.

Definition 2 (special properties of modules) *The following special properties of A-modules are important.*

- *An A-module N is* **finitely generated** *if it is spanned, over A, by finitely many of its elements.*
- *An A-module N is* **free** *of rank $n \in \mathbb{N}$ if it is of the form $N \simeq A^n := A^{\oplus n}$.*
- *An A-module N is* **projective** *if any of the following equivalent conditions hold*
  - *N is a* **direct summand of a free module**, *i.e. there exists another module N' such that $N \oplus N'$ is free.*
  - *N is the image $N \simeq \mathrm{im}(P)$ of a projection $P \in \mathrm{End}(A^n)$ on some free module.*
  - *N satisfies the lifting property*

$$\forall g, f : \qquad \begin{array}{ccc} & & M' \\ & {\scriptstyle \exists h} \nearrow & \downarrow \\ N & \xrightarrow{g} & M \end{array}$$

Fact 1 (Swan's theorem) *For X a real manifold and $A = C(X)$ the algebra of real functions on X, the sections of vector bundles on X are precisely the finitely-generated projective modules over $A = C(X)$:*

$$\mathrm{VectBun}(X) \xrightarrow{\simeq} A\mathrm{Mod}_{\mathrm{fin,proj}}$$
$$(E \to X) \mapsto \Gamma(E)\,.$$

We will come back to this special case.

There is an obvious notion of homomorphisms of A-modules.

Definition 3 *Given two A-modules N and N', an A-module homomorphism*

$$f : N \to N'$$

between them is a linear map that preseves the A action:

$$\begin{array}{ccccc} A \otimes N & \xrightarrow{\rho \otimes \mathrm{Id}} & \mathrm{End}(N) \otimes N & \xrightarrow{\mathrm{ev}} & N \\ \downarrow {\scriptstyle \mathrm{Id}_A \otimes f} & & & & \downarrow {\scriptstyle f} \\ A \otimes N' & \xrightarrow{\rho' \otimes \mathrm{Id}} & \mathrm{End}(N') \otimes N' & \xrightarrow{\mathrm{ev}} & N' \end{array} .$$

Definition 4 (the category of A-modules) *We write $A\mathrm{Mod}$ for the category whose objects are A-modules and whose morphisms are A-module homomorphisms.*

Structures and properties of AMod.

- AMod is an **abelian category**, hence we can do homological algebra inside AMod.

  Recall that this is equivalent to saying that that

  - it has a zero-object – this is the 0-dimensional A-module;
  - and it has all pullbacks and pushouts;
  - and all monomorphisms and epimorphisms are normal.

- AMod is **symmetric monoidal**. The tensor product

  $$\otimes_A : A\mathrm{Mod} \times A\mathrm{Mod} \to A\mathrm{Mod}$$

  is the ordinary tensor product of A-modules over A. The **tensor unit** is $I = A$ and the **symmetric braiding** is the obious one.

- AMod is **closed** with respect to the above monoidal structure. The internal hom

  $$\mathrm{hom} : A\mathrm{Mod}^{\mathrm{op}} \times A\mathrm{Mod} \to A\mathrm{Mod}$$

  sends any two A-modules to the vector space of A-module homomorphisms between them, equipped with an A-module structure in the obvious way.

- AMod **has duals**. The dual

  $$(-)^* : A\mathrm{Mod} \to A\mathrm{Mod}^{\mathrm{op}}$$

  is

  $$(-)^* = \mathrm{hom}(-, A)\,.$$

  The A-module V is of **finite rank** if $(V^*)^* \simeq V$.

  The full subcategory of finite rank modules we denote

  $$A\mathrm{Mod}_{\mathrm{fin}}\,.$$

- $A\mathrm{Mod}_{\mathrm{fin}}$ is **compact closed**, meaning that the internal hom exists and is

  $$\mathrm{hom}(V, W) \simeq V^* \otimes_A W\,.$$

Example. In the context of Swan's theorem, consider modules of function algebras given by sections $\Gamma(E)$ of vector bundles $E \to X$ over some space X.

Then:

- The dual module $V^* \simeq \Gamma(E^*)$ is the space of sections of the dual bundle.
- The tensor product $V \otimes_A W$ corresponds, under to the ordinary fiberwise tensor product of vector bundles:

  $$\Gamma(E) \otimes_{C(X)} \Gamma(E') \simeq \Gamma(E \otimes_{\mathrm{VectBun}} E')\,.$$

2.2 Categorical motivation: chain complexes as internal ω-categories.

One combinatorial model for higher dimensional homotopies are ω-categories (strict, globular or cubical). The nerve of an ω-category internal to AMod is a simplicial A-module.

The famous Dold-Kan correspondence says that by forgetting lots of face maps except one, and restricting it to the kernel of some of the other face maps, one obtains from a simplicial A-module a non-negatively graded chain complex of A-modules without loosing information.

Fact 2 (Dold-Kan correspondence) *Forming the normalized chain complex from a simplicial A-module is an equivalence of categories*

$$A\mathrm{Mod}^{\Delta^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Ch}^+_\bullet(A\mathrm{Mod}) \ .$$

This equivalence is just the first in a longer list.

Fact 3 (Brown and Higgins [1]) *Let $\mathcal{A}$ be an abelian category. Then the following categories, internal to $\mathcal{A}$, are all equivalent:*

- *simplicial objects*
- *chain complexes*
- *crossed complexes*
- *cubical sets with connections*
- *cubical ω-groupoids with connections*
- *globular ω-groupoids.*

Remark. There are one or two sign conventions that need to be fixed once and for all before dealing with complexes. With an eye towards maximal harmony with applications to the BV complex, we shall adopt the following convention

- All our complexes will be *co*chain complexes, meaning that the differentials *increase* the degree by one, with in general no restriction on the sign of the degree.
- Ordinary chain complexes are then recovered as cochain complexes of non-*positive* degree.

degree	+	0	-
interpretation		ordinary vector space	
		ω−vector space	ω−vector space
	ω−*co*vector space	ω−*co*vector space	

Table 2: **The interpretation of *co*chain complexes** in terms of higher order vector spaces.

2.3 Cochain complexes of A-modules

Definition 5 *We denote by* $\mathbf{Ch}^\bullet(A)$ *the category of* A *cochain complexes in* AMod.

Objects V *are cochain complexes of* A*-modules*

$$V^\bullet = (\cdots \longrightarrow V^{-2} \xrightarrow{d_V^{-1}} V^{-1} \xrightarrow{d_V^0} V^0 \xrightarrow{d_V^1} V^1 \xrightarrow{d_V^2} V^2 \longrightarrow \cdots),$$

$$d_V^{k+1} \circ d_V^k = 0 \quad \forall k \in \mathbb{Z}.$$

Morphisms $f^\bullet : V^\bullet \to W^\bullet$ *are cochain maps*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & V^{-2} & \xrightarrow{d_V^{-1}} & V^{-1} & \xrightarrow{d_V^0} & V^0 & \xrightarrow{d_V^1} & V^1 & \xrightarrow{d_V^2} & V^2 & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{1}} & & \\
\cdots & \longrightarrow & W^{-2} & \xrightarrow{d_W^{-1}} & W^{-1} & \xrightarrow{d_W^0} & W^0 & \xrightarrow{d_W^1} & W^1 & \xrightarrow{d_W^2} & W^2 & \longrightarrow & \cdots
\end{array}.$$

We assume all chain complexes to be nontrivial only in *finitely* many degrees. It is useful to distinguish the full subcategories

- $\mathbf{Ch}^-(A)$ of cochain complexes concentrated in *non-positive* degree;
- $\mathbf{Ch}^+(A)$ of cochain complexes concentrated in *non-negative* degree.

One may think of $\mathbf{Ch}^-(A)$ as ω-vector spaces and of $\mathbf{Ch}^+(A)$ as ω-co-vector spaces.

Using the notation

$$V_n := V^{-n}$$

we can neatly switch back and forth between the two pictures.

Remark.

- If we forget the differentials, i.e. if we look at cochain complexes with all differentials trivial (the 0-maps), then these are the same as $\mathbb{Z}$-graded A-modules.
- When we have nontrivial differentials, their nilpotency, $d^2 = 0$, necessarily imposes, as discussed below, on these graded vector spaces the structure of *super*vector spaces: the symmetric braiding is the nontrivial $\mathbb{Z}_2$-grading that introduces a sign whenever two odd-graded components are interchanged.

Structure and properties and of $\mathbf{Ch}^\bullet(A)$. We list some useful facts about $\mathbf{Ch}^\bullet(A)$.

- $\mathbf{Ch}^\bullet(A)$ is **symmetric monoidal** with the tensor product

  $$\otimes : \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

  defined by

$$V^\bullet \otimes W^\bullet = \; \left(\cdots \longrightarrow (V^\bullet \otimes W^\bullet)^n \xrightarrow{\quad d^{n+1}_{V^\bullet \otimes W^\bullet} \quad} (V^\bullet \otimes W^\bullet)^{n+1} \longrightarrow \cdots \right) \, .$$

$$\left(\cdots \longrightarrow \Big(\bigoplus_{k \in \mathbb{Z}} V^k \otimes_A W^{n-k} \Big) \xrightarrow{\bigoplus_{k\in\mathbb{Z}} \left(d_V^{k+1} \otimes_A \mathrm{Id}_{W^{n-k}} + (-1)^k \mathrm{Id}_{V^k} \otimes_A d_W^{n-k+1} \right)} \Big(\bigoplus_{k \in \mathbb{Z}} V^k \otimes_A W^{n-k+1} \Big) \longrightarrow \cdots \right)$$

  Remark. The signs appearing here are crucial. Their nature is fixed entirely by the requirement that the tensor product is again a chain complex, i.e. by the requirement that $(d_{V \otimes W})^2 = 0$. As we will see in the following, this will also imply that our modules are subject to the nontrivial symmetric braiding which introduces a sign whenever two odd-graded modules are interchanged. All this follows just from the nilpotency condition $d^2 = 0$.

  One way to understand the precise nature of the signs above is to note that when forming the tensor product $V \otimes W$, we obtain the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_W^m} & & \downarrow{\scriptstyle d_W^m} & & \\
\cdots & \xrightarrow{d_V^n} & V^n \otimes W^m & \xrightarrow{d_V^{n+1}} & V^{n+1} \otimes W^m & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_W^{m+1}} & & \downarrow{\scriptstyle d_W^{m+1}} & & \\
\cdots & \xrightarrow{d_V^n} & V^n \otimes W^{m+1} & \xrightarrow{d_V^{n+1}} & V^{n+1} \otimes W^{m+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

as an intermediate step. The squares commute, meaning that d_V and d_W commute. So we form

$$\tilde{d}_W := (-1)^{\deg_V} d_W$$

and then the nilpotent differential

$$d_{V\oplus W} := d_V + \tilde{d}_W \, .$$

The **tensor unit** is

$$I^\bullet := \left(\cdots \xrightarrow{d_I^{-1}} 0 \xrightarrow{d_I^0} A \xrightarrow{d_I^1} 0 \xrightarrow{d_I^2} \cdots \right)$$

$$\cdots \xrightarrow{0} 0 \xrightarrow{0} A \xrightarrow{0} 0 \longrightarrow \cdots$$

The **symmetric braiding**

$$\begin{array}{ccc} \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) & \xrightarrow{\quad\otimes\quad} & \mathbf{Ch}^\bullet(A) \otimes \mathbf{Ch}^\bullet(A) \\ {\scriptstyle\sigma}\searrow & \Downarrow{\scriptstyle b} & \nearrow{\scriptstyle\otimes} \\ & \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) & \end{array}$$

with

$$\sigma : \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A)$$

the exchange of factors is

$$b^n_{V^\bullet, W^\bullet} : \; \left(\bigoplus_k V^k \otimes_A W^{n-k}\right) \xrightarrow{\bigoplus_k (-1)^{k(n-k)}} \left(\bigoplus_k W^{n-k} \otimes_A V^k\right) \; .$$

The signs here ensure the required naturality

$$\begin{array}{ccc} \left(\bigoplus_k V^k \otimes_A W^{n-k}\right) & \xrightarrow{\bigoplus_k (-1)^{k(n-k)}} & \left(\bigoplus_k W^{n-k} \otimes_A V^k\right) \\ \Big\downarrow {\scriptstyle d_V \otimes_A \mathrm{Id} + (-1)^k \mathrm{Id} \otimes_A d_W} & & \Big\downarrow {\scriptstyle d_W \otimes_A \mathrm{Id} + (-1)^{n-k} \mathrm{Id} \otimes_A d_V} \\ \left(\bigoplus_k V^k \otimes_A W^{n-k+1}\right) & \xrightarrow{\bigoplus_k (-1)^{k(n-k+1)}} & \left(\bigoplus_k W^{n-k+1} \otimes_A V^k\right) \end{array}$$

- $\mathbf{Ch}^\bullet(A)$ is **enriched over** AMod

$$\mathrm{Hom} : \mathrm{Ch}^\bullet(A)^{\mathrm{op}} \times \mathrm{Ch}^\bullet(A)^{\mathrm{op}} \to A\mathrm{Mod}$$

- $\mathbf{Ch}^\bullet(A)$ is closed. So for all $V \in \mathbf{Ch}^\bullet(A)$ there is an internal hom functor

$$\mathrm{hom}(V,-) : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

right adjoint to

$$-\otimes V : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

meaning that

$$\mathrm{Hom}(U \otimes V, W) \simeq \mathrm{Hom}(U, \mathrm{hom}(V,W))$$

naturally in U and W.

This internal hom-complex $\mathrm{hom}(V,W)$ looks as follows:

$$\mathrm{hom}(V,W) := \quad (\cdots \longrightarrow \mathrm{hom}(V,W)^n \xrightarrow{d^{n+1}_{\mathrm{hom}(V,W)}} \mathrm{hom}(V,W)^{n+1} \longrightarrow \cdots)$$

$$= (\cdots \longrightarrow \bigoplus_k \mathrm{hom}_{A\mathrm{Mod}}(V^k, W^{k+n}) \xrightarrow{\bigoplus_k \left((d_W^{k+n+1}\circ -) - (-1)^n(-\circ d_V^k)\right)} \bigoplus_k \mathrm{hom}_{A\mathrm{Mod}}(V^k, W^{k+n+1}) \longrightarrow \cdots)$$

The differential $d_{\mathrm{hom}(V,W)}$ here can be understood from looking at the evaluation map

$$\mathrm{ev} : \mathrm{hom}(V,W) \otimes V \to W$$

which exists by general nonsense on internal homs. Let $f \in \oplus_k \mathrm{Hom}(V^k, W^{k+n})$ be any homogeneous element in the internal hom and $x \in V^m$. Write $f(x)$ for $\mathrm{ev}(f,x)$. Then the fact that ev is a cochain morphism says that

$$d_W(f(v)) = (d_{\mathrm{hom}(V,W)} f)(x) + (-1)^n f(d_V x)\,.$$

Solving this for $d_{\mathrm{hom}(V,W)} f$ yields the action of the differential as given above.

Remark. Notice that it is the space of *cocycles* in degree 0 of $\mathrm{hom}(V,W)$ that corresponds to the external $\mathrm{Hom}(V,W)$:

$$Z^0(\mathrm{hom}(V,W)) \simeq \mathrm{Hom}(V,W) = \ker((d_W \circ -) - (-\circ d_V))\,.$$

The nature of this differential in the internal hom complex will have strong significance in the context of applications to the BV formalism. There we will find, for par and tar the "parameter space" and the "target space" of an n-particle (an $(n-1)$-brane), respectively:

- $\mathrm{hom}(\mathrm{par},\mathrm{tar})^0$ is the space of all "worldvolume fields"
- $Z^0(\mathrm{hom}(\mathrm{par},\mathrm{tar}))$ is the space of all fields that solve the "classical equations of motion"
- $B^0(\mathrm{hom}(\mathrm{par},\mathrm{tar}))$ is the space of all fields that are gauge trivial.

Notice how the additional information obtained by passing from the external Hom to the internal matches with important physical information.

- $\mathbf{Ch}^\bullet(A)$ **has duals**. Since we have a tensor unit I and an internal hom, we have duality

$$(\cdot)^* : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)^{\mathrm{op}}$$

given by

$$V^* := \mathrm{hom}(V, I)\,.$$

Using the above one finds

$$\begin{aligned} V^* = \quad & \left(\cdots \longrightarrow (V^*)^n \xrightarrow{d_{V^*}^{n+1}} (V^*)^{n+1} \longrightarrow \cdots \right) \,. \\ = & \left(\cdots \longrightarrow (V^{-n})^* \xrightarrow{-(-1)^n (d_V^{-n})^*} (V^{-n-1})^* \longrightarrow \cdots \right) \end{aligned}$$

The unit $i : I \to V \otimes V^*$ is

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \,, \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \bigoplus_k i_{V^k}} & & \downarrow & & \\ \cdots & \longrightarrow & \bigoplus_k V^k \otimes_A (V^{k+1})^* & \longrightarrow & \bigoplus_k V^k \otimes_A (V^k)^* & \longrightarrow & \bigoplus_k V^k \otimes_A (V^{k-1})^* & \longrightarrow & \cdots \end{array}$$

while the counit $e : V^* \otimes V \to I$ is

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bigoplus_k V^k \otimes_A (V^{k+1})^* & \longrightarrow & \bigoplus_k V^k \otimes_A (V^k)^* & \longrightarrow & \bigoplus_k V^k \otimes_A (V^{k-1})^* & \longrightarrow & \cdots \,. \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \bigoplus_k e_{V^k}} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

$\square$

- Let $\mathbf{Ch}^\bullet_{\mathrm{fin}}(A)$ be the full subcategory on those chain complexes that are bounded (only finitely many entries are nonvanishing) and all whose entries satisfy $\mathrm{hom}_{A\mathrm{Mod}}(V^k, W^l) \simeq (V^k)^* \otimes_A W^l$.

  Then: $\mathbf{Ch}^\bullet_{\mathrm{fin}}(A)$ is **compact closed** meaning that the internal hom is isomorphic to a tensor product

$$\mathrm{hom}(V, W) \simeq W \otimes V^*\,.$$

Just for the fun of it, we now check that in detail.

By the finiteness assumption we have

$$\hom_{A\mathrm{Mod}}(V^k, W^{k-n}) \simeq (V^k)^* \otimes_A W^{k-n}\,.$$

Then we compute

$$
\begin{aligned}
&(V \otimes W^*)^* := \\
&\left(\cdots \longrightarrow ((V\otimes W^*)^*)^n \xrightarrow{\;d^{n+1}_{(V\otimes W^*)^*}\;} ((V\otimes W^*)^*)^{n+1} \longrightarrow \cdots\right) \\
&= \left(\cdots \longrightarrow ((V\otimes W^*)^{-n})^* \xrightarrow{\;(-1)^{n+1}(d^{-n}_{V\otimes W^*})^*\;} (V\otimes W^*)^{-n-1})^* \longrightarrow \cdots\right) \\
&= \left(\cdots \longrightarrow \bigoplus_k (V^k \otimes_A (W^*)^{-n-k})^* \xrightarrow{\;\bigoplus_k (-1)^{n+1}\left(d^k_V \otimes_A \mathrm{Id}_{(W^*)^{-n-k}} + (-1)^k \mathrm{Id}_{V^k} \otimes_A d^{-n-k}_{W^*}\right)^*\;} \bigoplus_k (V^k \otimes_A (W^*)^{-n-k-1})^* \longrightarrow \cdots\right) \\
&= \left(\cdots \longrightarrow \bigoplus_k (V^k \otimes_A (W^{n+k})^*)^* \xrightarrow{\;\bigoplus_k (-1)^{n+1}\left(d^k_V \otimes_A \mathrm{Id}_{(W^{n+k})^*} + (-1)^{n+1} \mathrm{Id}_{V^k} \otimes_A (d^{n+k+1}_{W})^*\right)^*\;} \bigoplus_k (V^k \otimes_A (W^{n+k+1})^*)^* \longrightarrow \cdots\right) \\
&= \left(\cdots \longrightarrow \bigoplus_k (V^k)^* \otimes_A W^{n+k} \xrightarrow{\;\bigoplus_k \left(\mathrm{Id}_{(V^k)^*} \otimes_A d^{n+k+1}_W + (-1)^{n+1} (d^k_V)^* \otimes_A \mathrm{Id}_{W^{n+k}}\right)\;} \bigoplus_k (V^k)^* \otimes_A W^{n+k+1} \longrightarrow \cdots\right) \\
&\simeq \left(\cdots \longrightarrow \bigoplus_k \hom_{A\mathrm{Mod}}(V^k, W^{k+n}) \xrightarrow{\;[d,-]\;} \bigoplus_k \hom_{A\mathrm{Mod}}(V^k, W^{k+n+1}) \longrightarrow \cdots\right) \\
&= \hom(V, W)\,.
\end{aligned}
$$

By the symmetry of $\mathbf{Ch}^\bullet(A)$, we have of course $W \otimes V^* \simeq V^* \otimes W$.

Computing the latter directly yields

$$V^*\otimes W := \quad (\cdots \longrightarrow (V^*\otimes W)^n \xrightarrow{d^{n+1}_{V^*\otimes W}} (V^*\otimes W)^{n+1} \longrightarrow \cdots)\ .$$

$$= (\cdots \longrightarrow \bigoplus_k (V^*)^k \otimes_A W^{n-k} \xrightarrow{\bigoplus_k \left(d^{k+1}_{V^*}\otimes_A \mathrm{Id}_{W^{n-k}} + (-1)^k \mathrm{Id}_{(V^*)^k}\otimes_A d_W^{n-k+1}\right)} \bigoplus_k (V^*)^k \otimes_A W^{(n-k+1)} \longrightarrow \cdots)$$

$$= (\cdots \longrightarrow \bigoplus_k (V^{-k})^* \otimes_A W^{n-k} \xrightarrow{\bigoplus_k (-1)^k \left(-(d_V^{-k})^*\otimes_A \mathrm{Id}_{W^{n-k}} + \mathrm{Id}_{(V^{-k})^*}\otimes_A d_W^{n-k+1}\right)} \bigoplus_k (V^{-k})^* \otimes_A W^{(n-k+1)} \longrightarrow \cdots)$$

$$= (\cdots \longrightarrow \bigoplus_k (V^{k})^* \otimes_A W^{n+k} \xrightarrow{\bigoplus_k (-1)^k \left(-(d_V^{k})^*\otimes_A \mathrm{Id}_{W^{k+n}} + \mathrm{Id}_{(V^{k})^*}\otimes_A d_W^{k+n+1}\right)} \bigoplus_k (V^{k})^* \otimes_A W^{(k+n+1)} \longrightarrow \cdots)$$

$$= (\cdots \longrightarrow \bigoplus_k \hom_{A\mathrm{Mod}}(V^k, W^{k+n}) \xrightarrow{\bigoplus_k (-1)^k \left((d_W^{k+n+1}\circ -) - (-\circ d_V^k)\right)} \bigoplus_k \hom_{A\mathrm{Mod}}(V^k, W^{k+n+1}) \longrightarrow \cdots)$$

The differential here looks superficially different from the one of $\hom(V,W)$. But the complex is indeed isomorphic to $\hom(V,W)$, as it should be: let $f : V^k \to W^{k+n}$ be a map of degree n. We find

$$\begin{array}{ccc}
f & \xmapsto{(-1)^k((d_W\circ -)-(-\circ d_V))} & (-1)^k(d_W\circ f - f\circ d_V) \\
\downarrow{\scriptstyle (-1)^{k(k+n)}} & & \downarrow{\scriptstyle (-1)^{k(k+n+1)}\oplus(-1)^{(k-1)(k+n)}} \\
(-1)^{(k+kn)} f & \xmapsto{(d_W\circ -)-(-1)^n(-\circ d_V)} & (-1)^{k+kn}(d_W\circ f) - (-1)^{kn+n+k} f\circ d_V
\end{array}\ ,$$

where the vertical arrows come from the braiding isomorphism $b_{V^*\otimes V}$.

- $\mathbf{Ch}^\bullet(A)$ has plenty of other nice structures. In particular, it naturally is a **model category**.

2.4 dg-Algebras and dg-coalgebras

The crucial (but simple) fact underlying most of what we are doing here is:

Observation 1 *Monoids in* $\mathbf{Ch}^\bullet(A)$*, i.e. cochain complexes* V *equipped with a product morphism*

$$\mu : V\otimes V \to V$$

and a unit morphism

$$i : I \to V$$

such that μ is associative

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{\mu \otimes \mathrm{Id}_V} & V \otimes V \\ {\scriptstyle \mathrm{Id}_V \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ V \otimes V & \xrightarrow{\mu} & V \end{array}$$

and unital

$$\begin{array}{ccc} I \otimes V & \longrightarrow & V \\ & {\scriptstyle i \otimes \mathrm{Id}_V} \searrow \quad \nearrow {\scriptstyle \mu} & \\ & V \otimes V & \end{array}$$

are precisely **differential graded algebras (dg-algebras)**.

Definition 6 (dg-algebra) *A dg-algebra is an associative graded algebra $(V, \cdot)$ equipped with a graded derivation*

$$d : V \to V$$

of degree +1 that squares to 0,

$$d^2 = 0\,.$$

Of ocurse this has a co-version:

Observation 2 *Comonoids in $\mathbf{Ch}^\bullet(A)$, i.e. cochain complexes V equipped with a coproduct morphism*

$$\delta : V \to V \otimes V$$

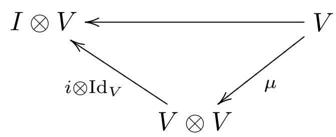
and a counit morphism

$$e : V \to I$$

such that δ is coassociative

$$\begin{array}{ccc} V \otimes V \otimes V & \xleftarrow{\delta \otimes \mathrm{Id}_V} & V \otimes V \\ {\scriptstyle \mathrm{Id}_V \otimes \delta} \uparrow & & \uparrow {\scriptstyle \delta} \\ V \otimes V & \xleftarrow[\delta]{} & V \end{array}$$

and counital



are precisely **codifferential graded coalgebras (cdg-coalgebras)**.

Definition 7 (cdg-coalgebra) *A cdg-coalgebra is a coassociative graded coalgebra $(V, \cdot)$ equipped with a graded coderivation*

$$D : V \to V$$

of degree +1 that squares to 0,

$$D^2 = 0 \,.$$

Definition 8 *We write*

$$\mathrm{Monoids}(\mathrm{Ch}^\bullet(A))$$

for the category of monoids internal to $\mathrm{Ch}^\bullet(A)$ *and*

$$\mathrm{CoMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of comonoids internal to $\mathrm{Ch}^\bullet(A)$.

We write

$$\mathrm{ComMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of commutative monoids internal to $\mathrm{Ch}^\bullet(A)$ *and*

$$\mathrm{CoComMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of cocommutative comonoids internal to $\mathrm{Ch}^\bullet(A)$.

2.4.1 The internal hom in dg-(co)algebras

We had seen that the internal hom $\mathrm{hom}_{\mathrm{Ch}^\bullet(A)}(X, Y)$ in cochain complexes, between two *positively* graded cochain complexes X and Y is itself, in general, no longer positively graded. A similar statement applies to the hom internal to (co)monoids in $\mathrm{Ch}^\bullet(A)$.

As discussed in 5, important examples of arbitrarily graded dg-(co)algebras arise by forming the internal hom betwen dg-(co)algebras with degrees of definite sign.

The closed structure on dg-coalgebras which I shall describe now I learned from Todd Trimble, who learned it from Jim Dolan. Todd tells me that the result may also have been known to Thomas Fox in the 80s.

Fact 4 (well known) *Let C be a symmetric monoidal category. Then the category* $\mathrm{CoComMonoids}(C)$ *has cartesian products given by the tensor product in* C*. For* X *and* Y *comonoids, the projections*

$$\begin{array}{ccc} & X\otimes Y & \\ {}^{p_X}\swarrow & & \searrow^{p_Y} \\ X & & Y \end{array} \quad := \quad \begin{array}{ccc} & X\otimes Y & \\ {}^{e_X\otimes Y}\swarrow & & \searrow^{X\otimes e_Y} \\ X & & Y \end{array}$$

are obtained using the counit and the diagonal

$$X \xrightarrow{\delta_X} X\otimes X$$

is simply the coproduct.

Theorem 1 (Dolan) *If C is symmetric monoidal closed such that* $\mathrm{CoComMonoids}(C)$ *has equalizers und such that the forgetful functor*

$$U : \mathrm{CoComMonoids}(C) \to C$$

has a right adjoint,

$$\mathrm{Cofree} : C \to \mathrm{CoComMonoids}(C)$$

then $\mathrm{CoComMonoids}(C)$ *is cartesian closed.*

Proof. The strategy is to show that the internal $\hom_{\mathrm{CoComMonoids}(C)}(X,Y)$ has to be the equalizer

$$\hom_{\mathrm{CoComMonoids}(C)}(X,Y) \hookrightarrow \mathrm{Cofree}(\hom_C(X,Y)) \overset{\Psi}{\underset{\Phi}{\rightrightarrows}} \mathrm{Cofree}(\hom_C(X,Y\otimes Y))\otimes \mathrm{Cofree}(\hom_C(X,I))$$

in $\mathrm{CoComMonoids}(C)$ of two morphisms Φ and Ψ. Then it exists by the assumption that C is such that $\mathrm{CoComMonoids}(C)$ has equalizers.

The two morphisms in question are, on the first tensor factor

$$\Psi_1 := \mathrm{Cofree}(\mathrm{Hom}(X,\delta_Y))$$

and

$$\begin{array}{c} \mathrm{Cofree}(\hom_C(X,Y)) \xrightarrow{\Phi_1} \mathrm{Cofree}(\hom_C(X,Y\otimes Y)) \xrightarrow{\pi} \hom_C(X,Y\otimes Y) \\ \mathrm{Cofree}(\hom_C(X,Y)) \xrightarrow{\delta} \mathrm{Cofree}(\hom_C(X,Y))^{\otimes 2} \xrightarrow{\pi\otimes\pi} \hom_C(X,Y)^{\otimes 2} \xrightarrow{\otimes_1} \hom_C(Y\otimes Y)^{X\otimes X} \xrightarrow{\hom_C(\delta_X, Z\otimes Z)} \hom_C(X,Y\otimes Y) \end{array}$$

□

2.4.2 Quasi-free dg-(co)algebras

We shall mainly be interested in dg-algebras that are free in a certain sense. These come from symmetric tensor powers.

Definition 9 *The* **symmetric tensor prodct** *of an object V in $\mathrm{Ch}^\bullet(A)$ with itself is*

$$\begin{aligned} V \wedge V &:= \ker(\mathrm{Id}_{V\otimes V} - b_{V,V}) \\ &= \mathrm{im}(\frac{1}{2}(\mathrm{Id}_{V\otimes V} + b_{V,V}))\,, \end{aligned}$$

where $b_{V,W}$ is the component of the symmetric braiding, described above.

Similarly the nth symmetric tensor power

$$\wedge^n V$$

is defined by symmetrizing, using $b_{V,V}$, over all $n!$ permutations.

Remark.

- Notice that for chain complexes concentrated in degree 0, the symmetric tensor product coincides with the usual symmetric tensor product of plain A-modules. For chain complexes with all differentials vanishing it corresponds to the graded symmetric product of the corresponding graded A-modules.

- The definition of $V \wedge V$ in terms of the image of the projector

  $$\mathrm{sym} := \frac{1}{2}(\mathrm{Id}_{V\otimes V} + b_{V,V})$$

  is convenient (see below), but does need to assume that we are working over a field not of characteristic 2.

Observation 3 **A graded-commutative dg-algebra** *is a monoid (V, μ) in $\mathbf{Ch}^\bullet(A)$ whose product factors through $V \wedge V$.*

$$\begin{array}{ccccc} V\otimes V & & \xrightarrow{\quad\mu\quad} & & V \\ {\scriptstyle \frac{1}{2}(\mathrm{Id}_{V\otimes V}+b_{V,V})}\downarrow & & & \nearrow{\scriptstyle\mu} & \\ V\wedge V & \hookrightarrow & V\otimes V & & \end{array}$$

Definition 10 *The* **tensor algebra** *over a complex V is the complex*

$$TV := \bigoplus_{n\in\mathbb{N}} \underbrace{V\otimes\cdots\otimes V}_{n} := I \oplus V \oplus (V\otimes V) \oplus \cdots\,.$$

equipped with the tautological monoid structure

The **symmetric tensor algebra** *over a complex V is the complex*

$$\Lambda^\bullet V := \bigoplus_{n\in\mathbb{N}} \wedge^n V = I \oplus V \oplus (V \wedge V) \oplus \cdots .$$

The monoid structure $\cdot : \Lambda V \otimes \Lambda V \to \Lambda V$ *on this is the one from above, composed with the graded symmetrization projector*

$$\mu : \; V^{\wedge k} \otimes V^{\wedge l} \xrightarrow{\text{sym}} V^{\wedge(k+l)} \; .$$

Here the infinite sum is defined to be the direct limit

$$\bigoplus_{n\in\mathbb{V}} \wedge^n V := \lim_{\to} \left(\bigoplus_{n=0}^{k} \wedge^n V \right) .$$

Example. Let V be an ordinary vector space, regarded as a chain complex concentrated in degree 0, with $A = k$ the ground field. Then

$$TV$$

is the ordinary tensor algebra over V,

$$\Lambda V$$

is the free symmetric tensor algebra (the **bosonic Fock space over** V) and

$$\Lambda(V[1])$$

is the (free graded-commutative) **Grassmann algebra** over V (**the fermionic Fock space over** V).

Example. Let $(\mathfrak{g}, [\cdot,\cdot])$ a finite dimensional Lie algebra over our ground field. Then the **Chevalley-Eilenberg algebra** $CE(\mathfrak{g})$ of that Lie algebra is the graded commutative dg-algebra obtained by equipping

$$\Lambda(\mathfrak{g}^*[1])$$

with the differential

$$d : \Lambda(\mathfrak{g}^*[1]) \to (\Lambda(\mathfrak{g}^*[1]) \wedge \Lambda(\mathfrak{g}^*[1]))[-1]$$

defined by

$$d|_{\mathfrak{g}^*[1]} := [\cdot,\cdot]^* .$$

The cohomology of the corresponding complex is, by definition, the Lie algebra cohomology of $\mathfrak{g}$ (with values in the trivial module).

Example. Let $(\mathfrak{g}, [\cdot,\cdot])$ a finite dimensional Lie algebra over our ground field. Then the **Weil algebra** $W(\mathfrak{g})$ of that Lie algebra is the graded commutative dg-algebra obtained by equipping

$$\Lambda(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$$

with the differential

$$d : \Lambda(\mathfrak{g}^*[1]) \to (\Lambda(\mathfrak{g}^*[1]) \wedge \Lambda(\mathfrak{g}^*[1]))[-1]$$

defined by

$$d|_{\mathfrak{g}^*[1]} := [\cdot,\cdot]^* + s^*$$

and

$$d(s^*(x)) := s^* dx$$

for all $x \in \mathfrak{g}^*[1]$ and with $s : \mathfrak{g}[2] \to \mathfrak{g}[1]$ the canonical isomorphism. The closed elements in $\Lambda\mathfrak{g}^*[2] \subset \Lambda(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$ are the symmetric invariant polynomials on $\mathfrak{g}$.

Remark. In order to put this into perspective, I make the following remark, without, at this point, trying to actually describe or explain any of these statements.

The Weil algebra $W(\mathfrak{g})$ arises from $CE(\mathfrak{g})$ in a universal way. All of the following are synonymous:

- $W(\mathfrak{g})$ is the **mapping cone** of the identity map on $CE(\mathfrak{g})$. .
- $W(\mathfrak{g})$ is the **homotopy quotient** of the identity map on $CE(\mathfrak{g})$.
- $W(\mathfrak{g})$ is the **weak cokernel** of the identity map on $CE(\mathfrak{g})$.

Moreover

- $CE(\mathfrak{g})$ plays the role of differential forms on G.
- $W(\mathfrak{g})$ plays the role of differential forms on EG.
- $\mathrm{inv}(\mathfrak{g})$, the graded commutative algebra of closed elements in $W(\mathfrak{g})|_{\wedge\mathfrak{g}^*[2]}$, plays the role of differential forms on BG.

And we have a canonical sequence

$$G \hookrightarrow EG \twoheadrightarrow BG$$

$$CE(\mathfrak{g}) \twoheadleftarrow W(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})$$

The following fact will be of importance:

Proposition 1

$$\Lambda V = \wedge^\infty (I \oplus V)$$

In particular

Corollary 1 *If $V \in \mathrm{bf}Ch^\bullet(A)$ contains in degree 0 just the tensor unit*

$$V^0 = A$$

then

$$\wedge^\infty V$$

naturally is a monoid in $\mathbf{Ch}^\bullet(A)$.

Remark. We shall be dealing with dg-algebras which are obtained from a complex V by forming $\wedge^\infty V$ and then extendending the differential on that from being co-unary to having higher co-arities.

(** This still sounds a little mysterious. It's really just supposed to convey the basic construction well familiar to dg-practitioners, but I hope to eventually say it in a nice abstract manner .**)

2.5 Various concepts from homological algebra

We will need the following standard constructions in homological algebra.

- The functor

  $$H^\bullet : \mathbf{Ch}^\bullet(X) \to \mathbf{Ch}^\bullet(X)$$

  maps each cochain complex to its cohomology

  $$H(\cdots \xrightarrow{d_V^n} V^n \xrightarrow[d_V]{n+1} V^{n+1} \longrightarrow \cdots) \quad := \quad (\cdots 0 \longrightarrow H^n(V) \xrightarrow{0} H^{n+1}(V) \longrightarrow \cdots)$$

- The **shift functors**

  $$[n] : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

  for all $n \in \mathbb{Z}$ give a $\mathbb{Z}$-action on $\mathbf{Ch}^\bullet(A)$.

  Notice the trivial but useful fact that for every complex V there is a canonical morphism

  $$V \xrightarrow{d_V} V[-1] \ .$$

- The **mapping cone** of a morphism

  $$f : V \to W$$

  of chain complexes is the chain complex

  $$(V \xrightarrow{f} W) := \left(V[1] \oplus W, d = \begin{pmatrix} d_{V[1]} & 0 \\ f[1] & d_W \end{pmatrix} \right) .$$

  The mapping cone is the "weak cokernel" of f, or "homotopy quotient": at behaves like W modulo the image of f.

- **Behaviour of cohomologies under the tensor product**

  Let V and W be objects in $\mathbf{Ch}^\bullet(A)$ with cohomologies $H(V)$ and $H(W)$, respectively. Then, in general, the cohomology of their tensor product is not the tensor products of their cohomologies

  $$H(V \otimes W) \not\simeq_{\text{i.g.}} H(V) \otimes H(W)\,.$$

  The failure of this isomorphism to exist is measured by

  $$\mathrm{Tor}^\bullet(V, W)\,.$$

  The **Künneth formula** says that we have an exact sequence of complexes

  $$0 \to H(V) \otimes H(W) \to H(V \otimes W) \to \mathrm{Tor}_1(H(V), H(W))[-1] \to 0\,.$$

  Here $\mathrm{Tor}_1^n(H(V), H(W))$ is the n-th cohomology group of any projective resolution $P_H(V)$ of $H(V)$ tensor $H(W)$:

  $$\mathrm{Tor}_1^n(H(V), H(W)) := H^n(P_{H(V)} \otimes H(W))\,.$$

  (** hope I got this right **)

- **long exact sequences in cohomology**

  Let

  $$0 \to A \to B \to C \to 0$$

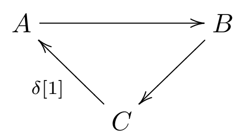
  be a short exact sequence in $\mathbf{Ch}^\bullet(A)$. Then we get a **connecting homomorphism** of cohomologies

  $$H(C) \to H(A)[-1]$$

  and hence a long exact sequence in cohomology

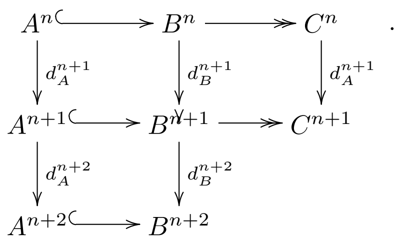
  $$\cdots A \to B \to C \to A[-1] \to B[-1] \to C[-1] \to A[-2] \to \cdots\,,$$

  also known as a triangle



  as follows:

  Consider the square of component maps of the above exact sequence of complexes.



Let c_n in C^n be closed. It comes from some element b_n in B^n. b_n need not be closed itself, but db_n has to be in the kernel of the map to C^{n+1}. Therefore there is a preimage a_{n+1} of db_n in A^{n+1}. Since db_n is closed and the map from A to B is an injection, also a_{n+1} needs to be closed. Its cohomology class is defined to be the image under the connecting homomorphism of the class of c_n.

3 Lie ∞-modules and their Chevalley-Eilenberg algebras

We recall ordinary Lie-Rinehart pairs and Lie ∞-algebras and their dual Chevalley-Eilenberg algebras. We formulate the Chevalley-Eilenberg algebra of a Lie-Rinehart pair in a suggetsive way and the propose the obvious generalization of that to be a Lie-Rinehart ∞-pair.

3.1 Lie-Rinehart pairs

Definition 11 *An ordinary Lie-Rinehart pair* $(\mathfrak{g}, B)$ *is a Lie algebra* $\mathfrak{g}$ *together with an associative algebra* B *such that both are modules over each other*

$$\rho : \mathfrak{g} \to \mathrm{Der}(A)$$

$$\mu : A \otimes_K \mathfrak{g} \to \mathfrak{g}$$

in a compatible way mimicking that of the archetypical Lie-Rinehart pair which is$(\mathfrak{g} = \Gamma(TX), B = C^\infty(X))$ *for* X *some smooth manifold.*

Remark. Lie-Rinehart pairs $(\mathfrak{g}, A)$ for which $A = C^\infty(X)$ for some smooth manifold X are precisely equivalent to Lie algebroids $E \to X$ over X. The Lie algebra $\mathfrak{g} = \Gamma(E)$ is that on the sections of E and the Lie action of $\mathfrak{g}$ on A is that coming from the anchor map of the Lie algebroid. The archetypical Lie-Rinehart pair $(\Gamma(TX), C^\infty(X))$ corresponds to the **tangent algebroid** of X, which is the differential version of the fundamental groupoid of X.

Definition 12 (The Chevalley-Eilenberg algebra of a Lie-Rinehart pair) *Let*

$$\mathfrak{g}^* := \mathrm{Hom}_{A\mathrm{Mod}}(\mathfrak{g}, A)[1]$$

be the dual, over A*, of the Lie algebra of a Lie-Rinehart pair* $(\mathfrak{g}, A)$*, regarded as being in degree 1. Let*

$$V := \ (0 \longrightarrow V^0 \xrightarrow{d_V^1} V^1 \longrightarrow 0)$$

$$(0 \longrightarrow A \xrightarrow{\rho(\cdot)(\cdot)} \mathfrak{g}^* \longrightarrow 0)$$

in $\mathbf{Ch}^\bullet(X)$ *and form*

$$\wedge^\infty V = (A \oplus \mathrm{Hom}_{A\mathrm{Mod}}(\mathfrak{g}, A)[1] \oplus \wedge^2_A \mathrm{Hom}_{A\mathrm{Mod}}(\mathfrak{g}, A)[1] \oplus \cdots,\ \Lambda d_V)$$

in $\mathbf{Ch}^\bullet(A)$.

Let

$$F : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(K)$$

be the forgetful functor from complexes of A-modules to mere complexes of vector spaces. The image $F(\wedge^\infty V)$ remembers the fact that it comes from complexes of A-modules in that there is a canonical monoid structure

$$\mu : F(\wedge^\infty V) \otimes_{\mathrm{Ch}^\bullet(k)} F(\wedge^\infty V) \to F(\wedge^\infty V)$$

defined componentwise by forming the wedge product over A

$$\mu : \; V^k \otimes_k V^l \xrightarrow{\mathrm{sym}_A} V^k \wedge_A V^l \; .$$

The differential induced by the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \otimes_K \mathfrak{g} \to \mathfrak{g}$ which acts as

$$d_{(\mathfrak{g},A)} : \; \mathfrak{g}^*[1] \xrightarrow{[\cdot,\cdot]^*} (\mathfrak{g}^*[1] \wedge_A \mathfrak{g}^*[1])[-1]$$

extends to a differential

$$d_{(\mathfrak{g},A)} : F(\wedge^\infty V) \to F(\wedge^\infty V)[-1]$$

which extends the differential on V in that we have a morphism

$$\begin{array}{c} (F(\wedge^\infty V), d_{(\mathfrak{g},A)}) \\ \big\downarrow \\ V \end{array}$$

in $\mathrm{Ch}^\bullet(k)$*. The complex $(F(\wedge^\infty V), d_{(\mathfrak{g},A)})$ is the Chevalley-Eilenberg complex of the Lie-Rinehart pair $(\mathfrak{g}, A)$.*

Remark. One point here deserves emphasis:

While it is crucial that $\mathfrak{g}$ is an A-module, the differential

$$d_{(\mathfrak{g},A)} : \; \mathfrak{g}^*[1] \xrightarrow{[\cdot,\cdot]^*} (\mathfrak{g}^*[1] \wedge_A \mathfrak{g}^*[1])[-1]$$

is *not* an A-module homomorphism. Rather, it is an A-module derivation. It is this fact which is encoded in the statement that $d_{(\mathfrak{g},A)}$ is actually a differential not on ΛV, but on $F(\Lambda V)$. That frees it from having to be an A-module homomorphism. But then the condition that we have the obvious monoidal structure on $F(\Lambda V)$ forces it to be an A-module derivation.

We think of the monoidal structure

$$\cdot : F(\Lambda V) \otimes_{\mathrm{Ch}^\bullet(K)} F(\Lambda(V)) \to F(\Lambda(V))$$

as being the **free graded commutative over** A.

Example (the tangent algebroid). Let X be a smooth manifold and $A = C^\infty(X)$ the algebra of smooth functions on X.

Let $\mathfrak{g}^* = \Omega^1(X)$, naturally regarded as being in degree 1, and $B = C^\infty(X)$ be two objects in $\mathrm{Ch}^\bullet(C^\infty(X))$ and form

$$\wedge^\infty(\mathfrak{g}^* \oplus B) = \wedge^\infty(\Omega^1(X) \oplus C^\infty(X))\,.$$

Notice that due to

$$\Omega^1(X) \wedge_{C^\infty(X)} \Omega^1(X) = \Omega^2(X)$$

this is nothing but $\Gamma(\Lambda^\bullet T^*X)$.

$\mathfrak{g}^*$ is equipped with the structure (dual to) a Lie 1-algebra where the co-binary differential

$$d : \mathfrak{g}^* \to (\mathfrak{g}^* \otimes \mathfrak{g}^*)[-1] = \Omega^2(X)$$

is just the deRham differential.

Hence

$$\begin{array}{c}(\wedge^\infty\Omega^1(X)), d_{dR})\\ \downarrow \\ (C(X) \stackrel{d_{\mathrm{dR}}}{\to} \Omega^1(x))\end{array}$$

is the deRham complex with the degree 0 part truncated. To get a Lie-Rinehart pair we need to find an extension of the differential

3.2 L_∞-algebras

Definition 13 *Let V be a positively graded vector space. An L_∞-algebra over V is a codifferential*

$$D : S^c(V) \to S^c(V)$$

on the free graded-commutative coalgebra over $S^c(V)$ V such that

- *the degree of D is -1*
- $D^2 = 0$.

If V is finite dimensional, then this is, dually, the same as a differential $d : \Lambda V^* \to \Lambda V^*$ defined by

$$d\omega = \omega(D(\cdot))$$

for all $\omega \in V^*$. This satisfies

- the degree of d is $+1$
- $d^2 = 0$.

Hence this is a graded-commutative dg-algebra, which is free as a graded commutative algebra. It generalizes the Chevalley-Eilenberg algebra from Lie algebras to L_∞-algebras.

We can try to say this entirely internal to the category of cochain complexes:

Let $V^* \in \mathbf{Ch}^\bullet(K)$ be a *positively* graded cochain complexes. Then (the dual of an) L_∞-structure on V^* is a complex

$$(\Lambda V^*, d)$$

which is equipped with an epimorphism

$$\begin{array}{c} (\Lambda V^*, d) \;\; . \\ \Big\downarrow{\scriptstyle p} \\ V^* \end{array}$$

Notice that d can be decomposed into is components of homogeneous co-arity:

$$d = d_0 + d_1 + d_2 + \cdots$$

with

$$d_k : V^* \to (\wedge^k V^*)[-1] \, .$$

The above projection is

$$p = \begin{cases} \mathrm{Id} & \text{on } \wedge^1 V^* \\ 0 & \text{else} \end{cases} \, .$$

Hence the co-unary component d_0 of d has to be the original differential on V^*.

Example. For each Lie algebra $\mathfrak{g}$ with a degree $(n+1)$-cocycle μ, we get an L_∞-algebra structure (dually)

$$(\mathfrak{g}_\mu)^* := (\Lambda(\mathfrak{g}^*[1] \oplus \mathfrak{u}(1)[n]), d)$$

by setting

$$d|_{\mathfrak{g}^*[1]} = d_{\mathrm{CE}(\mathfrak{g})}$$

and

$$db = \mu$$

for b the canonical basis of $\mathfrak{u}(1)[n]$.

For $\mathfrak{g}$ semisiple and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle on it, this is (the CE-alegebra dual to) the **String Lie 2-algebra**.

Similarly, one obtains L_∞-algebras $\mathrm{ch}_k(\mathfrak{g})$ for each invariant polynomial k on $\mathfrak{g}$ and $\mathrm{cs}_k(\mathfrak{g})$ for each transgression element interpolating between μ and k.

3.3 Lie-Rinehart ∞-pairs

We now give the obvious generalization of our definition of the Chevalley-Eilenberg algebra of a Lie-Rinehart pair.

Definition 14 *Let A be a commutative algebra and $\mathfrak{g}^*$ a* positively *graded object and*

$$B = \left(\cdots \longrightarrow B^{-2} \xrightarrow{d_B^{-1}} B^{-1} \xrightarrow{d_B^0} B^0 \longrightarrow 0\right)$$

$$\left(\cdots \longrightarrow B^{-2} \xrightarrow{d_B^{-1}} B^{-1} \xrightarrow{d_B^0} A \longrightarrow 0\right)$$

a non-positively *graded object in* $\mathbf{Ch}^\bullet(A)$*.*

A **Lie-Rinehart ∞-pair** *structure on $\mathfrak{g}^*$ and B is a differential*

$$\begin{array}{c} (F(\wedge_A^\infty(\mathfrak{g}^* \oplus B)), d_{(\mathfrak{g},B)}) \ , \\ \big\downarrow \\ \mathfrak{g}^* \oplus B \end{array}$$

respecting the free graded-commutative product over A

$$\mu : F(\wedge_A^\infty(\mathfrak{g}^* \oplus B)) \otimes_{\mathbf{Ch}^\bullet(K)} F(\wedge_A^\infty(\mathfrak{g}^* \oplus B)) \to F(\wedge_A^\infty(\mathfrak{g}^* \oplus B))$$

defined above.

Remark. Notice that

- An ω-vector B space is a *non-positive* cochain complex.
- The CE algebra $\mathfrak{g}^*$ of a Lie ∞-algebra is a *positively* graded cochain complex.
- The cochain complex
$$\mathfrak{g}^* \oplus B$$
of a Lie-Rinehart ∞-pair is hence in *arbitrary* degree.

Example. Our main example for this shall be the BV complex which we turn to in the following.

Example. (Lie ∞-algebra acting on itself) One crucial consistency check on our definitions is: every Lie ∞-algebra should be a module over itself. Compare the definition in [2].

So, given the cochain complex $\mathfrak{g}$ in non-positive degree with a Lie ∞-structure

$$d_2 : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

LR-∞	$\Lambda($	$\mathfrak{g}^*$	$\oplus$	A	$\oplus$	B	$)$
degree		$+$		0		$-$	
physics name		ghosts		fields		anti-ghosts anti-fields	

Table 3: **Lie-Rinehart ∞ pairs interpreted in the language of BV-complexes**

$$d_3 : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^* \wedge \mathfrak{g}^*$$

etc, we want to naturally extend the differential to

$$\Lambda(\mathfrak{g}^* \oplus \mathfrak{g})\,.$$

And we can indeed to so by defining

$$d_2 : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}^*$$

$$d_3 : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}^* \wedge \mathfrak{g}^*$$

etc. by dualizing on the input and on one output.

4 The BV-complex

The ordinary BV-complex is a Lie-Rinehart ∞-pair $(\mathfrak{g}, B)$ where $\mathfrak{g}$ is an ordinary Lie algebra ("of local symmetries") acting on a space X and where B is a **Koszul-Tate resolution** of functions on a quotient of X.

4.1 The Koszul complex

Definition 15 (dual Koszul complex) *For*

$$f : M \to A$$

a morphism of A-modules with M of rank n, let

$$V := \left(0 \longrightarrow V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0 \right)$$

$$\left(0 \longrightarrow M \xrightarrow{f} A \longrightarrow 0 \right)$$

in $\mathbf{Ch}^\bullet(A)$ *be the corresponding cochain complex in degree -1 and 0. Then*

$$K[f] := \wedge^n V$$

is the (dual) Koszul complex defined by f*.*

Example. The main example of interest in the BV context is this: Let $A = C(X)$ for X some manifold. Let TX be the tangent bundle over X and $M = \Gamma(TX)$ its space of sections. Let $S \in C(X)$ be any function on X and set $f = dS(\cdot)$.

$$V := \left(0 \longrightarrow V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0 \right) \;.$$

$$\left(0 \longrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \longrightarrow 0 \right)$$

In the special case that M happens to be a free A-module $M = A^n$ (e.g. sections of a trivial vector bundle in the above example), any tuple $(E_a \in A)_{a=1}^n$ of elements in A provides a morphism

$$E : A^n \to A$$

by matrix multiplication

$$E : \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} \mapsto [E_1, \cdots, E_n] \cdot \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = \sum_{k=1}^n a^k E_k$$

we can give the following equivalent definition of the (dual) Koszul complex.

Definition 16 *For $E_1, \cdots, E_n \in A$ a sequence of elements of A, let*

$$K[E_a] := \left(0 \longrightarrow K[E_a]^{-1} \xrightarrow{d^0_{K[E_a]}} K[E_a]^0 \longrightarrow 0\right) .$$

$$\left(0 \longrightarrow A \xrightarrow{\cdot E_a} A \longrightarrow 0\right)$$

The complex

$$K[E_1, \cdots, E_n] := K[E_1] \otimes \cdots \otimes K[E_n]$$

is the (dual) **Koszul complex** *associated with the (E_i).*

Example. For two elements $K[E_1, E_2]$ looks as follows.

$$K[E_1, E_2] = K[E_1] \otimes K[E_2] := \left(0 \longrightarrow A \xrightarrow{\left[\begin{smallmatrix} -E_2 \\ E_1 \end{smallmatrix}\right]} A^2 \xrightarrow{[E_1, E_2]} A \longrightarrow 0\right) .$$

Notice that the left A factor arises as $A \otimes_A A$.

Remarks.

- $$K[E_1, \cdots, E_n] := \left(\cdots \longrightarrow A^{\oplus n} \xrightarrow{\sum_{a=1}^n \cdot E_k} A \longrightarrow 0\right)$$

- The cohomology of the Koszul complex in degree 0 is the quotient $K[E_1, \cdots, E_n]$ form the quotient
$$A/(E_1, \cdots, E_n)A, .$$
Therefore, in the case that all other cohomology groups of the Koszul complex vanish, it provides a **resolution** of this quotient. Since all A-modules appearing in the Koszul complex are free, this resolution is necessarily *projective*.

- Therefore the cohomologies of the Koszul complex in non-vanishing degree measure the *interdependency* of the E_a. In particular, cohomology in degree -1 contains the *relations* among the E_a, namely tuples of elements $v \in A^{\oplus n}$ such that
$$v^a E_a = 0$$
modulo the *trivial relations*.

The idea that the Koszul complex measures the independency of the elements $(E_1, \cdots, E_n)$ is made precise by the following standard definition and fact.

Definition 17 (Regular sequence) *An element $E \in A$ is called* **regular** *if it is not a zero-divisor and if $A/EA \neq 0$. For M an A-module, E is called* M**-regular** *if it is not a zero-divisor and M/EM is nonzero.*

A sequence of elements $(E_1, \cdots, E_n \in A)$ is M-regular if E_k is $M/(E_1, \cdots, E_{k-1})M$-regular.

Example. Let $A = C^\infty(X)$ for some manifold X. Then a function $E \in A$ is regular, if there is no open set on which it vanishes. Let $E \to X$ be a vector bundle and $M = \Gamma(E)$ be its module of sections. Then, again, E is M-regular if it is just regular.

Fact 5 *For A a local ring and M a finitely generated module, any sequence $(E_1, \cdots, E_n \in A)$ is M-regular precisely if the Koszul complex $K[E_1, \cdots, E_n]$ is a resolution of $M/(E_1, \cdots, E_n)$.*

If the Koszul complex $K[E_1, \cdots, E_n]$ is not a resolution, we can read off from its failure of acyclicity the maximal regular sequence inside $(E_1, \cdots, E_n)$.

Fact 6 *If precisely the highest r cohomology groups of $K[E_1, \cdots, E_2]$ vanish, then the maximal regular sequence inside $(E_1, \cdots, E_2)$ has length r.*

More precisely, if

$$H^{-n+j}(K[E_1, \cdots, E_n]) = 0$$

for all $j < r$, while

$$H^{-n+r}(K[E_1, \cdots, E_n]) \neq 0$$

then every maximal regular sequence inside $(E_1, \cdots, E_n)$ has length r.

Remark. The local rings of relevance in the BV context are formal power series $K[[X_1, \cdots, X_n]]$.

Example. Let $A = K[[x_1, x_2]]$ be a power series in two variables. Then (x_1) is a regular sequence and $A/(x_1)A \simeq K[[x_2]]$. Compare this with the example 2 below.

Example. Continue the example $f = dS(\cdot)$ from above. Let $X \subset \mathbb{R}^n$ such that $TX = \mathbb{R}^n \times X$. Sections of TX are simply n-tuples of functions

$$s = \begin{bmatrix} s^1 \\ \vdots \\ s^n \end{bmatrix} \in M_{n\times 1}(C(X))\,.$$

Then for $S \in C(X)$ any function we have globally

$$dS = [dS_1, \cdots, dS_n] \in M_{1\times n}(C(X))\,.$$

4.2 The Tate construction: killing of cohomology groups

In practice it is often useful to resolve quotients $A/(E_1, \cdots, E_n)$ where the $(E_1, \cdots, E_n)$ do *not* form a regular sequence.

There is a canonical procedure, going back to John Tate (and used throughout rational homotopy theory, see the example below), to systematically "kill" all unwanted cohomology groups by introducing further generators.

Example. Let again

$$f : M \to A$$

be a morphism of A-modules with nontrivial kernel

$$\ker(f) \hookrightarrow M \; .$$

Then instead of using the 2-term complex

$$\begin{array}{ccccccccc} V := & (0 & \longrightarrow & V^{-1} & \xrightarrow{d_V^0} & V^0 & \longrightarrow & 0) \\ & (0 & \longrightarrow & M & \xrightarrow{f} & A & \longrightarrow & 0) \end{array}$$

consider the 3-term complex

$$\begin{array}{ccccccccccc} W := & (0 & \longrightarrow & W^{-2} & \xrightarrow{d_W^{-1}} & W^{-1} & \xrightarrow{d_W^0} & W^0 & \longrightarrow & 0) \; . \\ & (0 & \longrightarrow & \ker(f) & \hookrightarrow & M & \xrightarrow{f} & A & \longrightarrow & 0) \end{array}$$

anti ghosts	anti fields	fields

The introduction of the new term in degree -2 "kills" all unwanted cohomology in degree -1. Therefore, by construction, the cohomology of W is concentrated in degree 0

$$\begin{array}{ccccccccccc} H(W) := & (0 & \longrightarrow & H(W)^{-2} & \xrightarrow{d_{H(W)}^{-1}} & H(W)^{-1} & \xrightarrow{d_{H(W)}^0} & H(W)^0 & \longrightarrow & 0) \; . \\ & (0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A/\mathrm{im}(f) & \longrightarrow & 0) \end{array}$$

It might seem that forming now $\wedge^n W$ instead of $\wedge^n V$ produces a dg-algebra with no cohomology away from degree 0.

However, the Künneth formula tells us, for the cohomology of the tensor product of two complexes X and Y, that

$$0 \to \left(\bigoplus_{p+q=n} H(X)^p \otimes H^q(q) \right) \to H^n(X \otimes Y) \to \left(\bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \right) \to 0$$

is exact. This implies that even if the cohomologies of X and Y are both concentrated in degree 0, we get

$$H^{-1}(X \otimes Y) \simeq \mathrm{Tor}(H^0(X), H^0(Y)),$$

which may be nontrivial.

Example. Suppose that the non-positively graded complex V is a projective resolution of A/I in degree 0, $H^0(V) = A/I$, $H^{-n<0}(V) = 0$ for $I = (E_1, \cdots, E_n)$ some ideal generated by a regular sequence. For instance take $V = K[E_1, \cdots, E_n]$ to be the Koszul complex of that regular sequence.

Or, in our standard example, consider

$$V = \; (0 \longrightarrow V^{-2} \xrightarrow{d^{-1}_{V_{-1}}} V^{-1} \xrightarrow{d^0_V} V^0 \longrightarrow 0)$$

$$(0 \longrightarrow \ker(dS(\cdot)) \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \longrightarrow 0)$$

and assume that $\ker(dS(\cdot))$ is projective, i.e. sections of a vector bundle over X.

Then the following is true (see [Loday:Cyclic Homology, 3.4.7])

Fact 7 *The first Tor-algebra is*

$$\mathrm{Tor}^{-1}(A/I, A/I) \simeq I/I^2 .$$

The higher Tor-algebras are the exterior powers of this:

$$\mathrm{Tor}^{-\bullet}(A/I, A/I) \simeq \wedge^\bullet_{A/I}(I/I/2)$$

Remark. If $A = C(X)$, then $\wedge^\bullet(I/I^2)$ is the algebra of differential forms on X restricted the vanishing set of $(E_1, \cdots, E_n)$.

So we find in this case that even though the cohomology of V is concentrated in degree 0, the cohomology of $V \otimes V$ can be nontrivial in degree -1.

The Tate construction Let $(V, \cdot)$ be monoid in $\mathrm{Ch}^\bullet(A)$, hence a dg-algebra over A, concentrated in either non-positive or non-negative degree. Let us assume V is in non-positive degree for definiteness, as in our applications.

There is a systematic way to create from $(V, \cdot)$ a new dg-algebra $(V', \cdot)$ extending it

$$(V', \cdot) \longrightarrow\!\!\!\!\to (V, \cdot)$$

with the property that the cohomology of V' vanishes everywhere except in degree 0, where it coincides with the cohomology of V.

The procedure works by induction over the degree of the cohomology groups:

- Let $(V_{-k}, \cdot) \longrightarrow\!\!\!\!\to (V, \cdot)$ be a dg-algebra extending V such that

$$H^{-k<d<0}(V_{-k}) = 0\,.$$

- add an addition generator etc [need to rewrite this]

Using this procedure, one obtains the following

Fact 8 (Tate) *For I any ideal in $\mathcal{A}$ there exists a free acyclic dg-algebra X such that $H^0(X) = \mathcal{A}/I$.*

In other words: we can *always* find *some* resolution of a quotient A/I by a dg-algebra.

Remark. In the context of the BV formalism, it is for this reason that one is actually not primarily interested in Koszul complexes themselves: even if they fail to provide a resolution, using the Tate construction ("incorporating (possibly higher order)antighosts"), one always forms a resolution of the "shell".

Example (rational homotopy groups of spheres). Let $A = \mathbb{R}$ be the field of real numbers.

1. Suppose we want to build a graded-commutative dg algebra V with the only nontrivial cohomology group being $H^{2n+1}(V) = A$. Clearly, this is simply achieved by letting V be generated from a single degree $2n+1$ generator ω

$$V = V^{2n+1} = \langle\omega\rangle$$

   with $d\omega = 0$. Since,

$$\omega \wedge \omega = 0$$

   due to the fact that $2n+1$ is odd, this choice is consistent and no further generators need to be introduced.

2. But now consider the same situation for even degree: suppose the graded-commutative dg-algebra V has a single non-exact degree $2n$-generator ω with $d\omega = 0$. Then the cohomology in degree $2n$ is again A. But now also all elements of the form $\omega \wedge \omega \wedge \cdots \omega$ are non-vanishing and closed. In

order to remove the unwanted cohomology generated by these, we throw in another generator, λ, in degree $4n-1$, and set

$$d\lambda = \omega \wedge \omega \,.$$

This removes all the superfluous cohomologies: now all troublesome elements are exact.

$$\omega \wedge \omega \wedge \underbrace{\omega \wedge \cdots \wedge \omega}_{k \in \mathbb{N}} = d(\lambda \wedge \underbrace{\omega \wedge \cdots \wedge \omega}_{k}) \,.$$

Notice that

- the nontrivial homology groups of the n-sphere are

$$H_k(S^n, \mathbb{Z}) = \left\{ \begin{array}{cc} \mathbb{Z} & k = n, 0 \\ 0 & else \end{array} \right.$$

- the *rational* homotopy groups of the n-sphere are

$$\pi_k(S^{2n+1}) = \left\{ \begin{array}{cc} \mathbb{Q} & k = 2n+1 \\ 0 & else \end{array} \right.$$

$$\pi_k(S^{2n}) = \left\{ \begin{array}{cc} \mathbb{Q} & k = 2n, k = 4n-1 \\ 0 & else \end{array} \right.$$

.

This matches the pattern which we found for complexes with cohomology in a single degree, under the identification

quasi-free dg algebra	**rational top. space**
degree k cohomology	degree k chomology
degree n generators	rational homotopy group in degree n

Table 4: **The relation between dg-algebra and topological spaces** in terms of rational cohomology and homotopy groups.

4.3 The BV complex for the (-1)-brane

In practice the BV complex is applied in the context of local functions on some jet space.

But here we want to get rid of inessential technicalities as far as possible and try to extract the pure relevant structure. For that reason we study the BV complex in a setup where we are dealing with ordinary functions on some manifold.

Mathematically, this amounts to studying the critical points of an ordinary function by cohomological means. Physically it can fruitfully be thought of as the standard BV formalism applied to what is called a (-1)-brane: an object whose worldvolume is a single point.

Remark: the algebra of functions Some of the crucial statements about the cohomologies of the complexes we are about to consider depend sensitively on the precise nature of the function algebras over which we work.

For X a (real) manifold, we shall write $C(X)$ for an unspeified class of functions on X as long as it does not matter. When it matters, we will choose from

$$C(X) \rightsquigarrow \begin{cases} C^\omega(X) & \text{real analytic functions} \\ C^\infty(X) & \text{smooth functions} \end{cases}$$

4.3.1 The ingredients

Definition 18 *Let X be a smooth manifold and $S \in C^\infty(X)$ a smooth function on it. Denote by $\Gamma(TX)$ the space of smooth vector fields on X.*

Consider the 3-term chain complex

$$\begin{array}{ccccc} \ker(dS(\cdot)) & \hookrightarrow & \Gamma(TX) & \xrightarrow{dS(\cdot)} & C^\infty(X) \\ \\ -2 & & -1 & & 0 \end{array}$$

with the degrees as indicated. The corresponding 3-vector space we denote

$$WC(\Sigma) \in 3\mathrm{Vect}\,.$$

The physics terminology. The entities in the above definition are known in the physics literature under the following names.

- The space X is the **configuration space** or the **space of histories** (the difference need not concern us here).
- An point $x \in X$ is a **field configuration**.
- The function S is the **action**.
- The condition $dS = 0$ is the **equations of motion**.

- The space of critical points $\Sigma := \{x \in X \mid (dS)_x = 0\}$ is the **shell**.
- The elements of the space $\ker(dS(\cdot))$ are the **Noether identities**.

The notation $WC(\Sigma)$ is for smooth functions on the "*w*eak *s*hell". By construction, the cohomology of $WC(\Sigma)$ is concentrated in degree 0, where it is $C(X)/\mathrm{im}(dS(\cdot))$. Following the situation in physical examples, we assume that $\mathrm{im}(dS(\cdot))$ is the space of all smooth functions on X that vanish on Σ. (** When is this assumption satisfied?? **)

Observation 4 *Hence the above chain complex is a resolution for the space of on-shell functions, which is the way it is usually thought of. Passing from homotopical to n-categorical language this means: the 3-vector space $WC(\Sigma)$ is equivalent, as a 3-vector space, to the 1-vector space of on-shell functions.*

4.3.2 Symmetries and Noether identities

We are now going to give what is supposed to be the standard definition of *symmetries* and *Noether identities* as they appear in physics, adapted to the toy example we are looking at, where X is a mere manifold.

Definition 19 *Given $S \in C(X)$ as above, we say*

- *A (local)* **symmetry** *of S is a vector field $v \in \Gamma(TX)$ such that the Lie derivative*

$$\mathcal{L}_{\epsilon v} dS = 0$$

  for all $\epsilon \in C(X)$.

- *A* **Noether identity** *of S is a vector field v such that*

$$\mathcal{L}_v S = 0\,.$$

Remark. To see the connection of this definition to the definitions one might find in most of the physics literature notice that

- The above says that a local symmetry preserves the equations of motion. This corresponds to the more common requirement that the local symmetry preserves the Lagrangian up to a divergence. Compare with p.7 of [KazinskiLyakhovichSharapov:1993].
- To see that a Noether identity can be regarded as a vector preserving the action in our context, take the usual defintion and truncate jet space at 0th order everywhere. Compare with [StasheffFulpLada:2002].

In the same vein, the following plays the role of **Noether's second theorem** in the context of our toy example.

Proposition 2 (toy version of Noether's second theorem) *The space of local symmetries is isomorphic to that of Noether identities.*

Proof. Using Cartan's "magic formula" we have

$$
\begin{aligned}
\mathcal{L}_{\epsilon v} dS &= \epsilon \mathcal{L}_v dS + d\epsilon \wedge v(S) \\
&= \epsilon d(v(S)) + v(S) d\epsilon \,.
\end{aligned}
$$

Clearly, every Noether identity v is hence also a local symmetry. Conversely, if v is a local symmetry then from $\mathcal{L}_v dS = 0$ and using $\mathcal{L}_{\epsilon v} dS = 0$ in the above formula for all $\epsilon \in C(X)$ it follows that v is a Noether identity. ☐

$$
V = \quad \begin{array}{c} \ker(dS(\cdot)) \\ \| \\ C(X) \otimes \mathfrak{g} \end{array} \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \xrightarrow{\quad d_\rho \quad} C(X) \otimes \mathfrak{g}^*
$$

chain complex $V =$	$\ker(dS(\cdot)) = C(X)\otimes\mathfrak{g}$	$\Gamma(TX)$	$C(X)$	$C(X)\otimes\mathfrak{g}^*$
degree	-2	-1	0	1
	←— ass. 3-algebra		—→	←— Lie 1-algebra —→
Noether's second thm.	Noether identities			local symmetries
math guys	Tate	Koszul		Chevalley-Eilenberg
physics names	*antighosts*	*antifields*	*fields*	*ghosts*
antifield number = **ω-vector space dimension**	2	1	0	
ghost number = **ω-covector space dimension**				1

Table 5: **The structure of the BV complex** for the simple case where the action functional S is a mere function on a manifold. From the complex $V \in \mathbf{Ch}^\bullet(C(X))$ shown the full BV complex is obtained following def 14 by giving an extension of the differential on $F(\Lambda V)$ compatible with the one on ΛV induced by the one shown above.

4.3.3 Local symmetry Lie algebras

We now make the assumption that we have a "Lie algebra of local gauge symmetries".

Definition 20 *In the case that the space of local symmetries of S is of the form*

$$C(X) \otimes \mathfrak{g}$$

with $\mathfrak{g}$ a finite-dimensional Lie algebra equipped with an action

$$\rho : \mathfrak{g} \to \Gamma(TX)$$

we say that $\mathfrak{g}$ is the **gauge Lie algebra** *of S and that ρ are the* **gauge transformations of the fields**.

In that case we can extend our 3-term complex $WC(\Sigma)$ by the map

$$C(X) \xrightarrow{d_\rho} C(X) \otimes \mathfrak{g}^*$$

by setting

$$d_\rho : f \mapsto \rho(\cdot)(f)\,.$$

Remark. Notice that d_ρ is the differential of the Chevalley-Eilenberg complex that computes the Lie algebra cohomology of $\mathfrak{g}$ with values in the Lie module $C(X)$ restricted to degree 0.

Example 1 (the case where X is a principal bundle)

In the more well-behaved situations the local symmetries will act freely on our space X, and X will be a principal G-bundle.

So assume that the Lie group G acts on X such that $p : X \to X/G$ is a principal G-bundle. Let

$$S \in C(X)$$

be the pullback of a smooth function $S_G \in C(X/G)$ downstairs

$$S := p^* S_G$$

which has the property that it is not annihilated by any nontrivial vector field on X/G.

Then the local symmetries of S are precisely the *vertical* vector fields on the G-bundle X, namely sections

$$\Gamma_{\mathrm{vert}}(P) := \Gamma(\mathrm{Vert}(X)) \simeq C(X) \otimes_{\mathbb{R}} \mathfrak{g}$$

of the vector bundle of vertical vector fields

$$\mathrm{Vert}(TX) := \ker(dp) \simeq P \times \mathfrak{g}\,.$$

So in this case our complex is

$$(0 \longrightarrow B^{-2} \longrightarrow B^{-1} \longrightarrow C(X) \longrightarrow A \otimes \mathfrak{g} \longrightarrow 0) \ .$$

$$(0 \longrightarrow \Gamma_{\mathrm{vert}}(TX) \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \xrightarrow{d_\rho} C(X) \otimes_{\mathbb{R}} \mathfrak{g} \longrightarrow 0)$$

As a simple special case, consider the following example, which models essentially the harmonic oscillator with a circle worth of gauge degeneracies thrown in.

Example 2 (invariant function on the trivial circle bundle)

Let

$$X = \mathbb{R} \times S^1$$

the cylinder, thought of as the trivial circle bundle

$$p : X \to \mathbb{R}$$

and let the action $S \in C^\omega(X)$ be

$$S = p^*(x \mapsto x^2)\,.$$

Then

$$\ker(dS) = \ker(dp)$$

are the analytic vertical vector fields on S^1.

Since dS is just multiplication of (component) functions by x, we find that the on-shell functions are indeed precisely the quotient

$$C^\omega(\Sigma) \simeq \mathrm{coker}(dS(\cdot)) = C^\omega(X)/\mathrm{im}(dS(\cdot))\,.$$

Since furthermore

$$\ker(d_\rho) \simeq C^\omega(X/G)$$

we find that the cohomology of the above complex in degree 0 is precisely that of gauge-invariant on-shell functions

$$C^\omega(\Sigma/G) = C^\omega(\{0\}) = \mathbb{R}\,.$$

4.3.4 Extending to the full BV complex

With our complex $V \in \mathbf{Ch}^\bullet(C(X))$ given,

$$V = (\ 0 \longrightarrow \ker(dS(\cdot)) \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \xrightarrow{d_\rho} C(X) \otimes \mathfrak{g}^* \longrightarrow 0\ )$$

we now form

$$\wedge^\infty V$$

and extend the differential on

$$F(\wedge^\infty V)$$

such that the result is still a monoid in $\mathbf{Ch}^\bullet(K)$ with respect to the canonical monoidal structure induced from the fact that V comes from $\mathbf{Ch}^\bullet(C(X))$.

This "free-over-$C(X)$" differential graded-commutative algebra $(F(\wedge^\infty V), d)$ is then the BV complex describing our (-1)-brane.

In positive degree we require it to cover

$$\begin{array}{c} (F(\wedge^\infty V), d) \\ \downarrow \\ (F(\Lambda \mathfrak{g}^*), d_{\mathrm{CE}(\mathfrak{g})}) \end{array}$$

the Chevalley-Eilenberg algebra of our Lie algebra, such that we can address it as a Lie-Rinehart pair $(\mathfrak{g}, B)$ for $\mathfrak{g}$ the ordinary Lie algebra and B our Koszul-Tate resolution of on-shell functions.

Etc.

4.3.5 The dual complex and distribution-valued fields

We have addressed $C(X)$ as the space of fields. But more precisely it is the dual to the space of fields, since X is to be interpreted as the space of field configurations.

What should really be addressed as a physical field is hence an element dual to $C(X)$, hence a *linear functional*, a distribution, on $C(X)$.

Then we obtain the following statements

- A **delta-distribution field**

  $$\phi(x) : C(X) \to k$$

  $$\phi(x) : f \mapsto f(x)$$

  is closed precisely when $x \in X$ is a solution of the equations of motion, $dS_x = 0$. Because for all $v \in \Gamma(TX)$

  $$(d^*\phi(x))(v) = \phi(x)(dS(v)) = dS_x(v)$$

  and hence

  $$d^*\phi(x) = 0 \;\Leftrightarrow\; dS_x = 0\,.$$

- A **derivative of a delta-distribtion field**

  $$v(\phi(x)) : f \mapsto (v(f))(x) \qquad v \in \Gamma(TX)$$

  is exact precisely when $v \in \Gamma(TX)$ is a local symmetry. Because let $t \in \mathfrak{g}$ such that

  $$t \otimes \delta_x : C^\infty(X) \otimes \mathfrak{g}^* \to k\,,$$

  then for $f \in C(X)$

  $$d^*(t \otimes \delta_x) = (t \otimes \delta_x)(df) = (t \otimes \delta_x)(\rho(\cdot)(f)) = \rho(t)(f)|_x\,.$$

dual complex	$\mathcal{D}_0(X) \otimes_{C(X)} \Omega^1(X) \xleftarrow{d^*}$	$\mathcal{D}_0(X) \xleftarrow{d^*}$	$\mathcal{D}_0(X) \otimes \mathfrak{g}$
mathematical objects	distribution-valued 1-forms	distribution-valued functions	distribution- and Lie algebra-valued functions
physical interpretation	antifields	fields	ghosts
cohomology		cycles: Z^0 = on shell boundaries: B^0 = pure gauge	

Table 6: The dual complex knows about **distribution-valued physical fields**. Here $\mathcal{D}_p(X) := \mathrm{Hom}_k(\Omega^p(X), k)$ is the space of *p-currents.* (For $p = 0$ these are just distributions.)

4.4 Formulation as a Lie-Rinehart ∞-pair

We now make explicit the fact that the BV-complex is the CE-algebra of a Lie-Rinehart ∞-pair using our definition.

(** I haven't really stated that definition yet here, but it goes something like this

Definition 21 *A Lie-Rinehart ∞-pair over an (ordinary) algebra A is an L_∞-algebra structure on an ω-vector space $\mathfrak{g}$ which is in fact a complex of A-modules $\mathfrak{g} \in \mathbf{Ch}^\bullet_n(A)$, together with an ω-vector space $B \in \mathbf{Ch}^\bullet_-(A)$ which is such that in degree 0 it contains the tensor unit in $\mathbf{Ch}^\bullet(A)$,*

$$B^0 = A\,,$$

together with an L_∞-map

$$\rho : \mathfrak{g} \to \mathrm{End}(B)\,.$$

**)

4.4.1 The Lie 3-algebroid structure

We now show that

- The 3-vector space of weakly on-shell functions, $WC^\infty(\Sigma)$, naturally carries the structure of an associative 3-algebra.

- The action of the symmetries (the ghosts) on the fields, antifields and antighosts is actually an action of the Lie 1-algebra $\mathfrak{g}$ of symmetries on that associative 3-algebra by 3-algebra derivations.

Together with the obvious action of $WC^\infty(\Sigma)$ on $\mathfrak{g}$, this gives the structure of a Lie-Rinehart 3-pair which "resolves" the Lie-Rinehart 1-pair of on-shell functions acted on by gauge symmetries.

Thinking of Lie-Rinehart n-pairs as Lie n-algebroids, this means that we obtain a Lie 3-algebroid structure.

Definition 22 *Define a monoidal structure*

$$\mu : WC^\infty(\Sigma) \otimes WC^\infty(\Sigma) \to WC^\infty(\Sigma)$$

on the 3-vector space

$$WC^\infty(\Sigma) := \begin{array}{c} C^\infty(X) \otimes \mathfrak{g} \\ \big\downarrow \\ \Gamma(TX) \\ \big\downarrow{\scriptstyle dS(\cdot)} \\ C^\infty(X) \end{array}$$

by

$$\begin{array}{ccc} (\Gamma(TX)\otimes\Gamma(TX)) \oplus 3(C^\infty(X)\otimes C^\infty(X)\otimes\mathfrak{g}) & \xrightarrow{0\oplus\mu_0} & C^\infty(X)\otimes\mathfrak{g} \\ \big\downarrow & & \big\downarrow \\ 2(C^\infty(X)\otimes\Gamma(TX)) & \xrightarrow{l_0} & \Gamma(TX) \\ \big\downarrow & & \big\downarrow \\ C^\infty(X)\otimes C^\infty(X) & \xrightarrow{\mu_0} & C^\infty(X) \end{array},$$

where μ_0 is the ordinary product of functions and l_0 denotes the obvious left action of functions on sections.

Proposition 3 *This turns $WC^\infty(\Sigma)$ into a symmetric associative 3-algebra.*

Definition 23 *For each $t \in \mathfrak{g}$, define an endomorphism*

$$\delta^t : WC^\infty(\Sigma) \to WC^\infty(\Sigma)$$

$$\begin{array}{ccc}
\Gamma(TX) & \xrightarrow{\delta_1^t} & \Gamma(TX) \\
\Big\downarrow{\scriptstyle dS(\cdot)} & & \Big\downarrow{\scriptstyle dS(\cdot)} \\
C^\infty(X) & \xrightarrow{\delta_0^t} & C^\infty(X)
\end{array}$$

as

$$\delta_0^t : f \mapsto \rho(t)(f)$$

and

$$\delta_1^t : v \mapsto [\rho(t), v] \,.$$

Proposition 4 *This is indeed a morphism of 3-vector spaces and in fact a derivation with respect to the associative 3-algebra structure on $WC^\infty(\Sigma)$.*

Proof. The respect for the 3-vector space structure means that the above square indeed commutes, as it does: for every $v \in \Gamma(TX)$ we have

$$\begin{array}{ccc}
v & \stackrel{\delta_1^t}{\longmapsto} & [\rho(t), v] \\
\Big\downarrow{\scriptstyle dS(\cdot)} & & \Big\downarrow{\scriptstyle dS(\cdot)} \\
v(S) & \stackrel{\delta_0^t}{\longmapsto} & [\rho(t), v](S) = \rho(t)(v(S))
\end{array} \quad .$$

Notice that this makes crucial use of the fact that $\rho(t)$ is a local symmetry, which implies that $v(\rho(t)(S)) = 0$.

It is clear that δ_0^t is a derivation on $C^\infty(X)$. The derivation condition on δ_1^t is the commutativity of

$$\begin{array}{ccc}
\Gamma(TX) \otimes C^\infty(X) & \xrightarrow{\delta_1^t \otimes \mathrm{Id} + \mathrm{Id} \otimes \delta_0^t} & \Gamma(TX) \otimes C^\infty(X) \\
\Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle r} \\
\Gamma(TX) & \xrightarrow{\delta_1^t} & \Gamma(TX)
\end{array} \quad .$$

One checks in components that, indeed, for all $(v \otimes f) \in \Gamma(TX) \otimes C^\infty(X)$ we have

$$\begin{array}{ccc}
(v \otimes f) & \xmapsto{\delta_1^t \otimes \mathrm{Id} + \mathrm{Id} \otimes \delta_0^t} & ([\rho(t), v] \otimes f) + (v \otimes \rho(t)(f)) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
fv & \xmapsto{\delta_1^t} & f[\rho(t), v] + \rho(t)(f) v
\end{array}$$

I am too tired to write out the corresponding statements in degree 2.

□

5 Transgression and quantization

One of crucial points of [3], nicely reviewed in [4] is that the BV cochain complex with its generators in positive and negative degrees, should be thought of as coming from forming the internal hom between non-negatively graded cochain complexes in the world of arbitrary graded cochain complexes.

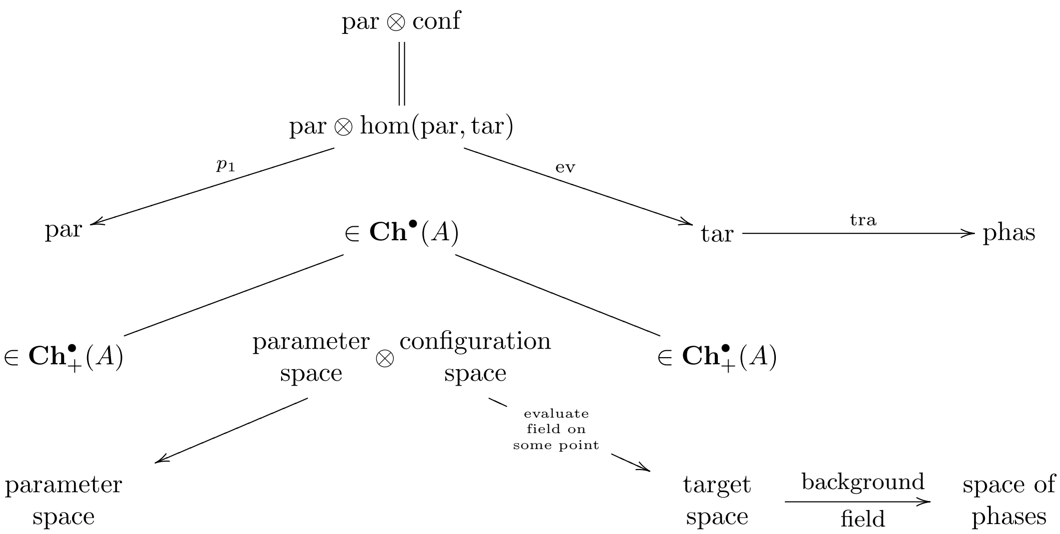


Table 7: **The setup of QFT** after passing from Lie ∞-groupoids to dg-algebras, using the Lie functor Lie : ∞Grpd → ∞LieAlg. This step suggests that the configuration space of fields needs to be formed using the internal hom in *arbitrarily graded* cochain complexes. As noticed by AKSZ, this induces the BV field-antifield formalism.

Slogan.

(AKSZ-)BV-formalism $\hat{=}$

when forming the space of fields
$\phi : \mathrm{par} \to \mathrm{tar}$
of the n-particle
par
propagating on target space
tar
replace the non-negatively graded hom
$\mathrm{hom}_{\mathbf{Ch}^\bullet(A)_+}(\mathrm{par}, \mathrm{tar})$
by the arbitrarily graded hom
$\mathrm{conf} := \mathrm{hom}_{\mathbf{Ch}^\bullet(A)}(\mathrm{par}, \mathrm{tar})$

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