Connections with values in Lie n-algebras

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Abstract

A connection on a principal G-bundle P may be conceived as a map

$$f^* : \mathfrak{g}^* \to \Omega^\bullet(P)$$

respecting the action of the Lie algebra $\mathfrak{g}$ on P. We want to generalize this from Lie algebras to Lie n-algebras.

Contents

1 Introduction

Let $P \to X$ be a G-principal bundle with right G-action

$$R : P \times G \to P .$$

Then the differential graded commutative algebra (DGCA for short) of differential forms on P

$$(\Omega^\bullet(P), d)$$

is naturally equipped with two families of derivations of degree -1 and 0, respectively.

- for every vector field $v \in \Gamma(TP)$ we have the interior product

$$i(v) : \Omega^\bullet(P) \to \Omega^\bullet(P)$$

- for every Lie algebra element $X \in \mathfrak{g}$ we have the Lie derivative

$$L_{R_* X} : \Omega^\bullet(P) \to \Omega^\bullet(P)$$

  along the vector field $R_* X \in \Gamma(TP)$ coming from the differential of the right action of G on P.

If we define analogous operations on the dual of the Lie algebra

$$\iota_X \omega := \omega(X)$$

and

$$L_X \omega := \mathrm{ad}^*_X \omega$$

for all $\omega \in \mathfrak{g}^*$ then a connection on P, regarded as a 1-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying certain compatibility conditions (the Ehresmann or Cartan-Ehresmann conditions, see 3.3.2 for a detailed discussion) may be regarded as a linear map

$$f^* : \mathfrak{g}^* \to \Omega^\bullet(P)$$

which respects both these derivations in the sense that

$$\iota_{R_* X} f^*(\omega) = f^*(\iota_X \omega) \tag{1}$$

and

$$L_{R_* X} f^*(\omega) = f^*(L_X \omega) . \tag{2}$$

Here we agree that f^* restricted to $\wedge^0 \mathfrak{g}^* \simeq \mathbb{R}$ returns a multiple of the constant 0-form on X.

So a connection, in this language, is two things:

- A $\mathfrak{g}$-valued 1-form on a covering space of base space (which here happens to be the total space of the bundle P itself).
- A compatibility condition on this 1-form, which ensures that the connection on the total space descends to a connection over base space.

Here we want to generalize this description of connections from Lie algebras to Lie n-algebras, following up on our discussion [2] and [3].

In order to keep track of various concepts involved when generalizing this way, it may be helpful to realize that the definition of a Cartan-Ehresmann connection as above is really just a special case of a general differential cocycle description of a connection. This relation is described in detail in 3.3.2.

Our approach consists of three steps.

- First we generalize the notion of a connection form on a total space P with values in a Lie algebra $\mathfrak{g}$ to connection forms with values in any Lie n-algebra $\mathfrak{g}_{(n)}$.

  In particular, we pass from mere linear maps between qfDGCAs to true morphisms of qfDGCAs by noticing that instead of using just the Lie n-algebra $\mathfrak{g}_{(n)}$ itself, we should instead use its Lie $(n+1)$-algebra of inner derivation

  $$\mathrm{inn}(\mathfrak{g}_{(n)})\,.$$

  Hence, a connection form on P with values in a Lie n-algebra $\mathfrak{g}_{(n)}$ is defined to be a morphism

  $$f : \mathrm{Vect}(X) \to \mathrm{inn}(\mathfrak{g}_{(n)})\,.$$

- Then we reinterpret the descent conditions (1) and (2) in this context. It turns out that due to the passgage to $\mathrm{inn}(\mathfrak{g}_{(n)})$ these two conditions unify to a single condition, namely that

  $$[\iota_X, f^*] = 0$$

  for all $X \in \mathfrak{g}$. Here the commutator is the obvious shorthand for

  $$\iota_{R_* X} \circ f^* - f^* \circ \iota_X = 0\,.$$

  This defines a descent condition of $\mathfrak{g}$-valued forms on a space P whenever there is an action

  $$R_* : \mathfrak{g} \to \Gamma(TP)\,.$$

- Set up this way, there is a rather obvious generalization to connections with values in arbitrary Lie n-algebras:

  given any Lie n-algebra $\mathfrak{g}_{(n)}$, it is acted on by a Lie (1-)algebra $\mathrm{inn}_n(\mathfrak{g}_{(n)})$ of generalized inner derivations. (For $n = 1$ we have $\mathrm{inn}_1(\mathfrak{g}) \simeq \mathfrak{g}$.)

  Therefore we can say that a $\mathfrak{g}_{(n)}$-valued form

  $$f : \mathrm{Vect}(P) \to \mathfrak{g}_{(n)}$$

  for P any spaces with a $\mathrm{inn}_n(\mathfrak{g}_{(n)})$-action

  $$R_* : \mathrm{inn}_n(\mathfrak{g}_{(n)}) \to \Gamma(TP)$$

  is compatible with this action if

  $$[\iota_X, f^*] = 0$$

  for all $X \in \mathrm{inn}_n(\mathfrak{g}_{(n)})$.

	Lie n-groupoids	$\xmapsto{\text{differentiation}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**quasi free differential graded commutative algebras** (qfDGCAs)
morphism	$\Sigma(\mathrm{INN}(G_{(n)}))$ $\uparrow F$ $\Pi_{n+1}(P)$		$\mathrm{inn}(\mathfrak{g}_{(n)})$ $\uparrow f$ $\mathrm{Vect}(P)$		$(\bigwedge^\bullet s\mathfrak{g}^*_{(n)}, d_{\mathrm{inn}(\mathfrak{g}_{(n)})})$ $\downarrow f^*$ $(\Omega^\bullet(P), d)$
description	smooth pseudofunctor from pair groupoid of X to inner automorphisms of structure Lie n-group $G_{(n)}$		morphism of Lie n-algebroids $\simeq$ n-term L_∞-algebras from tangent algebroid of X to inner derivation Lie $(n+1)$-algebra $\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})$		dual morphism of qfDGCAs

Table 1: **Parallel transport functors and their differentials.** Smooth parallel transport is a morphism of Lie n-groupoids, its differential is therefore a morphism of Lie n-algebras. The table restricts attention to transport on a cover space $P \to X$ with values in the structure Lie n-group (Lie n-algebra) itself. A compatibility condition then ensures that this descends to a connection on a nontrivial n-bundle on X. (For the relation between smooth pseudofunctors on the pair groupoid and smooth n-functors on the fundamental n-groupoid see the text.)

2 Connection forms with values in Lie n-Algebras

We review some aspects of the functorial description of bundles with connection [4]. This realizes connections as morphisms of Lie groupoids.

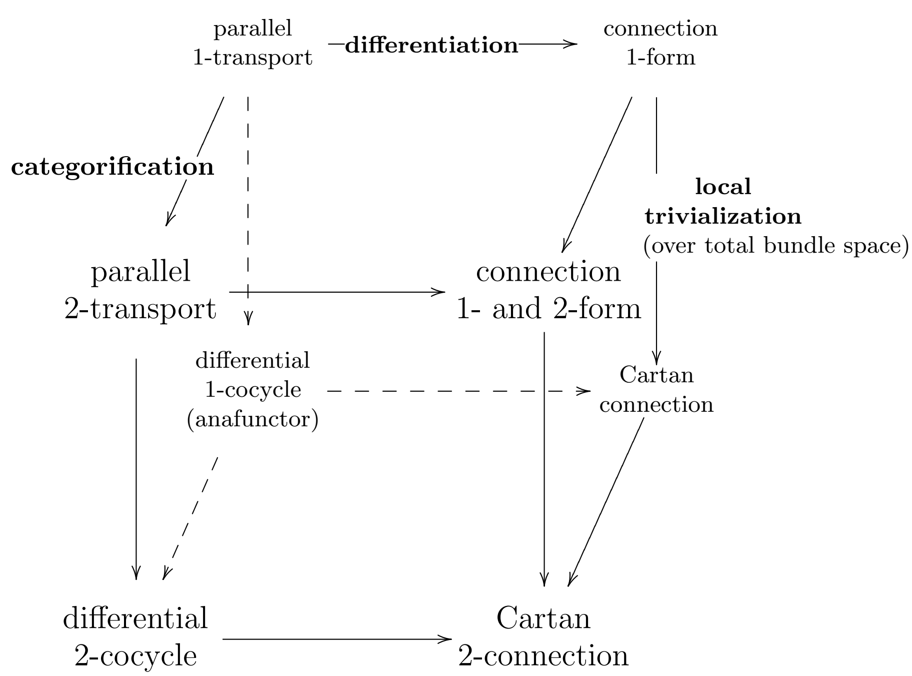


Figure 1: **Differentiation, categorification and local trivialization** are the three procedures relating parallel n-transport that play a role in the local description of n-connections with values in n-groups and Lie n-algebras. Categorification sends n-transport to $(n+1)$-transport. Differentiation sends functors on Lie groupoids to morphisms of Lie n-algebras. Local trivialization sends n-transport on globally defined n-paths to n-transport on local n-paths glued by descent data. The differential version of local trivialization yields Cartan connections if the trivialization is over the total space of the bundle itself.

By differentiating this, we arrive at the description of connections in terms of morphisms of Lie algebroids. In particular, by differentiating the functorial concept of a connection on a covering space with descent property, we arrive at the concept of Cartan-Ehresmann connections and their higher analogs.

2.1 Prelude: (pseudo)-functorial description of connections

A useful way to think of a smooth princpal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

with connection

$$\nabla$$

is to perceive it entirely in terms of its parallel transport.

2.1.1 Parallel Transport

The parallel transport induced by ∇ is a smooth functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to G\mathrm{Tor}\,,$$

where

$$\mathcal{P}_1(X)$$

is the path groupoid of X, whose morphisms are thin homotopy classes of paths in X.

In [4] it is discussed at length that

Theorem 1 *Such smooth functors*

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

are equivalent *to smooth G-bundles with connection.*

For $p : P \to X$ a bundle with connection over X, there is a cover

$$\pi : Y \to X$$

of X such that

Locally, such a functor looks like a smooth functor

$$\mathrm{triv}_\nabla : \mathcal{P}_1(Y) \to \Sigma G\,,$$

where $\pi : Y \to X$ is some surjective submersion, and where ΣG is the category with a single object and one morphism per element of the Lie group G.

2.1.2 Curvature

For simplicity, assume now and in the following that $Y \simeq \mathbb{R}^n$.

Let $\Pi_1(Y)$ be the fundantal groupoid of Y, whose morphisms are not just thin homotopy classes, but true homotopy classes of paths in Y.

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \twoheadrightarrow & \Pi_1(Y) \;. \\ \downarrow & & \downarrow \\ \mathcal{P}_2(Y) & \twoheadrightarrow & \Pi_2(Y) \\ \downarrow & & \downarrow \\ \mathcal{P}_3(Y) & \twoheadrightarrow & \Pi_3(Y) \end{array}$$

We have triv_∇ is flat precisely if it factors through $\Pi_1(X)$:

$$\begin{array}{ccc} \Sigma G & \xrightarrow{=} & \Sigma G \\ \uparrow{\scriptstyle (A)} & & \uparrow{\scriptstyle (A)}_{F_A=0} \\ \mathcal{P}_1(X) & \twoheadrightarrow & \Pi_1(Y) \end{array}$$

for $A \in \Omega^1(Y, \mathfrak{g})$.

Still, for many purposes, like ours here, it is more convenient to work with $\Pi_1(Y)$. This can be accomplished by passing to *pseudo*functors

pseudofunctor strict 2-functor

$$\begin{array}{ccc} \Sigma(\mathrm{INN}(G)) & \xrightarrow{=} & \Sigma(\mathrm{INN}(G)) \\ \uparrow{\scriptstyle (A)} & & \uparrow{\scriptstyle \mathrm{curv}_\nabla} \\ \Pi_1(X) & \longrightarrow & \Pi_2(X) \end{array}$$

for $A \in \Omega^1(Y, \mathfrak{g})$.

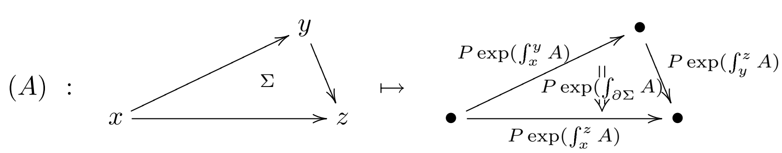


Figure 2: **Curvature is the "compositor"** of the parallel transport functor, when regarded as a psuedofunctor on $\Pi_1(Y)$. It measures how the parallel transport fails to respect the "flat" composition of homotopy classes of paths.

The great advantage of this pseudo functorial description of parallel transport is that it lends itself much easier to differentiation.

2.1.3 Bianchi identity

2.2 The differential description: morphisms of Lie algebroids

Above in 2.1 we described connection forms in terms of pseudofunctors between Lie n-groupoids

$$F : X \times X \to \Sigma G_{(n)} \,.$$

By differentiating these, we obtain morphisms of Lie n-algebroids

$$f : \mathrm{Vect}(X) \to \mathfrak{g}_{(n)} \,.$$

Here

$$\mathrm{Vect}(X) := \mathrm{Lie}(X \times X)$$

is the Lie algebroid corresponding to the pair Lie groupoid of the smooth space X. This is also known as the tangent algebroid.

Analogously,

$$\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})$$

is the Lie n-algebra of the Lie n-group $G_{(n)}$.

These objects are discussed in [2, 3].

By differentiating everything in sight, we obtain a Lie algebra analog of the three concepts parallel transport, curvature, Bianchi identity.

2.2.1 Connection

Definition 1 *For X a manifold and $\mathfrak{g}_{(n)}$ a Lie n-algebra, a* flat $\mathfrak{g}_n$-valued connection on X *is a morphism*

$$f : \mathrm{Vect}(X) \to \mathfrak{g}_{(n)} \,.$$

Definition 2 *A* general $\mathfrak{g}_n$-valued connection on X *is a morphism*

$$f : \mathrm{Vect}(X) \to \mathrm{inn}(\mathfrak{g}_{(n)}) \,.$$

Dually – and this will be the point of view we shall use mostly – a flat connection is a DGCA morphism

$$f^* : (\mathfrak{g}_{(n)})^* \to \Omega^\bullet(X) \,,$$

while a general connection is a DGCA morphism

$$f^* : (\mathrm{inn}(\mathfrak{g}_{(n))})^* \to \Omega^\bullet(X) \,.$$

Remark. While there is a deeper reason behind this particular definition, as indicated in **??**, here we shall just accept it and demonstrate in examples that it does make good sense.

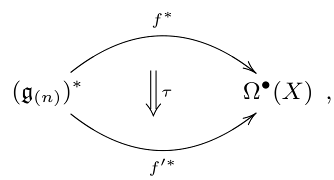
Definition 3 *All $\mathfrak{g}_{(n)}$-valued connections on X form an n-category*

$$Z_X(\mathfrak{g}_{(n)}) := \mathrm{Hom}((\mathfrak{g}_{(n)})^*, \Omega^\bullet(X)) \,.$$

Objects in that category are DGCA morphisms

$$(\mathfrak{g}_{(n)})^* \xrightarrow{f} \Omega^\bullet(X)\,,$$

morphisms are derivation homotopies



and so on.

2.2.2 Curvature

Let $\mathfrak{g}_{(n)}$ be represented by the qfDGCA $(\bigwedge^\bullet(sV)^*, d)$. Then a general connection with values in $\mathfrak{g}_{(n)}$, i.e. a morphism

$$f^* : (\mathrm{inn}(\mathfrak{g}_{(n)}))^* \to \Omega^\bullet(X)$$

is fully determined by its restriction

$$f^*|_{(sV)^*}$$

to the original graded vector space of $\mathfrak{g}_{(n)}$ and its restriction

$$f^*|_{(ssV)^*}$$

to the shifted copy of that vector space which appears in the definition of $\mathrm{inn}(\mathfrak{g}_{(n)})$.

This second restriction carries the information about the (higher) curvature of f^*.

Definition 4 *Given a $\mathfrak{g}_{(n)}$-valued connection f^* as above, the V-valued differential forms encoded by the dual of*

$$f^*|_{(ssV)^*} : (ssV)^* \to \Omega^\bullet(X)$$

are the curvature forms *of f^*.*

Remark. One might, alternatively, be tempted to consider, for a given Lie n-algebra $\mathfrak{g}_{(n)}$, general homomorphisms of graded commutative algebras

$$f^* : (\mathfrak{g}_{(n)})^* \to \Omega^\bullet(X)$$

and then define their *curvature* to be the *failure* of these to be morphisms of *differential* algebras.

This is essentially what is often done in Henri Cartan's algebraic analog of principal bundles or in the context of splittings of the Atiyah sequence of a principal bundle:

given any principal G-bundle $P \to X$, a connection on X may be regarded as a splitting ∇ of the sequence

$$0 \longrightarrow \mathrm{ad}P \longrightarrow TP/G \underset{\curvearrowleft \nabla}{\longrightarrow} TX \longrightarrow 0$$

of vector bundles over X. But there is a natural algebroid structure on all these vector bundles, and the morphism of vector bundles ∇ will be a morphism of algebroids if and only if its curvature vanishes.

However, as emphasized in [1], every splitting ∇ does yield a morphism of 2-algebroids

$$\begin{array}{ccccccccc} \mathrm{DER}(\mathrm{ad}P) & & & & & & & & \\ & \nwarrow^{(\nabla, F_\nabla)} & & & & & & & \\ & & 0 & \longrightarrow & \mathrm{ad}P & \longrightarrow & TP/G & \longrightarrow & TX \longrightarrow 0 \end{array} .$$

The nonvanishing curvature F_∇ now finds its place properly as one component of a morphism to a 2-algebroid.

One finds that locally, i.e. when we may assume that P is in fact a trivial bundle, the morphism (∇, F_∇) factors through the Lie 2-algebra $\mathrm{inn}(\mathfrak{g}) \subset \mathrm{DER}(\mathfrak{g})$. This then precisely recovers, as a special case, our definition of connection and curvature above. This is described in detail in **??**.

Remark. It follows that an n-connection has a curvature 2-form, a curvature 3-form, etc., up to a curvature $(n+1)$-form.

2.2.3 Bianchi Identity

By combining definition 1 with definition 2, we find

Fact. *Every n-connection is a flat $(n+1)$-connection.*

This flatness of every connection, one level higher, is the higher version of the *Bianchi identity.*

	n-connection $\to$	$\frac{\text{flat } (n+1)\text{-connection}}{n\text{-curvature}}$ $\to$	$\frac{\text{trivial } (n+2)\text{-connection}}{n\text{-Bianchi identity}}$
$n=1$	A	F_A	$d_A F_A$
$n=2$	(A,B)	(β, H)	$(d_A\beta, d_A H)$

Table 2: **DGCA morphisms from a Lie n-algebra to $\Omega^\bullet(X)$** turn out to always encode flat n-connections. However, these may be interpreted as the curvatures of non-flat $(n-1)$-connections. Their flatness then translates into the corresponding $(n-1)$-Bianchi identity.

3 Descent condition for connection forms with values in Lie n-algebras

3.1 Prelude: descent of Lie n-groupoid morphisms

3.1.1 General descent

Given a cover

$$\pi : Y \to X$$

of some space X, and given a connection form on Y, with values in a Lie group G, hence, according to 2.1, a smooth functor

$$\begin{array}{c} \mathcal{P}_1(Y) \\ \big\downarrow{\scriptstyle \mathrm{triv}} \\ \Sigma G \end{array}$$

one finds that the condition for this to descend to a G-bundle with connection on X, in that we may complete a square to the right

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \xrightarrow{\pi} & \mathcal{P}_1(X) \\ {\scriptstyle \mathrm{triv}}\big\downarrow & \swarrow\!\!\!\swarrow\, {\scriptstyle \sim} & \big\downarrow{\scriptstyle \mathrm{tra}} \\ \Sigma G & \xhookrightarrow{i} & G\mathrm{Tor} \end{array}$$

is that triv extends to a smooth functor

$$(\mathrm{triv}, g) : \mathcal{C}_\pi(Y) \to \Sigma G$$

on the "path pushout" groupoid $\mathcal{C}_\pi(Y)$, defined by the weak pushout square

$$\begin{array}{ccc} \mathcal{P}_1(Y^{[2]}) & \xrightarrow{\pi_1} & \mathcal{P}_1(Y) \\ {\scriptstyle \pi_2}\big\downarrow & \swarrow\!\!\!\swarrow\, {\scriptstyle \sim} & \big\downarrow \\ \mathcal{P}_1(Y) & \longrightarrow & \mathcal{C}_\pi(Y) \end{array} .$$

This groupoid is generated from paths in Y and "jumps" between patches, coming from points in $Y^{[2]}$, subject to the relation

$$\begin{array}{ccc} \pi_1(x) & \xrightarrow{\pi_1(\gamma)} & \pi_1(y) \\ \downarrow & & \downarrow \\ \pi_2(x) & \xrightarrow{\pi_2(\gamma)} & \pi_2(y) \end{array}$$

for all paths $x \xrightarrow{\gamma} y$ in $\mathcal{P}_1(Y^{[2]})$.

Extending triv to a functor (triv, g) on $\mathcal{C}_\pi(Y)$ is equivalent to specifying a transition function $g : Y^{[2]} \to G$ such that the differential cocycle condition is satisfied:

$$(\mathrm{triv}, \mathrm{g}) \;:\; \begin{array}{ccc} \pi_1(x) & \xrightarrow{\pi_1(\gamma)} & \pi_1(y) \\ \downarrow & & \downarrow \\ \pi_2(x) & \xrightarrow{\pi_2(\gamma)} & \pi_2(y) \end{array} \quad \mapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{P\exp(\int_{\pi_1(\gamma)} A)} & \bullet \\ {\scriptstyle g_{\pi_1(x),\pi_2(x)}}\downarrow & & \downarrow{\scriptstyle g_{\pi_1(y),\pi_2(y)}} \\ \bullet & \xrightarrow{P\exp(\int_{\pi_2(\gamma)} A)} & \bullet \end{array}$$

Since the canonical projection

$$\mathcal{C}_\pi(Y) \to \mathcal{P}_1(X)$$

is a surjective equivalence, we may regard (triv, g) as the comonent functor of a smooth *anafunctor*

$$F : \mathcal{P}_1(X) \to \Sigma G$$

(as disucssed by Makkai and Bartels, reviewed in our our context in [4]), given by a diagram of ordinary functors

$$|F| : \quad \begin{array}{ccc} \mathcal{C}_\pi(Y) & \xrightarrow{(\mathrm{triv}, g)} & \Sigma G \\ {\scriptstyle \sim}\downarrow & & \\ \mathcal{P}_1(X)\,. & & \end{array}$$

3.1.2 Descent from the total space of the bundle itself

One main point in our discussion is that

Cartan connections are special differential cocycles, namely those where the covering Y is taken to be the total space P of the G-bundle.

Every principal G-bundle $p : P \to X$ *canonically* trivializes when pulled back to its own total space. In the above description, this corresponds to setting

$$Y := P \,.$$

Then

$$Y^{[2]} \simeq Y \times G$$

and the cocycle $g : Y^{[2]} \to G$ is canonically given by the projection on the second factor. (This is spelled out in great detail in 3.3.2.)

In terms of the above diagrammatic description, this means that now $\mathcal{C}_\pi(Y)$ is the groupoid from paths in the total space and elements of G itself, satisfying

$$\begin{array}{ccc} x & \xrightarrow{\gamma} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ x \cdot g & \xrightarrow{\gamma \cdot g} & y \cdot g \end{array} \,.$$

For all paths $x \xrightarrow{\gamma} y$ in $\mathcal{P}_1(P)$ and all $g \in G$.

As our abuse of the notation "g" already makes inevitable, in this case now the transition function part of the anafunctor (triv, g) has to be the *identity* on G.

$$(\mathrm{triv}, \mathrm{g}) \;:\; \begin{array}{ccc} x & \xrightarrow{\gamma} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ x \cdot g & \xrightarrow{\gamma \cdot g} & y \cdot g \end{array} \quad \mapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{P\exp(\int_\gamma A)} & \bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[\mathrm{Ad}_g P\exp(\int_{\gamma\cdot g} A)]{} & \bullet \end{array} \,. \qquad (3)$$

This single diagram encodes the two Cartan conditions (1) and (2) in integrated form.

3.2 The integrated Cartan conditions

In (3) we identify the two Cartan conditions, as indicated in table 3.

The differentiation step is described in more detail in 3.3.

Notice that, in the integral as well as in the differential picture, the morphism property and the two Cartan conditions (in their integrated form) are not independent. This allows us in 3.3 to replace the two Cartan conditions by a single one, after ensuring that f^* is a qfDGCA-morphism by passing from $\mathfrak{g}$ to $\mathrm{inn}(\mathfrak{g})$.

Also notice that another diagrammatic way to formulate the integrated Cartan condition is as shown in figure 3.

This makes the commutator nature of the differential Cartan condition quite manifest. Compare with figure 4.

	Lie group picture	$\xmapsto{\text{differentiation}}$	**Lie algebra picture**
morphism property	functoriality of $\lvert F\rvert$		$[d, f^*] = 0$
first Cartan condition	$\lvert F\rvert : (\, x \xrightarrow{g} x \cdot g \,) \mapsto (\, \bullet \xrightarrow{g} \bullet \,)$		$[\iota_X, f^*] = 0$
second Cartan condition	$\lvert F\rvert : (\, x \cdot g \xrightarrow{\gamma \cdot g} y \cdot g \,) \mapsto (\, \bullet \xrightarrow{\mathrm{Ad}_g F(\gamma)} \bullet \,)$		$[L_X, f^*] = 0$

Table 3: **The two conditions on a Cartan connection** express the general cocycle property of a connection form for the special case that the covering space is the total space of the bundle itself. The table shows the integrated and the differential version of the cocycle condition, now interpreted as the two conditions on a Cartan connection.

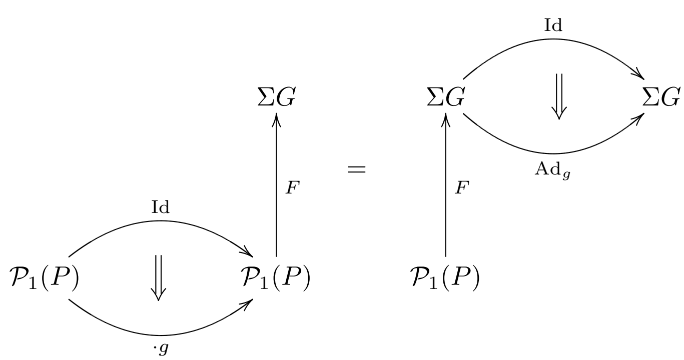


Figure 3: **Another diagrammatic form of the integrated Cartan condition.**

3.3 Descent of Lie n-algebra morphisms

3.3.1 Reformulation of the ordinary Cartan condition

We reformulate the descent conditions (1) and (2) for the case that we pass from $\mathfrak{g}$ to $\mathrm{inn}(\mathfrak{g})$, thereby allowing f^* to commute with the differential, even if the connection is not flat. This allows us to unify the two conditions to a single one, using the fact that if the connection 1-form A satisfies (1) then its curvature F_A satisfies

$$\iota_{r(X)} F_A = 0 \,.$$

Let $(S^c s\mathfrak{g}, D)$ be the L_∞ version of a Lie algebra $\mathfrak{g}$ and $(\bigwedge^\bullet s\mathfrak{g}^*, d_{\mathfrak{g}})$ the dual qfDGCA.

For any element $X \in \mathfrak{g}$, let

$$\iota_X : \bigwedge^\bullet s\mathfrak{g}^* \to \bigwedge^\bullet s\mathfrak{g}^*$$

denote the degree -1 derivation which forms the interior product with X.

The corresponding Lie derivative is the degree 0 derivation

$$L_X := [d_{\mathfrak{g}}, \iota_X] \,.$$

Definition 5 *We say that a manifold P has a $\mathfrak{g}$-action if there is a morphism of Lie algebras*

$$r : \mathfrak{g} \to \Gamma(TP) \,.$$

To generalize (1) and (2) from $\mathfrak{g}$ to $\mathrm{inn}(\mathfrak{g})$, we first need to extend the action of ι_X on $\bigwedge^\bullet s\mathfrak{g}^*$ to an action on $\bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$. We do this in the obvious trivial way:

Definition 6 *For any $X \in \mathfrak{g}$, let*

$$\iota_X : \bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*) \to \bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$

be the degree -1 derivation which acts by contraction with X on $\bigwedge^\bullet s\mathfrak{g}^*$ and which acts as 0 on $ss\mathfrak{g}^*$.

Using this definition, we still write L_X for the corresponding Lie derivative on $\mathrm{inn}(\mathfrak{g})^*$:

$$L_X := [d_{\mathrm{inn}(\mathfrak{g})}, \iota_X]$$

Definition 7 *Given a manifold P with a $\mathfrak{g}$-action $r : \mathfrak{g} \to \Gamma(P)$, and given a qfDGCA morphism*

$$f^* : (\bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \to (\Omega^\bullet(P), d) \,,$$

we say that f^* is compatible *with the $\mathfrak{g}$-action if*

$$\iota_{r(X)} \circ f^* = f^* \circ \iota_X$$

for all $X \in \mathfrak{g}$.

Proposition 1 *The $\mathfrak{g}$-valued 1-forms encoded by $\mathfrak{g}$-compartible morphisms $f^* : (\mathrm{inn}(\mathfrak{g}))^* \to \Omega^\bullet(P)$ are precisely the 1-forms satisfying the Cartan conditions (1) and (2).*

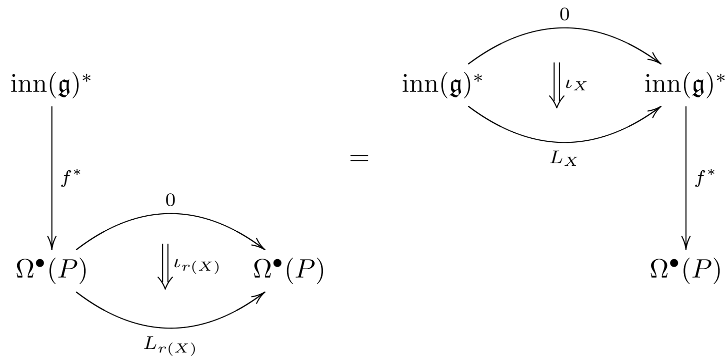


Figure 4: **Diagrammatic form of the Cartan condition.** Since the interior product ι_X is nothing but the homotopy which connects the corresponding Lie derivative L_X to the 0-derivation, we find that the two commutator conditions $[\iota_X, f^*] = 0$ and $[L_X, f^*] = 0$ on a Cartan connection are in fact just a single commutator condition. This works because with $\mathfrak{g}$ replaced by $\mathrm{inn}(\mathfrak{g})$ we have $[d, f^*] = 0$.

Proof. Restricted to $s\mathfrak{g}^*$ the compatibility condition $[\iota_X, f^*] = 0$ is precisely condition (1).

Restricted to $ss\mathfrak{g}^*$ it becomes

$$\iota_{r(X)} F_A = 0\,,$$

which follows from (1).

But since we also have $[d, f^*] = 0$ this now already implies that $[L_X, f^*] = [[d, \iota_X], f^*] = 0$.

Using the fact that

$$[d_{\mathrm{inn}(\mathfrak{g})}, \iota_X](\omega) = -\omega([X, \cdot])$$

for all $\omega \in s\mathfrak{g}^*$ and all $\omega \in ss\mathfrak{g}^*$ it follows that

$$[L_X, f^*] = 0$$

is equivalent to

$$L_{r(X)} A = \mathrm{ad}_X A$$

and

$$L_{r(X)} F_A = \mathrm{ad}_X F_A\,.$$

The first line here is condition (2), while the second line is implied by the first one. □

3.3.2 A careful walk through the derivation

The following is a detailed derivation of the Cartan condition from that of a differential cocycle on a principal bundle using just standard familiar concepts from differential geometry. Every step is elementary and nothing here should be news to anyone who ever thought about these issues in any detail. However, we find it helpful to make the relation relation between Cartan connections and differential cocycles manifest by going through the following reasoning. In particular since there are a couple of signs and conventions that tend to be tedious to disentangle.

So the following makes explicit the notion of a connection on a principal bundles

1. as a differential cocycle;
2. as a 1-form on the total space of the bundle;
3. as a morphism of DGCAs respecting a generalized Lie derivative and interior product .

The first point of view is the most general one, in a sense. It implies the second point of view by pulling a G-bundle back along itself. This has the advantage that transitions may be differentiated with respect to the right G-action, thus leading to the third perspective.

In the following we trace out this path from differential cocycles to Cartan connections in full detail.

Basic conventions on connection 1-forms. For $u : \mathbb{R} \to G$ a group-valued function on the line and $A \in \Omega^1(\mathbb{R}, \mathrm{Lie}(G))$ a Lie-algebra valued 1-form, the condition that u be *parallel* with respect to A is taken to be

$$du = -(R_u)_* \circ A\,,$$

where $R_g : G \to G$ denotes right multiplication with $g \in G$.

The convention here is such that for G a matrix group we can equivalently write

$$(d + A)u = 0\,.$$

Let u and v be parallel with respect to A and A', respectively, and normalized such that $u(0) = e$ and $v(0) = e$. We say that A and A' are related by a gauge transformation

$$g : \mathbb{R} \to G$$

if

$$v(x) = g(x)u(x)g(0)^{-1}$$

for all $x \in \mathbb{R}$. This then implies

$$A' = gAg^{-1} + gdg^{-1} = \mathrm{Ad}_g A + g^*\bar{\theta}\,,$$

or equivalently

$$A = g^{-1}A'g + g^{-1}dg = \mathrm{Ad}_{g^{-1}} A' + g^*\theta\,,$$

which we also write as

$$A \xrightarrow{g} A' \ .$$

Here θ denotes the left-invariant Maurer-Cartan form on G, while $\bar{\theta}$ denotes the right-invariant MC form.

Connections in terms of differential cocycles. One way to define a connection on a principal G-bundle $p : P \to X$ is to specify a trivialization

$$t : \pi^* P \xrightarrow{\sim} Y \times G$$

of the bundle on a 'cover'

$$\pi : Y \to X\,,$$

where π is a surjective submersion, together with a 1-form

$$A \in \Omega^1(Y, \mathrm{Lie}(G))$$

on that cover, which satisfies the glueing cocycle condition

$$\pi_2^* A = \mathrm{Ad}_g(\pi_1^* A) + g^* \bar{\theta}$$

on 'double intersections'

$$\begin{array}{ccc} Y^{[2]} & \xrightarrow{\pi_1} & Y \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\pi} & X \end{array}$$

of the cover.

Here $Y^{[2]}$ is the total space of the pull back of π over itself, i.e. $Y^{[2]} \subset Y \times Y$ consists of pairs (y_1, y_2) such that $\pi(y_1) = \pi(y_2)$, while $\theta \in \Omega^1(G, \mathrm{Lie}(G))$, which is the canonical Lie-algebra valued 1-form on G that sends any right-invariant vector field to its value at the identity and g is the transition function induced by the trivialization.

One familiar way to do this is to take $Y := \coprod U_\alpha$ an open cover of X by open contractible subspaces U_α. The principal bundle $p : P \to X$ restricts to trivial principal bundles $\pi^* P|_{U_\alpha}$ over each U_α. The transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

satisfy the 1-cocycle condition: $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ and act on the local connections by

$$A_\beta = g_{\alpha\beta} A_\alpha g_{\alpha\beta}^{-1} + g_{\alpha\beta} d g_{\alpha\beta}^{-1}.$$

Thus they do not fit togeher to give a global 1-form on X, but do define a global 1-form on P.

Trivialization of a principal G-bundle over itself. Every principal G-bundle $p : P \to X$ canonically trivializes over itself. This means that if we choose the surjective submersion Y to equal the total space of the bundle itself

$$Y := P$$

and accordingly set

$$\pi = p$$

then the pullback bundle

$$p^*P \simeq P \times_X P := P^{[2]}$$

is trivializable, where the canonical trivializing morphism

$$t : p^*P \xrightarrow{\sim} P \times G$$

sends $(p_1, p_2) \in P^{[2]}$ to $t(p_1, p_2) := (p_1, g_{p_1,p_2})$, where g_{p_1,p_2} is the unique group element whose action carries p_1 to p_2:

$$p_1 \cdot g_{p_1,p_2} = p_2 \, .$$

This implies that the transition function

$$g : P \times_X P \to G$$

acts as

$$g : (p_1, p_2) \mapsto g_{p_2,p_1}$$

i.e. by picking out the last factor of $P^{[2]} \simeq P \times G$ and inverting it.

In this case the corresponding differential cocycle is a 1-form

$$A \in \Omega^1(Y, \mathrm{Lie}(G)) = \Omega^1(P, \mathrm{Lie}(G))$$

JIM: $Y \neq P$ so what does $=$ here mean?

URS: In this example $Y = P$. That is what makes this example so special. I have now emphasized this a little more above.

on the total space of the bundle, satisfying

$$\pi_2^* A = \mathrm{Ad}_g(\pi_1^* A) + g^* \bar{\theta} \, .$$

where now

$$\begin{array}{ccc} P \times_X P \simeq P \times G & \xrightarrow{\pi_1 = p_1} & P \;\; , \\ \Big\downarrow{\scriptstyle \pi_2 = R} & & \\ P & & \end{array}$$

with $R : P \times G \to P$ denoting the right G-action on P.

Now let

$$h : \mathbb{R} \to G$$

and

$$\rho : \mathbb{R} \to P$$

be curves with tangents

$$v := \frac{d}{dt}\rho$$

and

$$\xi := \frac{d}{dt}h$$

and

$$X_\xi := \rho \frac{d}{dt} h \, .$$

We then have

$$\pi_2^* A(v,\xi) = R_h^* A(v) + A(X_\xi)$$

and

$$\pi_1^* A(v,\xi) = A(v)$$

and

$$g^*\bar{\theta}(v,\xi) = h^{-1} dh(\frac{d}{dt})$$

Therefore the gluing cocycle condition now reads

$$R_h^* A(v) + A(X_\xi) = \mathrm{Ad}_h^{-1} A(v) + h^*\theta(\frac{d}{dt})$$

or equivalently

$$A(\frac{d}{dt}(\rho h)) = \mathrm{Ad}_h^{-1}(A(\frac{d}{dt}\rho)) + \theta(\frac{d}{dt}h)\,.$$

Cartan-Ehresmann connection 1-Form on total Space of the Bundle. This last equation is equivalent to the two conditions

- $A : X_\xi \mapsto \xi$
- $A = \mathrm{Ad}_h(R_h^* A)$

on the 1-form

$$A \in \Omega^1(P, \mathrm{Lie}(G))$$

on the total space of the bundle, which is the standard way in which the connection 1-form on the total space of the bundle is defined.

One important difference between the trivialization over a good cover of open subspaces and the one over the bundle itself, as considered now, is that now we may differentiate the last equation with respect to h and make it live entirely in the world of Lie algebras.

We find

$$\frac{d}{dt}(\mathrm{Ad}_h^{-1} A) = \mathrm{ad}_\xi A = [\xi, \cdot] \circ A$$

and

$$\frac{d}{dt} R_h^* A = L_{X_\xi} A\,,$$

where L_{X_ξ} denotes the Lie derivative on differential forms.

Dual formulation of the Cartan connection 1-form. We may regard the 1-form $A \in \Omega^1(P, \mathrm{Lie}(G))$ as a special linear map

$$f : TP \to \mathfrak{g}\,.$$

Denote the dual map by

$$f^* := A^* : \mathfrak{g}^* \to \Omega^1(P)\,.$$

The first of the two conditions on A then becomes

$$f^*(\omega)(X_\xi) = \omega(\xi)$$

for all $\omega \in \mathfrak{g}^*$, which we can write as

$$i(X_\xi) f^*(\omega) = i(\xi)\omega,$$

to emphasize the role of the *inner derivation* $i(\ )$.

Similarly, the dual of the second condition reads

$$L_{X_\xi} f^*(\omega) = f^*(\omega([\xi, \cdot]))\,.$$

If we introduce the notation

$$L_\xi \omega := \omega([\xi, \cdot])$$

then this takes the form

$$L_{X_\xi} f^*(\omega) = f^*(L_\xi \omega)\,.$$

Connection as dual morphism respecting ι and L. Summarizing the above, we have found that a connection on a principal G-bundle P is encoded in a linear map

$$f^* : \mathfrak{g}^* \to \Omega^1(P)$$

that respects the action of the two derivations ι and L:

$$\iota_{R_* X} f^*(\omega) = \iota_X \omega \qquad (4)$$

$$L_{R_* X} f^*(\omega) = f(L_X \omega))\,, \qquad (5)$$

for all $X \in \mathrm{Lie}(G)$.

3.3.3 The n-Cartan condition.

Let $\mathfrak{g}_{(n)}$ be any Lie n-algebra. Denote by

$$\mathrm{inn}_n(\mathfrak{g}_{(n)})$$

its Lie algebra of generalized inner derivations, described in [2]. Notice that $\mathrm{inn}_n(\mathfrak{g}_{(n)})$ extends to a Lie $(n+1)$-algebra. But here we consider it just as an ordinary Lie algebra.

Definition 8 *For P a smooth space with a regular epimorphism*

$$p : P \to X$$

and $\mathfrak{g}_{(n)}$ a Lie n-algebra as above, we say that a Lie algebra morphism

$$r : \mathrm{inn}_n(\mathfrak{g}_{(n)}) \to \Gamma(TP)$$

is $\mathfrak{g}_{(n)}$-action *on P.*

Remark. At the moment we shall just take this definition for granted. Eventually we will show how this derives from differentiating n-connections on n-bundles conceived in terms of parallel transport n-functors, in direct analogy to the discussion in **??**.

Definition 9 *Given a space P with $\mathfrak{g}_{(n)}$-action r as above, and given a $\mathfrak{g}_{(n)}$-valued connection form*

$$f : \mathrm{Vect}(P) \to \mathrm{inn}(\mathfrak{g}_{(n)})$$

on P, we say that f is compatible with the the action r if

$$\iota_{r(X)} \circ f^* - f^* \circ \iota_X = 0$$

for all $X \in \mathrm{inn}_n(\mathfrak{g}_{(n)})$.

We abbreviate this condition as

$$[\iota_r, f^*] = 0\,.$$

Remark. For an ordinary Lie (1-)algebra $\mathfrak{g}_{(1)} = \mathfrak{g}$ we have $\mathrm{inn}_1(\mathfrak{g}) \simeq \mathfrak{g}$. Therefore the general definition here does reproduce the $n = 1$ case described in 3.3.1.

References

[1] D. Stevenson, major unpublished work

[2] U. Schreiber, J. Stasheff, Structure of Lie n-Algebras (in preparation)

[3] U. Schreiber, J. Stasheff, Zoo of Lie n-Algebras (in preparation)

[4] U. Schreiber, K. Waldorf, Parallel Transport and Functors