1-Bundles classifying 2-Bundles

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Abstract

In [1] a theorem was stated saying, essentially, that a 2-bundle (gerbe) for some 2-group $\mathcal{K}$ is classified by a $|\mathcal{K}|$-1-bundle – much like a G-(1-)bundle is classfied by a $|G|$-0-bundle (function). We here indicate a way to prove this theorem using functorial techniques.

The main point is to understand how transition bundles may descend to transition functions, and how the construction respects gauge transformations on both sides.

Strategy. The construction described in the following is supposed to be an application of the general reasoning of [3]. In particular, we try to make use of the theorem 2 stated there, which, roughly, says that a 2-functor may be reconstructed, up to equivalence, from the transitions between its local trivializations.

In the present context, the 2-functor in question is a 2-functor on the Čech 2-groupoid with values in a strict 2-group. This is the same as a nonabelian Čech 2-cocycle representing a 2-bundle. Over contractible open sets this 2-functor is trivializable, and the transitions between its trivializations define 1-functors from Čech 1-groupoids to the 2-group (now regarded as a 1-category), hence 1-cocycles representing crossed module transition bundles.

For the time being, I assume the reader to be familiar with section 3 of [2] (and possibly with [3]). I'll use the constructions and the notation from section 3.

So I denote by $\check{\mathrm{C}}_2(\mathcal{U})$ the Čech 2-groupoid of some good cover $\mathcal{U}$ of a space X by open sets. $\mathcal{K}$ is any strict 2-group.

Let E_0 be the trivial representative of the trivial 2-bundle, i.e. the 2-functor

$$E_0 : \check{\mathrm{C}}_2(\mathcal{U}) \to \mathcal{K}$$

which sends everything to the identity.

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Observation 1 *Let $E \in [\check{C}_2(\mathcal{U}), \mathcal{K}]$ be any representative of any $\mathcal{K}$-2-bundle on X. Any morphism*

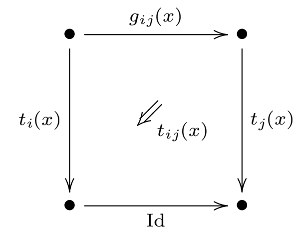
$$E \xrightarrow{t} E_0$$

is strictly invertible.

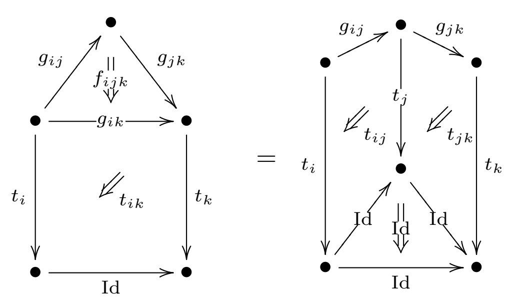
Proof. The pseudonatural transformation t is given by an assignment

$$\mathrm{Mor}_1\left(\check{\mathrm{C}}_2(\mathcal{U})\right) \quad \rightarrow \quad \mathrm{Mor}_2(\mathcal{K})$$

$$\Big((x,i) \longrightarrow (x,j) \Big) \quad \mapsto$$

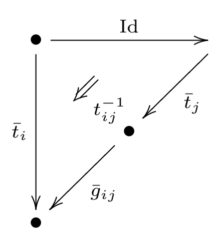


which makes all naturality tin can equations



hold. (The dependence of everything on x is and will be suppressed.)

Now denote by



the strict horizontal inverse of t_{ij}, for all i, j (and x).[1]

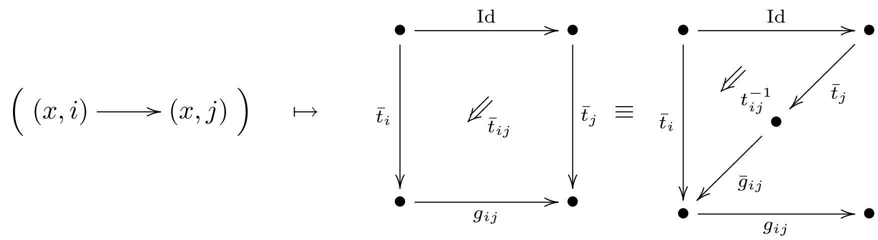
The pseudonatural transformation

$$E_0 \xrightarrow{\bar{t}} E$$

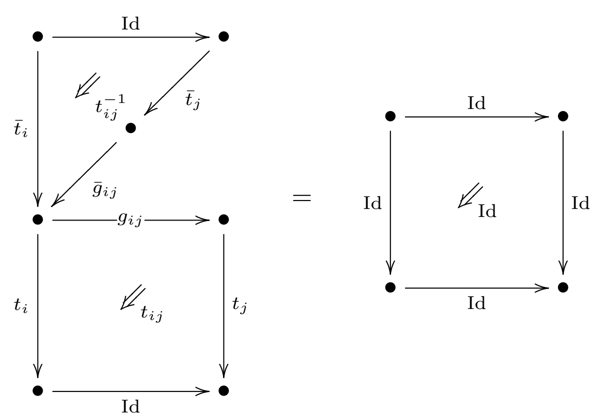
[1]This is not a typo. I am using t_{ij}^{-1} to really mean the *horizontal* inverse, i.e. $t_{ij}^{-1} = \alpha(\bar{t}_j g_{ji}(\bar{t}_{ij}))$ in this case, where overbars denote ordinary inverses of group elements. See example 1, where this is used.

defined by

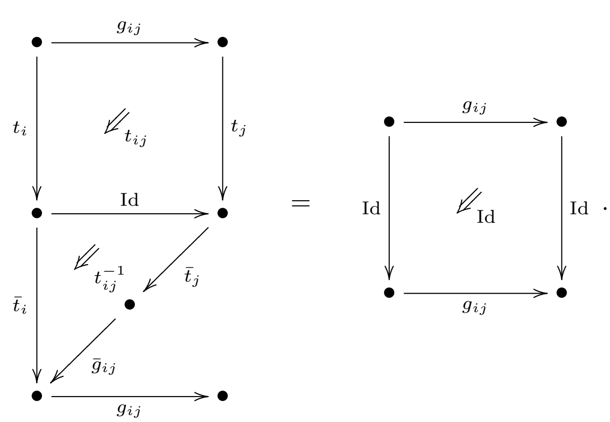
$$\mathrm{Mor}_1\left(\check{\mathrm{C}}_2(\mathcal{U})\right) \quad \to \quad \mathrm{Mor}_2(\mathcal{K})$$



is manifestly the strict inverse of t:



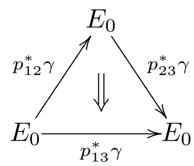
and



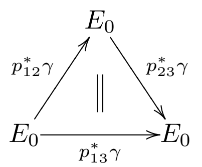
The naturality tin can equation for $\bar{t}$ is satisfied by construction. $\square$

The relevance of this fact for our present purpose is that it implies the following corollary:

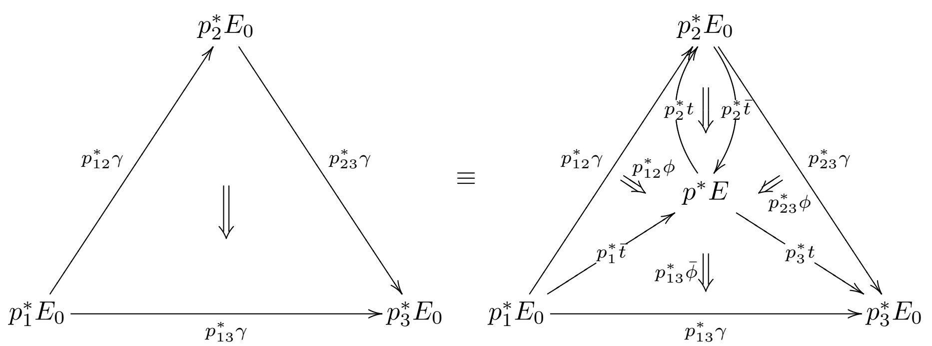
Observation 2 *We can always arrange that (i.e., we can always find a gauge such that) the 2-morphism in*



(on p. 16 of [2]) is the identity:



Proof. By the general construction (def. 4 of [3]) this 2-morphism is given by



The 2-morphism ϕ appearing here may, by construction, always be chosen to be the identity. Observation 2 tells us that also the 2-morphism on the top on the right hand side may be chosen to be the identity in our present context. Hence the 2-morphism on the left may be chosen to be the identity. □

In order to appreciate what this means, first recall observation 7 from [2], which says that automorphisms

$$E_0 \xrightarrow{\gamma} E_0$$

of the trivial representative of the trivial $\mathcal{K}$ 2-bundle are in bijection with representatives of $\mathcal{K}$ crossed module bundles,[2] simply because any such pseudonatural

[2]For the reader more comfortable with the language of (nonabelian) Čech cohomology we point out that this is equivalent to saying that Čech coboundaries from the trivial Čech 2-cocycle to itself are in bijection with Čech 1-cocycles corresponding to crossed module bundles. However, the main point we are trying to make here, while given by rather elementary reasoning in the diagrammtic 2-group-language given above, becomes, to my mind at least, rather obscure in the language of Čech cocycles with values in crossed modules.

transformation comes from an assignment

$$
\begin{array}{ccc}
\mathrm{Mor}_1\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right) & \to & \mathrm{Mor}_2\left(\mathcal{K}\right) \\
\left((x,i) \longrightarrow (x,j) \right) & \mapsto & \begin{array}{ccc} \bullet & \xrightarrow{\mathrm{Id}} & \bullet \\ {\scriptstyle h_i}\downarrow & \swarrow_{h_{ij}} & \downarrow{\scriptstyle h_j} \\ \bullet & \xrightarrow[\mathrm{Id}]{} & \bullet \end{array}
\end{array} ,
$$

satisfying a condition which says that the h_{ij} (or rather their inverses, if you like), are local cocycles defining a crossed module bundle. This becomes more manifest once we pass from 2-categories (2-groupoids in our case) with a single object back to monoidal (group-like, in our case) 1-categories. Then the above becomes simply a functor sending $(x,i) \longrightarrow (x,j)$ to $h_i \xleftarrow{h_{ij}} h_j$, as befits a transition of a crossed module bundle.

Finally, let us make the following observation explicit:

Observation 3 *Given automorphisms* $E_0 \xrightarrow{\gamma_1} E_0$ *and* $E_0 \xrightarrow{\gamma_2} E_0$ *of the trivial representative of the trivial 2-bundle, both corresponding to cocycles of crossed module bundles, as described above, their composition*

$$E_0 \xrightarrow{\gamma_1} E_0 \xrightarrow{\gamma_2} E_0$$

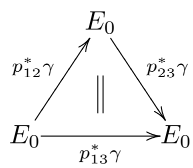
corresponds to the cocycle of the product of these two crossed module bundles.

Proof. It's a trivial statement in terms of diagrams. Composition of the two pseudonatural transformations corresponds to the pseudonatural transformation given by the assignment

$$
\left((x,i) \longrightarrow (x,j) \right) \quad \mapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{\mathrm{Id}} & \bullet \\ {\scriptstyle (h_1)_i}\downarrow & \swarrow_{(h_1)_{ij}} & \downarrow{\scriptstyle (h_1)_j} \\ \bullet & \xrightarrow{\mathrm{Id}} & \bullet \\ {\scriptstyle (h_2)_i}\downarrow & \swarrow_{(h_2)_{ij}} & \downarrow{\scriptstyle (h_2)_j} \\ \bullet & \xrightarrow[\mathrm{Id}]{} & \bullet \end{array} .
$$

This encodes precisely the product of transition data of crossed module bundles. (Crossed module bundles and their products are essentially *defined* by this.) □

All in all, we hence find that the identity 2-morphism



from above says nothing but that

$$H^{rs} \cdot H^{st} = H^{rt} ,$$

where H^{rs} is the cocycle representing our crossed module transition bundle on patch Y_{rj} (we choose to let $Y \to X$ be a good cover of X by open sets Y_{rj}) and where the product on the left is the product of (cocycles for) crossed module bundles.

Now, the main point is that each crossed module bundle cocycle H^{rs} corresponds (by taking the nerve of the underlying functor) to a choice of classifying function

$$q_{rs} : Y_{rs} \to |\mathcal{K}| .$$

Unless I am missing something, the above equation implies that these classifying functions are $|\mathcal{K}|$-1-cocycles:

$$q_{rs} q_{st} = q_{rt} .$$

Hence they define a $|\mathcal{K}|$-1-bundle on X, as desired.

I think by running the same resoning backwards one finds that every (cocycle for a) $|\mathcal{K}|$-1-bundle on X defines a (cocycle for a) $\mathcal{K}$-2-bundle on X.

It remains to be shown that gauge transformations of $|\mathcal{K}|$-1-bundle cocycles corresponds to those of their corresponding 2-bundle cocycles.

Observation 4 *Under the map between $\mathcal{K}$-2-bundles and $|\mathcal{K}|$-1-bundles ordinary gauge transformations on the 1-bundle side are in bijection with gauge transformation on the 2-bundle side which respect the special gauge in which map is constructed (namely the gauge where t and $\bar{t}$ are mutual strict inverses.)*

Proof. Consider two trivializations of our 2-bundle,

$$p^*E \xrightarrow{t} E_0$$

and

$$p^*E \xrightarrow{t'} E_0$$

together with their strict inverses

$$E_0 \xrightarrow{\bar{t}} p^*E$$

and

$$E_0 \xrightarrow{\bar{t}'} p^*E \ .$$

Due to the assumption of strict invertibility of the trivializations, we have morphisms

$$E_0 \xrightarrow{\rho} E_0$$

given by

$$\begin{array}{ccc} p_1^* E_0 & \xrightarrow{\gamma'} & p_2^* E_0 \\ {\scriptstyle p_1^*\rho}\downarrow & & \downarrow{\scriptstyle p_2^*\rho} \\ p_1^* E_0 & \xrightarrow[\gamma]{} & p_2^* E_0 \end{array} \quad \equiv \quad \begin{array}{ccccc} p_1^* E_0 & & \xrightarrow{\gamma'} & & p_2^* E_0 \\ & \searrow^{p_1^*\bar{t}'} & & \swarrow^{p_2^*\bar{t}'} & \\ & & p_1^* p^* E & & \\ & \swarrow_{p_1^* t} & & \searrow_{p_2^* t} & \\ p_1^* E_0 & & \xrightarrow[\gamma]{} & & p_2^* E_0 \end{array} \,.$$

On the left we have a diagram purely in automorphisms of the trivial representative of the trivial 2-bundle. Hence every arrow there defines a $\mathcal{K}$ crossed module bundle, with composition of arrows being the product of these representatives. After passing to realizations of nerves we hence obtain

$$\gamma'_{rs} \cdot \rho_s = \rho_r \cdot \gamma_{rs} \,.$$

This is precisely the coboundary relation for $|\mathcal{K}|$-1-bundles. Conversely, by running this argument backwards we find that every gauge transformation on the $|\mathcal{K}|$-1-bundle side corresponds to a gauge transformation on the $\mathcal{K}$-2-bundle side, preserving the condition that trivializations come together with their strict inverses. □

Example 1

Let's choose $Y \equiv \mathcal{U}$. Then we have to trivialize the Čech 2-cocycle $E : \check{\mathrm{C}}_2(\mathcal{U}) \to \mathcal{K}$ over every open patch U_r of $\mathcal{U}$.

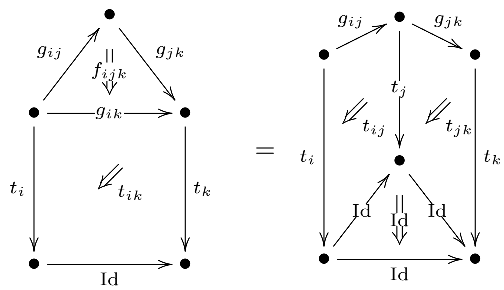
A canonical way to do so is by choosing the trivialization morphism

$$E|_r \xrightarrow{t_r} E_0$$

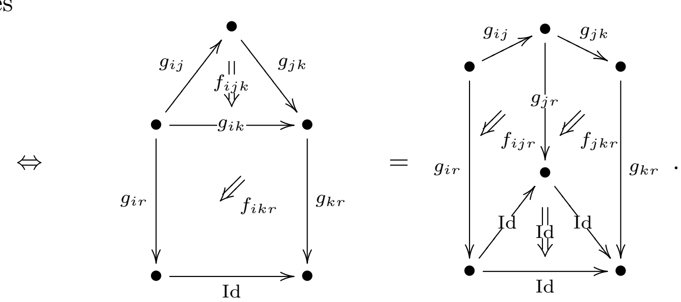
such that it is given by the assigment

$$\begin{array}{ccc} \mathrm{Mor}_1\left(\check{\mathrm{C}}_2(\mathcal{U}|U_r)\right) & \to & \mathrm{Mor}_2(\mathcal{K}) \\ \Big(((x,i),r) \longrightarrow ((x,j),r) \Big) & \mapsto & \begin{array}{ccc} \bullet & \xrightarrow{g_{ij}(x)} & \bullet \\ {\scriptstyle (t_r)_i(x)}\downarrow & \swArrow_{(t_r)_{ij}(x)} & \downarrow{\scriptstyle (t_r)_j(x)} \\ \bullet & \xrightarrow[\mathrm{Id}]{} & \bullet \end{array} \equiv \begin{array}{ccc} \bullet & \xrightarrow{g_{ij}(x)} & \bullet \\ {\scriptstyle g_{ir}(x)}\downarrow & \swArrow_{f_{ijr}(x)} & \downarrow{\scriptstyle g_{jr}(x)} \\ \bullet & \xrightarrow[\mathrm{Id}]{} & \bullet \end{array} \,, \end{array}$$

where the point is that on the right we have defined t_r in terms of the cocycle data available on U_{ijr}. With this definition the required naturality tin can equation



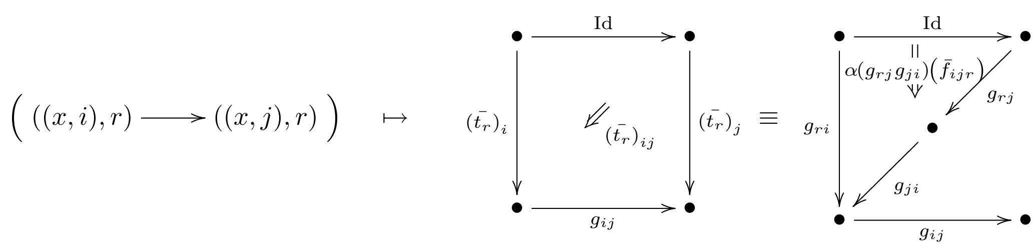
becomes



By staring at this for a second one realizes that this is nothing but the tetrahedron equation which is indeed satisfied by $f_{\dots}$. Therefore our t_r is indeed a pseudonatural transformation locally trivializing E.

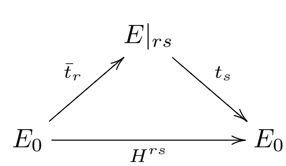
In order to find the transitions that we are after, we need to strictly invert t_r for each r. From observation 1 we know that we have to choose the assignment

$$\mathrm{Mor}_1\left(\check{\mathrm{C}}_2\left(\mathcal{U}|U_r\right)\right) \quad \to \quad \mathrm{Mor}_2\left(\mathcal{K}\right)$$



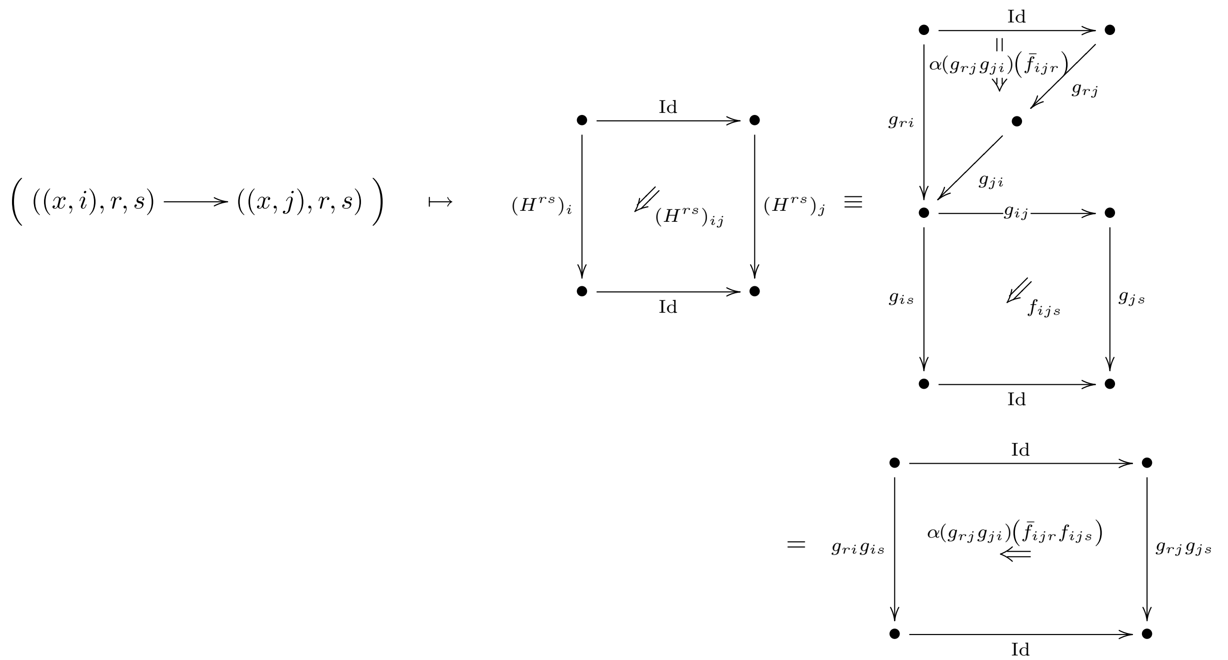
We assume that we have arranged that $g_{ij} = (g_{ji})^{-1}$ for all i, j and we indicate the inverse of morphisms and group elements by overbars.

By composing the 2-morphisms that we have constructed this way we find the transition



to be given by the assignment

$$\mathrm{Mor}_1\left(\check{\mathrm{C}}_2\left(\mathcal{U}|U_{rs}\right)\right) \quad\to\quad \mathrm{Mor}_2\left(\mathcal{K}\right)$$



This assignment is the same as a 1-functor

$$\begin{array}{ccc} \check{\mathrm{C}}_1\left(\mathcal{U}|U_{rs}\right)^{\mathrm{op}} & \to & \mathcal{K} \\ & & \\ ((x,i),r,s) & & g_{ri}g_{is} \\ \Big\downarrow & \mapsto & \Big\uparrow \alpha(g_{rj}g_{ji})\left(\bar{f}_{ijr}f_{ijs}\right) \, . \\ ((x,j),r,s) & & g_{rj}g_{js} \end{array}$$

This happens to be contravariant, due to the nature of pseudonatural transformations, but since everything is invertible this can equally be regarded as a covariant functor

$$\check{\mathrm{C}}_1\left(\mathcal{U}|U_{rs}\right) \quad\to\quad \mathcal{K}$$

$$
\begin{array}{ccc}
((x,i),r,s) & & g_{ri}g_{is} \\
\Big\downarrow & \mapsto & \Big\downarrow{\scriptstyle \alpha(g_{rj}g_{ji})\left(\bar{f}_{ijs}f_{ijr}\right)} \\
((x,j),r,s) & & g_{rj}g_{js}
\end{array} .
$$

(Notice that on the right we have now the inverse morphism.)

You might want to recall observation 7 in [2], and it particular its proof on p. 16 of [2], for why we indeed have functoriality here.

References

[1] B. Jurčo, *Crossed Module Bundle Gerbes; Classification, String Group and Differential Geometry*, math.DG/0510078

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