$\Sigma(\mathrm{Inn}(G_2))$-2-Transport

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Abstract

Transitions of smooth 2-transport with values in $\Sigma(\mathrm{Inn}(G_2))$ are studied. It is shown that these imply the differential nonabelian cocycles found in the study of nonabelian gerbes with connection.

A differential G-2-cocycle characterizing a G-bundle with connection is the same as a transition triangle for 1-transport with values in $\Sigma(\mathrm{Inn}(G)) = \Sigma(G \to G)$.

Here we pass from G to a strict 2-group $G_2 = (H \to G)$ and consider 2-transport and its transition tetrahedra with values in $\Sigma(\mathrm{Inn}(G_2))$.

$\Sigma(\mathrm{Inn}(G_2))$-2-transport turns out to be be similar to $\Sigma(G_2)$ transport, but admitting nonvanishing "fake" 2-form curvature.

Accordingly, we find that transitions of $\Sigma(\mathrm{Inn}(G_2))$-transport are slight generalizations of those found for $\Sigma(G_2)$-transport.

In fact, we find that these transitions do include the differential cocycles found in the theory of nonabelian gerbes (for $G_2 = \mathrm{Aut}(G)$) as a special case. We do however find somewhat more general relations which reduce to those found before only after restricting certain p-form data to vanish.

The first part of the following is concerned with understanding the 3-group $\mathrm{Inn}(G_2)$.

The second part involves writing down the naturality diagrams that define p-morphisms of $\Sigma(\mathrm{Inn}(G_2))$-2-transport functors, differentiating them and deriving the corresponding differential cocycle equations in terms of differential forms.

(Warning: as far as I can see, the following reproduces the known formulas found in the literature, except for one single term which appears in Aschieri-Jurčo in the transition law for d_{ij}. I don't see this one term appearing here. But quite possibly I have overseen something somewhere.)

The 3-Group $\mathrm{Inn}(G_2)$. For a given strict 2-group G_2 coming from the crossed module $(H \to G)$ we now define what we shall call the "inner" part of the automorphism 3-group $\mathrm{Aut}(G_2)$.

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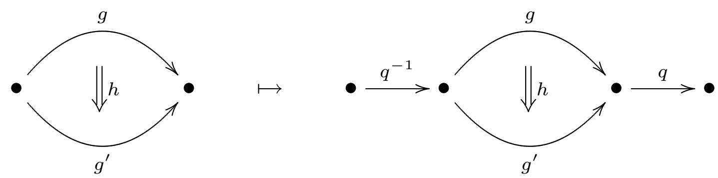
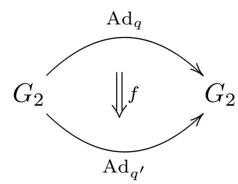
So we restrict attention to 1-automorphism of G_2 of the form

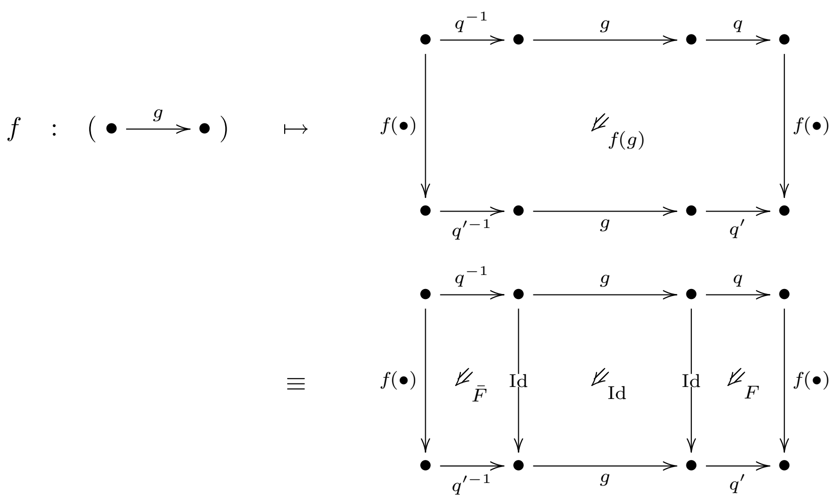
$$\mathrm{Ad}_q \quad : \quad G_2 \quad \to \quad G_2$$



labeled by elements $q \in G$. Between these we naturally have 2-morphisms

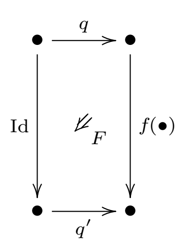


given by pseudonatural transformations which are represented by functorial assignments



for some $f \in G$ and $F \in H$. Here $\bar{F} = q\left(F^{-1}\right)$ and $f\,(g) = \bar{F}\,\bar{q}g\,(F)$.

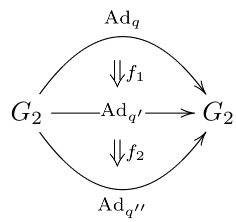
Notice how the 2-morphism is completely specified by



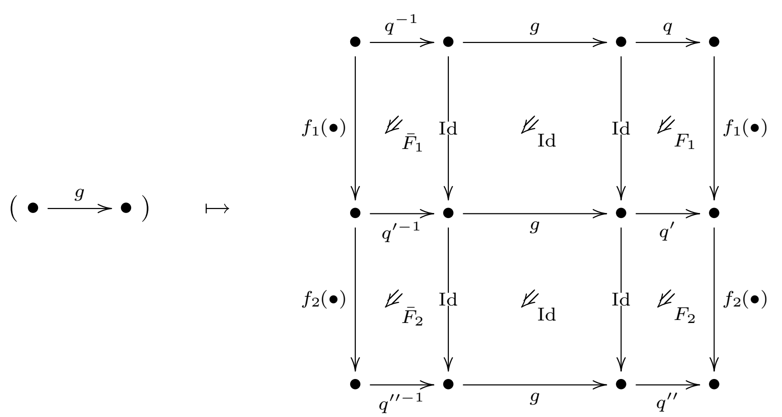
$$\in \mathrm{Mor}_2\left(\Sigma\left(G_2\right)\right).$$

It is the existence of the nontrivial $f(\bullet)$ which will be responsible for the appearance of nonvanishing fake curvature in proposition 1.

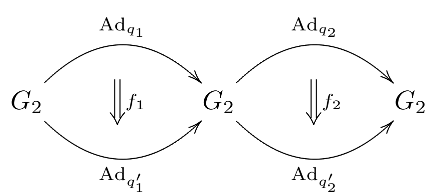
The vertical composition of two such 2-morphisms



is represented by the assignment

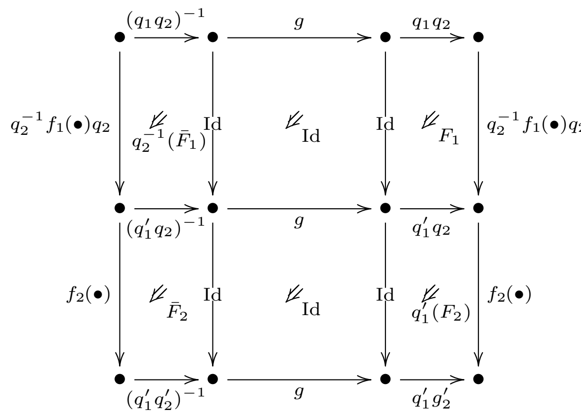


We find the horizontal composition



by whiskering with identity 2-cells and applying vertical composition. The result is

$f_1 \cdot f_2 \quad : \quad (\bullet \xrightarrow{g} \bullet) \qquad \equiv$



.

We write $q(F)$ for $\alpha(q)(F)$ whenever convenient.

If we uniquely label our 2-morphisms f by triples $(q,(f,F))$, where $q \in G$, $f \equiv f(\bullet) \in G$ and $F \in H$ then the above diagram translates into the product operation

$$(q_1,(f_1,F_1)) \cdot (q_2,(f_2,F_2)) = (q_1 q_2,\ (q_2^{-1} f_1 q_2 f_2,\ F_1\, q_1 f_1(F_2))\,. \tag{1}$$

Writing $(f,F) \equiv (\mathrm{Id},(f,F))$ and $q \equiv (q,(\mathrm{Id},\mathrm{Id}))$ we identify the semidirect product group

$$\tilde{H} \equiv \{(f,F) | f \in G, F \in H\}$$

with product

$$(f_1,F_1) \cdot (f_2,F_1) = (f_1 f_2,\ F_1\, f_1(F_2))\,,$$

as well as the group $\tilde{G} = G$.

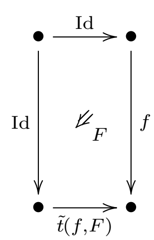
The latter acts by automorphisms on the former

$$\begin{aligned} \tilde{\alpha}(q)(f,F) &\equiv (q,(\mathrm{Id},\mathrm{Id}))\,(\mathrm{Id},(f,F))\,(q^{-1},(\mathrm{Id},\mathrm{Id})) \\ &= (qfq^{-1},\ q(F))\,. \end{aligned}$$

The target map

$$\tilde{t}(\mathrm{Id},(f,F)) \equiv t(F)\, f \tag{2}$$

defined by



is a group homomorphism $\tilde{H} \to \tilde{G}$.

We almost have a crossed module $\tilde{H} \to \tilde{G}$. One of the two consistency equations holds on the nose

$$\tilde{t}\,(\tilde{\alpha}\,(g)\,(f,F)) = g\tilde{t}\,(f,F)\,g^{-1}\,.$$

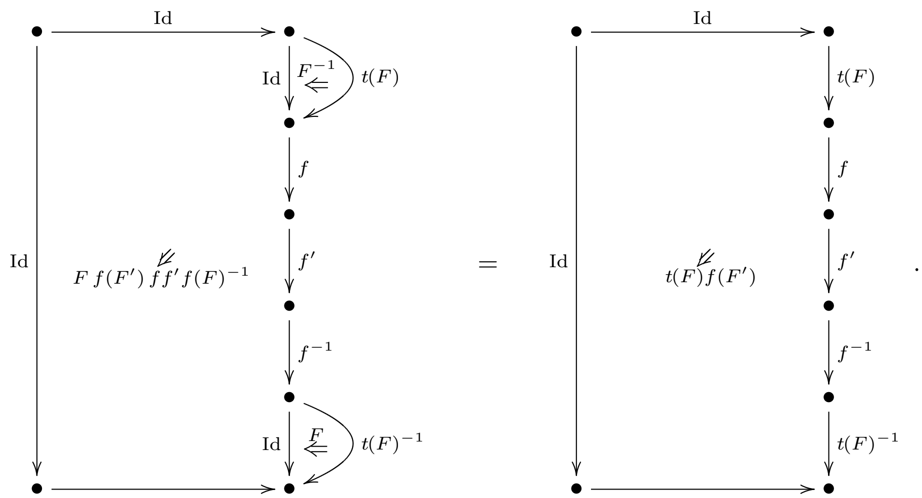
The other condition would require equality of

$$\tilde{\alpha}\left(\tilde{t}\,(f,F)\right)(f',F') = \left(t\,(F)\,f f' f^{-1} t(F)^{-1},\ t\,(F)\,f(F')\right)$$

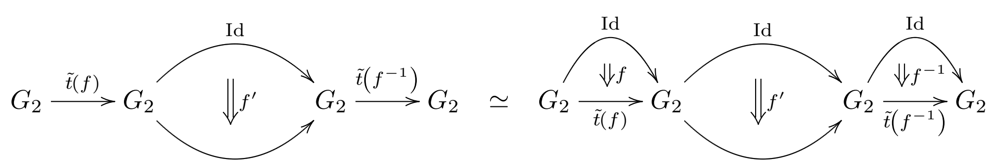
and

$$(f,F)(f',F')(f^{-1},f(F)^{-1}) = \left(f f' f^{-1},\ F\,f(F')\,f f' f(F)^{-1}\right)\ .$$

While not equal, both sides are isomorphic (related by an isomodification)

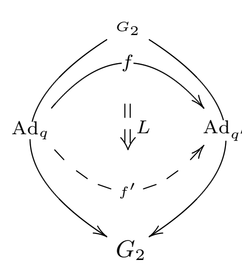


The failue of this relation to hold on the nose is due to the fact that it comes from the exchange law for 2-cells

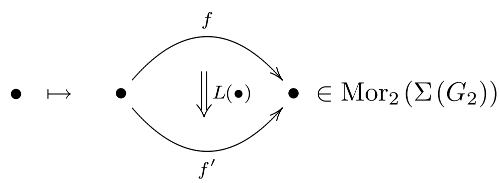


and this is not an identity in our 3-group $\mathrm{Aut}\,(G_2)$.

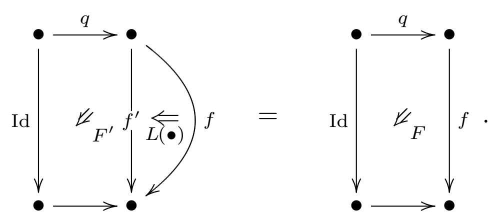
A 3-morphism



between inner 2-morphisms in $\Sigma(\mathrm{Aut}(G_2))$ is an invertible modification of pseudo-natural transformations f and f'. This is represented by an assignment



satisfying the equation



(3)

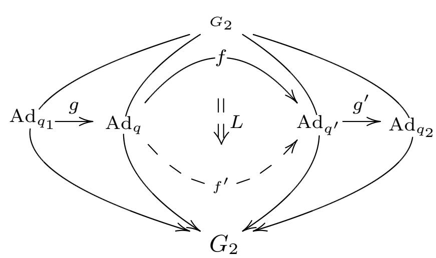
In terms of group elements this says that

$$L(\bullet) = q^{-1}(F'^{-1}F) \tag{4}$$

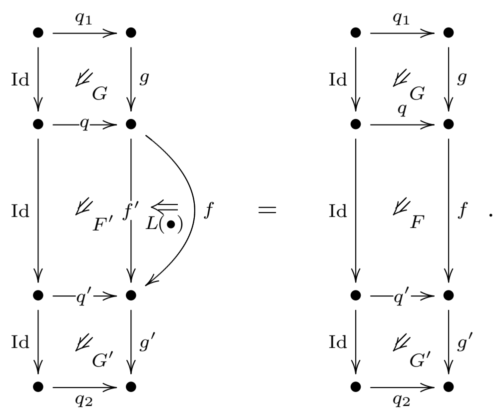
Hence there is a unique 3-morphism of this sort between any two parallel 2-morphisms f and f'.

Horizontal and vertical composition of 3-morphisms L is given by the respective composition of the $L(\bullet) \in \mathrm{Mor}_2(\Sigma(G_2))$.

There are two directions in which to whisker a 3-morphism. Whiskering of this sort



corresponds to



At the level of group elements this kind of whiskering hence acts as

$$L(\bullet) \mapsto g(L(\bullet)) \; . \tag{5}$$

Definition 1 *The sub-3-group of* $\mathrm{Aut}(G_2)$ *involving only the morphisms described above shall be called here* $\mathrm{Inn}(G_2)$*, the* **inner automorphism 3-group** *of* G_2*.*

Transitions of $\Sigma(\mathrm{Inn}(G_2))$-Transport.

Proposition 1 *Smooth 2-transport functors*

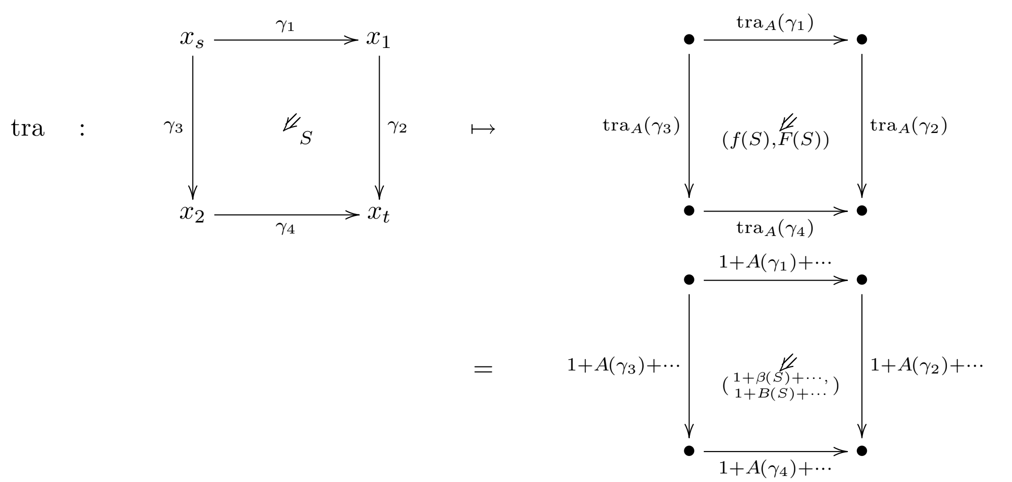
$$\mathrm{tra} : \mathcal{P}_2(U) \to \Sigma(\mathrm{Inn}(G_2))$$

are in bijection with pairs (A, B) *of differential forms* $A \in \Omega^1(U, \mathrm{Lie}(G))$*,* $B \in \Omega^2(U, \mathrm{Lie}(H))$*.*

Proof. The proof follows the same logic as for $\Sigma(G_2)$-2-transport.

First of all, the value on 1-morphism is given by 1-transport tra_A. The crucial difference now is that 2-morphisms are no longer labeled by H, but by

$\tilde{H} = (G \ltimes H)$.



This implies

$$\begin{aligned} 1 &= \mathrm{tra}_A(\partial S)\,\tilde{t}(f(S), F(S)) \\ &\overset{(2)}{=} \mathrm{tra}_A(\partial S)\,(f(S))^{-1} t(F(S)) \\ &= 1 + F_A(S) + t(B(S)) - \beta(S) + \cdots . \end{aligned}$$

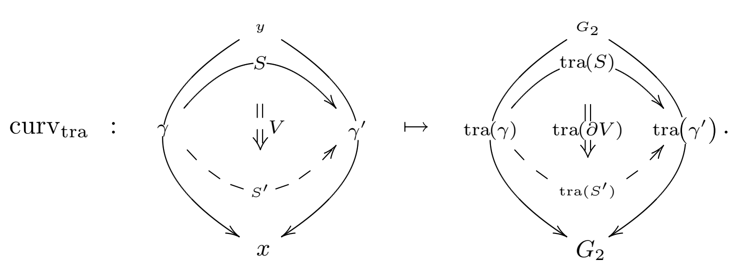
Hence

$$\beta = F_A + t(B)$$

is fixed by the choice of A and B. □

Definition 2 *The 2-form $\beta = F_A + t(B)$ is known as the* **fake curvature** *or* **2-form curvature** *of the connection (A, B).*

Since $\Sigma(\mathrm{Inn}(G_2))$ has a unique 3-morphisms between any pair of parallel 2-morphisms, we may find the **curvature 3-functor** of tra : $\mathcal{P}_2(X) \longrightarrow \Sigma(\mathrm{Inn}(G_2))$.

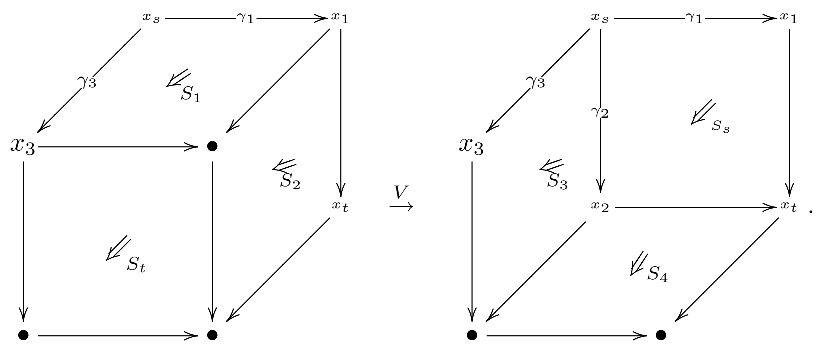


Proposition 2 *The third differential of* $\mathrm{curv}_{\mathrm{tra}_{A,B}}$ *is*

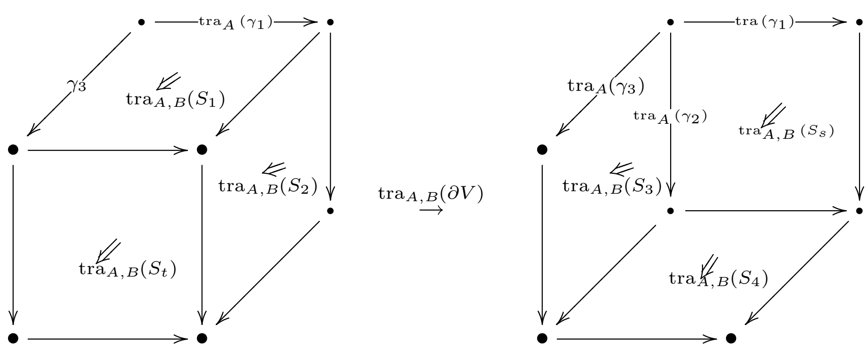
$$d\mathrm{curv}_3 = d_A B\,.$$

Proof. The computation is closely analogous to that for $\Sigma(G_2)$-2-transport and yields the same formula.

On a cube V, i.e. a 3-morphism in $\mathcal{P}_3^{\mathrm{cub}}(X)$ spanned by straight paths γ_1, γ_2 and γ_3:



we compute $\mathrm{curv}_{\mathrm{tra}_{A,B}}(V)$:



and expand to first order in the length of the three paths. In terms of the universal enveloping algebra of $\mathrm{Lie}(H)$ and $\mathrm{Lie}(G)$ we use $\mathrm{tra}_{A,B}(S) = (f(S), F(S)) = (1 + (\beta(S), B(S)) + \cdots)$. The required re-whiskering is read off from the above diagram and leads to actions of the form $\mathrm{tra}_A(\gamma)(B(S))$. Writing $B(\gamma_1, \gamma_2) = |\gamma_1||\gamma_2|B_{ij}$, etc, the term of order $|\gamma_1||\gamma_2||\gamma_3|$ in $\mathrm{Lie}(H)$ on the left is

$$|\gamma_1||\gamma_2||\gamma_3| \left(\partial_k B_{ij} + A_k(B_{ij}) + \partial_i B_{jk} + A_i(B_{jk})\right) ,$$

while on the right it is

$$|\gamma_1||\gamma_2||\gamma_3| \left(\partial_j B_{ik} + A_j(B_{ik})\right) .$$

The difference of both is the lowest order term of the H-component of $\mathrm{tra}(\partial V)$, namely

$$\mathrm{tra}_{A,B}(\partial V) = d_A B(\gamma_1, \gamma_2, \gamma_3) + \mathcal{O}|\gamma_n|^2 .$$

□

Definition 3 *The 3-form*

$$H = d_A B$$

from prop. 2 is known as the **curvature 3-form** *associated to* (A, B).

This is the same fomula as for $\Sigma(G_2)$-2-transport. The difference now is that the H here is subject to a more general Bianchi identiy and to more general transition laws.

In order to find these transition laws, we have, following the general principle of transition for n-transport, to compute 1- and 2-morphisms of our transport n-functors, i.e. pseudonatural transformations and modifications of these.

Proposition 3 *Smooth isomorphisms of* $\Sigma(\mathrm{Inn}(G_2))$*-2-transport*

$$\mathrm{tra}_{A,B} \xrightarrow{g} \mathrm{tra}_{A',B'}$$

are in bijection with quadruples (g, a, d, q)*, where* $g \in \Omega^0(U, G)$*,* $a \in \Omega^1(U, \mathrm{Lie}(H))$*,* $d \in \Omega^2(U, \mathrm{Lie}(H))$ *and* $q \in \Omega^1(U, \mathrm{Lie}(G))$*, that satisfy certain relations. Under the condition that* $q = 0$ *these relations read*

$$gA'a^{-1} + gdg^{-1} = A + t(a)$$

and

$$B = g(B') + F_a + d\,,$$

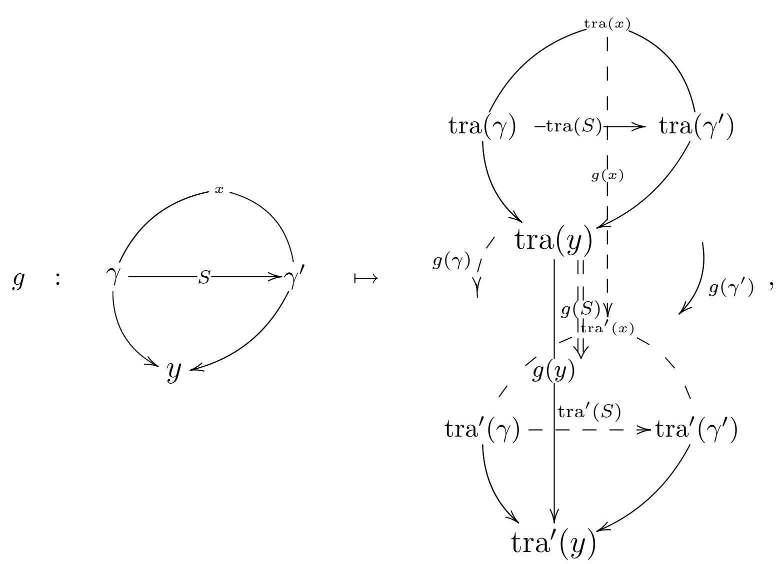
where

$$F_a = da + a \wedge a + A(a)\;.$$

Proof. A morphism of $\Sigma(\mathrm{Inn}(G_2))$-2-transport

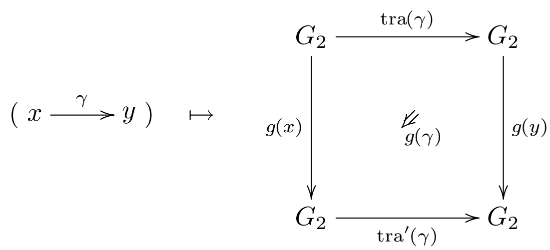
$$\mathrm{tra} \xrightarrow{g} \mathrm{tra}'$$

is represented by a 2-functorial assignment

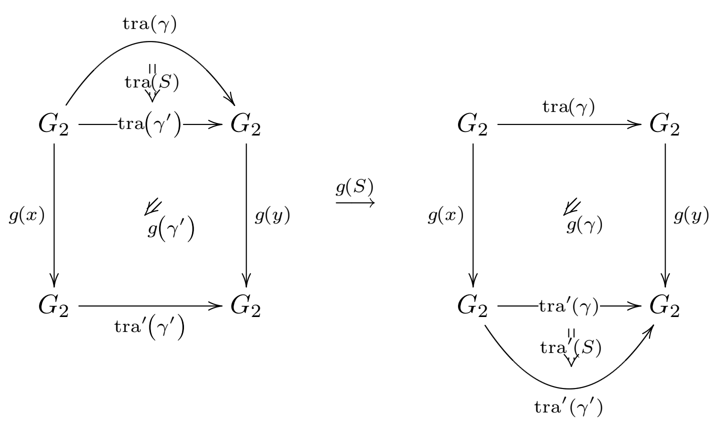


where the 3-morphism $g(S)$ has to make a certain diagram in $\Sigma(\mathrm{Inn}(G_2))$ 3-commute. But since $\Sigma(\mathrm{Inn}(G_2))$ has unique 3-morphisms between given source and target 2-morphisms, there is no an extra condition here.

Taking this apart, we have a 1-functorial assignment



giving rise to a unique 3-morphism



in $\Sigma(\mathrm{Inn}(G_2))$.

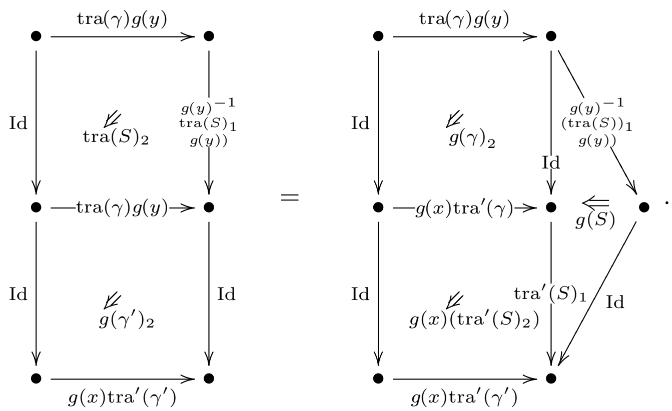
If we write $f = (f_1, f_2)$ for elements in $\tilde{H} = G \ltimes H$, then the mere existence of the 2-morphism $g(\gamma)$ is equivalent to

$$g(x)\,\mathrm{tra}'(\gamma) = t\left(g(\gamma)_2\right)\mathrm{tra}(\gamma)\,g(y)\,g(\gamma)_1\ .$$

Except for the factor $g(\gamma)_1$, this formula is the same as for $\Sigma(G_2)$-2-transport. Expanding all factors as before and setting $g(\gamma)_1 = 1 + q(\gamma) + \mathcal{O}\left(|\gamma|^2\right)$ yields

$$gA'g^{-1} + gdg^{-1} = A + t(a) + q\,.$$

Next, the existence of $g(S)$ says, according to (3), that



This means in terms of group elements that

$$g(\gamma')_2 \mathrm{tra}\,(S)_2 = g\,(x)\,(\mathrm{tra}'(S)_2) g\,(\gamma)_2 \;\; \hat{g}\,(S) \ ,$$

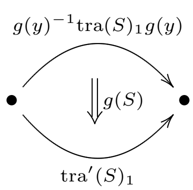
where we abbreviate

$$\hat{g}\,(S) = \mathrm{tra}\,(\gamma)\,g\,(y)\,(g\,(S)) \ . \tag{6}$$

Again, this is essentially the same as for $\Sigma\,(G_2)$-2-transport, with the only difference being the appearance of the $\hat{g}\,(S)$-factor. Expanding this as $\hat{g}\,(S) = 1 + d\,(S) + \cdots$ in the universal enveloping algebra of $\mathrm{Lie}\,(H)$ yields the transition law for B familiar from $\Sigma\,(G_2)$-2-transport, but including the contribution by d:

$$B = g\,(B') + F_a + d\,.$$

Finally notice that the existence of

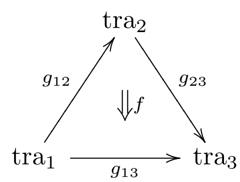


itself implies the transformation law for the fake curvature

$$g\beta' g^{-1} = \beta + t(d)\,.$$

□

Proposition 4 *Smooth 2-isomorphisms*



of smooth 1-isomorphisms (with $q = 0$) of smooth $\Sigma(\mathrm{Inn}(G_2))$-transport are in bijection with triples $(f, f_1, \tilde{f})$, where $f \in \Omega^0(U, H)$, $f_1 \in \Omega(U, G)$ and $\tilde{f} \in \Omega^1(U, H)$, which satisfy certain equations. In the case where $f_1 = \mathrm{Id}$ and $\tilde{f} = 0$ these equations are

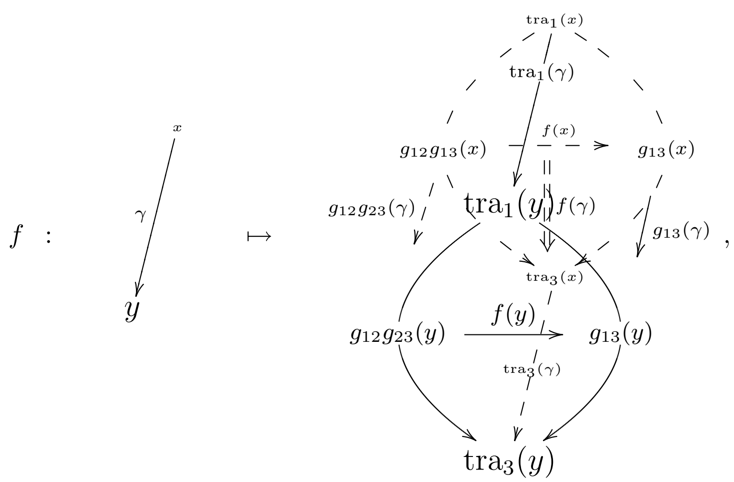
$$t(f)g_{12}g_{23} = g_{13}\,,$$

$$a_{12} + g_{12}(a_{23}) = f a_{13} f^{-1} + f\, df^{-1} + f^{-1} A_1(f)\ .$$

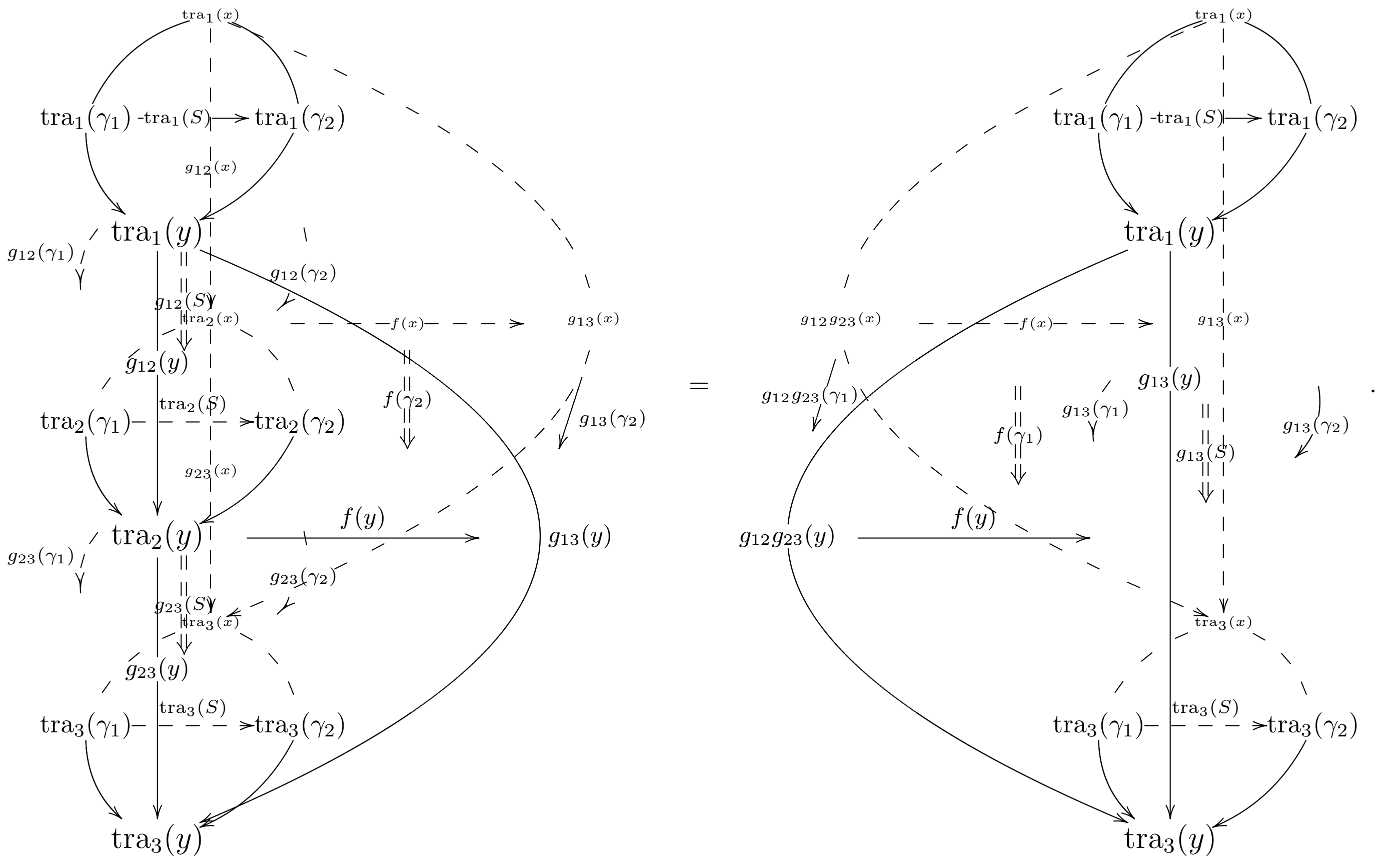
and

$$d_{12} + g_{12}(d_{23}) = f^{-1} d_{13} f\,.$$

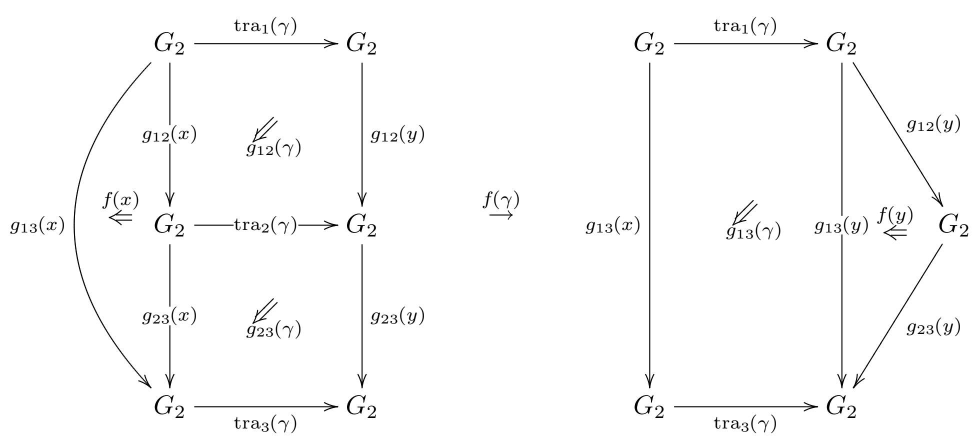
Proof. The 2-isomorphism f is represented by a 1-functorial assignment



which satisfies



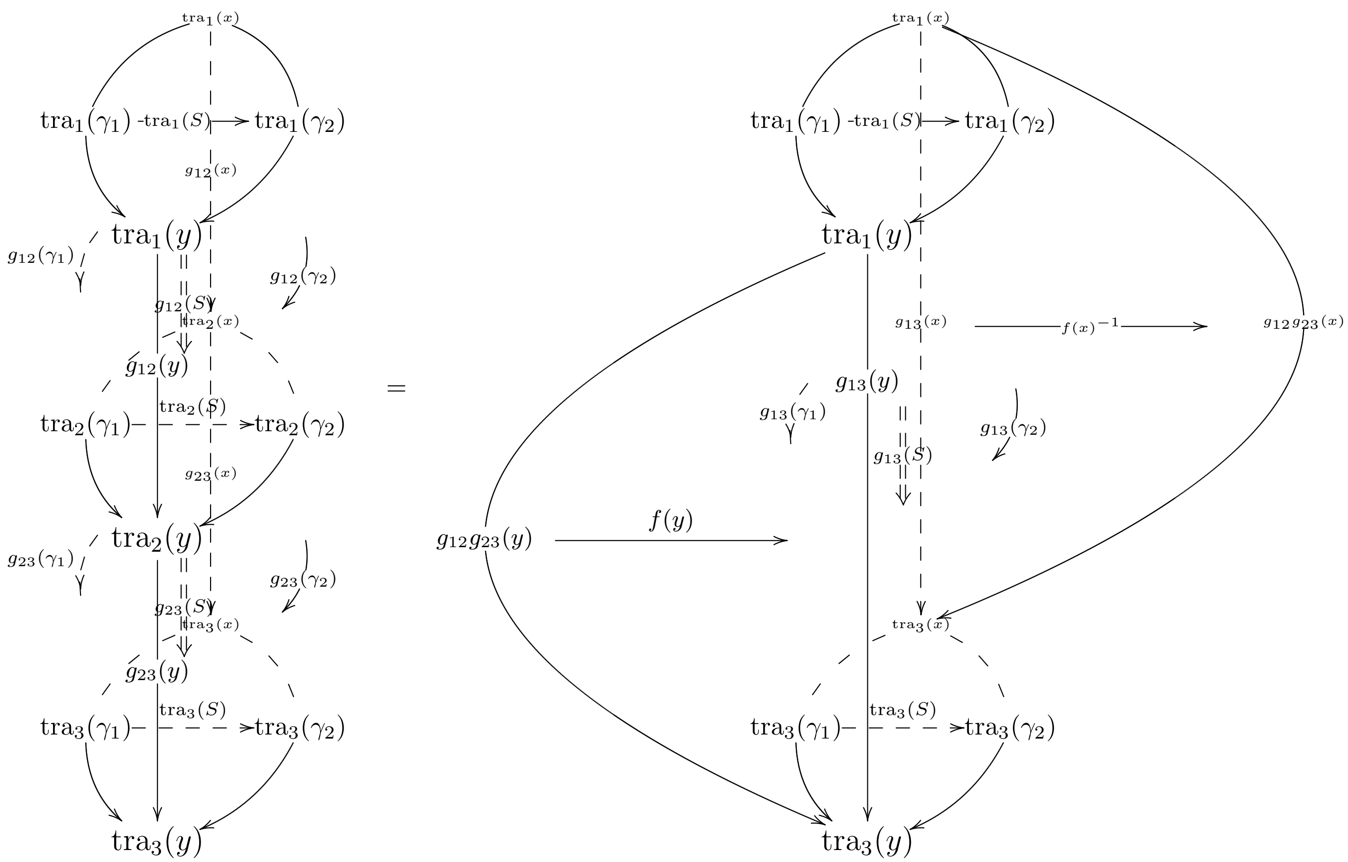
Slicing $f(\gamma)$ open, it looks like



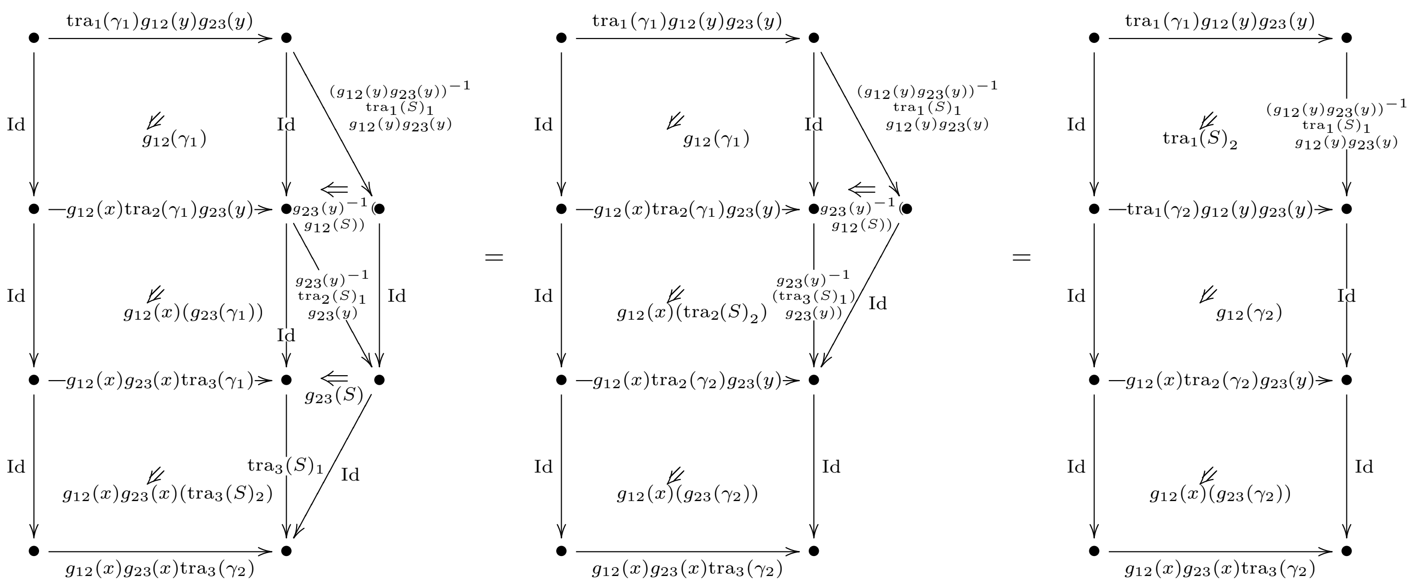
We had already restricted attention to the case $(g_{ij}(\gamma))_1 = \mathrm{Id}$. We shall now furthermore assume that also $(f(x))_1 = \mathrm{Id}$ and that $f(\gamma) = \mathrm{Id}$.

The last assumption makes the above isomorphism an identity, which is then equivalent to the respective equation found for $\Sigma(H \to G)$-transport.

Moreover, with $f(\gamma)$ being the identity, the above compatibility of f with g simplifies to



In the sense of (3) the 3-morphism on the left is given by



It is given by the group element (the $L(\bullet)$ in (3)) equal to

$$g_{23}(S)\, g_{23}(y)^{-1}(g_{12}(S))\,.$$

Notice that in terms of $\hat{g}(S) = 1 + d(S) + \cdots$ (6) this equals

$$\begin{aligned} & g_{23}(x)^{-1}(1+d_{23}(S))(g_{12}(x)g_{23}(x))^{-1}(1+d_{12}(S)) + \cdots \\ = \;& (g_{12}(x)g_{23}(x))^{-1}(1+d_{12}(S)+g_{12}(x)(d_{23}(S))) + \cdots\,. \end{aligned}$$

The 3-morphisms on the right is analogously given simply by

$$g_{13}(y)\,.$$

In terms of $\hat{g}_{13}(S) = 1 + d_{13}(S) + \cdots$ this is

$$\begin{aligned} & g_{13}(x)^{-1}(1+d_{13}(S)) + \cdots \\ = \;& (g_{12}(x)g_{23})^{-1}\left(g_{12}(x)g_{23}(x)g_{13}(x)^{-1}\left(1+d_{d_{13}(S)}\right)\right) + \cdots\,. \end{aligned}$$

Hence equating both sides yields

$$d_{12} + g_{12}(d_{23}) = g_{12}(x)g_{23}(x)g_{13}(x)^{-1}\left(d_{13}\right)\,.$$

If we assume that $g_{ij}(x) = g_{ji}(x)^{-1}$ and use that $g_{13} = t(f)g_{12}g_{23}$ then this is equivalent to

$$d_{12} + g_{12}(d_{23}) = f^{-1}d_{13}f\,.$$

□