Groupoid symmetry of general relativity

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Abstract

Notes taken in a talk by Christian Blohmann at Goettingen, Nov. 24. 2008, extended Born-Hilbert Seminar *Higher and graded structures in differential geometry*

there used to be a question mark here, now asnwered, recetn results,

1 The problem

first part on explaining the problem

4-manifold X and Lorentzian metric g, vacuum Einstein equations say that the metric is Ricc-flat: $\mathrm{Ric}(g) = 0$

often one needs to formulate this as an initial value problem

(predictions, in numerical relativity, or if one wants to quantize)

so single out on X a Cauchy hypersurface Σ (which is oriented, spacelike, codimension 1)

assign a direction for time flow, i.e choose a vector field on the Cauchy surface

canonical choice: take n to be the unit normal vector field $g(n,n) = -1$

extend this by exponential map

integrate $\Rightarrow$ flow of Gaussian time

flow from $-\tau$ to τ now gives a cylinder $[-\tau, \tau] \times \Sigma$

the metric on this will look like $g = \gamma(t) - dt^2$

γ is a path of metrics in $\mathrm{Met}(\Sigma)$

one can regard this as the result of a choice of gauge fixing.

now how to describe the dynamics for γ?

nicest way: by an action principle

$$S^{\mathrm{field}}(g) = \int\limits_{\Sigma \times [-\tau,\tau]} R(g)\mathrm{vol}_g$$

where $R(g)$ is the scalar curvature of g.

$$S^{\mathrm{path}}(g) := S^{\mathrm{field}}(\gamma - dt^2) = \int\limits_{-\tau}^{\tau} L(\gamma(t), \dot{\gamma}(t))dt + \text{boundary term}$$

the boundary term contains all terms containing $\ddot{\gamma}$

here

$$L(\gamma, \dot{\gamma}) = \int_\Sigma (R(\gamma) + \frac{1}{4}\mathrm{Tr}_\gamma \dot{\gamma}^2 - \frac{1}{4}(\mathrm{Tr}_\gamma \dot{\gamma})\mathrm{vol}_g$$

variational principle for this $\Rightarrow$ Euler-Lagrange equations on $\mathrm{Met}(\Sigma) \overset{\text{Legendre transformation}}{\Leftrightarrow} (\gamma, \dot{\gamma}) \leftrightarrow (\gamma, \pi)$

yields Hamiltonian vector field on $T^*M\mathrm{Met}(\Sigma)$

Proposition: from earliest days of general relativity:

$\mathrm{Ric}(g) = 0 \Leftrightarrow$ Euler-Lagrange equations + constraints (because we set up variationa problem after making a gauge choice)

first constraint:

$$C_{\mathrm{energy}} = -R(\gamma) + \mathrm{Tr}_\gamma \pi^2 - \frac{1}{2}(\mathrm{Tr}_\gamma)^2 = 0$$

easy calculation $\dot\gamma = -\frac{1}{2}$second-fundamental-form

and there is the momentum constraint:

$$C_{\mathrm{momentum}} = -2\mathrm{div}_\gamma \pi$$

the constraints have to hold at every point $x \in \Sigma$.

remember in gauge theories: constraints are the momenta of the action of the gauge group

first parameterize constraints by a vector space

$C_{(X,\phi)} = \int_\Sigma \{\gamma(X, C_{\mathrm{momentum}}) + \phi C_{\mathrm{energy}}\}\mathrm{vol}_g$

where $(X,\phi) \in \Gamma(T\Sigma) \times C\Sigma$

now from [Katz 1962] and [deWitt 1967] we get the Poisson brackets

$$\{C_{(X,\phi)}, C_{(Y,\psi)}\} = C_{[X,Y]+\phi \mathrm{grad}_\gamma \psi - \psi \mathrm{grad}_\gamma \phi, X\cdot\phi - Y\cdot\psi}$$

so there is something strange about these brackets: index on the right depends on bracket

one good aspect: the constraint surface is coisotropic

bad aspect: the brackets do not close (since on the right we are pluggin in a vector field that depends on γ, which is not what the vector fields on the left are like)

so why not fix γ? that would seem to yield a bundle of Lie algebras parameterized by γ ...

but then the Jacobi identity is no longer satisfied:

so this is *not* a bundle of Lie algebras!

conclusion: the constraints are not the momenta of a group action

since this is joint work with Weinstein and Fernandes one can guess what the conclusion will be:

the constraints are moments of a groupoid action

2 Solution

idea: Cauchy surfaces

$$\mathcal{E}(\Sigma, X)\{i : \Sigma \to X \mathrm{embedding}\}$$

$$\begin{array}{ccc} \mathrm{Diff}(\Sigma) & \longrightarrow & \mathcal{E}(\Sigma, X) \\ & & \downarrow \\ & & \mathcal{H}(\Sigma, X) \end{array}$$

the bottom is hypersurfaces diffeomorphic to Σ

$$\mathcal{DH} = (\mathcal{E}(\Sigma, X) \times \mathcal{E}(\Sigma, X))/\mathrm{Diff}(\sigma)$$

observations:

$\mathrm{Diff}(X) \hookrightarrow \mathrm{Bisections}(\mathcal{DH})$

by conjugation we get $\mathrm{Diff}(X)$-action on $\mathcal{E}(\Sigma, X)$ descends to groupoid

big question: how does this groupoid act:

how does $\mathcal{DH}$ act on "metric information"

locally: push-forward of metric $\Rightarrow$ no action

globally: assume g on X

$$S \xrightarrow{\phi} S$$

$$\gamma = g|_S$$

$$\phi\gamma = g|_{S'}$$

both make no good sense here, so let's consider
"middle ground"

Definition : A Σ-blink ("Augenblick", "clin d' oeil") s the isometry class of a germ of a metric in a neighbourhood of a hypersurface.
let $\mathcal{B}\Sigma$ be the "space" of blinks

Proposition: (fix embedded Cauchy hypersurface then) Every blink has a unique Gaussian representative on $\Sigma \times [-\tau, \tau]$
meaning that $g = \gamma(t) - dt^2$
notice that if everything is analytic then these blinks are just the infinity-jets of the path $\gamma(t)$
how do we equip the space of blinks with a manifold structure?
extend $\phi; S \to S'$ to $\tilde{\phi}$

$$\tilde{\phi} \circ \Phi_t^n \simeq \Phi_t^{n'} \circ \tilde{\phi}$$

"condition of gaussian extendability" here Φ_t^n is the flow of the vector field n
what's the Lie algebroid equivalence?
Element of Lie algebroid is given by $(X_0, \phi_0) \in \Gamma T(\Sigma) \times C\Sigma$

Proposition: for v a vector field on $U = \Sigma \times [-\tau, \tau]$

$$\iota_n \mathcal{L}_v \gamma = 0$$

then:
every vector field $X_0 + \phi_0 n$ supported on $\Sigma \times \{0\}$ has a unique extension to a vector field $v = X + \phi n$ satisfying the condition of gaussian extension
let $X + \phi n$, $Y + \psi n$ be two gaussian vector fields satisfying gaussian extension property
then

$$[X + \phi n, Y + \psi n] = ([X, Y] + \phi \mathrm{grad}_\gamma \psi - \psi \mathrm{grad}_\gamma \phi) + (X \cdot \phi - Y \cdot \psi) n$$

so now we have a geometric interpretation of the original constraint brackets!

Definition: extrinsic Lie algebroid

$$\mathcal{A}_{\mathrm{ex}}\Sigma = \Gamma(TX) \times C\Sigma \times \mathcal{B}\Sigma$$

anchor is:

$$\rho(X_0, \phi_0, \gamma) = \mathcal{L}_{X+\phi n} g = \mathcal{L}_X \gamma = \mathcal{L}_X \gamma + \phi \dot{\gamma}$$

left summand in last term is the *shift* the other one is the *lapse*
so **answer**: the strange brackets are the Lie brackets of this Lie algebroid.
constraints:
view the Euler-Lagrane equations $\simeq$ as vector fields on $T\mathrm{Met}\Sigma$

$$\Phi^{\mathrm{EL}} : T\mathrm{Met}\Sigma \to \mathcal{B}\Sigma$$

$$(\gamma_0, \dot{\gamma}_0) \mapsto \text{solution of EL equations}$$

observation:
Φ^{EL} is an injective immersion
Theorem: The anchor ρ_{ex} of $\mathcal{A}_{\mathrm{ex}}\Sigma$ is tangent to

$$\Phi^{\mathrm{EL}}(T\mathrm{Met}\Sigma)$$

$$(\Phi^{\mathrm{EL}})^*\mathcal{A}_{\mathrm{ex}}\Sigma =: \mathcal{A}_{\mathrm{in}}\Sigma \simeq \Gamma TX \times C\Sigma \times T\mathrm{Met}(\Sigma)$$

is a Lie algebroid

Main result: theorem:
let $(X, \phi) \in \Gamma TX \times C\Sigma$ be viewed as a constant section of $\mathcal{A}_{\mathrm{in}}\Sigma$
then the anchor $\rho_{\mathrm{in}}(X, \phi)$ is a hamiltonian vector field generated by $C_{(X,\phi)}$