Groupoid symmetry of general relativity

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Abstract

Notes taken in a talk by Christian Blohmann at Goettingen, Nov. 24. 2008, extended Born-Hilbert Seminar Higher and graded structures in differential geometry

there used to be a question mark here, now answered, recent results,

1 The problem

first part on explaining the problem
4-manifold $X$ and Lorentzian metric $g$, vacuum Einstein equations say that the metric is Ricc-flat:
$\text{Ric}(g) = 0$

often one needs to formulate this as an initial value problem
(predictions, in numerical relativity, or if one wants to quantize )
so single out on $X$ a Cauchy hypersurface $\Sigma$ (which is oriented, spacelike, codimension 1)
assign a direction for time flow, i.e choose a vector field on the Cauchy surface
canonical choice: take $n$ to be the unit normal vector field $g(n,n) = -1$
extend this by exponential map
integrate $\Rightarrow$ flow of Gaussian time
flow from $-\tau$ to $\tau$ now gives a cylinder $[-\tau, \tau] \times \Sigma$
the metric on this will look like $g = \gamma(t) - dt^2$
$\gamma$ is a path of metrics in $\text{Met}(\Sigma)$
one can regard this as the result of a choice of gauge fixing.
now how to describe the dynamics for $\gamma$?
nicest way: by an action principle

$$S^{\text{field}}(g) = \int_{\Sigma \times [-\tau, \tau]} R(g) \text{vol}_g$$

where $R(g)$ is the scalar curvature of $g$.

$$S^{\text{path}}(g) := S^{\text{field}}(\gamma - dt^2) = \int_{-\tau}^\tau L(\gamma(t), \dot{\gamma}(t)) dt + \text{boundary term}$$

the boundary term contains all terms containing $\ddot{\gamma}$
here

$$L(\gamma, \dot{\gamma}) = \int_{\Sigma} (R(\gamma) + \frac{1}{4} \text{Tr}_\gamma \dot{\gamma}^2 - \frac{1}{4} (\text{Tr}_\gamma \dot{\gamma}) \text{vol}_g$$

variational principle for this $\Rightarrow$ Euler-Lagrange equations on $\text{Met}(\Sigma)$
Legendre transformation $\leftrightarrow$ $(\gamma, \pi)$
yields Hamiltonian vector field on $T^* M \text{Met}(\Sigma)$
**Proposition:** from earliest days of general relativity:

\[ \text{Ric}(g) = 0 \iff \text{Euler-Lagrange equations} + \text{constraints (because we set up variationa problem after making a gauge choice)} \]

first constraint:

\[ C_{\text{energy}} = -R(\gamma) + \text{Tr}_\gamma \pi^2 - \frac{1}{2}(\text{Tr}_\gamma)^2 = 0 \]

easy calculation \( \dot{\gamma} = -\frac{1}{2} \text{second-fundamental-form} \)

and there is the momentum constraint:

\[ C_{\text{momentum}} = -2 \text{div}_\gamma \pi \]

the constraints have to hold at every point \( x \in \Sigma \).

remember in gauge theories: constraints are the momenta of the action of the gauge group

first parameterize constraints by a vector space

\[ C(x, \phi) = \int_\Sigma \{ \gamma(X, C_{\text{momentum}}) + \phi C_{\text{energy}} \} \text{vol}_g \]

where \((X, \phi) \in \Gamma(T\Sigma) \times C\Sigma\)

now from [Katz 1962] and [deWitt 1967] we get the Poisson brackets

\[ \{ C(x, \phi), C(y, \psi) \} = C([X, Y] + \phi \text{grad}_\gamma \psi - \psi \text{grad}_\gamma \phi, X \cdot \phi - Y \cdot \psi) \]

so there is something strange about these brackets: index on the right depends on bracket
one good aspect: the constraint surface is coisotropic
bad aspect: the brackets do not close (since on the right we are pluggin in a vector field that depends on \( \gamma \), which is not what the vector fields on the left are like)
so why not fix \( \gamma \)? that would seem to yield a bundle of Lie algebras parameterized by \( \gamma \) ...
but then the Jacobi identity is no longer satisfied:
so this is *not* a bundle of Lie algebras!

**conclusion:** the constraints are not the momenta of a group action

since this is joint work with Weinstein and Fernandes one can guess what the conclusion will be:

*the constraints are moments of a groupoid action*

### 2 Solution

**idea:** Cauchy surfaces

\[ \mathcal{E}(\Sigma, X)\{ i : \Sigma \to X \text{embedding} \} \]

\[ \text{Diff}(\Sigma) \xrightarrow{\text{Diff}(\Sigma)} \mathcal{E}(\Sigma, X) \]

\[ \mathcal{H}(\Sigma, X) \]

the bottom is hypersurfaces diffeomorphic to \( \Sigma \)

\[ \mathcal{D} \mathcal{H} = (\mathcal{E}(\Sigma, X) \times \mathcal{E}(\Sigma, X))/\text{Diff}(\sigma) \]

**observations:**

- \( \text{Diff}(X) \leftarrow \text{Bisections}(\mathcal{D} \mathcal{H}) \)
- by conjugation we get \( \text{Diff}(X) \)-action on \( \mathcal{E}(\Sigma, X) \) descends to groupoid

big question: how does this groupoid act:

- how does \( \mathcal{D} \mathcal{H} \) act on “metric information”
- locally: push-forward of metric \( \Rightarrow \) no action
- globally: assume \( g \) on \( X \)
both make no good sense here, so let’s consider “middle ground”

**Definition**: A \( \Sigma \)-blink (“Augenblick”, “clin d’ oeil”) is the isometry class of a germ of a metric in a neighbourhood of a hypersurface. 

let \( B \Sigma \) be the “space” of blinks

**Proposition**: (fix embedded Cauchy hypersurface then) Every blink has a unique Gaussian representative on \( \Sigma \times [-\tau, \tau] \) 
meaning that \( g = \gamma(t) - dt^2 \) notice that if everything is analytic then these blinks are just the infinity-jets of the path \( \gamma(t) \)
how do we equip the space of blinks with a manifold structure?
extend \( \phi; S \to S' \) to \( \tilde{\phi} \)

\[ \tilde{\phi} \circ \Phi^n \simeq \Phi'^n \circ \tilde{\phi} \]

“condition of gaussian extendability” here \( \Phi^n \) is the flow of the vector field \( n \)
what’s the Lie algebroid equivalence?
Element of Lie algebroid is given by \( (X_0, \phi_0) \in \Gamma T(\Sigma) \times C\Sigma \)

**Proposition**: for \( v \) a vector field on \( U = \Sigma \times [-\tau, \tau] \)

\[ \iota_v \mathcal{L}_v \gamma = 0 \]
then:

every vector field \( X_0 + \phi_0 n \) supported on \( \Sigma \times \{0\} \) has a unique extension to a vector field \( v = X + \phi n \) satisfying the condition of gaussian extension

let \( X + \phi n, Y + \psi n \) be two gaussian vector fields satisfying gaussian extension property 
then 
\[ [X + \phi n, Y + \psi n] = ([X, Y] + \phi \text{grad}, \psi - \psi \text{grad}, \phi) + (X \cdot \phi - Y \cdot \psi) n \]

so now we have a geometric interpretation of the original constraint brackets!

**Definition**: extrinsic Lie algebroid 

\[ \mathcal{A}_{\text{ex} \Sigma} = \Gamma(TX) \times C\Sigma \times B\Sigma \]
anchor is:

\[ \rho(X_0, \phi_0, \gamma) = \mathcal{L}_{X + \phi n} g = \mathcal{L}_X \gamma = \mathcal{L}_X \gamma + \phi \gamma \]
left summand in last term is the *shift* the other one is the *lapse*

so **answer**: the strange brackets are the Lie brackets of this Lie algebroid.

**constraints**:

view the Euler-Lagrane equations \( \simeq \) as vector fields on \( T\text{Met}\Sigma \)

\[ \Phi^{\text{EL}} : T\text{Met}\Sigma \to B\Sigma \]
\[ (\gamma_0, \dot{\gamma}_0) \mapsto \text{solution of EL equations} \]
observation:
\( \Phi^{EL} \) is an injective immersion  

**Theorem:** The anchor \( \rho_{ex} \) of \( \mathcal{A}_{ex} \Sigma \) is tangent to  
\[
\Phi^{EL}(T\text{Met}\Sigma)
\]

\[
(\Phi^{EL})^{*} \mathcal{A}_{ex} \Sigma =: \mathcal{A}_{in} \Sigma \simeq \Gamma TX \times C \Sigma \times T\text{Met}(\Sigma)
\]
is a Lie algebroid

**Main result: theorem:**  
let \( (X, \phi) \in \Gamma TX \times C \Sigma \) be viewed as a constant section of \( \mathcal{A}_{in} \Sigma \)  
then the anchor \( \rho_{in}(X, \phi) \) is a hamiltonian vector field generated by \( C(X, \phi) \)