Abstract

Transitions of smooth 2-transport with values in $\Sigma(\text{Inn}(G_2))$ are studied. It is shown that these imply the differential nonabelian cocycles found in the study of nonabelian gerbes with connection.

A differential $G$-2-cocycle characterizing a $G$-bundle with connection is the same as a transition triangle for 1-transport with values in $\Sigma(\text{Inn}(G)) = \Sigma(G \to G)$.

Here we pass from $G$ to a strict 2-group $G_2 = (H \to G)$ and consider 2-transport and its transition tetrahedra with values in $\Sigma(\text{Inn}(G_2))$.

$\Sigma(\text{Inn}(G_2))$-2-transport turns out to be similar to $\Sigma(G_2)$ transport, but admitting nonvanishing “fake” 2-form curvature.

Accordingly, we find that transitions of $\Sigma(\text{Inn}(G_2))$-transport are slight generalizations of those found for $\Sigma(G_2)$-transport.

In fact, we find that these transitions do include the differential cocycles found in the theory of nonabelian gerbes (for $G_2 = \text{Aut}(G)$) as a special case. We do however find somewhat more general relations which reduce to those found before only after restricting certain $p$-form data to vanish.

The first part of the following is concerned with understanding the 3-group $\text{Inn}(G_2)$.

The second part involves writing down the naturality diagrams that define $p$-morphisms of $\Sigma(\text{Inn}(G_2))$-2-transport functors, differentiating them and deriving the corresponding differential cocycle equations in terms of differential forms.

(Warning: as far as I can see, the following reproduces the known formulas found in the literature, except for one single term which appears in Aschieri-Jurčo in the transition law for $d_{ij}$. I don’t see this one term appearing here. But quite possibly I have overseen something somewhere.)

The 3-Group $\text{Inn}(G_2)$. For a given strict 2-group $G_2$ coming from the crossed module $(H \to G)$ we now define what we shall call the “inner” part of the automorphism 3-group $\text{Aut}(G_2)$.

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So we restrict attention to 1-automorphism of $G_2$ of the form

\[ \text{Ad}_q : G_2 \to G_2 \]

labeled by elements $q \in G$. Between these we naturally have 2-morphisms

\[ \text{Ad}_q \]

\[ \text{Ad}_{q'} \]

given by pseudonatural transformations which are represented by functorial assignments

\[ f : (\bullet \xrightarrow{g} \bullet) \mapsto f(\bullet) \]

\[ \equiv f(\bullet) \]

for some $f \in G$ and $F \in H$. Here $\bar{F} = q(F^{-1})$ and $f(g) = \bar{F} \bar{q} g(F)$.

Notice how the 2-morphism is completely specified by

\[ 1d \]

\[ \varnothing_F \]

\[ f(\bullet) \in \text{Mor}_2(\Sigma(G_2)) \].
It is the existence of the nontrivial $f(\bullet)$ which will be responsible for the appearance of nonvanishing fake curvature in proposition 1.

The vertical composition of two such 2-morphisms

\[
\begin{array}{c}
\text{Ad}_q \\
\downarrow f_1 \\
G_2 \xrightarrow{\text{Ad}_q} G_2 \\
\downarrow f_2 \\
\text{Ad}_{q''}
\end{array}
\]

is represented by the assignment

\[
(\bullet \rightarrow g \rightarrow \bullet) \mapsto (\bullet \rightarrow q^{-1} \rightarrow g \rightarrow q \rightarrow \bullet, f_1(\bullet) \downarrow F_1 \uparrow \text{Id} \downarrow F_2, f_2(\bullet) \downarrow F_1 \uparrow \text{Id} \downarrow F_2, f_1(\bullet))
\]

We find the horizontal composition

\[
\begin{array}{c}
\text{Ad}_{q_1} \\
\downarrow f_1 \\
G_2 \xrightarrow{\text{Ad}_{q_1}} G_2 \\
\downarrow f_2 \\
\text{Ad}_{q_2}
\end{array}
\]

by whiskering with identity 2-cells and applying vertical composition. The result
We write \( q(F) \) for \( \alpha(q)(F) \) whenever convenient.

If we uniquely label our 2-morphisms \( f \) by triples \((q, (f, F))\), where \( q \in G \), \( f \equiv f(\bullet) \in G \) and \( F \in H \) then the above diagram translates into the product operation

\[
(q_1, (f_1, F_1)) \cdot (q_2, (f_2, F_2)) = (q_1 q_2, (q_2^{-1} f_1 q_2 f_2, F_1 q_1 f_1(F_2))) .
\]

Writing \((f, F) \equiv (\text{Id}, (f, F))\) and \( q \equiv (q, (\text{Id}, \text{Id})) \) we identify the semidirect product group

\[
\tilde{H} \equiv \{(f, F) | f \in G, F \in H\}
\]

with product

\[
(f_1, F_1) \cdot (f_2, F_1) = (f_1 f_2, F_1 f_1(F_2)) ,
\]

as well as the group \( \tilde{G} = G \).

The latter acts by automorphisms on the former

\[
\tilde{\alpha}(q)(f, F) \equiv (q, (\text{Id}, \text{Id})) (\text{Id}, (f, F)) (q^{-1}, (\text{Id}, \text{Id})) = (q f q^{-1}, q(F)) .
\]

The target map

\[
\tilde{t}(\text{Id}, (f, F)) \equiv t(F) f
\]

defined by

\[
\tilde{t}(\text{Id}, (f, F)) \equiv t(F) f
\]
is a group homomorphism \( \tilde{H} \to \tilde{G} \).

We almost have a crossed module \( \tilde{H} \to \tilde{G} \). One of the two consistency equations holds on the nose

\[
\tilde{t}(\tilde{\alpha}(g)(f, F)) = g\tilde{t}(f, F)g^{-1}.
\]

The other condition would require equality of

\[
\tilde{\alpha}(\tilde{t}(f, F)) (f', F') = (t(f)f f^{-1}t(F)^{-1}, t(F)f(F'))
\]

and

\[
(f, F)(f', F')(f^{-1}, F^{-1}) = (ff f^{-1}, F f(F') ff f^{-1})
\]

While not equal, both sides are isomorphic (related by an isomodification)

\[
\begin{aligned}
\text{Id} & \rightarrow \rightarrow \\
\downarrow \downarrow & \rightarrow \rightarrow \\
F f(F') & \Rightarrow f' f(F')^{-1} \\
\downarrow \downarrow & \rightarrow \rightarrow \\
f f^{-1} & \rightarrow \rightarrow \\
\text{Id} & \rightarrow \rightarrow \\
\downarrow \downarrow & \rightarrow \rightarrow \\
\tilde{t}(f) & \rightarrow \rightarrow \\
\end{aligned}
\]

The failure of this relation to hold on the nose is due to the fact that it comes from the exchange law for 2-cells

\[
G_2 \xrightarrow{\tilde{t}(f)} G_2 \xrightarrow{f'} G_2 \xrightarrow{\tilde{t}(f^{-1})} G_2 \simeq G_2 \xrightarrow{\tilde{t}(f)} G_2 \xrightarrow{f} G_2 \xrightarrow{\tilde{t}(f^{-1})} G_2 ,
\]

and this is not an identity in our 3-group \( \text{Aut}(G_2) \).

A 3-morphism

\[
\begin{aligned}
\text{Ad}_g & \parallel L \\
\text{Ad}_{f'} & \parallel \\
\end{aligned}
\]
between inner 2-morphisms in $\Sigma(\text{Aut}(G_2))$ is an invertible modification of pseudo-natural transformations $f$ and $f'$. This is represented by an assignment

$$
\bullet \mapsto \bullet \int L(\bullet) \bullet \in \text{Mor}_2(\Sigma(G_2))
$$

satisfying the equation

$$
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\xrightarrow{\bullet \downarrow \bullet}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\xrightarrow{\bullet \downarrow \bullet} =
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\xrightarrow{\bullet \downarrow \bullet}
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\xrightarrow{\bullet \downarrow \bullet}.
\tag{3}
$$

In terms of group elements this says that

$$
L(\bullet) = q^{-1}(F'^{-1}F)
\tag{4}
$$

Hence there is a unique 3-morphism of this sort between any two parallel 2-morphisms $f$ and $f'$.

Horizontal and vertical composition of 3-morphisms $L$ is given by the respective composition of the $L(\bullet) \in \text{Mor}_2(\Sigma(G_2))$.

There are two directions in which to whisker a 3-morphism. Whiskering of this sort
At the level of group elements this kind of whiskering hence acts as

\[ L(\bullet) \mapsto g(L(\bullet)). \]  

(5)

**Definition 1** The sub-3-group of \( \text{Aut}(G_2) \) involving only the morphisms described above shall be called here \( \text{Inn}(G_2) \), the **inner automorphism 3-group** of \( G_2 \).

**Proposition 1** Smooth 2-transport functors

\[ \text{tra} : \mathcal{P}_2(U) \to \Sigma(\text{Inn}(G_2)) \]

are in bijection with pairs \((A, B)\) of differential forms \( A \in \Omega^1(U, \text{Lie}(G)) \), \( B \in \Omega^2(U, \text{Lie}(H)) \).

**Proof.** The proof follows the same logic as for \( \Sigma(G_2) \)-2-transport.

First of all, the value on 1-morphism is given by 1-transport \( \text{tra}_A \). The crucial difference now is that 2-morphisms are no longer labeled by \( H \), but by
\[ \tilde{\mathcal{H}} = (G \ltimes H) . \]

\[
\begin{array}{c}
\text{tra} : \\
\gamma_1 \downarrow \downarrow \downarrow \downarrow \downarrow \gamma_3 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \gamma_2 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \gamma_4 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \gamma_1 \\
x \rightarrow (f(S), F(S)) \\
\end{array}
\]

This implies
\[
1 = \text{tra}_A(\partial S) \tilde{t}(f(S), F(S))
\]
\[
\overset{(2)}{=} \text{tra}_A(\partial S) (f(S))^{-1} t(F(S))
\]
\[
= 1 + F_A(S) + t(B(S)) - \beta(S) + \cdots .
\]

Hence
\[
\beta = F_A + t(B)
\]
is fixed by the choice of \(A\) and \(B\). \(\Box\)

**Definition 2** The 2-form \(\beta = F_A + t(B)\) is known as the **fake curvature** or **2-form curvature** of the connection \((A, B)\).

Since \(\Sigma(\text{Inn}(G_2))\) has a unique 3-morphisms between any pair of parallel 2-morphisms, we may find the **curvature 3-functor** of \(\text{tra} : \mathcal{P}_2(X) \rightarrow \Sigma(\text{Inn}(G_2))\).

**Proposition 2** The third differential of \(\text{curv}_{\text{tra}, A, B}\) is
\[
d\text{curv}_3 = dA \cdot B .
\]
Proof. The computation is closely analogous to that for \( \Sigma(G_2) \)-2-transport and yields the same formula.

On a cube \( V \), i.e. a 3-morphism in \( \mathcal{P}_3^{\text{cube}}(X) \) spanned by straight paths \( \gamma_1 \), \( \gamma_2 \) and \( \gamma_3 \):

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & & & & \gamma_1 \ar[dl] \ar[dr] & \gamma_2 \ar[dr] & \gamma_3 \ar[dl] \ar[dr] & & & \cr & & & & x_5 & & & & x_3 & & x_1 \cr & & & & & S_1 & & & S_2 & & S_3 \cr & & & & & & & & & & \cr & & & & & & & & & & \cr & & & & & & & & & & \cr \end{array}
\end{array}
\end{array}
\]

we compute \( \text{curv}_{\text{tra}_{A,B}}(V) \):

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & & & & \text{tra}_A(\gamma_1) \ar[dl] \ar[dr] & \text{tra}_A(\gamma_2) \ar[dr] & \text{tra}_A(\gamma_3) \ar[dl] \ar[dr] & & & \cr & & & & x_5 & & & & x_3 & & x_1 \cr & & & & & \text{tra}_{A,B}(S_1) & & & \text{tra}_{A,B}(S_2) & & \text{tra}_{A,B}(S_3) \cr & & & & & & & & & & \cr & & & & & & & & & & \cr & & & & & & & & & & \cr \end{array}
\end{array}
\end{array}
\]

and expand to first order in the length of the three paths. In terms of the universal enveloping algebra of \( \text{Lie}(H) \) and \( \text{Lie}(G) \) we use \( \text{tra}_{A,B}(S) = (f(S), F(S)) = (1 + (\beta(S), B(S)) + \ldots) \). The required re-whiskering is read off from the above diagram and leads to actions of the form \( \text{tra}_{A}(\gamma)(B(S)) \). Writing \( B(\gamma_1, \gamma_2) = |\gamma_1| |\gamma_2| B_{ij}, \) etc, the term of order \(|\gamma_1| |\gamma_2| |\gamma_3| \) in \( \text{Lie}(H) \) on the left is

\[|\gamma_1| |\gamma_2| |\gamma_3| (\partial_k B_{ij} + A_k(B_{ij}) + \partial_i B_{jk} + A_i(B_{jk})) ,\]

while on the right it is

\[|\gamma_1| |\gamma_2| |\gamma_3| (\partial_j B_{ik} + A_j(B_{ik})) .\]

The difference of both is the lowest order term of the \( H \)-component of \( \text{tra}(\partial V) \), namely

\[\text{tra}_{A,B}(\partial V) = d_A B(\gamma_1, \gamma_2, \gamma_3) + O|\gamma_n|^2 .\]

\( \square \)
**Definition 3** The 3-form

\[ H = dA \]

from prop. 2 is known as the **curvature 3-form** associated to \((A, B)\).

This is the same formula as for \(\Sigma(G_2)\)-2-transport. The difference now is that the \(H\) here is subject to a more general Bianchi identity and to more general transition laws.

In order to find these transition laws, we have, following the general principle of transition for \(n\)-transport, to compute 1- and 2-morphisms of our transport \(n\)-functors, i.e. pseudonatural transformations and modifications of these.

**Proposition 3** Smooth isomorphisms of \(\Sigma(\text{Inn}(G_2))\)-2-transport

\[ \text{tra}_{A,B} \xrightarrow{g} \text{tra}_{A',B'} \]

are in bijection with quadruples \((g, a, d, q)\), where \(g \in \Omega^0(U, G)\), \(a \in \Omega^1(U, \text{Lie}(H))\), \(d \in \Omega^2(U, \text{Lie}(H))\) and \(q \in \Omega^1(U, \text{Lie}(G))\), that satisfy certain relations. Under the condition that \(q = 0\) these relations read

\[ gA'a^{-1} + gdg^{-1} = A + t(a) \]

and

\[ B = g(B') + F_a + d, \]

where

\[ F_a = da + a \wedge a + A(a). \]

Proof. A morphism of \(\Sigma(\text{Inn}(G_2))\)-2-transport

\[ \text{tra} \xrightarrow{g} \text{tra}' \]

is represented by a 2-functorial assignment
where the 3-morphism $g(S)$ has to make a certain diagram in $\Sigma(\text{Inn}(G_2))$ commute. But since $\Sigma(\text{Inn}(G_2))$ has unique 3-morphisms between given source and target 2-morphisms, there is no an extra condition here.

Taking this apart, we have a 1-functorial assignment

\[
( x \xrightarrow{\gamma} y ) \mapsto \begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

giving rise to a unique 3-morphism

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\text{tra}(\gamma)} & G_2 \\
\downarrow^{g(x)} & & \downarrow^{g(y)} \\
G_2 & \xrightarrow{\text{tra}'(\gamma)} & G_2
\end{array}
\]

in $\Sigma(\text{Inn}(G_2))$.

If we write $f = (f_1, f_2)$ for elements in $\hat{H} = G \ltimes H$, then the mere existence of the 2-morphism $g(\gamma)$ is equivalent to

\[
g(x) \text{tra}(\gamma) = t(g(\gamma)_2) \text{tra}(\gamma) g(y) g(\gamma)_1.
\]

Except for the factor $g(\gamma)_1$, this formula is the same as for $\Sigma(G_2)$-2-transport. Expanding all factors as before and setting $g(\gamma)_1 = 1 + g(\gamma) + O(|\gamma|^2)$ yields

\[
gA'g^{-1} + gdg^{-1} = A + t(a) + q.
\]
Next, the existence of \( g(S) \) says, according to (3), that

\[
\text{Id} \quad \text{tra}(\gamma)g(y) \quad \text{Id} \quad \text{Id} \quad g(\gamma)_{2} \quad \text{Id} \quad \text{Id} \quad \text{Id} \quad \text{Id} \quad \text{Id} \quad \text{Id} \quad g(x)\text{tra}'(\gamma)
\]

\[
= \quad \text{Id} \quad g(\gamma)_{2} \quad \text{Id} \quad g(x)\text{tra}'(\gamma)
\]

This means in terms of group elements that

\[
g(\gamma')_{2}\text{tra}(S)_{2} = g(x) \left( \text{tra}'(S)_{2}g(\gamma)_{2} \right) \hat{g}(S),
\]

where we abbreviate

\[
\hat{g}(S) = \text{tra}(\gamma)g(y)g(S).
\]

(6)

Again, this is essentially the same as for \( \Sigma(G_{2}) \)-2-transport, with the only difference being the appearance of the \( \hat{g}(S) \)-factor. Expanding this as \( \hat{g}(S) = 1 + d(S) + \cdots \) in the universal enveloping algebra of \( \text{Lie}(H) \) yields the transition law for \( B \) familiar from \( \Sigma(G_{2}) \)-2-transport, but including the contribution by \( d \):

\[
B = g(B') + F_{a} + d.
\]

Finally notice that the existence of

\[
g(y)^{-1}\text{tra}(S)_{1}g(y)
\]

itself implies the transformation law for the fake curvature

\[
g\beta'g^{-1} = \beta + t(d).
\]

\( \square \)
Proposition 4 Smooth 2-isomorphisms

of smooth 1-isomorphisms (with \( q = 0 \)) of smooth \( \Sigma(\text{Inn}(G_2)) \)-transport are in bijection with triples \((f, f_1, \tilde{f})\), where \( f \in \Omega^0(U, H) \), \( f_1 \in \Omega(U, G) \) and \( \tilde{f} \in \Omega^1(U, H) \), which satisfy certain equations. In the case where \( f_1 = \text{Id} \) and \( \tilde{f} = 0 \) these equations are

\[
 t(f)g_{12}g_{23} = g_{13},
\]

\[
 a_{12} + g_{12}(a_{23}) = f a_{13} f^{-1} + f df^{-1} + f^{-1} A_1(f)
\]

and

\[
 d_{12} + g_{12}(d_{23}) = f^{-1} d_{13} f.
\]

Proof. The 2-isomorphism \( f \) is represented by a 1-functorial assignment
which satisfies

Slicing \( f(\gamma) \) open, it looks like

We had already restricted attention to the case \((g_{ij}(\gamma))_1 = \text{Id}\). We shall now furthermore assume that also \((f(x))_1 = \text{Id}\) and that \(f(\gamma) = \text{Id}\).
The last assumption makes the above isomorphism an identity, which is then equivalent to the respective equation found for $\Sigma(H \to G)$-transport.

Moreover, with $f(\gamma)$ being the identity, the above compatibility of $f$ with $g$ simplifies to
In the sense of (3) the 3-morphism on the left is given by

\[ g_{23}(y) g_{23}(y)^{-1} (g_{12}(S)) . \]

Notice that in terms of \( g(S) = 1 + d(S) + \cdots \) (6) this equals

\[
g_{23}(x)^{-1} (1 + d_{23}(S))(g_{12}(x) g_{23}(x))^{-1} (1 + d_{12}(S)) + \cdots
\]

\[
= (g_{12}(x) g_{23}(x))^{-1} (1 + d_{12}(S) + g_{12}(x)(d_{23}(S))) + \cdots .
\]

The 3-morphisms on the right is analogously given simply by

\[ g_{13}(y) . \]

In terms of \( g_{13}(S) = 1 + d_{13}(S) + \cdots \) this is

\[
g_{13}(x)^{-1} (1 + d_{13}(S)) + \cdots
\]

\[
= (g_{12}(x) g_{23})^{-1} (g_{12}(x) g_{23}(x) g_{13}(x)^{-1} (1 + d_{13}(S))) + \cdots .
\]

Hence equating both sides yields

\[ d_{12} + g_{12}(d_{23}) = g_{12}(x) g_{23}(x) g_{13}(x)^{-1} (d_{13}) . \]

If we assume that \( g_{ij}(x) = g_{ji}(x)^{-1} \) and use that \( g_{13} = t(f) g_{12} g_{23} \) then this is equivalent to

\[ d_{12} + g_{12}(d_{23}) = f^{-1} d_{13} f . \]

\[ \Box \]