Simplical n-categories to n-categories

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Fix some notion of "space" and some notion of path n-groupoid $\mathcal{P}_n(X)$ of a space X. For instance "space" could mean manifold and $\mathcal{P}_n(X)$ could denote the strict n-groupoid of thin homotopy classes of globular n-paths. But the precise details do not matter for the following discussion.

If a surjection

$$\pi : \; Y \twoheadrightarrow X$$

is regular, then all the fiberwise products

$$Y^{[n]} := \underbrace{Y \times_X Y \times_X \cdots \times Y}_{n}$$

exist again as spaces, and we get get the simplicial space

$$Y^\bullet := (\ldots Y^{[3]} \rightrightarrows\!\!\!\!\to Y^{[[2]} \rightrightarrows Y \;) \,.$$

This happens to be the nerve of a category, namely the Čech groupoid over Y whose objects are the points of Y, and which has a unique morphism for every ordered pair of points in the same fiber of Y.

Now we can apply $\mathcal{P}_n : \mathrm{Spaces} \to n\mathrm{Cat}$ to $Y^\bullet$ to obtain the simplicial n-category

$$\mathcal{P}_n(Y^\bullet) := (\cdots \mathcal{P}_n(Y^{[3]}) \rightrightarrows\!\!\!\!\to \mathcal{P}_n(Y^{[[2]}) \rightrightarrows \mathcal{P}_n(Y) \;) \,.$$

What is the analog of the Čech groupoid now? It should be an n-groupoid whose k-morphisms are l-morphisms $P_l(Y^{[k-l+1]})$ of $\mathcal{P}_n(Y^{[k-l+1]})$.

Hence from the bisimplical set obtained by passing to the nerve of all our n-groupoids

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\cdots P_2(Y^{[3]}) & \rightrightarrows\!\!\!\!\to & P_2(Y^{[2]}) & \rightrightarrows & P_2(Y) \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots P_1(Y^{[3]}) & \rightrightarrows\!\!\!\!\to & P_1(Y^{[2]}) & \rightrightarrows & P_1(Y) \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\cdots P_0(Y^{[3]}) & \rightrightarrows\!\!\!\!\to & P_0(Y^{[2]}) & \rightrightarrows & P_0(Y)
\end{array}$$

we want to, *somehow*, obtain a mere simplicial set

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\cdots P_2(Y^{[3]}) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & P_2(Y^{[2]}) & \rightrightarrows & P_2(Y) \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots P_1(Y^{[3]}) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & P_1(Y^{[2]}) & \rightrightarrows & P_1(Y) \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\cdots P_0(Y^{[3]}) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & P_0(Y^{[2]}) & \rightrightarrows & P_0(Y)
\end{array}
$$

whose set of 0-simplices is

$$P_0(Y)\,,$$

whose set of 1-simplices is generated from

$$P_1(Y)\,, P_0(Y^{[2]})\,,$$

modulo some relations, whose set of 2-simplices is generated from

$$P_2(Y)\,, P_1(Y^{[2]})\,, P_0(Y^{[3]})$$

modulo some relations. And so on.
Question: What, if any, is the name of the abstract construction achieving this?