notes taken in a talk by
W. v. Suijlecom

Representation of Feynman graphs on Gerstenhaber algebras at MPI, Bonn, 1st July 2008, conference: *The manifold geometries of QFT.*

aim: connect Feynman graphs to Gerstenhaber algebras

Contents

0.1 Feynman graphs, Hopf algebras

Feynman grapghs are built from certain types of edges and vertices

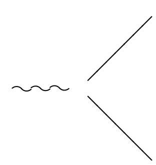
photon edge (propagator) ~~~~~~~~

electron edge (propagator) ————

label the set of possible edges by

$$\{e_i\}_i$$

interaction vertex



label the set of possible interaction vertices by

$$\{v_i\}_i$$

Examples: (QED)

[the usual example diagrams]

Example (QCD)

[the usual example diagrams]

Restrict attention to "1PI"= 1-particle irreducible graphs: those graphs which cannot be cut in two by cutting a single edge.

Residue

$$\mathrm{res}(\Gamma)$$

of a graph: remember only the outer edges and regard the entire graph as a single vertex (this gives in general vertices not of the elementary form, but these won't appear later on)

Connes and Kreimer found a nice structure on these graphs, the Connes-Kreimer Hopf algebra:

Definition 1 (KC) *H is the commutative algebra generated over* $\mathbb{C}$ *by 1PI graphs with coproduct*

$$\Delta : H \to H \otimes H$$
$$\Delta : \Gamma \mapsto \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \,,$$

where the sum is over disjoint unions of 1PI graphs with residue being an elementary vertex.

0.2 Structure of Hopf algebras

all Hopf algebras are commutative hence all dual to some group

Hopf algebra: characters form a group

a character is an algebra homomorphism

$$g \in \mathrm{Hom}(H, \mathbb{C})$$

these form a group with the convolution product

$$g \star g'(h) = \langle g \otimes g', \Delta(h) \rangle$$

our H is a graded Hopf algebra in two ways:

- loop order
$$H = \bigoplus_{l=0}^{\infty} H^l$$
- number of vertices
$$H = \bigoplus_{n_1, \cdots, n_k} H^{(n_1, \cdots, n_k)}$$

define Green functions:

$$G^{v_i} := 1 + \sum_{\mathrm{res}(\Gamma) = v_i} \frac{\Gamma}{|Aut(\Gamma)|}$$

$$G^{e_j} := 1 - \sum_{\mathrm{res}(\Gamma) = v_i} \frac{\Gamma}{|Aut(\Gamma)|}$$

These Green functions do not by themselves generate a Hopf subalgebra but we can project on the graded parts

$$Y_{v_i} = \frac{G^{v_i}}{\prod_{j=1}^{N} (G^e)^{N_j(v_i)/2}}$$

Theorem 1 *The elements* Y_{v_i} *and* G^{e_j} *do generate a Hopf subalgebra* H' *with dual group*

$$\mathrm{Hom}(H', \mathbb{C}) \simeq \left(\mathbb{C}[[\lambda_1, \cdots, \lambda_k]]^\times \rtimes \mathrm{Diff}(\mathbb{C}^k, 0) \right) \,.$$

Theorem 2 *The ideal* $J = \langle Y_{v_i}^{N(v_j)-2} - Y_{v_j}^{N(v_i)-2} \rangle$ *in* H' *is a Hopf ideal*

$$\mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \simeq \left(\mathbb{C}[[\lambda]]^\times \right)^N \rtimes \mathrm{Diff}(\mathbb{C}, 0) \,.$$

0.3 BV-formalism (Gerstenhaber algebra)

Batalin-Vilkovisky formalism in Yang-Mills gauge theories
gauge group with Lie algebra $\mathfrak{g}$, generators T^a
bigraded vector space with basis
gauge field $A \in \Omega^1 \otimes \mathfrak{g}$, $A = A^a_\mu dx^\mu \otimes T_a$
add new field:
ghost field: $\omega \in \Omega^0 \otimes \mathfrak{g}[-1]$
$\bar{\omega} \in \Omega^0 \otimes \mathfrak{g}[-1]$
auxiliary field $h \in \Omega^0 \otimes \mathfrak{g}$
these fields constitute a section of a bundle

$$E = \Lambda^1(\mathfrak{g}) \oplus \Lambda^0(\mathfrak{g}[-1]) \oplus \Lambda^0(\mathfrak{g}[1]) \oplus \Lambda^0(\mathfrak{g})$$

antifields:
$A^\ddagger, \omega^\ddagger, \bar{\omega}^\ddagger, h^\ddagger$ section of

$$E^\ddagger = \Lambda^1(\mathfrak{g}[1]) \oplus \Lambda^0(\mathfrak{g}) \oplus \Lambda^0(\mathfrak{g}[2]) \oplus \Lambda^0(\mathfrak{g}[1])$$

$\mathrm{deg}\phi^\ddagger = -\mathrm{deg}\phi - 1$
anti-bracket: degree 1 bracket defined on generators to be

$$(\phi^a(x), \phi^\ddagger_b(y)) = \delta^a_b \delta(x - y)$$

and zero otherwise

Definition 2 *Local functionals are integrals of polynomials in fields, antifields and their derivatives.*

$$F = F(E \oplus E^\ddagger)$$

Example: Yang-Mills action

$$S_{\mathrm{YM}} = \int \mathrm{tr}\,(-F_A \wedge \star F_A)$$

$$F_A := dA + \frac{g}{2}[A \wedge A]$$

invariant under $A \to dX + g[A, X]$, $X \in \Omega^0 \otimes \mathfrak{g}$

Gauge fixing:

$$S_{\mathrm{gf}} = S_{\mathrm{gf}}(A, \omega, \bar{\omega}, h)$$

Gauge invariance $\to$ BRST invariance of

$$S_{\mathrm{YM}} + S_{\mathrm{gf}}$$

$$s\,(S_{\mathrm{YM}} + S_{\mathrm{gf}}) = 0$$

Another way to write this is

$$(S, S) = 0$$

where now

$$S = S_{\mathrm{YM}} + S_{\mathrm{gf}} + \sum_{a=1}^{\mathrm{rk} f} \int \mathrm{tr}((s\phi_a) \star \phi_a^{\ddagger})$$

Comodule Gerstenhaber algebras we'll construct comodules over our Hopf algebra:

to each vertex assign

$$v_i \mapsto \lambda_i$$

λ_i called a coupling constant $i \in \{1, \cdots, k\}$

$$e_j \mapsto \phi_j, \phi_j^{\ddagger}$$

(field, antifield)

then the antibracket

$$(\phi_i(x), \phi_j^{\ddagger}(y)) = \delta_{ij}\delta(x - y)$$

makes

$$A := F([\phi_1, \phi_1^{\ddagger}, \cdots]) \otimes \mathbb{C}[[\lambda_1, \cdots, \lambda_k]]$$

a Gerstenhaber algebra (graded algebra with Lie bracket of degree 1)

Proposition 1 *A is a Gerstenhaber algebra comodule over H*

$$e : H \to A \otimes H$$

Consequences: the characters $\mathrm{Hom}(H', \mathbb{C})$ act on A

Consider $S \in A$

$$S = \sum_{j=1}^{N} \int dx\ \bar{\phi}_j(x)(D_j\phi_j\phi_j)(x) + \sum_{i=1}^{k} \lambda_i \int dx\ m_i(v_i)(x)$$

(recall that representation of group is same as comodule of its Hopf algebra)

Consider the ideal $I = \langle (S, S) \rangle$

$$I = \langle \lambda_i - g^{N(v_i)-2} \rangle$$

where $g = \lambda_j$ with $\mathrm{val}(v_j) = 3$

form quotient A/I

G the group of characters

Theorem 3

$$G^I \subset G$$

is dual to H'/J

$$G^I = (\mathbb{C}[[f]]]^\times)^N \rtimes \mathrm{Diff}(\mathbb{C}, 0)$$

discussion

I asked: so what is the main point in words? and then proposed the following summary: from Connes-Kreimer it follows that Feynman graphs give a Hopf algebra and that one can form that ideal A/I. The question is what forming that ideal means physically. The answer here is: it corresponds to restriction to the case that $(S, S) = 0$, i.e. to imposing the master BV-equation.

Walter Suijlekom: yes, essentially [or so, no guarantee]