On 2-Representations and 2-Vector Bundles

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1 Introduction.

There is a body of material on 2-representations of strict 2-groups and the associated 2-vector bundles with 2-connection that I have thought about, various parts of which

- exist and only need to be written up cleanly
- or exist in more or less rough outline and need to be worked out in detail
- or are conjectures for which a good amount of evidence exists but no proof yet.

Here I try to summarize this material and indicate what needs to be done where.

The main point is this:

Claim. There is a canonical 2-representation ρ of any strict 2-group G_2 on $\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}$. This allows to consider 2-vector transport 2-functors $\mathrm{tra} : \mathcal{P}_2(X) \to 2\mathrm{Vect}$ that are locally ρ-trivializable. This gives a notion of ρ-associated 2-vector bundles with connection.

For the simple case $G_2 = \Sigma U(1)$ these 2-functors have local semi-trivializations which are line bundle gerbes with connection.

It seems that everything goes through entirely analogously for the String 2-group $G_2 = \mathrm{String}_k(G)$. We naturally obtain a notion of String bundle with 2-conneciton this way. The resulting 2-functor is superficially different from the String connection 2-functor proposed by Stolz and Teichner, but both share a couple of striking similarities.

1.1 Fibers for 2-Vector Bundles.

Given any 2-group G_2, which we may regard as a 2-groupoid ΣG_2 with a single object, and given any notion of 2-vector spaces, living in the 2-category 2Vect, a 2-functor

$$\rho : \Sigma G_2 \to 2\mathrm{Vect}$$

is a *2-representation* of G_2.

Just as there are different flavors of 1-vector spaces (real, complex, etc.) there are various flavors of 2-vector spaces that one can imagine. One is that introduced by Kapranov and Voevodsky, living in KV2Vect.

2-representations on Kapranov-Voevodsky 2-vector spaces have been considered in the literature. In particular, Kapranov and Ganter have initiated the study of 2-representations of *discrete* 2-groups (2-groups that are really just 1-groups) by weak 2-functors on KV2Vect.

There are indications that these kind of representations form an interesting 2-category in their own right and play a crucial role in applications to Dijkgraaf-Witten theory.

Then there are the 2-vector spaces introduced by Baez and Crans, living in BC2Vect. These have been shown to be the right home for the categorification of the concept of Lie algebras. Thus they are certainly intimately related to the theory of Lie 2-groups. On the other hand, the killer application for 2-representations on BC2Vect apparently remains to be identified.

In any case, it seems that neither KV2Vect nor BC2Vect provide the right model for fibers of interesting classes of 2-vector bundles that apper in nature.

Baas, Dundas, Rognes, Richter, Kro and others have computed the classifying spaces of KV and of BC 2-vector bundles. The former turns out to be the algebraic K-theory of the ordinary K-theory ring spectrum. The other turns out to be two copies of the ordinary K-theory spectrum.

While not uninteresting, this falls considerably short of the original expectation which motivated these studies: that the classifying space of 2-vector bundles should be related to the tmf spectrum.

This indicates that neither the KV nor the BC flavor of 2-vector spaces are the ones that serve, generally, as fibers of interesting classes of 2-vector bundles.

From the point of view of 2-representation theory this is not really surprising: like ordinary vector bundles are associated by ordinary *faithful* representations to ordinary principal bundles, the representations of general 2-groups on KV2Vect and BC2Vect are typically far from being faithful.

This indicates that a larger class of 2-vector spaces may be needed for application in 2-vector bundle theory.

A hint towards in which direction this generalization is to be sought comes from basic facts about line bundle gerbes as well as from the work by Stolz and Teichner on String bundles.

Whatever concept of 2-vector bundle one finds, line bundle gerbes should certainly be examples of rank-1 2-vector bundles. And string bundles are supposed to be related to tmf roughly like spin bundles – via the Dirac operators acting on their sections – are related to ordinary K-theory.

Therefore it is remarkable, that both line bundle gerbes as well as String-bundles can be realized as *bundles of algebras*: algebras of compact operators in the former and von Neumann type III factors in the latter case.

Moreover – and this is the crucial aspect – in both cases a 2-connective structure may sensibly be defined, whose parallel surface transport is a 2–functor that sends points in base space to the algebra fiber over that point, which sends paths in base space to bimodules for the correspond endpoint algebras, and which sends surfaces in base space to bimodule homomorphisms.

For String bundles this 2-functorial description is one of the key constructions by Stolz-Teichner. For line bundle gerbes an analogous description of connective structure also exists, even though it may so far not have received due attention.

These two examples suggest that ordinary algebras may play the role of 2-vector spaces. This becomes evident once we notice that there is a canonical chain of inclusions

$$\mathrm{KV2Vect} \hookrightarrow \mathrm{Bim} \hookrightarrow {}_{\mathrm{Vect}}\mathrm{Mod}$$

and that Vect-module categories are an obvious categorification of ordinary vector spaces, which are K-modules, for K some field.

The 2-category ${}_{\mathrm{Vect}}\mathrm{Mod}$, in its entirety, is hard to get one's hands on. The image of Bim inside ${}_{\mathrm{Vect}}\mathrm{Mod}$ under the above inclusion is much more accessible. And still, this is considerably larger than the image of the inclusion of KV 2-vector spaces.

Finally, for every strict 2-group there is a canonical and essentially *monomorphic* 2-representation on Bim.

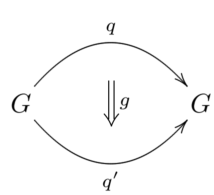
(And in writing this I realize that I need to figure out what precisely "essentially monomorphic" is supposed to mean here. This is one among a collection of aspects of my discussion here which should eventually be worked out in more detail.)

1.2 The canonical 2-representation.

Recall that the automorphism 2-group $\mathrm{AUT}(G)$ of any ordinary group is nothing but the auto-Hom-category of the category ΣG:

$$\Sigma\mathrm{AUT}(G) := \mathrm{Aut}_{\mathrm{Cat}}(\Sigma G)\,.$$

A 2-cell here looks like



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where $g, g' \in \mathrm{Aut}(G)$ are elements of the ordinary automorphism group of G and where $g \in G$ labels a natural transformation between these, given by

$$q' = \mathrm{Ad}_g \circ q\,.$$

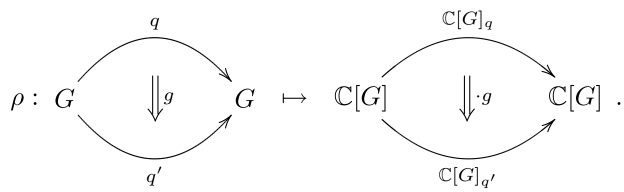
Notice that we have a functor

$$\mathrm{Grp} \to \mathrm{Bim}$$

which sends groups to their group algebras and group homomorphisms to the induced algebra homomorphism. This in fact extends to a 2-functor in that it provides us canonically with a 2-representation

$$\rho : \Sigma \mathrm{AUT}(G) \to \mathrm{Bim}$$

acting as



Here $\mathbb{C}[G]_q$ denotes the $\mathbb{C}[G]$-bimodule which, as an object, is $\mathbb{C}[G]$ itself, with the right action on itself twisted by q.

An analogous construction goes through for any other strict 2-group G_2, coming from any crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

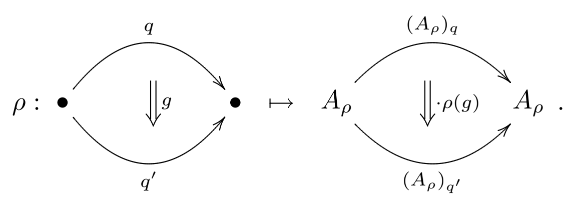
of ordinary groups H and G. Moreover, for any faithful ordinary representation

$$\rho_1 : \Sigma H \to \mathrm{Vect}$$

we can replace the group algebra by the algebra of operators

$$A_\rho := < \mathrm{im}(\rho_1) >$$

generated by the image of ρ and still obtain a 2-representation $\rho : \Sigma G_2 \to \mathrm{Bim}$, now given by



1.3 Line Bundle Gerbes and Rank-1 2-Vector Bundles.

A very simple example of the above kind of 2-representations is obtained by setting

$$G_2 = \Sigma U(1)$$

coming from the crossed module

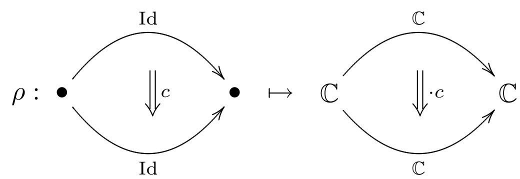
$$U(1) \longrightarrow 1$$

and using the standard representation

$$\rho_1 : \Sigma U(1) \to \mathrm{Vect}$$

of $U(1)$ on $\mathbb{C}$.

The corresponding 2-representation looks a little boring



but serves as a great toy example for more sophisticated scenarios.

For X any base manifold and $\mathcal{P}_2(X)$ the path 2-groupoid of X, we say that a 2-vector bundle with connection on X is a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to 2\mathrm{Vect}\,.$$

This 2-vector bundle is *smooth* and *associated* via ρ to a $\Sigma U(1)$-principal 2-bundle precisely if there is a surjective submersion $\pi : Y \to X$ such that tra fits into a square of the form

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{\pi_*} & \mathcal{P}_2(X) \\ {\scriptstyle \mathrm{triv}}\downarrow & \swarrow\!\!\swarrow {\scriptstyle \sim} & \downarrow{\scriptstyle \mathrm{tra}} \\ \Sigma^2 U(1) & \xrightarrow{\rho} & 2\mathrm{Vect} \end{array} ,$$

and that the corresponding transition data is smooth. Here ρ is the canonical 2-rep from above and we agreed that for our present purpose we set $2\mathrm{Vect} := \mathrm{Bim}$.

Notice that this condition implies in particular that every fiber of tra is equivalent, in Bim, to $\mathbb{C}$. But this means nothing but that the corresponding bundle here is a *bundle of algebras of compact operators*, $K(H)$, on a Hilbert space H. The automorphism group of $K(H)$ happens to be $PU(H)$. Notice that this group is precisely the classifying space of our structure 2-group

$$PU(H) \simeq |\Sigma U(1)|\,.$$

By working out everything in more detail, one finds indeed that smoothly locally ρ-trivializable 2-vector transport functors are equivalent to line bundle gerbes with connection ("and curving").

(I have spelled this out in detail for the case that the Hilbert space H is assumed to be finite, which then corresponds to gerbes whose class is pure torsion. I expect that after appropriately taking care of the relevant analysis, everything straightforwardly goes through also for the general case. Doing this carefully and in detail is one thing that needs to be done.)

1.4 String Bundles.

For G any simple, simply connected and compact Lie group, and $k \in \mathbb{Z}$ any level (class in $H^3(G)$), there exists the Fréchet-Lie 2-group $\mathrm{String}_k(G)$ coming from the crossed module

$$(\hat{\Omega}_k G \longrightarrow PG) ,$$

where PG is the group of based paths in G and $\hat{\Omega}_k G$ the level-k central extension of based loops in G.

While in a way a much more sophisticated example than the previous one, notice that we may think of this as nothing but the combination of the ordinary group G with a central part that looks like $\Sigma U(1)$, in that we have the exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\hat{\Omega} G \to \Omega G) & \longrightarrow & \mathrm{String}_k(G) & \longrightarrow & (1 \to G) & \longrightarrow & 1 \\ & & & & \Big\downarrow = & & & & \\ 1 & \longrightarrow & (U(1) \to 1) & \longrightarrow & \mathrm{String}_k(G) & \longrightarrow & (\Omega G \to PG) & \longrightarrow & 1 \end{array}$$

of strict 2-groups.

Therefore the above experience with line 2-bundles makes us want to consider "standard" representations of $\hat{\Omega}_k G$ and then construct the corresponding canonical 2-rep

$$\rho : \Sigma \mathrm{String}_k(G) \to \mathrm{Bim} .$$

Since now everything is infinite dimensional, one needs to check a couple of technical things in order to verify that everything does go through as expected.

The "standard" representation of $\hat{\Omega}_k G$ is the corresponding highest-weight representation

$$\rho_1 : \Sigma \hat{\Omega}_k G \to \mathrm{Hilb} .$$

When generating the algebra A_ρ from that, it matters in which sense one completes. The natural choice is to take A to be the von Neumann algebra obtained as the double commutant (in $B(H)$, for H the Hilbert space of the highest weight rep) of $\mathrm{im}(\rho_1)$.

Moreover, since now everything lives in Hilbert spaces, the tensor product of bimodules needs to take care of the necessary completions. The right way to do this is known as Connes fusion of bimodules. We get a 2-category

$$\mathrm{Bim}_{\mathrm{vN}}$$

this way, whose objects are von Neumann algebras and whose morphisms are Hilbert spaces equipped with a bimodule structure, their composition being given by Connes fusion.

This follows Stolz and Teichner in their discussion of String bundles. The question is then if our construction of the canonical 2-rep coming from the highest weight rep ρ_1 goes through in $\mathrm{Bim}_{\mathrm{vN}}$ as it did in Bim.

Indications are that this indeed is the case.

(I have talked to two experts about this, and both indicated that this should work. One thing to notice is that the relation

$$A_q \otimes_A A_{q'} \simeq A_{q' \circ q}$$

familiar from ordinary bimodules remains true in $\mathrm{Bim}_{\mathrm{vN}}$. Combined with the results by Stolz and Teichner on how PG acts on $A = \mathrm{im}(\rho_1)''$ this should imply that the canonical 2-rep on $\mathrm{Bim}_{\mathrm{vN}}$ does exist.)

So let me assume that we do have a 2-representation

$$\rho : \Sigma\mathrm{String}_k(G) \to \mathrm{Bim}_{\mathrm{vN}} .$$

Then a $\mathrm{String}_k(G)$-*asssociated 2-vector bundle* with connection is a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}_{\mathrm{vN}}$$

which admits a smooth local ρ-trivialization for the above ρ.

There are now three statements that deserve to be investigated, which are **conjectures** to varying degree:

- Principal $\mathrm{String}_k(G)$-2-bundles are equivalent to $|\mathrm{String}_k(G)|$-1-bundles, where $|\mathrm{String}_k(G)|$ is the topological String-1-group discussed by Stolz and Teichner.

- Forgetting the connection (restricting tra to constant paths) $\mathrm{String}_k(G)$-asssociated 2-vector bundles are canonically equivalent to the algebra bundles associated to principal String bundles considered by Stolz and Teichner.

- Smoothly locally ρ-trivializabel 2-transport $\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}_{\mathrm{vN}}$ is in fact equivalent to the notion of String connection proposed by Stolz and Teichner.

The first statement is a consequence of the fact [BaezCransStevensonSchreiber] that $|\mathrm{String}_k(G)|$ is indeed a model for the string 1-group and the fact that G_2-2-cocycles are equivalent to $|G_2|$-1-cocycles. The latter proposition has been stated by Jurco. (This statement can therefore probably be regarded as already proven, but notice that Baez and Stevenson are working on writing up a more detailed proof than has appeared so far.)

The second statement should be the analog of the similar situation for line bundle gerbes: there we found that $|G_2| \simeq PU(H)$ is the automorphism group of an algebra $K(H)$, which allowed to canonically associated $K(H)$-bundles to principal $|G_2|$-bundles. Analogously, but somewhat more subtle, Stolz and Teichner realize $|\mathrm{String}_k(G)|$ as the automorphism group of the von Neumann algebra $\mathrm{im}(\rho_1)''$ discussed above, and use this to associate a von-Neumann-algebra bundle with every principal String 1-bundle. So we have two natural ways of associating a von Neumann-algebra bundle to the class of a String bundle and it would be weird if these were not equivalent. (But this remains to be checked in detail.)

Finally, for the third statement I can so far only list the following evidence:

- If the second statment is true, than tra would be the natural notion of connection on a String-bundle, hence one would expect it to be equivalent to any other sensible such notion.

- In fact, also Stolz-Teichner's string connection is a 2-functor from 2-paths to $\mathrm{Bim}_{\mathrm{vN}}$, whose image over points is the algebra fiber of the algebra bundle mentioned in the second statement. So if the second statement is true, then it would seem weird if the two 2-functors in the third wouldn't be closely related.

- There is a curious property of our $\mathrm{String}_k(G)$-2-transport tra which is (somewhat unfortunately) known as **fake flatness**: it implies that when of the image

$$\mathrm{tra} : \; x \underset{\gamma'}{\overset{\gamma}{\Downarrow \Sigma}} y \;\mapsto\; A_x \underset{\mathrm{tra}(\gamma')}{\overset{\mathrm{tra}(\gamma)}{\Downarrow \mathrm{tra}(\Sigma)}} A_y$$

  of a 2-path under tra both $\mathrm{tra}(\gamma)$ and $\mathrm{tra}(\gamma')$ are given, then there is only a circle worth of possible choices for $\mathrm{tra}(\Sigma)$. (Because the kernel of the homomorphism $\hat{\Omega}_k(G) \to PG$ is $U(1)$.)

  Remarkably, the 2-functor considered by Stolz-Teichner enjoys exactly this same proprty (p. 70 of [StolzTeichner:What is an elliptic Object?]).

In summary, if the canonical 2-rep ρ of the String 2-group goes through in $\mathrm{Bim}_{\mathrm{vN}}$ as expected (which has been checked but needs to be spelled out in detail) then ρ-asscociated 2-transport in our sense yields a natural notion of connection on String bundles which, while the definitions differ, has a striking similarity to the proposal by Stolz and Teichner.