Quantum 2-States: Sections of 2-vector bundles

Urs Schreiber

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Abstract

Quantization of point particles is a process that sends a vector bundle to its space of sections ("states"), and a connection on the vector bundle to an action on this space of states.

This situation can be categorified. Suitable sections of line-2-bundles ($\simeq$ line bundle gerbes) describe states of open strings.

Over the endpoints of the string, such a 2-section amounts to a choice of gerbe module. This is known as a "D-brane".

Contents

1. **sections and states**

   (a) for applications in physics, we want to take a vector bundle and form its space of sections (= space of states H)

   (b) a connection on the vector bundle allows to do parallel transport of sections, its differential is the covariant derivative

   (c) from the covariant derivative one obtains a Laplace operator and from that a functor $\Sigma(\mathbb{R}) \to \mathrm{Aut}(H)$

   (d) moral: quantization sends a parallel transport functor on a bundle to a propagation functor on the space of states

      if we want to categorify this, we need to formulate the above in suitable terms

   (e) a dictionary for arrow-theoretic quantum mechanics; main idea: to get a section of an n-bundle with connection, pull it back to configuration space and pick a morphism from the trivial bundle to this pullback (essentially: section=trivialization)

2. **2-vector bundles**

   (a) hence we need to know what a 2-vector bundle with connection is – notice how an associated bundle with connection is particularly easy in anafunctor language: simply postcompose with representation

   (b) so all we need is to understand reps of 2-groups: should be 2-functor from $\Sigma(G_2)$ to 2Vect

   (c) what is "2Vect"? as we will later see, for us a useful definition is "module categories for abelian monoidal categories"

   (d) three examples: Baez-Crans, Kapranov-Voevodsky, Bimodules

   (e) the canonical rep of any strict 2-group on bimodules

   (f) plug this canonical rep into 2-anafunctor description of Sigma(U(1))-2-transport to obtain a line 2-bundle with connection

   (g) this defines a structure with transition line bundles with connection: a line bundle gerbe

3. **sections of 2-vector bundles**

   (a) now put all this together to see what sections of line-2-bundles are and how their parallel transoport looks like:

      parameter space is the category $\{a \to b\}$ (the open string) configuration space are maps from this into the cover 2-category of the parallel transport 2-anafunctor

   (b) use the dictionary from above to write dowm the diagram that defines a section of a line-2-bundle

(c) over a and b this defines a gerbe module, over $a \to b$ a morphism between these gerbe modules

(d) terminology: physicists call a gerbe module a "D-brane"; it is something that the ends of open strings may sit on

(e) to understand the parallel transport of these sections it helps to make the global picture of our 2-anafunctor manifest: our line 2-bundle 2-anafunctor is locally equivalent to a 2-functor with values in bimodules of compact operators

(f) remark: this generalizes to higher rank vector 2-bundles: Stolz-Teichner's string connections are of this form - just replace $G_2 = \Sigma(U(1))$ by $G_2 = \mathrm{String}(G)$ and its canonical 2-rep

(g) finally the disk diagram: a diagram depicting a 2-section coming in, propagating along a strip, and a 2-section coming out – this yields the formula for gerbe disk holonomy

1 Sections and States

Given a notion of parallel transport, there are three different things one may want to do to it:

Local trivialization expresses a globally defined functor in terms of an anafunctor.

Categorification passes from a 1-functor that sends paths to fiber morphisms to a 2-functor that sends 2-paths to morphisms of fibers that are objects in a 2-category.

These two steps are discussed in my other talk. Here I shall try to indicate, from the point of view of functorial parallel transport, a tiny aspect of a third operation that is of interest: *quantization*.

kinematics	**dynamics**
vector bundle $V \to X$	connection ∇
space of states	evolution operator
H	$U(t) : H \to H$
objects	morphisms
space of sections	path integral
straightforward	subtle

Table 1: **Quantization** involves a kinematical and a dynamical aspect.

Given an n-bundle with connection, the kinematical aspect of quantization should involve **finding the space of sections of the n-bundle and finding the action of parallel transport on these sections**.

This I discuss for line-2-bundles ($\simeq$ line bundle gerbes) with connection.

1.1 Ordinary quantization

The usual setup is this:

On a Riemannian space X we have a hermitean vector bundle $V \to X$ with connection ∇. The space of smooth sections

$$\Gamma(V)$$

of this vector bundle models the "space of states" of a particle "charged under" this vector bundle.

The Riemannian structure on X together with the hermitean structure on V induce a scalar product on $\Gamma^2(V) \subset \Gamma(V)$: the space of square integrable sections. Completing with respect to this scalar product yields the Hilbert space

$$H = \bar{\Gamma}^2(V)\,.$$

The connection ∇ on V gives rise to a differential operator

$$\nabla : H \to H$$

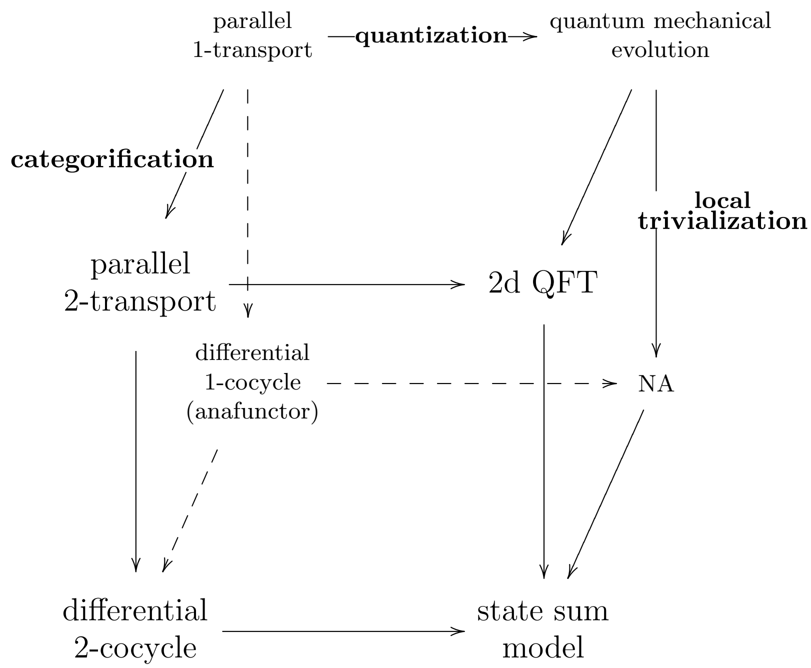


Figure 1: **Quantization, categorification and local trivialization** are the three procedures relating n-vector n-transport that play a role in the local description of n-dimensional quantum field theory. Categorification sends n-transport to $(n+1)$-transport. Quantization sends n-transport on n-paths in configuration space to n-transport on abstract n-paths (parameter space). Local trivialization sends n-transport on globally defined n-paths to n-transport on local n-paths glued by n-transitions.

on this space: the covariant derivative.

Composed with its Hilbert space adjoint

$$\nabla^\dagger$$

we obtain the covariant Laplace operator

$$\Delta = \nabla^\dagger \circ \nabla\,.$$

This induces a representation of the additive group of real numbers

$$U : t \mapsto \exp(it\Delta)$$

on H. This models the "time evolution" of a state in H over a period of time t.

If we regard the vector bundle with connection as a parallel transport functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to \mathrm{Vect}$$

then it pays to regard time evolution as a functor

$$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Hilb}$$

from 1-dimensional Riemannian cobordisms to Hilbert spaces. In fact, this is a morphism of symmetric monoidal categories with duals.

Moral. *Quantization is a process that takes a parallel transport functor and sends it to a transport functor acting on its space of sections.*

parallel transport on V	**quantization** $\longmapsto$	time evolution on $\Gamma^2(V)$
$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$		$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Aut}(\Gamma^2(V))$

Table 2: Quantization is an operation that sends **parallel transport functors to propagation functors** (modelling "time evolution") acting on the space of sections of that bundle.

1.2 Arrow-theoretic reformulation

I want to describe the analog of the above quantization procedure for the case where the vector bundle with connection is replaced by a 2-vector bundle with connection.

In order to do so, it is helpful to reformulate everything in an arrow-theoretic way that lends itself to categorification.

target space	$\mathrm{tar} = \mathcal{P}_1(X)$	paths in base space
background field	$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to \mathrm{Vect}$	parallel transport in a vector bundle with connection
parameter space	$\mathrm{par} = \{\bullet\}$	the discrete category on a single object
configuration space	$\mathrm{conf} = \mathrm{Disc}(X) \subset [\mathrm{par}, \mathcal{P}_1(X)]$	the space of inequivalent configurations of the particle in target space
space of phases	$\mathrm{phas} = [\mathrm{par}, \mathrm{Vect}]$	parallel transport of sections takes values here
abstract space of states	$\mathrm{tra}_* : \mathrm{conf} \to \mathrm{phas}$	the bundle transgressed to configuration space
concrete space of states	$\mathrm{Hom}(\mathbb{1}, \mathrm{tra}_*)$	a section is a generalized object of the transgressed bundle

Table 3: The **arrow theory of quantum mechanics** of a particle coupled to a vector bundle with connection.

With "parameter space" par the trivial category on a single object, as in the above table, we simply have

$$[\mathrm{par}, \mathcal{P}_1(X)] \simeq \mathcal{P}_1(X)\,.$$

But the idea is that varying the specification of par allows us to seamlessly model the propagation of entities richer than than the single point particle.

Parameter space and configuration space. For instance, the simplest more interesting example is that where

$$\mathrm{par} = \{a, b\}$$

is the discrete category on two elements. This models a system consisting not of one, but of two point particles propagating on X.

In this case configuration space $\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_1(X)]$ would be

$$\mathrm{conf} \simeq \mathrm{Disc}(X \times X)\,.$$

In general, we will take conf to be that subcategory of $[\mathrm{par}, \mathcal{P}_1(X)]$ that contains all morphisms that relate configurations which we want to regard as equivalent.

Configuration space for orbifolds. For example, the particle might be propagating on an orbifold O. This is best thought of as the corresponding action Lie groupoid $\mathcal{G}_O$. In this case we would take "target space"

$$\mathrm{tar} = \mathcal{P}_1(\mathcal{G}_O)$$

to be the category of paths *in* the groupoid. This is generated from paths in $\mathrm{Obj}(\mathcal{G}_O)$ together with morphisms of $\mathcal{G}_O$ modulo some natural relations.

In this case, we would want to consider the configuration where the particle sits at $x \in \mathrm{Obj}(\mathcal{G}_O)$ to be equivalent to that where it sits at $y \in \mathrm{Obj}(\mathcal{G}_O)$ if x and y are connected by a morphism of $\mathcal{G}_O$. So we would set

$$\mathrm{conf} = \mathcal{G}_O \subset [\mathrm{par}, \mathcal{P}_1(\mathcal{G}_O)]\,.$$

Configuration space for covers. Of general interest is the special case of the above situation, where the Lie groupoid in question is that coming from a cover

$$U \to X$$

of base space by open contractible sets. Paths in the corresponding groupoid

$$U^{[2]} \rightrightarrows U$$

form a category $\mathcal{P}_1(U^{[2]})$ that covers $\mathcal{P}_1(X)$ in such a way that every path has at most one lift with given source and target.

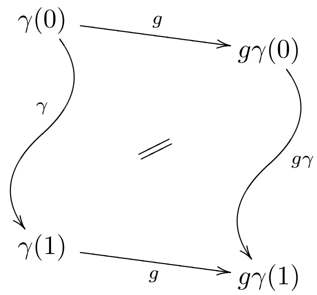


Figure 2: If **target space is an orbifold**, we can model it by the corresponding action Lie groupoid. A path *in* this groupoid is a combination of paths in the object space combined with jumps between points related by the group action. As configurations, $\gamma(0)$ and $g\gamma(0)$ are regarded as equivalent.

This is the situation encountered when the parallel transport functor is conceived as an anafunctor

$$\begin{array}{ccccc} \mathcal{P}_1(U^{[2]}) & \xrightarrow{(\mathrm{tra}_U, g)} & \Sigma(G) & \xrightarrow{\rho} & \mathrm{Vect} \\ \downarrow & & & & \\ \mathcal{P}_1(X) & & & & \end{array} \;.$$

The concrete space of sections. We know beforehand what the sections of an ordinary vector bundle are. The above arrow-theory should reproduce that.

So let configuration space be

$$\mathrm{conf} = \mathrm{Disc}(X)\,.$$

Then

$$\mathrm{tra}_* : \mathrm{conf} \to \mathrm{Vect}$$

is simply the restriction of tra to constant paths. This functor simply assigns to each point in target space the fiber above it:

$$\mathrm{tra}_* : \mathrm{Id}_x \mapsto \mathrm{Id}_{V_x}\,.$$

The tensor unit transport functor on conf is

$$\mathbb{1} : \mathrm{Id}_x \mapsto \mathrm{Id}_{\mathbb{C}}\,.$$

Therefore a natural transformation

$$e : \mathbb{1} \to \mathrm{tra}$$

is nothing but a morphims

$$e_x : \mathbb{C} \to V_x$$

for each $x \in X$. This is indeed nothing but a section of V.

We should get an equivalent result when we pass from the functor $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ to that on the cover $(\mathrm{tra}_U, g) : \mathcal{P}_1(U^{[2]}) \to \Sigma(G)$.

By the above, the transgressed functor

$$\mathrm{tra}_* : \; U^{[2]} \xrightarrow{g} \Sigma(G) \xrightarrow{\rho} \mathrm{Vect}$$

sends every transition morphism to the corresponding transition between the trivialized fibers:

$$\mathrm{tra}_* : \left(\; (x,i) \longrightarrow (x,j) \; \right) \mapsto \left(\; \mathbb{C}^n \xrightarrow{\rho(g_{ij}(x))} \mathbb{C}^n \; \right).$$

Now, a section

$$e : \mathbb{1} \to \mathrm{tra}_*$$

is a morphism

$$e_i(x) : \mathbb{C} \to \mathbb{C}^n$$

for each x in each patch U_i of the cover such that all these diagrams commute:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} \\ {\scriptstyle e_i(x)}\downarrow & & \downarrow{\scriptstyle e_j(x)} \\ \mathbb{C}^n & \xrightarrow[\rho(g_{ij}(x))]{} & \mathbb{C}^n \end{array} \quad .$$

This is nothing but a global section, as in the previous example, expressed with respect to the chosen local trivialization.

2 2-Vector bundles

The payoff of our efforts is that now it is easy to take the arrow theory discussed so far and consider it internal to 2-categories.

Principal 2-bundles with connection I have already discussed in my other talk. Now we need associated 2-bundles. This means we need to understand some basics of

- 2-vector spaces;
- representations of 2-groups on 2-vector spaces;
- 2-vector bundles associated to principal 2-bundles.

global transport on X	**local transport on cover $U \to X$**	
$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$	$(\mathrm{tra}_U, g) : \mathcal{P}_1(U^{[2]}) \to \Sigma(G) \to \mathrm{Vect}$	the transport functor
$\mathrm{conf} = \mathrm{Disc}(X)$	$\mathrm{conf} = U^{[2]}$	the configuration space
$\mathrm{tra}_* : \mathrm{Id}_x \mapsto \mathrm{Id}_{V_x}$	$\mathrm{tra}_* : \begin{array}{l} (\ (x,i) \longrightarrow (x,j)\ ) \\ \mapsto (\ \mathbb{C}^n \xrightarrow{\rho(g_{ij}(x))} \mathbb{C}^n\ ) \end{array}$	the transgressed transport
$e(x) : \mathbb{C} \to V_x$	$e_i(x) : \mathbb{C} \to \mathbb{C}^n$	the notion of section
	$\begin{CD} \mathbb{C} @>{\mathrm{Id}}>> \mathbb{C} \\ @V{e_i(x)}VV @VV{e_j(x)}V \\ \mathbb{C}^n @>>{\rho(g_{ij}(x))}> \mathbb{C}^n \end{CD}$	the gluing condition

Table 4: Summary of the way the arrow-theory reproduces the correct notion of **section of a vector bundle** with connection for the two cases where the vector bundle is encoded in a global transport functor on target space and as a local functor on a cover, respectively.

2.1 2-Vector spaces.

A vector space is a module for a field. The concept of field is hard to categorify. But fields are rings, and the categorification of a ring is an abelian monoidal category.

2.2 Representations of 2-groups

For G an ordinary group and $\Sigma(G)$ its suspension, i.e. the category with a single object and G worth of morphisms, a representation ρ of G is a functor

$$\rho : \Sigma(G) \to \mathrm{Vect}\,.$$

Accordingly, we say that for G_2 a 2-group and $\Sigma(G_2)$ the corresponding 2-groupoid with a single object that a 2-functor

$$\tilde{\rho} : \Sigma(G_2) \to {}_{\mathcal{C}}\mathrm{Mod}$$

is a $\mathcal{C}$-linear 2-representation.

A useful example is the 2-representation of a strict 2-group

$$G_2 = (t : H \to G)$$

induced from an ordinary representation ρ of H.

$$\tilde{\rho} : \Sigma(G_2) \to \mathrm{Bim}(\mathrm{Vect})\,.$$

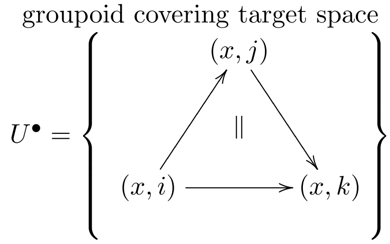
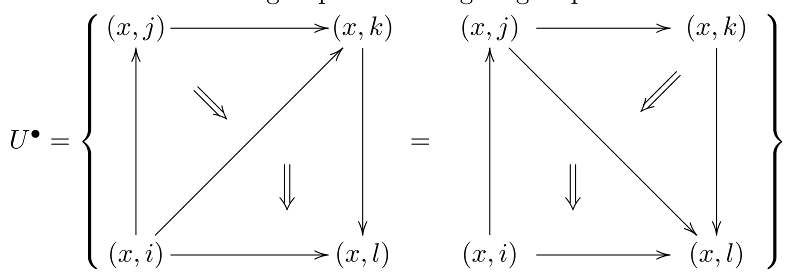
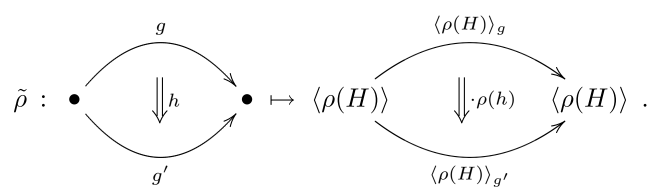
1-transport	**2-transport**
domain: path groupoid $\mathcal{P}_1(X)$	domain: 2-path 2-groupoid $\mathcal{P}_2(X)$
codomain: vector spaces Vect	codomain: 2-vector spaces 2Vect
structure group: G	structure 2-group: G_2
representation $\rho : \Sigma(G) \to \mathrm{Vect}$	2-representation $\rho : \Sigma(G_2) \to \mathrm{2Vect}$
trivial vector bundles with connection smooth functors: $\mathcal{P}_1(X) \longrightarrow \Sigma(G) \xrightarrow{\rho} \mathrm{Vect}$	trivial 2-vector bundle with connection smooth 2-functor: $\mathcal{P}_2(X) \longrightarrow \Sigma(G_2) \xrightarrow{\rho} \mathrm{2Vect}$
groupoid covering target space 	2-groupoid covering target space 
vector bundles with connection smooth anafunctors: $\mathcal{P}_1(U^\bullet) \longrightarrow \Sigma(G) \xrightarrow{\rho} \mathrm{Vect}$, $\; \mathcal{P}_1(U^\bullet) \xrightarrow{p} \mathcal{P}_1(X)$	2-vector bundle with connection smooth 2-anafunctor: $\mathcal{P}_2(U^\bullet) \longrightarrow \Sigma(G_2) \xrightarrow{\rho} \mathrm{2Vect}$, $\; \mathcal{P}_2(U^\bullet) \xrightarrow{p} \mathcal{P}_2(X)$

Table 5: On the left, our description of bundles with connection in terms of parallel transport functors. On the right our categorification of this situation.

Let $\langle\rho(H)\rangle$ be the algebra generated by the image of ρ and let $\langle\rho(H)\rangle_g$ for $g \in G$ be the $\langle\rho(H)\rangle$ bimodule which is $\langle\rho(H)\rangle$ itself as an object, with the right action twisted by g. Then $\tilde\rho$ is given by



We shall only need a very simple special case of this, namely the representation of the 2-group

$$\Sigma(U(1)) = (U(1) \to 1)$$

linear algebra	2-linear algebra
field or ring K	abelian monoidal category $\mathcal{C}$
K-vector space	$\mathcal{C}$-module category
K-linear map	$\mathcal{C}$-linear functor

Table 6: **Categorified linear algebra** is the theory of module categories for abelian monoidal categories.

abelian monoidal category $\mathcal{C}$	examples for $\mathcal{C}$-module categories
$\mathrm{Disc}(K)$	Baez-Crans 2-vector spaces: categories internal to Vect_K
Vect_K	Kapranov-Voevodsky 2-vector spaces: categories of the form $(\mathrm{Vect}_K)^n \simeq \mathrm{Mod}_{K^n}$
$\mathcal{C}$	categories of $\mathcal{C}$-internal algebra modules in the image of the canonical embedding $\mathrm{Bim}(\mathcal{C}) \xrightarrow{\subset} {}_{\mathrm{Vect}_K}\mathrm{Mod}$ $\left(A \underset{N'}{\overset{N}{\Downarrow \rho}} B\right) \mapsto \left(\mathrm{Mod}_A \underset{?\otimes_A N'}{\overset{?\otimes_A N}{\Downarrow ?\otimes \rho}} \mathrm{Mod}_B\right)$

Table 7: Various flavors of **2-vector spaces**.

induced from the standard representation of $U(1)$:

$$\tilde{\rho} : \left(\bullet \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\Downarrow c \in U(1)}} \bullet\right) \mapsto \left(\mathbb{C} \underset{\mathbb{C}}{\overset{\mathbb{C}}{\Downarrow \cdot c}} \mathbb{C}\right) .$$

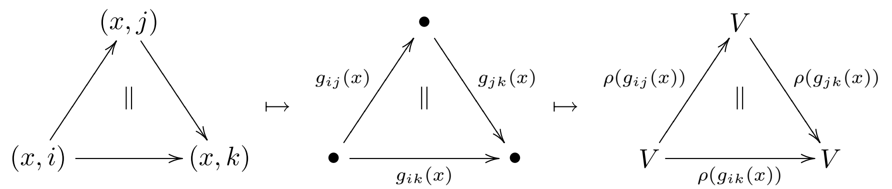
2.3 Associated 2-bundles

Given a principal G-bundle P and a representation G on a vector space V, one can form the total space of the associated vector bundle by forming the quotient

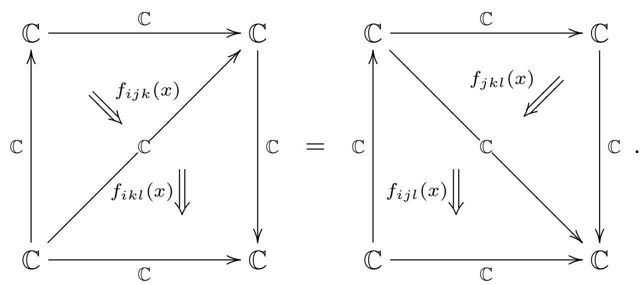
$$P \times_G V .$$

Categorifying this description is subtle, since it involves 2-coequalizers.

What is much simpler is the description of associated bundles in terms of local transitions. There it simply amounts to postcomposing the respective anafunctor with the representation:



This categorifies easily. We find that a **line-2-bundle**, coming from the canonical 2-rep of $\Sigma(U(1))$, is given by a 2-anafunctor that labels transition tetrahedra like this:



Bundle gerbes with connection. The above 2-anafunctor of the form

$$\begin{array}{ccccc} \mathcal{P}_2(U^\bullet) & \longrightarrow & \Sigma(\Sigma(U(1))) & \xrightarrow{\tilde{\rho}} & \mathrm{Bim}(\mathrm{Vect}) \\ {\scriptstyle p}\downarrow & & & & \\ \mathcal{P}_2(X) & & & & \end{array}$$

can be regarded as the end result of a local trivialization process of several steps, descending along this chain of injections:

$$\Sigma(\Sigma(U(1))) \xrightarrow[\subset]{\tilde{\rho}} \Sigma(1d\mathrm{Vect}_{\mathbb{C}}) \xrightarrow[\subset]{} \mathrm{Bim}_{\mathrm{FinRnk}}(\mathrm{Vect}_{\mathbb{C}}) \ .$$

Notice that $\mathbb{C}$, regarded as an algebra over itself, is (Morita-)equivalent to any algebra of finite-rank operators on a complex vector space.

Side remark. Therefore, in the world of 2-vector bundles with respect to the flavor of 2-vector spaces given by $\mathrm{Bim}(\mathrm{Vect}_{\mathbb{C}})$, the most general line-2-bundle with connection is a 2-functor with values in $\mathrm{Bim}_{\mathrm{FinRnk}}(\mathrm{Vect}_{\mathbb{C}})$.

Given any principal $\mathrm{PU}(H)$-bundle with connection on X, the fact that $PU(H)$ is the automorphism group of the algebra of finite-rank operators on H gives us canonically an associated algebra bundle with connection. This is encoded in a 1-functor $\mathcal{P}_1(X) \to \mathrm{Bim}_{\mathrm{FinRnk}}(\mathrm{Vect}_{\mathbb{C}})$. A smooth choice of lift of the $\mathrm{Lie}(\mathrm{PU}(H))$-valued curvature at each point to $\mathrm{Lie}(U(H))$ gives an extension to a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}_{\mathrm{FinRnk}}(\mathrm{Vect}_{\mathbb{C}})\,.$$

3 Sections of 2-vector bundles

Now that we know what a line-2-bundle is like, we can plug this into our arrow theory of quantum mechanics and find out what sections into this bundle are like.

3.1 Open string coupled to a line-2-bundle

For this, we have to specify a suitable parameter space. Of particular interest is the parameter space given by the category

$$\mathrm{par} = \{a \to b\}$$

that consists of two objects with a single nontrivial morphism between these.

This choice allows us to consider parallel transport over surfaces with the topology of disk:

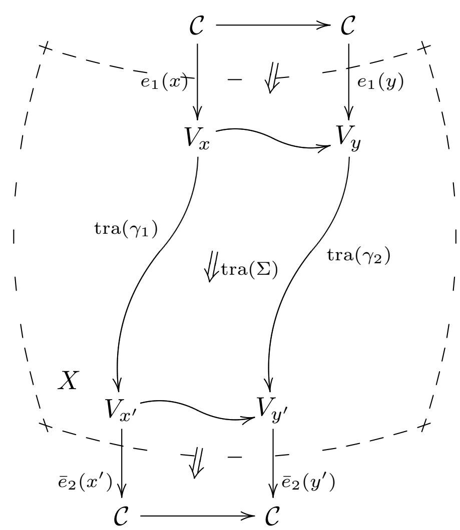


Figure 3: The parameter space $\{a \to b\}$ models "open strings" that trace out disk-like surfaces as they propagate through target space.

Of all maps

$$\mathrm{par} \to \mathcal{P}_2(U^\bullet)$$

we take those to be gauge equivalent that project to the same path in X.

Configuration space

$$\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_2(U^\bullet)]$$

contains all morphisms describing such gauge equivalences.

parameter space par	interpretation
$\{\bullet\}$	single point particle
$\{a, b\}$	two point particles
$\{a \to b\}$	open string
$\Sigma(\mathbb{N}) = \{a \to a\}$	closed string
$a \overset{\to}{\underset{\to}{\Downarrow}} b$	disk-shaped membrane
$a \overset{\to}{\underset{\to}{\Downarrow}} a$	cylinder-shaped membrane

Table 8: Different **choices of parameter space categories** and the corresponding physics interpretation.

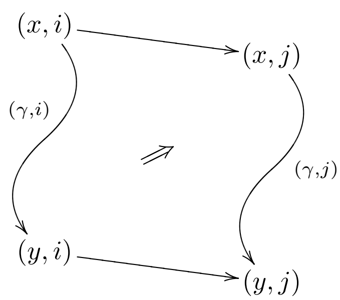


Table 9: A **morphism in the configuration space of the open string** relating two string configurations that differ only by a gauge transformation.

Notice that for par $= \{a \to b\}$ this configuration space is a lot like that describing two 1-particles $\{a, b\}$, but now encoding the information that these two points are connected by a string.

The inclusion

$$\{a, b\} \xrightarrow{\subset} \{a \to b\}$$

identifies the two endpoints of the open strings as two point particles.

In order to understand sections of the line-2-bundle over the open string it therefore helps to first study their behavious over these endpoints only. This amounts to pulling back our 2-bundle with connection, encoded in the 2-anafunctor (tra_U, g, f), all the way to the parameter space of the 2 point particles

$$\{a, b\} \xrightarrow{\subset} \{a \to b\} \xrightarrow{\gamma} \mathrm{conf} \xrightarrow{\subset} [\mathrm{par}, \mathcal{P}_2(U^\bullet)] \xrightarrow{(\mathrm{tra}, g, f)_*} [\mathrm{par}, 2\mathrm{Vect}] \ .$$

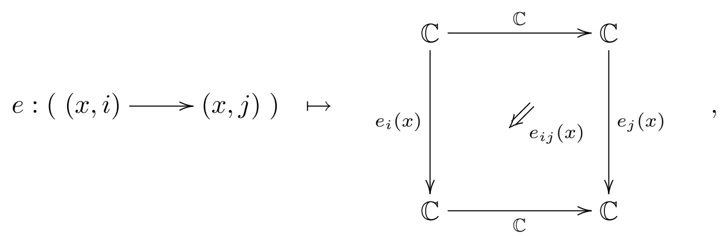
But the configuration space of two point particles a and b propagating on $\mathcal{P}_2(U^\bullet)$

is simply $U^\bullet \times U^\bullet$. A section e of a line-2-bundle over the endpoint of the string is therefore a morphism

$$e : \mathbb{1}|_{U^\bullet} \to \mathrm{tra}_*|_{U^\bullet} .$$

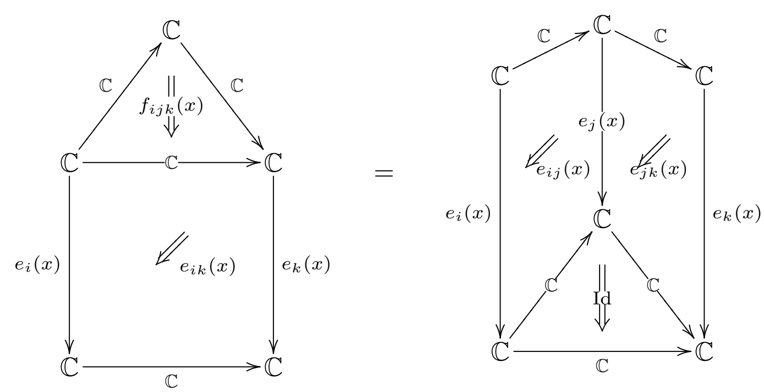
3.2 Gerbe modules and D-branes

What is such a morphism like? Being a morphism of 2-functors, it is a pseudo-natural transformation. This means e is determined by an assignment



for each point x in a double intersection of the cover, where $e_i(x)$ and $e_j(x)$ are $\mathbb{C}$-bimodules, hence vector spaces, and where $e_{ij}(x) : e_j(x) \to e_i(x)$ is a morphism of $\mathbb{C}$-bimoudles, hence a linear map.

The consistency condition this assignment has to satisfy is



for all x in triple overlaps of the cover.

If you like formulas better, think of this equivalently as saying that

$$e_{ij} \circ e_{jk} = f_{ijk} e_{ik} .$$

It follows that the section e of our line-2-bundle is, over the endpoints of the open string, much like an ordinary vector bundle, but one whose transition cocycle involves a certain "twist" which is measured by the cocycle data of the line-2-bundle.

Such structures are equivalently known as

- twisted vector bundles

- gerbe modules
- twisted representations of $U^{[2]}$
- D-branes with Chan-Paton bundles .

In conclusion, we find that

Proposition 1 *A section of a line-2-bundle ($\simeq$ line bundle gerbe) with respect to the open string $\{a \to b\}$ is a D-brane over a, another D-brane over b together with a morphism of D-branes over $a \to b$.*