Simplical \( n \)-categories to \( n \)-categories

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Fix some notion of “space” and some notion of path \( n \)-groupoid \( P_n(X) \) of a space \( X \). For instance “space” could mean manifold and \( P_n(X) \) could denote the strict \( n \)-groupoid of thin homotopy classes of globular \( n \)-paths. But the precise details do not matter for the following discussion.

If a surjection

\[
\pi : Y \rightarrow X
\]

is regular, then all the fiberwise products

\[
Y^{[n]} := \underbrace{Y \times_X \cdots \times_X Y}_n
\]

exist again as spaces, and we get get the simplicial space

\[
Y^\bullet := (\cdots Y^{[3]} \rightarrow Y^{[2]} \rightarrow Y).\]

This happens to be the nerve of a category, namely the Čech groupoid over \( Y \) whose objects are the points of \( Y \), and which has a unique morphism for every ordered pair of points in the same fiber of \( Y \).

Now we can apply \( P_n : \text{Spaces} \rightarrow n\text{Cat} \) to \( Y^\bullet \) to obtain the simplicial \( n \)-category

\[
P_n(Y^\bullet) := (\cdots P_n(Y^{[3]}) \rightarrow P_n(Y^{[2]}) \rightarrow P_n(Y)).\]

What is the analog of the Čech groupoid now? It should be an \( n \)-groupoid whose \( k \)-morphisms are \( l \)-morphisms \( P_l(Y^{[k-l+1]}) \) of \( P_n(Y^{[k-l+1]}) \).

Hence from the bisimplicial set obtained by passing to the nerve of all our \( n \)-groupoids

\[
\cdots \rightarrow P_2(Y^{[3]}) \rightarrow P_2(Y^{[2]}) \rightarrow P_2(Y) \\
\cdots \rightarrow P_1(Y^{[3]}) \rightarrow P_1(Y^{[2]}) \rightarrow P_1(Y) \\
\cdots \rightarrow P_0(Y^{[3]}) \rightarrow P_0(Y^{[2]}) \rightarrow P_0(Y)
\]

we want to, somehow, obtain a mere simplicial set.
whose set of 0-simplices is

\[ P_0(Y), \]

whose set of 1-simplices is generated from

\[ P_1(Y), P_0(Y^[2]), \]

modulo some relations, whose set of 2-simplices is generated from

\[ P_2(Y), P_1(Y^[2]), P_0(Y^[3]) \]

modulo some relations. And so on.

**Question:** What, if any, is the name of the abstract construction achieving this?